Quenched mass transport of particles towards a target

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Abstract

We consider the stochastic target problem of finding the collection of initial laws of a mean-field stochastic differential equation such that we can control its evolution to ensure that it reaches a prescribed set of terminal probability distributions, at a fixed time horizon. Here, laws are considered conditionally to the path of the Brownian motion that drives the system. We establish a version of the geometric dynamic programming principle for the associated reachability sets and prove that the corresponding value function is a viscosity solution of a geometric partial differential equation. This provides a characterization of the initial masses that can be almost-surely transported towards a given target, along the paths of a stochastic differential equation. Our results extend [16] to our setting.

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1 Introduction

Stochastic target problems are optimization problems in which the controller looks for the values x of a state process $X^{t,x,\nu}$ at time t, so that it can reach some given set K at a given terminal time T, by choosing an appropriate control ν. Namely, the objective is to characterize the reachability sets

$$V(t) = \Big\{x \in \mathbb{R}^d : \quad X_T^{t,x,\nu} \in K \text{ for some admissible control } \nu\Big\} \tag{1.1}$$

for $t \in [0, T]$. Such optimization problems were first studied in [17] and [16] in which the function $v(t, x) = 1 - \mathbb{1}_{V(t)}(x)$ is shown to solve a Hamilton-Jacobi-Bellman equation, in the viscosity solution sense. The main motivation of [16, 17] is the so-called super-replication problem, in financial mathematics: the controller looks for possible initial endowments such that there exists an investment strategy allowing the terminal wealth to satisfy a super-hedging constraint, almost-surely (see e.g. [8]). But, the range of applications is obviously much wider.

Another important type of stochastic target problems concerns the case where the terminal constraint is imposed on the mean value of a function of the controlled process. In this case the reachability sets take the following form:

$$V_\ell(t) = \Big\{x \in \mathbb{R}^d : \quad \mathbb{E}[\ell(X_T^{t,x,\nu})] \geq 0 \text{ for some admissible control } \nu\Big\}, \tag{1.2}$$

for $t \in [0, T]$. This type of constraints is also common in financial applications. Indeed, the super-replication price is usually too high to be accepted by buyers. This is a motivation for relaxing the a.s. super-hedging criteria by only asking that $X^{t,x,\nu} \in K$ holds, for instance, with a (high) probability $p < 1$. In this case, the function ℓ takes the form $\ell(x) = \mathbb{1}_K(x) - p$. For $p = 1$, one retrieves (1.1). This approach was introduced in [9] and further developped in [3] where the authors take advantage of the martingale representation theorem to transform the constraint given in terms of the mean value into an almost-sure constraint.

One of the motivations of this paper is to study the stochastic target problem (1.2) in the case of a mean-field (or McKean-Vlasov) controlled diffusion:

$$X_s^{t,\chi,\nu} = \chi + \int_t^s b_u(X_u^{t,\chi,\nu}, \mathbb{P}_{X_u^{t,\chi,\nu}}, \nu_u)du + \int_t^s \sigma_u(X_u^{t,\chi,\nu}, \mathbb{P}_{X_u^{t,\chi,\nu}}, \nu_u)dB_u,$$

where $\mathbb{P}_{X_u^{t,\chi,\nu}}$ is the marginal law of $X_u^{t,\chi,\nu}$ under $\mathbb{P}$, B is a standard Brownian motion and χ is an independent random variable whose distribution can be interpreted as the initial repartition of a population. This type of stochastic target problems can be embedded into a more general class of problems involving the conditional laws given the Brownian path. Indeed, using the martingale representation theorem as in [3], the constraint in (1.2) can be rewritten as

$$\mathbb{E}_B[\ell(X_T^{t,\chi,\nu})] - \int_t^T \alpha_s dB_s \geq 0 \text{ for some control } \nu \text{ and } \alpha ,$$

where $\mathbb{E}_B$ denotes the conditional expectation given B. In particular, if we define the control $\bar{\nu} = (\nu, \alpha)$ and the controlled process $\bar{X}^{t,(\chi,0),\bar{\nu}} = (X^{t,\chi,\nu}, \int_t^\cdot \alpha dB)$, this reads

$$L\big(\mathbb{P}^B_{\bar{X}_T^{t,(\chi,0),\bar{\nu}}}\big) \geq 0 \text{ for some control } \bar{\nu},$$

in which $\mathbb{P}^B_\zeta$ denotes the conditional law of a random variable ζ given B, and

$$L(\mu) \;=\; \int_{\mathbb{R}^d\times\mathbb{R}} (\ell(x) - y)\mu(dx, dy).$$

These considerations suggest to study a general constraint:

$$\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G \text{ for some admissible control } \nu,$$

in which $X^{t,\chi,\nu}$ is now defined by

$$X_s^{t,\chi,\nu} = \chi + \int_t^s b_u(X_u^{t,\chi,\nu}, \mathbb{P}^B_{X_u^{t,\chi,\nu}}, \nu_u)du + \int_t^s \sigma_u(X_u^{t,\chi,\nu}, \mathbb{P}^B_{X_u^{t,\chi,\nu}}, \nu_u)dB_u, \quad (1.3)$$

G is a Borel subset of probability measures and χ is the (random) initial position.

This general formulation is of importance on its own right as it is related to the probabilistic analysis of large scale particle systems, e.g. polymers in random media, in which one is interested in the behavior of particles conditionally on the environment. This is also known as 'quenched' behaviors/properties (quenched law of large numbers, quenched large deviations etc.), which is in general different from the so-called 'annealed' behaviors obtained by averaging over the underlying random environment (see e.g. [2, 10, 14] and the references therein). For diffusion processes, quenching boils down to making the drift and diffusion coefficients dependent on the conditional marginal law given the environment, while annealing corresponds to the case where the coefficients depend on the unconditional marginal law (see e.g. [14]). We therefore coin the term quenched diffusion instead of conditional diffusion to refer to SDEs of the form (1.3). For our stochastic target problem, the constraint $\mathbb{P}^B_{X_T} \in G$ imposed on the conditional law of the diffusion process is a quenched property for the underlying process.

One can also further identify the inital condition χ as a law μ. Then, our problem can be interpreted as a transport problem. What are the collection of initial distributions μ of a population of particles, that all have the same dynamics, such that the terminal repartition $\mathbb{P}^B_{X_T^{t,\chi,\nu}}$, given the environnement modelled by the Brownian path B, satisfies a certain constraint ? This amounts to asking what kind of masses can be transported along the SDE so as to reach a certain set, almost-surely, at T:

$$\mathcal{V}(t) = \Big\{\mu : \ \exists(\chi,\nu) \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\}. \tag{1.4}$$

The rest of the paper is organized as follows. In Section 2, we describe in details the quenched controlled diffusion. We provide some (expected) existence and stability results, together with a conditioning property. Section 3 is devoted to the detailed presentation of the quenched stochastic target problem (1.4). We prove that it admits a geometric dynamic programming principle. This is the main result of the paper. Then, one can combine the technologies developped in [4, 6] and [16] to derive in Section 4 the associated Hamilton-Jacobi-Bellman equation, which extends the main result of [16] to our context. In Section 5, we comment on the choice of the class of controls, and provide an interpretation in terms of control of the law of a population of particles.

2 Quenched mean-field SDE

We first describe our probabilistic setting. The d-dimensional Brownian motion is constructed on the canonical space in a usual way. More precisely, given a fixed time horizon $T > 0$, we let Ω° denote the space of continuous $\mathbb{R}^d$-valued functions on $[0,T]$, starting at 0, and let $\mathbb{F}^\circ = (\mathcal{F}^\circ_t)_{t\le T}$ denote the filtration generated by the canonical process $B(\omega^\circ) := \omega^\circ$, $\omega^\circ \in \Omega^\circ$. We set $\mathcal{F}^\circ = \mathcal{F}^\circ_T$ and we endow $(\Omega^\circ, \mathcal{F}^\circ)$ with the Wiener measure $\mathbb{P}^\circ$. Later on, $\bar{\mathbb{F}}^\circ = (\bar{\mathcal{F}}^\circ_t)_{t\le T}$ will denote the $\mathbb{P}^\circ$-completion of $\mathbb{F}^\circ$.

In order to model the initial repartition of the population, we let $\Omega^1 := [0,1]^d$ be endowed with its Borel σ-algebra $\mathcal{F}^1 := \mathcal{B}([0,1]^d)$ and the Lebegues measure $\mathbb{P}^1$. It supports the $[0,1]^d$-uniformly distributed random variable $\xi(\omega^1) = \omega^1$, $\omega^1 \in \Omega^1$. We then define the product filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ by setting $\Omega := \Omega^\circ \times \Omega^1$, $\mathbb{P} = \mathbb{P}^\circ \otimes \mathbb{P}^1$, $\mathcal{F} = \mathcal{F}_T$ where $\mathbb{F} = (\mathcal{F}_t)_{t\le T}$ is the completion of $(\mathcal{F}^\circ_t \otimes \mathcal{F}^1)_{t\le T}$. From now on, any identity involving random variables has to be taken in $\mathbb{P}$-a.s. sens. We canonically extend the random variable ξ and the process B on Ω by setting $\xi(\omega) = \xi(\omega^1)$ and $B(\omega) = B(\omega^\circ)$ for any $\omega = (\omega^\circ, \omega^1) \in \Omega$. We still denote by $\mathbb{F}^\circ$ the filtration generated by the extended process B on Ω. Note that it follows from [12, Theorem 6.15 and Proposition 7.7] applied to the process $(t,\omega) \in [0,T]\times\Omega \mapsto \xi(\omega) + B_t(\omega)$ that $\mathbb{F}$ is right continuous.

Given a random variable $Y \in \mathbf{L}_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ (resp. $Y \in \mathbf{L}_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$), we let $\mathbb{P}^B_Y$ (resp. $\mathbb{E}_B[Y]$) denote a regular conditional law (resp. expectation) under $\mathbb{P}$ of the random variable Y given $(B_t)_{t\le T}$ on $\mathbb{R}^d$. In particular we have the following identifications

$$
\begin{aligned}
\mathbb{P}^B_Y(A,\omega) &= \mathbb{P}^1_{Y(\omega^\circ,.)}(A) && (2.5)\\
\mathbb{E}_B\big[Y\big](\omega) &= \mathbb{E}^1\big[Y(\omega^\circ,.)\big] && (2.6)
\end{aligned}
$$

for any $\omega = (\omega^\circ, \omega^1) \in \Omega$ and any $A \in \mathcal{B}(\mathbb{R}^d)$. Here, $\mathbb{E}^1$ denotes the expectation under $\mathbb{P}^1$ and $\mathbb{P}^1_{Y(\omega^\circ,.)}$ denotes the law under $\mathbb{P}^1$ of the random variable defined on Ω^1 by $Y(\omega^\circ,.)(\omega^1) = Y(\omega^\circ,\omega^1)$. We let $\mathcal{P}(\mathrm{S})$ denote the space of probability measures on a Borel space $(\mathrm{S},\mathcal{B}(\mathrm{S}))$, and define

$$\mathcal{P}_2 \ := \ \left\{\mu \in \mathcal{P}(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)) \text{ s.t. } \int_{\mathbb{R}^d} |x|^2\mu(dx) < +\infty\right\} .$$

In the above, $|x|$ is the Euclidean norm of x. This space is endowed with the 2-Wasserstein distance defined by

$$\mathcal{W}_2(\mu,\mu') := \inf\Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\pi(dy,dy) \ : \ \pi \in \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^d)) \\ \text{s.t. } \ \pi(\cdot\times\mathbb{R}^d) = \mu \text{ and } \pi(\mathbb{R}^d\times\cdot) = \mu'\Big\}^{\frac{1}{2}} ,$$

for $\mu,\mu' \in \mathcal{P}_2$. For later use, we also define the collection $\mathcal{P}_2^{\bar{\mathbb{F}}^\circ}$ of $\bar{\mathbb{F}}^\circ$-adapted continuous $\mathcal{P}_2$-valued processes.

Let now U be a closed subset of $\mathbb{R}^q$ for some $q \geq 1$ and denote by $\mathcal{U}$ the collection of U-valued $\mathbb{F}$-progressive processes. This will be the set of controls. Let $\bar{\mathcal{T}}^\circ$ denote the set of $[0,T]$-valued $\bar{\mathbb{F}}^\circ$-stopping times. Given $\theta \in \bar{\mathcal{T}}^\circ$ and $\chi \in \mathbf{X}^2_\theta := \mathbf{L}^2(\Omega,\mathcal{F}_\theta,\mathbb{P};\mathbb{R}^d)$, $\nu \in \mathcal{U}$, and $(b,a) : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2\times\mathrm{U} \mapsto \mathbb{R}^d\times\mathbb{R}^{d\times d}$, we let $X^{\theta,\chi,\nu}$ denote the solution of

$$X = \mathbb{E}[\chi|\mathcal{F}_{\theta\wedge\cdot}] + \int_\theta^{\theta\vee\cdot} b_s\big(X_s,\mathbb{P}^B_{X_s},\nu_s\big)ds + \int_\theta^{\theta\vee\cdot} a_s\big(X_s,\mathbb{P}^B_{X_s},\nu_s\big)dB_s, \quad (2.7)$$

in which (b,a) is assumed to be continuous, bounded and satisfy:

(H1) There exists a constant L such that

$$|b_t(x,\mu,\cdot) - b_t(x',\mu',\cdot)| + |a_t(x,\mu,\cdot) - a_t(x',\mu',\cdot)| \leq L\Big(|x-x'| + \mathcal{W}_2(\mu,\mu')\Big)$$

for all $t \in [0,T]$, $x,x' \in \mathbb{R}^d$ and $\mu,\mu' \in \mathcal{P}_2$.

The term $\mathbb{E}[\chi|\mathcal{F}_{\theta\wedge\cdot}]$ in (2.7) allows to define X as a continuous adapted process on $[0,T]$, which is done for convinience of notations. One could obviously only consider the process on $[\![\theta,T]\!]$.

Remark 2.1. *Note that the controls can depend on the initial value of χ. One could also restrict to $\bar{\mathbb{F}}^\circ$-progressive processes, see Section 5 for a discussion.*

The above condition ensures as usual that a unique strong solution to (2.7) can indeed be defined.

Proposition 2.1. *For all* $\theta \in \bar{\mathcal{T}}^\circ$, $\nu \in \mathcal{U}$ *and* $\chi \in \mathbf{X}^2_\theta$, (2.7) *admits a unique strong solution* $X^{\theta,\chi,\nu}$, *and it satisfies*

$$\mathbb{E}\Big[\sup_{[0,T]} |X^{\theta,\chi,\nu}|^2\Big] < +\infty\ . \tag{2.8}$$

Moreover, for all $(t,\chi,\nu) \in [0,T] \times \mathbf{X}^2_t \times \mathcal{U}$, *if* $t_n \to t$, $\chi_n \to \chi$ *in* $\mathbf{L}_2$ *with* $\chi_n \in \mathbf{X}^2_{t_n}$ *for all* n, *and* $(\nu^n)_n \subset \mathcal{U}$ *converges to* ν $dt \times d\mathbb{P}$*-a.e., then*

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{W}_2(\mathbb{P}^B_{X_T^{t_n,\chi_n,\nu^n}}, \mathbb{P}^B_{X_T^{t,\chi,\nu}})^2] = 0. \tag{2.9}$$

Proof. 1. The estimate (2.8) is a consequence of the boundedness of (b,a).
2. Existence follows from a similar fixed point argument as in [11] (see also [18]). Since we work in a slightly different context, we provide the proof for completeness.
2.a. Let $\mathbb{C}$ denote the space of continuous $\mathbb{R}^d$-valued maps on $[0,T]$ endowed with the sup-norm topology. For $\hat{Q}, \hat{P} \in \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and $t \le T$, we set

$$\begin{aligned} D_t(\hat{P},\hat{Q}) := \inf \Big\{ \int_{\mathbb{C}\times\mathbb{C}} \sup_{0\le s\le t} |Y_s^{\hat{P}} - Y_s^{\hat{Q}}|^2\ \hat{R}(dY_s^{\hat{P}}, dY_s^{\hat{Q}}) :&\\ \hat{R} \in \mathcal{P}_2(\mathbb{C}\times\mathbb{C}, \mathcal{B}(\mathbb{C}\times\mathbb{C}))&\\ \text{s.t. } \ \hat{R}(\cdot\times\mathbb{C}) = \hat{P} \text{ and } \hat{R}(\mathbb{C}\times\cdot) = \hat{Q} \Big\}^{\frac{1}{2}}.& \end{aligned}$$

We write $\hat{P} \in \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ if

$$\|\hat{P}\|_{\mathcal{P}_2(\mathbb{C},\mathcal{B}(\mathbb{C}))} := D_T(\hat{P}, \hat{\delta}_0) < \infty,$$

where $\hat{\delta}_0$ is the measure putting mass equal to 1 to the constant path 0. If $\hat{Q} \in \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ has time marginals $(\hat{Q}_s)_{s\le T}$ then

$$\mathcal{W}_2(\hat{Q}_t, \hat{Q}_s)^2 \le \int_{\mathbb{C}} |Y_t - Y_s|^2 \hat{Q}(dY)$$

so that $\mathcal{W}_2(\hat{Q}_t, \hat{Q}_s) \to 0$ as $s \to t$, by dominated convergence. Hence, $(\hat{Q}_s)_{s\le T}$ is continuous.

2.b. Let $\mathbf{S}_2$ denote the set of continuous adapted $\mathbb{R}^d$-valued processes Z such that $\|Z\|_{\mathbf{S}_2} := \mathbb{E}[\sup_{[0,T]} |Z|^2]^{\frac{1}{2}} < \infty$. Let $\mathbf{L}_2(\Omega^\circ; \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C})))$ be the collection of random variables defined on Ω° and with values in $\mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, with finite norm $\mathbb{E}[\|\cdot\|^2_{\mathcal{P}_2(\mathbb{C},\mathcal{B}(\mathbb{C}))}]^{\frac{1}{2}}$. Let Φ be the map that to $\bar{Q} \in \mathbf{L}_2(\Omega^\circ; \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C})))$ associates $\mathbb{P}^B_{X^{\bar{Q}}} \in \mathbf{L}_2(\Omega^\circ; \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C})))$ in which $\mathbb{P}^B_{X^{\bar{Q}}}(\omega^\circ)$ is a regular conditional law of $X^{\bar{Q}}$ given $\omega^\circ \in \Omega^\circ$ with $X^{\bar{Q}}$ defined as the solution of

$$X^{\bar{Q}} = \mathbb{E}[\chi|\bar{\mathcal{F}}^\circ_{\theta\wedge\cdot}] + \int_\theta^{\theta\vee\cdot} b_s\big(X^{\bar{Q}}_s, \bar{Q}_s, \nu_s\big)ds + \int_\theta^{\theta\vee\cdot} a_s\big(X^{\bar{Q}}_s, \bar{Q}_s, \nu_s\big)dB_s,$$

and where $\bar{Q}_s(\omega^\circ)$ is the s-marginal of $\bar{Q}(\omega^\circ)$ for $\omega^\circ \in \Omega^\circ$. It follows from 2.a. that $\mathbb{P}^B_{X^{\bar{Q}}}(\omega^\circ)$ has continuous path, for $\mathbb{P}^\circ$-a.e. $\omega^\circ \in \Omega^\circ$. By repeating the arguments in [11, Proof of Proposition 2], see also 3. below, we obtain that Φ is contracting. Since $\mathbf{L}_2(\Omega^\circ; \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C})))$ is complete, it follows that Φ admits a fix point $\bar{Q}$.

3. It remains to prove our last estimate. The Lipschitz continuity and boundedness of (b, a) combined with Burkholder-Davis-Gundy inequality implies that one can find $C > 0$, that only depends on (b, a), such that

$$\begin{aligned}
&\mathbb{E}[\sup_{[0,s]} |X^{t,\chi,\nu} - X^{t_n,\chi_n,\nu^n}|^2] \\
\le& C(|t - t_n| + \mathbb{E}[|\chi - \chi_n|^2]) \\
&+ C\mathbb{E}\left[\int_0^s \sup_{[0,r]} |X^{t,\chi,\nu} - X^{t_n,\chi_n,\nu^n}|^2 + \mathcal{W}_2^2(\mathbb{P}^B_{X_r^{t,\chi,\nu}}, \mathbb{P}^B_{X_r^{t_n,x_n,\nu^n}})dr\right] \\
&+ C\mathbb{E}\left[\int_0^s |b_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu_r) - b_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu^n_r)|^2 dr\right] \\
&+ C\mathbb{E}\left[\int_0^s |a_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu_r) - a_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu^n_r)|^2 dr\right].
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}[\mathcal{W}_2^2(\mathbb{P}^B_{X_r^{t,\chi,\nu}}, \mathbb{P}^B_{X_r^{t_n,x_n,\nu^n}})] &\le \mathbb{E}[D_r^2(\mathbb{P}^B_{X^{t,\chi,\nu}}, \mathbb{P}^B_{X^{t_n,x_n,\nu^n}})] \\
&\le \mathbb{E}[\sup_{[0,r]} |X^{t,\chi,\nu} - X^{t_n,x_n,\nu^n}|^2],
\end{aligned}$$

by Gronwall's Lemma we obtain (for a different constant $C > 0$)

$$
\begin{aligned}
&\mathbb{E}[\mathcal{W}_2^2(\mathbb{P}^B_{X_T^{t,\chi,\nu}}, \mathbb{P}^B_{X_T^{t_n,x_n,\nu^n}})] \\
&\leq \mathbb{E}[\sup_{[0,T]} |X^{t,\chi,\nu} - X^{t_n,x_n,\nu^n}|^2] \\
&\leq C(|t - t_n| + \mathbb{E}[|\chi - \chi_n|^2]) \\
&\quad + C\mathbb{E}\left[\int_0^T |b_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu_r) - b_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu_r^n)|^2 dr\right] \\
&\quad + C\mathbb{E}\left[\int_0^T |a_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu_r) - a_r(X_r^{t,\chi,\nu}, \mathbb{P}^B_{X_r^{t,\chi,\nu}}, \nu_r^n)|^2 dr\right].
\end{aligned}
$$

The function (b, a) being continuous and bounded, the required result follows. $\square$

In the sequel, we denote by ${}^t\omega^\circ$ the element $(\omega^\circ_{s\wedge t})_{s\in[0,T]}$ for $\omega^\circ \in \Omega^\circ$ and $t \in [0, T]$. We note that the solution can also be defined ω^1 by ω^1. More precisely, we have the following.

Proposition 2.2. *Fix $\theta \in \bar{\mathcal{F}}^\circ$, $\chi \in \mathbf{X}_\theta^2$ and $\nu \in \mathcal{U}$. Let X^Q be the solution of (2.7) with $Q = (Q_s)_{s\leq T} \in \mathcal{P}_2^{\bar{\mathbb{F}}^\circ}$ in place of $(\mathbb{P}^B_{X_s})_{s\leq T}$. Then, there exists Borel measurable maps $\mathrm{x} : \Omega^\circ \times \Omega^1 \to \mathbb{R}^d$ and $\mathrm{u} : [0, T] \times \Omega^\circ \times \Omega^1 \to \mathrm{U}$ such that $\mathrm{x} = \chi$ $\mathbb{P}$-a.s. and $\nu_\cdot = \mathrm{u}_\cdot({}^\cdot B, \xi)$ $dt \times \mathbb{P}$-a.e. on $[0, T] \times \Omega$, such that, for all stopping time τ, $X^{Q,\omega^1}_{\tau\vee\theta} = X^Q_{\tau\vee\theta}(\cdot, \omega^1)$ $\mathbb{P}^\circ$-a.s. for $\mathbb{P}^1$-a.e. $\omega^1 \in \Omega^1$, in which X^{Q,ω^1} solves*

$$
\begin{aligned}
X^{Q,\omega^1} =& \mathbb{E}[\mathrm{x}(B, \omega^1)|\mathcal{F}_{\cdot\wedge\theta}] + \int_\theta^{\theta\vee\cdot} b_s\big(X_s^{Q,\omega^1}, Q_s, \mathrm{u}_s({}^sB, \omega^1)\big)ds \\
&+ \int_\theta^{\theta\vee\cdot} a_s\big(X_s^{Q,\omega^1}, Q_s, \mathrm{u}_s({}^sB, \omega^1)\big)dB_s.
\end{aligned}
$$

Moreover, the map $\omega^1 \in \Omega^1 \mapsto X^{Q,\omega^1}_{\tau\vee\theta} \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbf{L}_2(\Omega^\circ, \mathcal{F}^\circ_T, \mathbb{P}^\circ; \mathbb{R}^d))$ is measurable.

Proof. The existence of the Borel maps x and u is standard, and it is not difficult to prove that $\omega^1 \in \Omega^1 \mapsto X^{Q,\omega^1}_\tau \in \mathbf{L}_2(\Omega^\circ, \mathcal{F}^\circ_T, \mathbb{P}^\circ; \mathbb{R}^d)$ is measurable because a and b are continuous and bounded. Standard estimates then show that $\mathbb{E}[|X^{Q,\xi}_{\tau\vee\theta} - X^Q_{\tau\vee\theta}|^2|\xi] = 0$. $\square$

For later use, we now show that the law of $(X^{t,\chi,\nu}, B)$ actually only depends on the joint law of $(\chi, \nu, {}^tB)$.

Proposition 2.3. *Let* $\mathrm{x} : \Omega^\circ \times \Omega^1 \to \mathbb{R}^d$ *and* $\mathrm{u} : [0,T] \times \Omega^\circ \times \Omega^1 \to \mathrm{U}$ *be Borel maps such that* $\chi := \mathrm{x}({}^tB, \xi) \in \mathbf{X}_t^2$ *and* $\nu := \mathrm{u}.(B, \xi) \in \mathcal{U}$. *Let* $\bar{\xi}$ *and* $\bar{\xi}'$ *be* $[0,1]^d$*-valued* $\mathcal{F}_t$*-measurable and set* $\bar{\chi} := \mathrm{x}({}^tB, \tilde{\xi})$ *and* $\bar{\nu} := \mathrm{u}.(B, \bar{\xi}')$. *Assume that* $(\chi, \nu_{\cdot\vee t}, {}^tB)$ *and* $(\bar{\chi}, \bar{\nu}_{\cdot\vee t}, {}^tB)$ *have the same law. Then,* $(X^{t,\chi,\nu}, B)$ *and* $(X^{t,\bar{\chi},\bar{\nu}}, B)$ *have the same law.*

Proof. When the coefficients (b, a) of the stochastic differential equation do not depend on the marginal conditional law but are $\mathbb{F}$-progessive, the result follows from the same arguments as in the proof of [7, Theorem 3.3]. In their case, the conditioning is made with respect to ${}^tB$, in our case it has to be done with respect to $({}^tB, \xi)$, where ξ is independent of B, so that the equation can actually be solved conditionally to ξ, see Proposition 2.2. Given the fixed point procedure used in Step 2. of the proof of Proposition 2.1 above, one can then find a sequence $(\hat{P}^n)_{n\geq 1} \subset \mathbf{L}_2(\Omega^\circ, \mathcal{P}_2(\mathbb{C}, \mathcal{B}(\mathbb{C})))$ such that both $\hat{P}^n \to \mathbb{P}^B_{X^{t,\chi,\nu}}$ and $\hat{P}^n \to \mathbb{P}^B_{X^{t,\bar{\chi},\bar{\nu}}}$ as $n \to \infty$. □

3 The stochastic target problem: alternative formulations and geometric dynamic programming principle

Our aim is to provide a characterization of the set of initial measures for the conditional law of the initial condition χ given B such that the conditional law of $X_T^{t,\chi,\nu}$ given B belongs to a fixed closed subset G of $\mathcal{P}_2$:

$$\mathcal{V}(t) \;=\; \Big\{\mu \in \mathcal{P}_2 :\ \exists(\chi,\nu) \in \mathbf{X}_t^2 \times \mathcal{U} \text{ s.t. } \mathbb{P}^B_\chi = \mu \ \text{ and } \ \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\}.$$

Before to go on, let us first show that χ in the definition of $\mathcal{V}(t)$ can be replaced by any random variable $\chi' \in \mathbf{X}_t^2$ such that $\mathbb{P}^B_{\chi'} = \mu$. Apart from showing that only the distribution μ matters (which is a desirable property if we think in terms of mass transportation), this will be of important use later on to provide a geometric dynamic programming principle for $\mathcal{V}$.

Proposition 3.4. *A measure $\mu \in \mathcal{P}_2$ belongs to $\mathcal{V}(t)$ if and only if for all $\chi \in \mathbf{X}_t^2$ such that $\mathbb{P}_\chi^B = \mu$ there exists $\nu \in \mathcal{U}$ for which $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$.*

Proof. Let $\tilde{\mathcal{V}}(t)$ denote the collection of measures $\mu \in \mathcal{P}_2$ such that for all $\chi \in \mathbf{X}_t^2$ satisfying $\mathbb{P}_\chi^B = \mu$ there exists $\nu \in \mathcal{U}$ for which $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$. Clearly, $\tilde{\mathcal{V}}(t) \subset \mathcal{V}(t)$. We now prove the reverse inclusion. Let $\mu \in \mathcal{V}(t)$ and consider $(\chi, \nu) \in \mathbf{X}_t^2 \times \mathcal{U}$ such that $\mathbb{P}_\chi^B = \mu$ and $\mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G$. Since ν is $\mathbb{F}$-progressive, it is, up to modification, of the form

$$\nu_s(\omega^\circ, \omega^1) \;\; = \;\; \mathrm{u}(s, {}^sB(\omega^\circ), \xi(\omega^1)) \; , \quad s \in [t, T] \; ,$$

with u a Borel map. Recall that $\xi = (\xi_1, \ldots \xi_d)$ is the random variable on Ω defined by $\xi(\omega^\circ, \omega^1) = \omega^1$ for $\omega = (\omega^\circ, \omega^1) \in \Omega$. Let $F_1(\cdot; x, {}^t\omega^\circ)$ denote a (regular) conditional cumulated distribution of ξ_1 given $(\chi, {}^tB) = (x, {}^t\omega^\circ)$, and define $F_i(\cdot; y_1, \ldots, y_{i-1}, x, {}^t\omega^\circ)$ as the (regular) conditional cumulated distribution of ξ_i given $(\xi_1, \ldots, \xi_{i-1}, \chi, {}^tB) = (y_1, \ldots, y_{i-1}, x, {}^t\omega^\circ)$ for $i \geq 2$. Given $\bar{\chi} \in \mathbf{X}_t^2$ such that $\mathbb{P}^B_{\bar{\chi}} = \mu$, we then set $\bar{\xi}_1 := F_1^{-1}(F_1(\xi_1;\, \chi, {}^tB);\, \bar{\chi}, {}^tB)$ and $\bar{\xi}_i := F_i^{-1}(F_i(\xi_i;\, \xi_1, \ldots, \xi_{i-1},\, \chi, {}^tB);\, \bar{\xi}_1, \ldots,\, \bar{\xi}_{i-1}, \bar{\chi}, {}^tB)$ for $i \geq 2$. Set now $\bar{\nu} := u\mathbb{1}_{[0,t)} + \mathbb{1}_{[t,T]}\mathrm{u}(\cdot, {}^{\cdot}B, \bar{\xi}) \in \mathcal{U}$, for some $u \in \mathrm{U}$. Then, $(\bar{\chi}, \bar{\nu}_{t\vee\cdot}, B)$ and $(\chi, \nu_{t\vee\cdot}, B)$ have the same law, and Proposition 2.3 implies that $\mathbb{P}^B_{X_T^{t,\chi,\nu}} = \mathbb{P}^B_{X_T^{t,\bar{\chi},\bar{\nu}}}$ so that the latter belongs to G, thus proving that $\mathcal{V}(t) \subset \tilde{\mathcal{V}}(t)$, by arbitrariness of $\bar{\chi}$. □

Before to state the dynamic programming principle, let us provide the following measurable selection lemma. We define the subset $\mathcal{G}$ of $[0, T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ by

$$\mathcal{G} \;\; := \;\; \Big\{(t, \chi) \in [0, T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d) : \; \exists \nu \in \mathcal{U} \text{ s.t. } \; \mathbb{P}^B_{X_T^{t,\chi,\nu}} \in G\Big\} \; .$$

From now on, we consider $\mathcal{U}$ as a subset of $\mathbf{L}_2([0, T] \times \Omega, dt \times d\mathbb{P}; \mathrm{U})$ endowed with its strong topology.

Lemma 3.1. *For any probability measure $\mathfrak{P}$ on $[0, T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$, there exists a measurable map $\vartheta : \; \mathcal{G} \to \mathcal{U}$ such that*

$$\mathbb{P}^B_{X_T^{t,\chi,\vartheta(t,\chi)}} \;\; \in \;\; G$$

for $\mathfrak{P}$-a.e. $(t,\chi)\in\mathcal{G}$. Moreover, for each $(t,\chi)\in\mathcal{G}$, $\vartheta(t,\chi)$ can be chosen to be progressive w.r.t. $\mathbb{F}_{[t,T]} := (\sigma((B_{r\vee t}-B_t)_{t\le r\le s},\xi))_{s\in[t,T]}$.

Proof. It follows from (2.9) of Proposition 2.1 that the set

$$\mathcal{J} := \{(t,\chi,\nu)\in[0,T]\times\mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)\times\mathcal{U}:\ \mathbb{P}^B_{X_T^{t,\chi,\nu}}\in G\}$$

is closed. Then, the Jankov-von Neumann Theorem (see [1, Proposition 7.49]), ensures that there exists an analytically measurable function $\tilde{\vartheta}$: $[0,T]\times\mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)\to\mathcal{U}$ such that

$$(t,\chi,\tilde{\vartheta}(t,\chi))\in\mathcal{J}\ \text{ for all } (t,\chi)\in\mathcal{G}\ .$$

Since any analytically measurable map is also universally measurable, the existence of ϑ follows from [1, Lemma 7.27]. It remains to prove our last claim. Let u be a progessive measurable map such that $\mathrm{u}_s(\omega^\circ,\omega^1)=\vartheta(t,\chi)_s$ for $s\in[t,T]$. For $s\in[0,T]$, $\mathrm{w},\mathrm{w}'\in\Omega^\circ$, set $\mathrm{w}\otimes_s\mathrm{w}' := \mathrm{w}_{\cdot\wedge s}+(\mathrm{w}'_{\cdot\vee s}-\mathrm{w}'_s)$. Define $\nu_s^{\omega^\circ}(\tilde{\omega}^\circ,\omega^1) := \mathrm{u}_s(\omega^\circ\otimes_t\tilde{\omega}^\circ,\omega^1)$. Then, one can find $\omega^\circ\in\Omega^\circ$ such that $\mathbb{P}^B_{X_T^{t,\chi,\nu^{\omega^\circ}}}(\tilde{\omega}^\circ)\in G$ for $\mathbb{P}^\circ$-a.e. $\tilde{\omega}^\circ\in\Omega^\circ$, see [7, Theorem 5.4] and Proposition 2.2. The control ν^{ω° is progressive w.r.t. $\mathbb{F}_{[t,T]}$. □

We can now state the dynamic programming principle.

Theorem 3.1. *Fix $t\in[0,T]$ and $\theta\in\bar{\mathcal{T}}^\circ$ with values in $[t,T]$. Then,*

$$\mathcal{V}(t)=\Big\{\mu\in\mathcal{P}_2:\ \exists(\chi,\nu)\in\mathbf{X}_t^2\times\mathcal{U}\ s.t.\ \ \mathbb{P}^B_\chi=\mu\ \ and\ \ \mathbb{P}^B_{X_\theta^{t,\chi,\nu}}\in\mathcal{V}(\theta)\Big\}.$$

Proof. Denote by $\hat{\mathcal{V}}(t)$ the right hand side of the equality.
1. We first prove the inclusion $\mathcal{V}(t)\subset\hat{\mathcal{V}}(t)$. Fix $\mu\in\mathcal{V}(t)$. Then, there exists $(\chi,\nu)\in\mathbf{X}_t^2\times\mathcal{U}$ and $\tilde{\Omega}^\circ\in\mathcal{F}^\circ$ such that $\mathbb{P}^\circ(\tilde{\Omega}^\circ)=1$, $\mathbb{P}^B_\chi=\mu$ and $\mathbb{P}^B_{X_T^{t,\chi,\nu}}\in G$ on $\tilde{\Omega}^\circ$. For $\tilde{\omega}^\circ\in\tilde{\Omega}^\circ$, we define $(\chi^{\tilde{\omega}^\circ},\nu^{\tilde{\omega}^\circ})$ by

$$\chi^{\tilde{\omega}^\circ}(\omega)=X^{t,\chi,\nu}_{\theta(\tilde{\omega}^\circ)}(\tilde{\omega}^\circ,\omega^1)\quad,\quad \nu_s^{\tilde{\omega}^\circ}(\omega)=\nu_s(\tilde{\omega}^\circ\oplus_{\theta(\tilde{\omega}^\circ)}\omega^\circ,\omega^1)\ ,\quad s\in[0,T]$$

for all $\omega=(\omega^\circ,\omega^1)\in\Omega$. Note that $\chi^{\tilde{\omega}^\circ}\in\mathbf{X}^2_{\theta(\tilde{\omega}^\circ)}$, $\mathbb{P}^B_{\chi^{\tilde{\omega}^\circ}}=\mathbb{P}^B_{X_\theta^{t,\chi,\nu}}(\tilde{\omega}^\circ)$ and $\nu^{\tilde{\omega}^\circ}\in\mathcal{U}$ for all $\tilde{\omega}^\circ\in\tilde{\Omega}^\circ$. Moreover, it follows from [7, Theorem 5.4] and

Proposition 2.2 that $X_T^{\theta(\tilde{\omega}^\circ),\chi^{\tilde{\omega}^\circ},\nu^{\tilde{\omega}^\circ}}$ has the same law as $X_T^{t,\chi,\nu}$ given $W_{\cdot\wedge\theta} = \tilde{\omega}^\circ_{\cdot\wedge\theta(\tilde{\omega}^\circ)}$, for $\mathbb{P}^\circ$-a.e. $\tilde{\omega}^\circ \in \Omega^\circ$. Since $\mathbb{P}^B_{X_T^{t,\chi,\nu}}(\omega^\circ) \in G$ for $\omega^\circ \in \tilde{\Omega}^\circ$, it follows that $\mathbb{P}^B_{X_\theta^{t,\chi,\nu}}(\tilde{\omega}^\circ) = \mathbb{P}^B_{\chi^{\tilde{\omega}^\circ}} \in \mathcal{V}(\theta(\tilde{\omega}^\circ))$ for all $\tilde{\omega}^\circ \in \tilde{\Omega}^\circ$. Therefore $\mu \in \hat{\mathcal{V}}(t)$.

2. We now prove the inclusion $\hat{\mathcal{V}}(t) \subset \mathcal{V}(t)$. Fix $\mu \in \hat{\mathcal{V}}(t)$ and $(\chi, \nu) \in \mathbf{X}_t^2 \times \mathcal{U}$ such that $\mathbb{P}^B_\chi = \mu$ and $\mathbb{P}^B_{X_\theta^{t,\chi,\nu}} \in \mathcal{V}(\theta)$. It follows from Proposition 3.4 that $\big(\theta(\omega^\circ), X^{t,\chi,\nu}_{\theta(\omega^\circ)}(\omega^\circ, .)\big) \in \mathcal{G}$, for $\mathbb{P}^\circ$-a.e. $\omega^\circ \in \Omega^\circ$. Let $\mathfrak{P}$ be the probability measure induced by $\omega^\circ \mapsto \big(\theta(\omega^\circ), X^{t,\chi,\nu}_{\theta(\omega^\circ)}(\omega^\circ, .)\big)$ on $[0,T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$. By Lemma 3.1, there exists a measurable map ϑ such that $\mathbb{P}^B_{X_T^{t',\chi',\vartheta(t',\chi')}} \in G$ $\mathbb{P}^\circ$-a.s. for $\mathfrak{P}$-a.e. $(t', \chi') \in \mathcal{G}$. Since $\vartheta(t', \chi')$ can be chosen in the filtration $\mathbb{F}_{[t',T]}$ to which ${}^{t'}B$ is independent, $\mathbb{P}^B_{X_T^{t',\chi',\vartheta(t',\chi')}}$ is measurable with respect to $\sigma(B_{\cdot\vee t'} - B_{t'})$. Hence, there exists null sets N and $\tilde{N}$ such that

$$\mathbb{P}^B_{X_T^{\alpha(\omega^\circ,\cdot)}}(\tilde{\omega}^\circ) \in G \quad \text{for } \omega^\circ \notin N \text{ and } \tilde{\omega}^\circ \notin \tilde{N},$$

where

$$\alpha(\omega^\circ, \cdot) := (\theta(\omega^\circ), X_\theta^{t,\chi,\nu}(\omega^\circ, \cdot), \vartheta(\theta(\omega^\circ), X_\theta^{t,\chi,\nu}(\omega^\circ, \cdot)).$$

It remains to define the process $\bar{\nu} \in \mathcal{U}$ by

$$\bar{\nu}(\omega) \;=\; \nu(\omega)\mathbb{1}_{[0,\theta(\omega^\circ))} + \vartheta(\theta(\omega^\circ), X_\theta^{t,\chi,\nu}(\omega^\circ, \cdot))(\omega)\mathbb{1}_{[\theta(\omega^\circ),T]}\,, \tag{3.10}$$

and observe that $X_T^\alpha = X_T^{t,\chi,\bar{\nu}}$, to conclude that $\mu \in \mathcal{V}(t)$. □

4 The dynamic programming partial differential equation

Let $\mathrm{v}:\ [0,T] \times \mathcal{P}_2 \to \mathbb{R}$ be the indicator function of the complement of the reachability set $\mathcal{V}$:

$$\mathrm{v}(t, \mu) \;=\; 1 - \mathbb{1}_{\mathcal{V}(t)}(\mu)\,, \quad (t, \mu) \in [0,T] \times \mathcal{P}_2. \tag{4.11}$$

The aim of this section is to provide a characterization of v as a (discontinuous) viscosity solution of a fully non-linear second order parabolic partial differential equation, in the spirit of [16]. Given Theorem 3.1, this follows from combining the technologies developped in [4, 6] and [16].

4.1 Derivatives on the space of probability measures and Itô's lemma

We first recall here the notions of derivative with respect to a probability measure that has been introduced by Lions, see the lecture notes [4], and further developed in [6], to our context.

For a function $w: \mathcal{P}_2 \to \mathbb{R}$, we define its lifting as the function W from $\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ to $\mathbb{R}$ such that

$$W(X) \;\; = \;\; w(\mathbb{P}_X)\,, \qquad \text{for all } X \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)\,.$$

We then say that w is Fréchet differentiable (resp. $\mathcal{C}^1$) on $\mathcal{P}_2$ if its lift W is (resp. continuously) Fréchet differentiable on $\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$. If it exists, the Fréchet derivative $DW(X)$ of W at $X \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ can be identified by Riez Theorem to an element of $\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ and admits a representation of the form

$$DW(X) \;\; = \;\; \partial_\mu w(\mathbb{P}_X)(X)$$

for some measurable map $\partial_\mu w(\mathbb{P}_X): \mathbb{R}^d \to \mathbb{R}^d$, that we call the derivative of w at $\mathbb{P}_X$. We have $\partial_\mu w(\mu) \in \mathbf{L}_2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$ for $\mu \in \mathcal{P}_2$. In the case where $x \in \mathbb{R}^d \mapsto \partial_\mu w(\mu)(x)$ is differentiable at x, given $\mu \in \mathcal{P}_2$, we denote by $\partial_x \partial_\mu w(\mu)(x)$ the corresponding gradient.

Following [6, Section 3.3], we say that w is partially $\mathcal{C}^2$ if it is differentiable on $\mathcal{P}_2$ and if, for each $\mu \in \mathcal{P}_2$, there exists a continuous version of the map $x \in \mathbb{R}^d \mapsto \partial_\mu w(\mu)(x)$ such that

- the map $(\mu, x) \mapsto \partial_\mu w(\mu)(x)$ is continuous at any $(\mu, x) \in \mathcal{P}_2 \times \mathbb{R}^d$ such that $x \in \text{supp}(\mu)$,
- for any $\mu \in \mathcal{P}_2$, the map $x \mapsto \partial_\mu w(\mu)(x)$ is continuously differentiable and the map $(\mu, x) \mapsto \partial_x \partial_\mu w(\mu)(x)$ is continuous at any $(\mu, x) \in \mathcal{P}_2 \times \mathbb{R}^d$ such that $x \in \text{supp}(\mu)$.

Under the additional assumption that W is twice continuously Fréchet differentiable, D^2W can be identified by Riez Theorem as a self-adjoint operator on $\mathbf{L}_2(\Omega^1,\ \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$. We identify it as above to a map $(x, x') \in$

$\mathbb{R}^d \times \mathbb{R}^d \mapsto \partial_\mu^2 w(\mu)(x, x')$. For later use, note that we also have the following identification by [5, Remark 6.4]

$$\mathbb{E}\left[D^2 W(X)(YZ)YZ^\top\right] \;=\; \mathbb{E}\left[\mathrm{Tr}\left(\partial_x\partial_\mu w(\mu)(X)YY^\top\right)\right] \qquad (4.12)$$

for any random variables $Y \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^{d\times d})$, $Z \sim N(0, I_d)$ and Z independent of (X, Y) (to be defined on an enlarged probability space).

From now on, we define $\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2)$ as the set of continuous functions $w:\ [0,T]\times\mathcal{P}_2 \to \mathbb{R}$ such that $w(t,\cdot)$ is partially $\mathcal{C}^2$ for all $t\in[0,T]$, $\partial_t w$ exists and is continuous on $[0,T]\times\mathcal{P}_2$, $\partial_\mu w$ and $\partial_x\partial_\mu w$ are continuous at any (t,μ,x) such that $x\in\mathrm{supp}(\mu)$ and

$$\sup_{t\in[0,T],\ \mu\in\mathcal{K}} \int_{\mathbb{R}^d}\Big[\big|\partial_t w(t,\mu)\big| + \big|\partial_\mu w(t,\mu)(x)\big|^2 + |\partial_x\partial_\mu w(t,\mu)(x)|^2\Big]\mu(dx) < \infty \qquad (4.13)$$

for any compact subset $\mathcal{K}$ of $\mathcal{P}_2$.

Proposition 4.5. *Let $w\in\mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2)$. Given $(t,\chi,\nu)\in[0,T]\times\mathbf{X}_t\times\mathcal{U}$, set $X = X^{t,\chi,\nu}$. Then,*

$$\begin{aligned}
w(s,\mathbb{P}^B_{X_s}) = w(t,\mathbb{P}^B_\chi)&\\
&+\int_t^s \mathbb{E}_B\left[\partial_t w(r,\mathbb{P}^B_{X_r}) + \partial_\mu w(r,\mathbb{P}^B_{X_r})(X_r)b_r(X_r,\mathbb{P}^B_{X_r},\nu_r)\right]dr\\
&+\frac{1}{2}\int_t^s \mathbb{E}_B\left[\mathrm{Tr}\left(\partial_x\partial_\mu w(r,\mathbb{P}^B_{X_r})(X_r)a_r a_r^\top(X_r,\mathbb{P}^B_{X_r},\nu_r)\right)\right]dr\\
&+\int_t^s \mathbb{E}_B\left[\partial_\mu w(r,\mathbb{P}^B_{X_r})(X_r)a_r(X_r,\mathbb{P}^B_{X_r},\nu_r))\right]dB_r
\end{aligned}$$

for all $s\in[t,T]$.

Proof. The proof follows from similar arguments as in [6] and we only mention the main ideas. Since $\chi\in\mathbf{X}_t$ and $\nu\in\mathcal{U}$, we can find Borel maps x and u such that $\chi = \mathrm{x}(B,\xi)$ $\mathbb{P}$-a.s. and $\nu = \mathrm{u}(\cdot,^\cdot B,\xi)$, up to modification. We first enlarge the space Ω by considering the space $\tilde\Omega = \Omega^\circ\times\tilde\Omega^1$ where $\tilde\Omega^1 = (\Omega^1)^{\mathbb{N}}$. We endow this space with the completion $\tilde{\mathcal{F}}$ of the σ-algebra $\mathcal{F}^\circ\otimes(\mathcal{F}^1)^{\otimes\mathbb{N}}$ and the probability measure $\tilde{\mathbb{P}} = \mathbb{P}^\circ\otimes(\mathbb{P}^1)^{\otimes\mathbb{N}}$. We define on this space the sequence of random variables $(\xi^\ell)_{\ell\ge 1}$ and we extend (ξ, B) in a canonical way by setting

$$\xi = \xi^1 \,,\ \xi^\ell(\omega) = \tilde\omega^1_\ell \ \text{ for } \ell \in \mathbb{N} \,,\ B(\omega) \ = \ \omega^\circ,$$

for all $\omega = (\omega^\circ, (\tilde\omega^1_\ell)_{\ell\in\mathbb{N}}) \in \tilde\Omega$. Note that $(\xi^\ell)_{\ell\in\mathbb{N}}$ is an *i.i.d.* sequence, independent of B. We then set $(\chi^\ell, \nu^\ell) := (\mathrm{x}(B), \mathrm{u}(\cdot; B, \xi^\ell))$, for $\ell \ge 1$, and define X^ℓ as the solution on $[t,T]$ of

$$X^\ell = \chi^\ell + \int_t^\cdot b^\ell_s ds + \int_t^\cdot a^\ell_s dB_s,$$

in which $(b^\ell, a^\ell) = (b,a)(X^\ell, \mathbb{P}^B_{X^1}, \nu^\ell)$. It follows from Proposition 2.2 that $(X^\ell_r)_{\ell\ge1}$ is a sequence of i.i.d. random variables given $(B_{r'})_{r'\le T}$, for each $r\in[t,s]$. Set $\bar\mu^N_r := \frac{1}{N}\sum_{\ell=1}^N \delta_{X^\ell_r}$ for $t\le r\le s$.

1. We first assume that w is fully $\mathcal{C}^2$ in the sense of [6, p17], that is $(\mu, v) \mapsto (\partial_\mu w(\mu)(v), \partial_v\partial_\mu w(\mu)(v), \partial^2_\mu w(\mu)(v))$ is continuous, and that w, $\partial_\mu w$, $\partial_x\partial_\mu w$ and $\partial^2_\mu w$ are bounded and uniformly continuous. Then, it follows from [6, Proposition 3.1] combined with Itô's Lemma that

$$\begin{aligned} w(s,\bar\mu^N_s) \ &= \ w(t,\bar\mu^N_t) + \int_t^s \partial_t w(r,\bar\mu^N_r)dr + \frac{1}{N}\sum_{\ell=1}^N \int_t^s \partial_\mu w(r,\bar\mu^N_r)(X^\ell_r) b^\ell_r dr \\ &\quad + \frac{1}{N}\sum_{\ell=1}^N \int_t^s \partial_\mu w(r,\bar\mu^N_r)(X^\ell_r) a^\ell_r dB_r \\ &\quad + \frac{1}{2N}\sum_{\ell=1}^N \int_t^s \mathrm{Tr}\left[\partial_x\partial_\mu w(r,\bar\mu^N_r)(X^\ell_r) a^\ell_r (a^\ell_r)^\top\right] dr \\ &\quad + \frac{1}{2N^2}\sum_{\ell=1}^N \int_t^s \mathrm{Tr}\left[\partial^2_\mu w(r,\bar\mu^N_r)(X^\ell_r, X^\ell_r) a^\ell_r (a^\ell_r)^\top\right] dr. \end{aligned}$$

We now take the expectation given $(B_{r'})_{r'\le T}$ on both sides and use [15, Corollaries 2 and 3 of Theorem 5.13] and [13, Lemma 14.2] together with the fact that the quadruplets $(\bar\mu^N_r, X^\ell_r, b^\ell_r, a^\ell_r)_{\ell\le N}$ have all the same law given

$(B_{r'})_{r'\le T}$, for $t \le r \le s$, to obtain

$$\begin{aligned}
\mathbb{E}_B[w(s,\bar\mu_s^N)] &= \mathbb{E}_B[w(t,\bar\mu_t^N)] + \int_t^s \mathbb{E}_B\left[\partial_t w(r,\bar\mu_r^N) + \partial_\mu w(r,\bar\mu_r^N)(X_r^1)b_r^1\right] dr \\
&\quad + \int_t^s \mathbb{E}_B\left[\partial_\mu w(r,\bar\mu_r^N)(X_r^1)a_r^1)\right] dB_r \\
&\quad + \frac{1}{2}\int_t^s \mathbb{E}_B\left[\mathrm{Tr}\left(\partial_x\partial_\mu w(r,\bar\mu_r^N)(X_r^1)a_r^1(a_r^1)^\top\right)\right] dr \\
&\quad + \frac{1}{2N}\int_t^s \mathbb{E}_B\left[\mathrm{Tr}\left(\partial_\mu^2 w(r,\bar\mu_r^N)(X_r^1,X_r^1)a_r^1(a_r^1)^\top\right)\right] dr.
\end{aligned}$$

We then use the fact that $\mathcal{W}_2(\bar\mu_r^N, \mathbb{P}^B_{X_r^1}) \to 0$ a.s. as $N \to \infty$ for all $r \in [t,s]$. This is a consequence of [11, Lemma 4] and the fact that $(X_r^\ell)_{\ell\ge 1}$ is a sequence of i.i.d. random variables given $(B_{r'})_{r'\le T}$. Since all the involved maps are assumed to be bounded and continuous, one can take the limit as $N \to \infty$ in the above to obtain

$$\begin{aligned}
w(s,\mathbb{P}^B_{X_s^1}) &= w(t,\mathbb{P}^B_{X^1}) + \int_t^s \mathbb{E}_B\left[\partial_t w(r,\mathbb{P}^B_{X_r^1}) + \partial_\mu w(r,\mathbb{P}^B_{X_r^1})(X_r^1)b_r^1\right] dr \\
&\quad + \int_t^s \mathbb{E}_B\left[\partial_\mu w(r,\mathbb{P}^B_{X_r^1})(X_r^1)a_r^1)\right] dB_r \\
&\quad + \frac{1}{2}\int_t^s \mathbb{E}_B\left[\mathrm{Tr}\left(\partial_x\partial_\mu w(r,\mathbb{P}^B_{X_r^1})(X_r^1)a_r^1(a_r^1)^\top\right)\right] dr.
\end{aligned} \tag{4.14}$$

2. The validity of (4.14) can be extended to the case where w is just assumed to be fully $\mathcal{C}^2$ by following the molifying argument of [6, Proposition 3.4] whenever the condition (4.13) holds, recall that (b,a) is bounded. The extension to a partially $\mathcal{C}^2$ function then follows from the same considerations as in the proof of [6, Theorem 3.5]. □

Later on, we shall need to use this Itô's formula at the level of the lift W of a function w. From now on, we say that $W : [0,T] \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d) \to \mathbb{R}$ is $\mathcal{C}_b^{1,2}$ if it is the lifting function of a map $w \in \mathcal{C}_b^{1,2}([0,T]\times\mathcal{P}_2)$. Given a random variable $X \in \mathbf{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$, we define $W(t,X)$ as the random variable $\omega^\circ \in \Omega^\circ \mapsto W(t,X(\omega^\circ,\cdot))$ where $X(\omega^\circ,\cdot)$ is now a random variable on $\mathbf{L}^2(\Omega^1,\mathcal{F}^1,\mathbb{P}^i;\mathbb{R}^d)$. We use the same convention for $DW(t,X(\omega^\circ,\cdot))$ and $D^2W(t,X(\omega^\circ,\cdot))$. As an immediate corollary of Proposition 4.5 and (4.12), we have the following. From now on Z denotes a d-dimensional Gaussian vector $N(0, I_d)$ independent of (B, ξ), whose existence is ensured, up to increasing the probability space.

Corollary 4.1. *Let $W : [0,T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d) \to \mathbb{R}$ be $\mathcal{C}_b^{1,2}$. Given $(t, \chi, \nu) \in [0,T] \times \mathbf{X}_t \times \mathcal{U}$, set $X = X^{t,\chi,\nu}$. Then,*

$$
\begin{aligned}
W(s, X_s) = W(t,\chi)& \\
&+ \int_t^s \mathbb{E}_B \left[\partial_t W(r, X_r) + DW(r, X_r) b_r(X_r, \mathbb{P}^B_{X_r}, \nu_r)\right] dr \\
&+ \frac{1}{2}\int_t^s \mathbb{E}_B \left[D^2 W(r, X_r)(X_r)(a_r Z)(a_r Z)^\top (X_r, \mathbb{P}^B_{X_r}, \nu_r)\right] dr \\
&+ \int_t^s \mathbb{E}_B \left[DW(r, X_r) a_r(X_r, \mathbb{P}^B_{X_r}, \nu_r))\right] dB_r
\end{aligned}
$$

for all $s \in [0,T]$.

4.2 Viscosity solution characterization

We aim at proving that v solves a Hamilton-Jacobi-Bellman equation of the form

$$
-\partial_t w(t,\mu) + H\big(t, \mu, \partial_\mu w(t,\mu), \partial_\mu \partial_x w(t,\mu)\big) \;=\; 0\,,
$$

in the sense that the lifting function $V : [0,T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d) \to \mathbb{R}$ of v is solution on $[0,T) \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ of

$$
-\partial_t W(t,\chi) + \mathcal{H}\big(t, \chi, DW(t,\chi), D^2 W(t,\chi)\big) \;=\; 0\,. \tag{4.15}
$$

Before to define the operator $\mathcal{H}$, let us recall that $W : [0,T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ is extended to $[0,T] \times \mathbf{L}_2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ by defining $W(t, X)$ as the random variable $\omega^\circ \in \Omega^\circ \mapsto W(t, X(\omega^\circ, \cdot))$, and let us introduce the set $S(\mathbf{L}_2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d))$ of self-adjoint operators on $\mathbf{L}_2(\Omega,\, \mathcal{F},\, \mathbb{P}; \mathbb{R}^d)$.

Then, $\mathcal{H}$ is defined as $\mathcal{H}_0$ where, for $\varepsilon \geq 0$,

$$
\begin{aligned}
\mathcal{L}^u_t(\chi, P, Q) &:= \mathbb{E}_B\Big[b_t^\top(\chi, \mathbb{P}_\chi, u)P + \frac{1}{2} Q\big(a_t(\chi, \mathbb{P}_\chi, u)Z\big) a_t(\chi, \mathbb{P}_\chi, u)Z\Big] \\
\mathcal{H}_\varepsilon(t, \chi, P, Q) &:= \sup_{u \in \mathcal{N}_\varepsilon(t,\chi,P)} \Big\{ -\mathcal{L}^u_t(\chi, P, Q)\Big\} \\
\mathcal{N}_\varepsilon(t, \chi, P) &:= \Big\{u \in \mathbf{L}_0(\Omega, \mathcal{F}, \mathbb{P}; \mathrm{U}) \;:\; |\mathbb{E}_B[a_t(\chi, \mathbb{P}_\chi, u)P]| \leq \varepsilon\Big\},
\end{aligned}
$$

for $t \in [0,T]$, $u \in \mathbf{L}_0(\Omega,\mathcal{F},\mathbb{P};\mathrm{U})$, $\chi, P \in \mathbf{L}_2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ and $Q \in S(\mathbf{L}_2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d))$.

Since neither V nor $\mathcal{H}$. are a-priori continuous, we define V_* and V^* as the lower-semicontinous and upper-semicontinuous enveloppes of V, and let $\mathcal{H}^*$ and $\mathcal{H}_*$ be defined as the relaxed upper- and lower-semilimits as $\varepsilon \to 0$.

We say that V_* is a viscosity supersolution (resp. V^* is a subsolution) of (4.15) if for any $(t,\chi) \in [0,T] \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ and any $\mathcal{C}_b^{1,2}$ function Φ on $[0,T] \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ such that

$$
\begin{aligned}
(V_* - \Phi)(t,\chi) &= \min_{[0,T]\times\mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)} (V_* - \Phi) \\
(\text{ resp. } (V^* - \Phi)(t,\chi) &= \max_{[0,T]\times\mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)} (V^* - \Phi) \text{)}
\end{aligned}
$$

we have

$$
\begin{aligned}
-\partial_t\Phi(t,\chi) + \mathcal{H}^*\big(t,\chi,D\Phi(t,\chi),D^2\Phi(t,\chi)\big) &\geq 0 \\
(\text{resp. } -\partial_t\Phi(t,\chi) + \mathcal{H}_*\big(t,\chi,D\Phi(t,\chi),D^2\Phi(t,\chi)\big) &\leq 0 \text{)} .
\end{aligned}
$$

If V_* is a supersolution and V^* is a subsolution, we say that V is a discontinuous solution.

We are now ready to state the viscosity property of the function V. This requires the following continuity assumption on the set $\mathcal{N}$.

(H2): Let $\mathcal{O}$ be an open subset of $[0,T] \times \mathbf{L}_2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \times \mathbf{L}_2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ such that $\mathcal{N}_0 \neq \emptyset$ on $\mathcal{O}$. Then, for every $\varepsilon > 0$, $(t_0,\chi_0,P_0) \in \mathcal{O}$ and $u_0 \in \mathcal{N}_0(t_0,\chi_0,P_0)$, there exists an open neighborhood $\mathcal{O}'$ of (t_0,χ_0,P_0) and a measurable map $\hat{u} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Omega^1 \to \mathrm{U}$ such that:
(i) $\mathbb{E}_B[|\hat{u}_{t_0}(\chi_0,P_0,\xi) - u_0|] \leq \varepsilon$.
(ii) There exists $C > 0$ for which

$$
\mathbb{E}[|\hat{u}_t(\chi,P,\xi) - \hat{u}_t(\chi',P',\xi)|^2] \leq C\mathbb{E}[|\chi-\chi'|^2 + \mathcal{W}_2^2(\mathbb{P}_P,\mathbb{P}_{P'})]
$$

for all $(t,\chi,P),(t,\chi',P') \in \mathcal{O}'$.
(iii) $\hat{u}_t(\chi,P,\xi) \in \mathcal{N}_0(t,\chi,P)$ $\mathbb{P}^\circ - a.e.$, for all $(t,\chi,P) \in \mathcal{O}'$.

We also strengthen **(H1)** by the following additional condition.

(H1') There exist a constant C and a function $m : \mathbb{R}_+ \to \mathbb{R}$ such that $m(t) \to 0$ as $t \to 0$ and

$$|b_t(x,\mu,u) - b_{t'}(x,\mu,u')| + |a_t(x,\mu,u) - a_{t'}(x,\mu,u')| \quad \leq \quad m(t-t') + C|u-u'|.$$

for all $t,t' \in [0,T]$, $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2$ and $u,u' \in \mathrm{U}$.

Theorem 4.2. *Under **(H1)** and **(H1')** the function V_* is a viscosity supersolution of* (4.15)*. If in addition **(H2)** holds, then V^* is a viscosity subsolution of* (4.15)*.*

Proof. **Part I.** Supersolution property. Fix $(t_0,\chi_0) \in [0,T) \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ and a $\mathcal{C}_b^{1,2}$ test function Φ on $[0,T) \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ such that

$$(V_* - \Phi)(t_0,\chi_0) \quad = \quad \min_{[0,T]\times\mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)} (V_* - \Phi) \quad = \quad 0\ .$$

We must prove that

$$-\partial_t\Phi(t_0,\chi_0) + \mathcal{H}^*\big(t_0,\chi_0,D\Phi(t_0,\chi_0),D^2\Phi(t_0,\chi_0)\big) \quad \geq \quad 0\ . \qquad (4.16)$$

1. Suppose that the function V is constant in a neighborhood of (t_0,χ_0). Then $\Phi(t_0,\chi_0)$ is a local maximum of Φ and therefore

$$\partial_t\Phi(t_0,\chi_0) \leq 0\ , \quad D\Phi(t_0,\chi_0) = 0 \quad \text{and} \quad D^2\Phi(t_0,\chi_0) \leq 0\ . \qquad (4.17)$$

Hence, $\mathcal{N}_0(t_0,\chi_0,D\Phi(t_0,\chi_0)) = \mathbf{L}_0(\Omega,\mathcal{F},\mathbb{P};\mathrm{U})$ and

$$-\partial_t\Phi(t_0,\chi_0) + \mathcal{H}_0\big(t_0,\chi_0,D\Phi(t_0,\chi_0),D^2\Phi(t_0,\chi_0)\big) \quad \geq \quad 0\ ,$$

so that (4.16) is satisfied.

2. We now consider the complementary case: $V_*(t_0,\chi_0) = 0$. Let $(t_n,\chi_n)_{n\geq 1}$ be a sequence of $[0,T) \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ converging to (t_0,χ_0) and such that

$$V(t_n,\chi_n) \quad = \quad 0\ , \qquad \text{for all}\ \ n \geq 1. \qquad (4.18)$$

We argue by contradiction and suppose that

$$-\partial_t\Phi(t_0,\chi_0) + \mathcal{H}^*\big(t_0,\chi_0,D\Phi(t_0,\chi_0),D^2\varphi(t_0,\chi_0)\big) \quad =: \quad -2\eta\ .$$

for some $\eta > 0$. Define

$$\tilde{\Phi}(t,\chi) \;\; = \;\; \Phi(t,\chi) - |t-t_0|^2 - \mathbb{E}\big[\big|\chi - \chi_0\big|^2\big]^2$$

for $(t,\chi) \in [0,T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$. Then,

$$(\partial_t \tilde{\Phi}, D\tilde{\Phi}, D^2\tilde{\Phi})(t_0,\chi_0) = (\Phi, D\Phi, D^2\Phi)(t_0,\chi_0),$$

and we can find $\varepsilon > 0$ and an open ball $B_\varepsilon(t_0,\chi_0)$ such that

$$-\eta \geq -\,\partial_t \tilde{\Phi}(t,\chi) - \mathcal{L}^u_t(\chi, D\tilde{\Phi}(t,\chi), D^2\tilde{\Phi}(t,\chi)) \tag{4.19}$$

for any $(t,\chi) \in B_\varepsilon(t_0,\chi_0)$ and any $u \in \mathcal{N}_\varepsilon(t,\chi,D\Phi(t,\chi))$. Let $\partial_p B_\varepsilon(t_0,\chi_0) := \{t_0+\varepsilon\} \times cl(B_\varepsilon(\chi_0)) \cup [t_0, t_0+\varepsilon) \times \partial B_\varepsilon(\chi_0)$ denote the parabolic boundary of $B_\varepsilon(t_0,\chi_0)$ and observe that

$$\zeta \;\; := \;\; \min_{\partial_p B_\varepsilon(t_0,\chi_0)} (V_* - \tilde{\Phi}) \;\; > \;\; 0\,. \tag{4.20}$$

In view of (4.18), we can find a control $\nu^n \in \mathcal{U}$ such that

$$\mathbb{P}^B_{X^n_t} \;\; \in \;\; G\,,$$

where $X^n = X^{t_n,\chi_n,\nu^n}$. We then define the stoping times

$$\theta_n(\omega^\circ) \;\; = \;\; \inf\Big\{ s \geq t_n \;:\; \big(s, X^n_s(\omega^\circ,.)\big) \notin B_\varepsilon(t_0,\chi_0) \Big\}\,, \quad \omega^\circ \in \Omega^\circ\,.$$

By Theorem 3.1, $V(\cdot, X^n_\cdot) = 0$ on $[t_n,T]$, so that $-\tilde{\Phi}(\cdot, X^n) \geq 0$ on $[t_n,T]$ and $-\tilde{\Phi}(\theta_n, X^n_{\theta_n}) \geq \zeta$ by (4.20). Let us set $\beta_n := -\tilde{\Phi}(t_n,\chi_n)$ and define

$$\begin{aligned} \alpha^n_t :=& \mathbb{E}_B[\partial_t \tilde{\Phi}(t,X^n_t) + \mathcal{L}^{\nu^n_t}_t(X^n_t, D\tilde{\Phi}(t,X^n_t), D^2\tilde{\Phi}(t,X^n_t))], \\ \rho^n :=& -\,\alpha^n \mathbb{1}_{A_n}\,,\; \psi^n := -\mathbb{E}_B\big[a(X^n, \mathbb{P}^B_{X^n}, \nu^n) D\tilde{\Phi}(\cdot, X^n)\big] \end{aligned}$$

with

$$A_n := \Big\{ t \in [t_n,\theta_n] \;:\; -\alpha^n_t > -\eta \Big\}.$$

Applying Corollary 4.1 and Remark 4.2 to $\tilde{\Phi}(., X^n)$, we then get that $M^n_{\theta_n} \geq 0$ where

$$M^n := \beta_n - \zeta + \int_{t_n}^{\cdot} \rho^n_t dt + \int_{t_n}^{\cdot} \psi^n_t dB_t \geq \beta_n - \zeta \geq -\frac{1}{2}\zeta, \tag{4.21}$$

for n large. By (4.19),

$$\left|\mathbb{E}_B\left[a_t(X_t^n, \mathbb{P}^B_{X_t^n}, \nu_t^n) D\tilde{\Phi}(t, X_t^n)\right]\right| \;\; > \;\; \varepsilon\ , \quad \text{for } t \in A_n,$$

and we can define the positive $\bar{\mathbb{F}}^\circ$-local martingale L^n by

$$L_t^n \;\; = \;\; 1 - \int_{t_n}^t L_s^n \rho_s^n |\psi_s^n|^{-2} \psi_s^n dB_s\ , \quad t \geq t_n\ .$$

The coefficients a and b being bounded, L^n is a true martingale. In view of (4.21), $L^n M^n$ is a non-negative local martingale that is bounded from below by a martingale. Therefore, it is a super-martingale and

$$0 \leq \mathbb{E}[L^n_{\theta_n} M^n_{\theta_n}] \leq L^n_{t_n} M^n_{t_n} = M^n_{t_n} = \beta_n - \zeta\ .$$

Sending n to ∞, we get a contradiction since $\beta_n \to 0$.

Part II. Subsolution property. Fix $(t_0, \chi_0) \in [0, T) \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ and a $\mathcal{C}_b^{1,2}$ function $\Phi : [0, T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)) \to \mathbb{R}$ such that

$$(V^* - \Phi)(t_0, \chi_0) \;\; = \;\; \max_{[0,T]\times\mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)} (V^* - \Phi). \tag{4.22}$$

We have to prove that

$$-\partial_t \Phi(t_0, \chi_0) + \mathcal{H}_*\big(t_0, \chi_0, D\Phi(t_0, \chi_0), D^2\Phi(t_0, \chi_0)\big) \;\; \leq \;\; 0\ .$$

We distinguish two cases.

1. Suppose that $V^*(t_0, \chi_0) = 0$. Then, we deduce from (4.22) that

$$\partial_t \Phi(t_0, \chi_0) \;\; \geq \;\; 0\ , \quad D\Phi(t_0, \chi_0) \;\; = \;\; 0 \quad \text{and} \quad D^2\Phi(t_0, \chi_0) \;\; \geq \;\; 0\ . \tag{4.23}$$

Let $(\varepsilon_n, t_n, \chi_n, P_n, Q_n)_{n\geq 1} \subset [0, 1] \times [0, T] \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d) \times \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d) \times S(\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d))$ be a sequence converging to $(0, t_0, \chi_0, D\Phi(t_0, \chi_0), D^2\Phi(t_0, \chi_0))$ such that

$$\mathcal{H}_{\varepsilon_n}(t_n, \chi_n, P_n, Q_n) \;\; \to \;\; \mathcal{H}_*(t_0, \chi_0, D\Phi(t_0, \chi_0), D^2\Phi(t_0, \chi_0))\ . \tag{4.24}$$

It follows from (4.23) that

$$\begin{aligned}
&\lim_{n\to+\infty} \mathcal{H}_{\varepsilon_n}(t_n, \chi_n, P_n, Q_n) \\
&\leq \lim_{n\to+\infty} -\frac{1}{2} \inf_{u\in\mathbf{L}_0(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathrm{U})} \mathbb{E}\Big[Q_n(a_{t_n}(\chi_n, \mathbb{P}_{\chi_n}, u)Z) a_{t_n}(\chi_n, \mathbb{P}_{\chi_n}, u)Z\Big].
\end{aligned}$$

Since a is continuous and bounded, it follows from the convergence of Q_n to $D\Phi(t_0, \chi_0)$ that

$$\lim_{n\to+\infty} \inf_{u\in\mathbf{L}_0(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathrm{U})} \mathbb{E}\Big[Q_n(a_{t_n}(\chi_n, \mathbb{P}_{\chi_n}, u)Z)a_{t_n}(\chi_n, \mathbb{P}_{\chi_n}, u)Z\Big] = \inf_{u\in\mathbf{L}_0(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathrm{U})} \mathbb{E}\Big[D^2\Phi(t_0, \chi_0)(a_{t_0}(\chi_0, \mathbb{P}_{\chi_0}, u)Z)a_{t_0}(\chi_0, \mathbb{P}_{\chi_0}, u)Z\Big].$$

Combining the above leads to

$$\lim_{n\to+\infty} \mathcal{H}_{\varepsilon_n}(t_n, \chi_n, P_n, Q_n) \leq -\frac{1}{2} \inf_{u\in\mathbf{L}_0(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathrm{U})} \mathbb{E}\Big[D^2\Phi(t_0, \chi_0)(a_{t_0}(\chi_0, \mathbb{P}_{\chi_0}, u)Z)a_{t_0}(\chi_0, \mathbb{P}_{\chi_0}, u)Z\Big] ,$$

so that (4.23) and (4.24) lead to

$$-\partial_t\Phi(t_0, \chi_0) + \mathcal{H}_*(t_0, \chi_0, D\Phi(t_0, \chi_0), D^2\Phi(t_0, \chi_0)) \leq 0 .$$

2. Suppose now that $V^*(t_0, \chi_0) = 1$. We argue by contradiction and suppose that

$$-\partial_t\Phi(t_0, \chi_0) + \mathcal{H}_*\big(t_0, \chi_0, D\Phi(t_0, \chi_0), D^2\Phi(t_0, \chi_0)\big) =: 4\eta > 0 .$$

Since the left hand-side is finite and $\mathcal{N}_0 \subset \mathcal{N}_\varepsilon$ for $\varepsilon \geq 0$, there exists an open neighborhood $\mathcal{O}$ of $(t_0, \chi_0, D\Phi(t_0, \chi_0))$ such that $\mathcal{N}_0 \neq \emptyset$ on $\mathcal{O}$ and there exists $u_0 \in \mathcal{N}_0(t_0, \chi_0, D\Phi(t_0, \chi_0))$ such that

$$-\partial_t\Phi(t_0, \chi_0) - \mathcal{L}_{t_0}^{u_0}\big(t_0, \chi_0, D\Phi(t_0, \chi_0), D^2\Phi(t_0, \chi_0)\big) \geq 2\eta .$$

Then, **(H2)** implies that for any $\varepsilon > 0$ there exists an open neighborhood $\mathcal{O}'$ of $(t_0, \chi_0, D\Phi(t_0, \chi_0))$ and a measurable map $\hat{u} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Omega^1 \to \mathrm{U}$ such that:
(i) $\mathbb{E}_B[|\hat{u}_{t_0}(\chi_0, P_0, \xi) - u_0|] \leq \varepsilon$
(ii) There exists $C > 0$ for which

$$\mathbb{E}[|\hat{u}_t(\chi, P, \xi) - \hat{u}_t(\chi', P', \xi)|^2] \leq C\mathbb{E}[|\chi - \chi'|^2 + \mathcal{W}_2^2(\mathbb{P}_P, \mathbb{P}_{P'})]$$

for all $(t, \chi, P), (t, \chi', P') \in \mathcal{O}'$.
(iii) $\hat{u}_t(\chi, P, \xi) \in \mathcal{N}_0(t, \chi, P)$ $\mathbb{P}^\circ - a.e.$, for all $(t, \chi, P) \in \mathcal{O}'$.

Define

$$\tilde{\Phi}(t,\chi) \;\; = \;\; \Phi(t,\chi) + |t-t_0|^2 + \mathbb{E}_B\big[|\chi-\chi_0|^2\big]^2 \, ,$$

for $(t,\chi) \in [0,T] \times \mathbf{L}_2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$. Then,

$$(\partial_t\tilde{\Phi}, D\tilde{\Phi}, D^2\tilde{\Phi})(t_0,\chi_0) = (\partial_t\Phi, D\Phi, D^2\Phi)(t_0,\chi_0).$$

The above combined with **(H1)-(H1')** shows that we can find some $\varepsilon > 0$ such that

$$-\partial_t\tilde{\Phi}(t,\chi) - \mathcal{L}_t^{\hat{u}_t(\chi,D\tilde{\Phi}(t,\chi),\xi)}(\chi, D\tilde{\Phi}(t,\chi), D^2\tilde{\Phi}(t,\chi)) \geq \eta \tag{4.25}$$

for all $(t,\chi) \in B_\varepsilon(t_0,\chi_0)$.

Let now $(t_n,\chi_n)_{n\geq 1}$ be a sequence of $[0,T] \times \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$ such that

$$\big(t_n,\chi_n,V(t_n,\chi_n)\big) \;\; \to \;\; \big(t_0,\chi_0,V^*(t_0,\chi_0)\big) \, , \tag{4.26}$$

and consider the solution X^n of (2.7) starting from χ_n at t_n and associated to the feedback control $\hat{\nu}^n := \hat{u}_.(X^n, D\tilde{\Phi}(.,X^n),\xi)$. The fact that X^n is well-defined is guaranteed by (ii) above, this is obtained by a straightforward extension of Proposition 2.1. We then define the stopping times θ_n by

$$\theta_n(\omega^\circ) \;\; = \;\; \inf\big\{s \geq t_n \; : \; (s, X_s^n(\omega^\circ,.) \notin B_\varepsilon(t_n,\chi_n)\big\} \, , \quad \omega^\circ \in \Omega^\circ \, .$$

Letting

$$-\zeta := \max_{\partial_p B_\varepsilon(t_0,\chi_0)} (V^* - \tilde{\Phi}) \;\; < \;\; 0 \, ,$$

we have $(V-\Phi)(\theta_n, X_{\theta_n}^n) \leq -\zeta$.

We then apply Corollary 4.1 and Remark 4.2, to deduce from (iii) and (4.25) that $\tilde{\Phi}(\theta_n, X_{\theta_n}^n) \leq \tilde{\Phi}(t_n,\chi_n)$ which implies $V(\theta_n, X_{\theta_n}^n) \leq \tilde{\Phi}(t_n,\chi_n) - \zeta$. Since $\tilde{\Phi}(t_n,\chi_n) \to 1$, we have $V(\theta_n, X_{\theta_n}^n) < 1$ for n large enough, which contradicts Theorem 3.1. □

Remark 4.2. *In the above proof, we needed to apply the chain rule formula to the map* W^2 *where* $W : \; \chi \in \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d) \mapsto \mathbb{E}[|\chi-\chi_0|^2]$ *for some* $\chi_0 \in \mathbf{L}_2(\Omega^1,\mathcal{F}^1,\mathbb{P}^1;\mathbb{R}^d)$. *Note that it is not lift-invertible. On the other hand,* *the fact that the chain rule formula of Proposition 4.5 hold can be verified by simple computations and the use of [15, Corollaries 2 and 3 of Theorem 5.13] and [13, Lemma 14.2]. Namely, given $(t, \chi, \nu) \in [0,T] \times \mathbf{X}_t \times \mathcal{U}$, set $X = X^{t,\chi,\nu}$. Then,*

$$
\begin{aligned}
W(X_s) =& W(\chi) + \int_t^s \mathbb{E}_B \left[2(X_r - \chi_0)^\top b_r(X_r, \mathbb{P}^B_{X_r}, \nu_r) \right] dr \\
& + \frac{1}{2} \int_t^s \mathbb{E}_B \left[2 I_d (a_r Z)(a_r Z)^\top (X_r, \mathbb{P}^B_{X_r}, \nu_r) \right] dr \\
& + \int_t^s \mathbb{E}_B \left[2(X_r - \chi_0)^\top a_r(X_r, \mathbb{P}^B_{X_r}, \nu_r)) \right] dB_r \\
=& W(\chi) + \int_t^s \mathbb{E}_B \left[DW(r, X_r) b_r(X_r, \mathbb{P}^B_{X_r}, \nu_r) \right] dr \\
& + \frac{1}{2} \int_t^s \mathbb{E}_B \left[D^2 W(r, X_r)(X_r)(a_r Z)(a_r Z)^\top (X_r, \mathbb{P}^B_{X_r}, \nu_r) \right] dr \\
& + \int_t^s \mathbb{E}_B \left[DW(r, X_r) a_r(X_r, \mathbb{P}^B_{X_r}, \nu_r)) \right] dB_r.
\end{aligned}
$$

The same obviously holds for W^2 by application of the usual Itô's formula.

We end this section with the derivation of the boundary condition at the terminal time T. To this end, let us define the function $g = 1 - \mathbb{1}_{\bar{G}}$ where

$$
\bar{G} \quad = \quad \left\{ \chi \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d) \ : \ \mathbb{P}_\chi \in G \right\}.
$$

Notice that $\bar{G}$ is a closed subset of $\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$ since G is closed for $\mathcal{W}_2$. Hence,

$$
g^* = 1 - \mathbb{1}_{\mathrm{int}(\bar{G})} \ , \ g_* = 1 - \mathbb{1}_{\bar{G}},
$$

where g^* and g_* stand for the upper and lower semi-continuous envelopes of g respectively.

Theorem 4.3. *Under **(H1)**, the function V satisfies*

$$
V^*(T, .) \quad = g^* \quad \textit{and} \quad V_*(T, .) \quad = g_*
$$

on $\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$.

Proof. (i) We first prove that $V^*(T,.) = g^*$. Since $V(T,.) = g$, we have $V^*(T,.) \geq g^*$. For the reverse inequality, we argue by contradiction and suppose that $1 = V^*(T,\chi) > g^*(\chi) = 0$ for some $\chi \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$. Since $g^*(\chi) = 0$, we know that $\chi \in \mathrm{int}(\bar{G})$. Let $(t_n, \chi_n)_n$ be a sequence such that $(t_n, \chi_n, V(t_n, \chi_n)) \to (T, \chi, 1)$. Fix some $u_0 \in \mathrm{U}$ and denote by X^{t_n,χ_n,u_0} the solution to (2.7) starting from χ_n at t_n and controlled by the constant processes $\nu = u_0$. Then, $X_T^{t_n,\chi_n,u_0} \in \bar{G}^c$, after possibly considering a subsequence. Sending n to ∞, we obtain that χ belongs to the closure of $\bar{G}^c$, which is a contradiction.
(ii) We now prove that $V_*(T,.) = g_*$. Since $V(T,.) = g$ we have $V_*(T,.) \leq g_*$. Again the reserve inequality is proved by contradiction. Suppose that $0 = V_*(T,\chi) < g_*(\chi) = 1$ for some $\chi \in \mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$. Since $g_* = g$, we know that $\chi \in \bar{G}^c$. Let $(t_n, \chi_n)_n$ be a sequence such that $(t_n, \chi_n, V(t_n, \chi_n)) \to (T, \chi, 0)$. Then, up to taking a subsequence, there exists $\nu^n \in \mathcal{U}$ such that $X_T^{t_n,\chi_n,\nu_n} \in \bar{G}$. Since a and b are continuous bounded and $\bar{G}$ is closed in $\mathbf{L}_2(\Omega^1, \mathcal{F}^1, \mathbb{P}^1; \mathbb{R}^d)$, we deduce that $\chi \in \bar{G}$ by sending n to ∞, which is a contradiction. $\square$

5 Additional remark on the choice of controls

In the above sections, the collection $\mathcal{U}$ of controls permits to take into account the exact value of the initial random variable χ, it is $\mathbb{F}$-progressively measurable. If we think in terms of controlling a population of particles which initial distribution is the law of χ, this means that we allow to have a different control for each of the particles. One could also consider the case where the control belongs to the subclass $\mathcal{U}^\circ$ of controls in $\mathcal{U}$ that are only $\bar{\mathbb{F}}^\circ$-progressively measurable. This would mean that the control of each particle does not depend on the position of each particle but only of the conditional repartition of the whole population of particles given B.
This can be treated in a similar way as the case we considered above. In particular, the result of Proposition 3.4 becomes trivial, see Proposition 2.3. In (3.10), the control ν will be $\bar{\mathbb{F}}^\circ$-progressively measurable and the map ϑ will take values in $\mathcal{U}^\circ$, so that $\bar{\nu}$ will actually be $\bar{\mathbb{F}}^\circ$-progressively measurable since the argument $X_\theta^{t,\chi,\nu}(\omega^\circ, \cdot)$ only enters as a random variable (not as the

value of the random variable). As for the first part of the proof of Theorem 3.1, the construction will just be simpler. Then, Theorem 3.1 actually holds for the class $\mathcal{U}^\circ$ as well. As for the PDE characterization of Theorem 4.2, we only have to replace $\mathcal{N}_\varepsilon(t,\chi,P)$ by $\{u \in \mathrm{U} \,:\, |\mathbb{E}_B[a_t(\chi,\mathbb{P}_\chi,u)P]| \le \varepsilon\}$, which changes the definition of $\mathcal{H}^*$ and $\mathcal{H}_*$ accordingly. Up to this modification, the proof is the same.

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