BSDEs with weak terminal conditions

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Problem formulation

- Given Ψ and m, find the minimal solution (Y, Z) to

$$Y_t \geq Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]$$

satisfying

$$E\left[\Psi(Y_T)\right] \geq m.$$

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□ "Weak terminal condition" : no fixed terminal condition, but a constraint in expectation.

□ Can look at it forward : stochastic target problem under controlled loss (B., Elie and Touzi 2009)

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$\Rightarrow$ Generalize previous results of B., Elie and Touzi (09) and Föllmer and Leukert (99,00).

Generalization and Standard assumptions

□ Given $\mu \in L^0(\mathcal{F}_\tau, [0,1])$, let $\Gamma(\tau, \mu)$ denote the set of super-solutions Y of

$$Y_{t\vee\tau} \geq Y_T + \int_{t\vee\tau}^T g(s, Y_s, Z_s)ds - \int_{t\vee\tau}^T Z_s dW_s, \quad t \in [0, T]$$

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□ Assumptions on Ψ : For $\mathbb{P}-$ a.e. $\omega \in \Omega$, $y \in \mathbb{R} \mapsto \Psi(\omega, y)$ is non-decreasing and valued in $[0,1] \cup \{-\infty\}$, its right-inverse $\Phi : \Omega \times [0,1] \mapsto [0,1]$ is measurable.

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□ Assumptions on g : Predictable for fixed (y, z) and uniformly Lipschitz in (y, z).

Problem reduction

- Let $\mathbf{A}_{\tau,\mu}$ be the set elements $\alpha \in \mathbf{H}_2$ such that

$$M^{(\tau,\mu),\alpha} := \mu + \int_{\tau}^{\tau \vee \cdot} \alpha_s dW_s \text{ takes values in } [0,1]$$

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and let $(Y^{(\tau,\mu),\alpha}, Z^{(\tau,\mu),\alpha})$ be the solution of

$$Y_{t\vee\tau} = \Phi(M_T^{(\tau,\mu),\alpha}) + \int_{t\vee\tau}^{T} g(s, Y_s, Z_s)ds - \int_{t\vee\tau}^{T} Z_s dW_s, \quad t \in [0,T]$$

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☐ **Proposition :**

$$Y \in \Gamma(\tau,\mu) \;\Leftrightarrow\; Y = Y^{(\tau,\mu),\alpha} \text{ for some } \alpha \in \mathbf{A}_{\tau,\mu}$$

and

$$\operatorname{essinf} \Gamma(\tau,\mu) = \mathcal{Y}_\tau(\mu) := \operatorname{essinf}\{Y_\tau^{(\tau,\mu),\alpha}, \alpha \in \mathbf{A}_{\tau,\mu}\}$$

Representation for the minimal supersolution

□ Given $m_o \in [0,1]$ fixed, we can look at the minimal condition under each path $M^\alpha := M^{(0,m_o),\alpha}$, $\alpha \in \mathbf{A}_{0,m_o} =: \mathbf{A}_0$:

$$\mathcal{Y}^\alpha_\cdot := \mathcal{Y}_\cdot(M^\alpha_\cdot)$$

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□ **Theorem** : For $\alpha \in \mathbf{A}_0$, $\mathcal{Y}^\alpha$ is indistinguishable from a làdlàg g-submartingale, and

(i) $\mathcal{Y}^\alpha_{\tau_1} = \operatorname{ess}\inf_{\bar{\alpha} \in \mathbf{A}^\alpha_{\tau_1}} \mathcal{E}^g_{\tau_1,\tau_2}[\mathcal{Y}^{\bar{\alpha}}_{\tau_2}]$, for each $\tau_1 \leq \tau_2 \in \mathcal{T}$.

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Under the additional assumption that

$$m \in [0, 1] \mapsto \Phi(\omega, m) \text{ is continuous for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

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(ii) $\mathcal{Y}^\alpha$ is indistinguishable from a càdlàg g-submartingale, for each $\alpha \in \mathbf{A}_0$.

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(ii) $\mathcal{Y}^\alpha$ is indistinguishable from a càdlàg g-submartingale, for each $\alpha \in \mathbf{A}_0$.

(iii) There exists a (unique non-anticipating) family $(\mathcal{Z}^\alpha, \mathcal{K}^\alpha)_{\alpha \in \mathbf{A}_0} \subset \mathbf{H}_2 \times \mathbf{K}_2$ s.t.

$$\begin{aligned}
\mathcal{Y}^\alpha &= \Phi(M_T^\alpha) + \int_\cdot^T g(s, \mathcal{Y}_s^\alpha, \mathcal{Z}_s^\alpha) ds - \int_\cdot^T \mathcal{Z}_s^\alpha dW_s + \mathcal{K}^\alpha - \mathcal{K}_T^\alpha \\
\mathcal{K}_{\tau_1}^\alpha &= \operatorname{ess\,inf}_{\bar\alpha \in \mathbf{A}_{\tau_1}^\alpha} E\left[\mathcal{K}_{\tau_2}^{\bar\alpha} | \mathcal{F}_{\tau_1}\right], \ \forall\, \tau_1 \le \tau_2 \in \mathcal{T}.
\end{aligned}$$

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□ Rem : It g and Φ are convex a.s., then the essinf over α is achieved by some $\hat{\alpha}$ and $\mathcal{Y}^{\hat{\alpha}} = \operatorname{essinf} \Gamma(\cdot, M^{\hat{\alpha}})$.

Continuity in the μ-parameter

A modulus of continuity :

$$Err_t(\eta) := \operatorname{esssup}\left\{\mathcal{R}_t(M, M') : M, M' \in \mathbf{L}_0([0,1]), E_t[|M - M'|^2] \leq \eta\right\},$$

in which

$$\mathcal{R}_t(M, M') := |\mathcal{E}_t^g[\Phi(M)] - \mathcal{E}_t^g[\Phi(M')]|.$$

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□ **Proposition :** Fix $t < T$, $\mu_1, \mu_2 \in \mathbf{L}_0([0,1], \mathcal{F}_t)$. Then,

$$|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)| \le Err_t(\Delta(\mu_1, \mu_2)) + Err_t(\Delta(\mu_2, \mu_1)),$$

where

$$\Delta(\mu_i, \mu_j) := (1 - \frac{\mu_i}{\mu_j})\mathbf{1}_{\{\mu_i < \mu_j\}} + \frac{\mu_i - \mu_j}{1 - \mu_j}\mathbf{1}_{\{\mu_i > \mu_j\}}, \ i, j = 1, 2.$$

Continuity in the μ-parameter

Moreover, on $\{\mu_1 = 0\}$:

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and, on $\{\mu_1 = 1\}$:

$$\begin{aligned}
&|\mathcal{Y}_t(\mu_1) - \mathcal{Y}_t(\mu_2)| \\
&\leq \operatorname{esssup}\left\{\mathcal{R}_t(1, M) \ : \ M \in \mathbf{L}_0([0,1]) \ , \ \ E_t[|1 - M|^2] \leq 1 - \mu_2\right\}.
\end{aligned}$$

Convexity and convexification in the μ-parameter

Definition [$\mathcal{F}_t$-convexity]

(i) $D \subset \mathbf{L}_\infty(\mathbb{R}, \mathcal{F}_t)$ is $\mathcal{F}_t$-convex if $\lambda\mu_1 + (1-\lambda)\mu_2 \in D$, $\forall$ $\mu_1, \mu_2 \in D$ and $\lambda \in \mathbf{L}_0([0,1], \mathcal{F}_t)$.

(ii) Let D be a $\mathcal{F}_t$-convex subset of $\mathbf{L}_\infty(\mathbb{R}, \mathcal{F}_t)$. A map $\mathcal{J} : D \mapsto \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t)$ is said to be $\mathcal{F}_t$-convex if

$$\mathrm{Epi}(\mathcal{J}) := \{(\mu, Y) \in D \times \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t) : \; Y \geq \mathcal{J}(\mu)\}$$

is $\mathcal{F}_t$-convex.

(iii) Let $\mathrm{Epi}^c(\mathcal{Y}_t) = \mathcal{F}_t\text{-conv}(\mathrm{Epi}(\mathcal{Y}_t))$ and $\overline{\mathrm{Epi}}^c(\mathcal{Y}_t)$ its closure in $\mathbf{L}_2$. Then,

$$\mathcal{Y}_t^c(\mu) := \mathrm{essinf}\{Y \in \mathbf{L}_2(\mathbb{R}, \mathcal{F}_t) : (\mu, Y) \in \overline{\mathrm{Epi}}^c(\mathcal{Y}_t)\}$$

is the $\mathcal{F}_t$-convex envelope of $\mathcal{Y}_t$.

Convexity and convexification in the μ-parameter

□ **Proposition :** Assume that $\mathcal{Y} = \mathcal{Y}_{\cdot +}$. Then, the map

$$\mu \in \mathbf{L}_0([0,1], \mathcal{F}_t) \mapsto \mathcal{Y}_{t*}(\mu)$$

is $\mathcal{F}_t$-convex, for all $t < T$, where

$$\mathcal{Y}_{t*}(\mu) := \lim_{\varepsilon \to 0} \operatorname{essinf}\{\mathcal{Y}_t(\mu') : \ |\mu' - \mu| \le \varepsilon, \ \mu' \in \mathbf{L}_0([0,1], \mathcal{F}_t)\},$$

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- Rem : Note that
  - $\mathcal{Y} = \mathcal{Y}_{\cdot+}$ when Φ is a.s. continuous.

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- **Proposition :** Assume Φ deterministic and its convex envelope $\hat{\Phi}$ continuous on $[0,1]$. Then,

$$\mathcal{Y}^{\alpha}_{T-} = \hat{\Phi}(M^{\alpha}_T) \ \text{ and } \ \mathcal{Y}^{\alpha}_{\tau} = \operatorname{ess}\inf_{\alpha' \in \mathbf{A}^{\alpha}_{\tau}} \mathcal{E}^{g}_{\tau}\left[\hat{\Phi}(M^{\alpha'}_T)\right],$$

for all $\alpha \in \mathbf{A}_0$ and $\tau \in \mathcal{T}$ such that $\tau < T$.

Duality and optimal control in standard form

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□ Fenchel transforms :

$$\tilde{\Phi}(\omega, l) := \sup_{m \in [0,1]} \left(ml - \Phi(\omega, m) \right)$$

and

$$\tilde{g}(\omega, t, u, v) := \sup_{(y,z) \in \mathbb{R} \times \mathbb{R}^d} \left(yu + z^\top v - g(\omega, t, y, z) \right).$$

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□ Λ = predictable λ s.t. $\lambda_t(\omega) \in \mathrm{dom}(\tilde{g}(\omega, t, \cdot))$ $\mathrm{Leb} \times \mathbb{P}$-a.e.

Duality and optimal control in standard form

□ **Dual optimal control problem :**

Set for $\lambda = (\nu, \vartheta)$

$$L_t^\lambda = 1 + \int_0^t L_s^\lambda \nu_s ds + \int_0^t L_s^\lambda \vartheta_s dW_s$$

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and

$$\mathcal{X}_0(l) := \inf_{\lambda \in \Lambda} \mathrm{X}_0^{l,\lambda}$$

where

$$\mathrm{X}_0^{l,\lambda} := E\left[\int_0^T L_s^\lambda \tilde{g}(s, \lambda_s) ds + L_T^\lambda \tilde{\Phi}(l/L_T^\lambda)\right].$$

Duality and optimal control in standard form

□ **Theorem :** Under "good" assumptions (in particular existence in one of the two problems) :

$$\mathcal{Y}_0(m) = \sup_{l>0}(ml - \mathcal{X}_0(l))$$

and

$$\mathcal{X}_0(l) = \sup_{m>0}(ml - \mathcal{Y}_0(m))$$

+ standard explicit relations between the optimizers.

Duality and optimal control in standard form

□ **Rem** : In the quantile hedging problem for the BS model :

$$\Phi(\omega, m) = mg(S_T(\omega)) \ , \ g(\omega, y, z) = z\mu/\sigma.$$

In particular,

$$\tilde{\Phi}(\omega, l) = [l - g(S_T(\omega))]^+ \ , \ \tilde{g}(\omega, u, v) = +\infty \mathbf{1}_{\{(u,v) \neq (0,\mu/\sigma)\}}$$

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It follows that

$$\mathcal{X}_0(l) = E\left[L^o_T[l/L^o_T - g(S_T(\omega))]^+\right]$$

with

$$L^o_t = 1 + \int_0^t L^o_s(\mu/\sigma)dW_s.$$

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