First time to exit of a continuous Itô process: general moment estimates and $\mathbf{L}_1$-convergence rate for discrete time approximations

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Abstract

We establish general moment estimates for the discrete and continuous exit times of a general Itô process in terms of the distance to the boundary. These estimates serve as intermediate steps to obtain strong convergence results for the approximation of a continuous exit time by a discrete counterpart, computed on a grid. In particular, we prove that the discrete exit time of the Euler scheme of a diffusion converges in the $\mathbf{L}_1$ norm with an order 1/2 with respect to the mesh size. This rate is optimal.

Key words: Exit time; Strong approximation; Euler scheme.

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1 Introduction

This paper is motivated by the study of the strong convergence rate of the discrete time approximation of the first exit time θ of a process Z from a non-empty open subset $\mathcal{O}$.

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The interest for numerical discretization of diffusion processes dates back to the sixties, see [Mar55, Mul56] and [KP95] for general references. Different approaches can be used to approximate the first exit time of a diffusion. We briefly recall them for the sake of completeness and to make clear the contribution of this paper.

a. By the very nature of the problem, space discretization schemes naturally appear. The first version is based on the Walk On Sphere (WOS) schemes introduced in [Mul56]. In the Brownian motion case one simulates its position by the first hitting time of a ball contained in the domain and centered at the starting point: the position is uniformly distributed on the sphere and thus straightforward to sample. The sampled point is then used as a new starting point. One repeats the above procedure until one gets close enough to the boundary of $\mathcal{O}$. For a time-homogeneous diffusion process X the scheme is modified using small balls and an Euler-Maruyama approximation. In [Mil96, Mil98] strong error estimates on the *exit position* X_θ are proved, assuming in particular that the domain $\mathcal{O}$ is convex and that the diffusion coefficient satisfies a uniform ellipticity condition. These results do not include an approximation of the exit time θ. Weak approximation results – i.e. for $\mathbb{E}[\varphi(X_\theta)]$ with φ continuous and bounded – are established in [Mil97].

b. For polygonal domains moving from spheres to spheres may not be suitable because of the corners. One has to replace balls by parallelepipeds (tensor products of intervals). Exit times from parallelepipeds are easy to sample. Faure [Fau92] was probably the first one who developed these ideas. In [MT99] these ideas are further analyzed for diffusion processes with time-dependency by exploiting small parallelepipeds. Strong error estimates of the exit position and the exit time are established: the order of convergence of the exit time approximation is $1-\varepsilon$ with respect to the space step (for any $0<\varepsilon<1$), i.e. equivalently $\frac{1}{2}-\varepsilon$ (for any $0<\varepsilon<1/2$) with respect to the time step, see [MT99, Theorem 8.2]. Here again, convexity of $\mathcal{O}$ and strong ellipticity were assumed. Related simulations are discussed in [ZLD10]. Extensions to non-small parallelepipeds are investigated in [DL06].

c. To maintain a certain simplicity of the simulation, one can alternatively perform the usual Euler scheme on a grid π with deterministic time step $|\pi|$ and stop when it exits $\mathcal{O}$. This is a crude approximation, nevertheless the simplest and quickest to use: this is why it has gained much interest in the applied probability community. It results in an order of weak convergence equal to $\frac{1}{2}$ with respect to $|\pi|$, see [Gob00, GM10]. Interestingly, it is shown in [GM07] that this order of weak convergence remains valid for general Itô processes, far beyond the usual diffusion framework in which one can rely on PDE tools to decompose the error. The strong convergence of the exit time is stated in [GM05, Theorem 4.2] but without speed. Finally, note that different techniques can be used to speed-up the convergence in the weak sense: sampling the continuous time exit using diffusion bridge techniques [Bal95, Gob00, BC02] (possibly with local modifications of the boundary [Gob01, BP06] or exponential-time stepping [JL05]) or using discrete exit times combined with an inward shifting of the boundary [GM10]. To our knowledge, no strong error estimates are available for these schemes.

As a matter of fact, until recently only little was known about the rate of $\mathbf{L}_1$ convergence of the discrete exit time of an Euler scheme of a diffusion towards the exit time of the exact diffusion, although there are important fields where the $\mathbf{L}_1$ criterion is the only relevant one. As examples let us mention the approximation of backward stochastic differential equations considered in a domain [BM09] and the multi level Monte Carlo methods [Hei01, Gil08]. In [BM09, Theorem 3.1] the authors prove that the convergence rate of the discrete exit time of the Euler scheme is of order $\frac{1}{2} - \varepsilon$ with respect to $|\pi|$ (for any $0 < \varepsilon < 1/2$). Because of the aforementioned applications the question whether one can take $\varepsilon = 0$ in the previous estimate has been raised. Also, their arguments are restricted to finite time horizons and the question whether they could be extended to an infinite time horizon was open.

In this paper we answer these questions to the positive: the discrete exit time of an Euler scheme converges at the rate $1/2$ in the $\mathbf{L}_1$ norm, even if the time horizon is unbounded, see Theorem 3.7. In the same theorem we show that the stopped process converges at the rate $1/4$ in $\mathbf{L}_2$. Theorem 3.7 follows from an abstract version stated in Theorem 3.1, which we establish in a non-Markovian setting in the spirit of [GM07]. As a first step of our analysis we provide general controls on the expected time to exit in terms of the distance to the boundary, see Theorems 2.3 and 2.4 below. They are established both for continuous exit times and for discrete exit times, i.e. the latter are restricted to take values on a discrete grid. Essentially, we only use a mild non-characteristic boundary type condition and a uniform bound on the conditional expected times to exit. The fact that, as opposed to most of the papers quoted above, we analyze situations with unbounded time horizon in a $\mathbf{L}_\infty$ sense is delicate because the usual finite-time error estimates, e.g. on Euler schemes, blow up exponentially with respect to the time horizon.

In fact our results allow to address much more general problems than the first exit time approximations for Markovian stochastic differential equations. In terms of applications, many optimal stopping, impulse control, singular control or optimal monitoring problems have solutions given by the hitting times of a domain $\mathcal{O}$ by a state process Z, see e.g. [S07], [BL84], [ShSo94], [N90, Fu11, GL14]. In practice, the process Z is only monitored in discrete time and one needs to know how well these hitting times will be approximated by counterparts computed on a finite grid. In terms of modeling, there is also an increasing need in non-Markovian or infinite dimensional settings, in which there is no clear connection between exit times and PDEs with Dirichlet boundary conditions. A typical example is the HJM framework for interest rates, see [HJM92], but this can more generally refer to path-dependent SDEs, see e.g. [Bu00], or to stochastic evolution equations on Banach spaces, see e.g. [GyMi05].

The variety of possible applications motivates the abstract setting of Section 2 and Section 3.1 in which we provide our general moment and approximation estimates on the first exit time of a process Z from a domain $\mathcal{O}$. This process does not need to be neither Markov, nor finite dimensional, we only impose an Itô dynamic for the distance to the boundary and assume that it satisfies a non-characteristic type boundary condition, see Assumption **(P)**. In this general setting, we prove in particular that

$$\mathbb{E}\left[|\theta - \theta^{\pi}|\right] = O(|\pi|^{\frac{1}{2}})$$

where θ is the first exit time of Z, and θ^{π} is its counterpart computed on a time grid π, with modulus $|\pi|$, see Theorem 3.1 applied to $Z = X = \bar{X}$. The result remains true when an extra approximation is made on Z and the corresponding distance process converges in $\mathbf{L}_1$ at a rate $1/2$. We shall check our general assumptions in details only for the application to the first exit time approximation of SDEs, see Section 3.

We would like to insist on the fact that, even in the simpler context of a Markovian SDE, the advantage of the abstract results of Section 2 is that they can be applied simultaneously and without extra effort to the original diffusion process and to its Euler scheme. We are not aware of any specific proof that would simplify and shorten our argumentation when using the particular setting of Markovian SDEs.

The paper is organized as follows: In Section 2 we introduce a general set-up followed by the statement of our quantitative results on the moments of the first time to exit. The proof of the main results, Theorems 2.3 and 2.4, is split into several subsections. We first establish general Freidlin type inequalities on moments of exit times, which will be controlled in terms of the probability of sub-harmonic paths in Section 2.4. Estimates on this probability yield to the proof of Theorem 2.3, that applies to continuous exit times. A final recursion argument is needed to pass from continuous exit times to discrete exit times, see Section 2.6. The application to the exit time approximation error is discussed in Section 3, first in an abstract setting, then for the solution of a stochastic differential equation whose exit time is estimated by the discrete exit time of its Euler scheme.

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a d-dimensional Brownian motion W. We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right-continuous completion of the natural filtration induced by W. The symbol $\mathcal{T}$ denotes the set of stopping times that are finite a.s. We write $\mathbb{E}_\tau$ and $\mathbb{P}_\tau$ for the conditional expectation and probability, respectively, given $\mathcal{F}_\tau$. Inequalities between random variables are usually understood in the a.s.-sense without mentioning it. Finally, given a vector $a \in \mathbb{R}^d$ or a matrix $A \in \mathbb{R}^{m\times n}$, the notation $|a|$ and $|A|$ stands for the Euclidean and the Hilbert-Schmidt norm, respectively.

2 Moment estimates for continuous and discrete exit times

The main results of this section are Theorems 2.3 and 2.4. They are the basis to prove Theorem 3.1, which is the main result of the paper in its abstract form.

2.1 Assumptions

Let $(\mathcal{Z}, d_{\mathcal{Z}})$ be a metric space equipped with the Borel σ-algebra generated by the open sets. In the following we fix an open set $\mathcal{O}$ of $\mathcal{Z}$ with

$$\emptyset \neq \mathcal{O} \subsetneq \bar{\mathcal{O}} \subsetneq \mathcal{Z},$$

in which $\bar{\mathcal{O}}$ denotes the closure of $\mathcal{O}$, and let $(Z_t)_{t\geq 0}$ be a continuous $\mathbb{F}$-adapted $\mathcal{Z}$-valued process starting in $Z_0 \equiv z_0 \in \mathcal{O}$.

The two main results of this section concern estimates on the time taken by the process Z to reach the boundary of $\mathcal{O}$, where the corresponding exit time takes values in a set π which either coincides with $\mathbb{R}_+$ or equals to a countable subset of $\mathbb{R}_+$, that can be thought to be the discretisation points in time of an approximation scheme. Therefore the standing assumption of this section is that either

(a) $\pi = \mathbb{R}_+$,

(b) or π consists of a strictly increasing sequence $0 = t_0 < t_1 < t_2 < \cdots$ with $\lim_n t_n = \infty$ and $|\pi| = \sup_{n\geq 1} |t_n - t_{n-1}| \leq 1$.

In both cases, we set

$$\phi_t := \max\{s \in \pi \ : \ s \leq t\} \text{ and } \phi_t^+ := \min\{s \in \pi \ : \ s \geq t\}, \tag{1}$$

which are the closest points in π to the left and to the right of t.

Our first assumption concerns the path regularity of the process Z. To simplify the notation, we set

$$\gamma(t,s) := d_{\mathcal{Z}}(Z_t, Z_s), \ \ t,s \geq 0. \tag{2}$$

Assumption (Z) (Regularity of Z along π). *There is a locally bounded map $\kappa : \mathbb{R}_+ \times (0,\infty) \mapsto \mathbb{R}_+$ such that*

$$\mathbb{P}_\tau \left[\sup_{\tau \leq t \leq \tau + T} \gamma(t, \phi_t \vee \tau) > \rho \right] \leq \kappa(T,\rho)|\pi|$$

for all $\tau \in \mathcal{T}$, $T \geq 0$, and $\rho > 0$.

Although the condition **(Z)** is - so far - a condition on a single fixed time-net π, we require the upper bound in a form of a product $\kappa(T,\rho)|\pi|$. As shown in Lemma A.1 below, this is a typical form that is also required in our later computations. Our next set of assumptions concerns the behaviour of the process Z close to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$.

Assumption (P) (Distance process $\delta(Z)$). *There exist $L \geq 1$ and an L-Lipschitz function $\delta : \mathcal{Z} \mapsto \mathbb{R}$ such that $\delta > 0$ on $\mathcal{O}$, $\delta = 0$ on $\partial\mathcal{O}$, and $\delta < 0$ on $\bar{\mathcal{O}}^c$. In addition, the process $P := \delta(Z)$ admits the Itô process decomposition*

$$P_t = P_0 + \int_0^t b_s ds + \int_0^t a_s^\top dW_s \tag{3}$$

for $t \geq 0$, where

(i) *(P,b,a) is a predictable process with values in $[-L,L]^{d+2}$,*

(ii) *there is a fixed $r \in (0, L^{-3}/4)$ and a set $\Omega_r \in \mathcal{F}$ of measure one, such that $|P_t(\omega)| \vee \gamma(t,\phi_t)(\omega) \leq r$ implies that $|a_t(\omega)| \geq 1/L$ whenever $\omega \in \Omega_r$ and $t \geq 0$.*

Before we continue, let us comment on the latter assumptions.

Remark 2.1. (a) The process $P = \delta(Z)$ measures the algebraic distance of Z to the boundary $\partial\mathcal{O}$ in terms of the function δ. The existence of a signed distance δ that is 1-Lipschitz can be checked in various settings easily (starting from the usual distance one can check whether for all segments $[x,y] = \{z \in \mathcal{Z} : d_{\mathcal{Z}}(x,z) + d_{\mathcal{Z}}(z,y) = d_{\mathcal{Z}}(x,y)\}$ with $x \in \mathcal{O}$ and $y \in (\bar{\mathcal{O}})^c$ the intersection $[x,y] \cap \partial\mathcal{O}$ is non-empty), and it can be modified outside a suitable neighborhood of $\partial\mathcal{O}$ in order to be uniformly bounded.

(b) The Itô decomposition (3) may implicitly impose additional smoothness assumptions on $\partial\mathcal{O}$: for instance, if Z is an $\mathbb{R}^d$-valued Itô process, then P is also an Itô process provided that the domain is C^2 with compact boundary, see [GM07, Proposition 2.1]. Hence, the condition (i) is not too restrictive.

(c) The coefficients b and a may depend on π. This will be the case in Section 3.2 when our abstract results will be applied to an Euler scheme.

(d) The condition (ii) is a uniform non-characteristic boundary condition. It ensures that the fluctuation of the paths of Z are not tangential to the boundary. When Z solves a SDE with diffusion coefficient $\sigma(\cdot)$, i.e. $a_t^\top = D\delta(Z_t)\sigma(Z_t)$, see Section 3.2, then the natural non-characteristic boundary condition is

$$|D\delta(z)\sigma(z)| \geq 1/L \quad \text{if} \quad |\delta(z)| \leq r, \tag{4}$$

i.e. $|a_t| \geq 1/L$ if $|P_t| \leq r$. In the case of an Euler scheme $\bar{Z}$, see (17), we have $\bar{a}_t^\top = D\delta(\bar{Z}_t)\sigma(\bar{Z}_{\phi_t})$ and $\bar{P}_t = \delta(\bar{Z}_t)$. The natural condition (4) is no more sufficient to ensure that $|\bar{a}_t| \geq 1/L$ if $|\bar{P}_t| \leq r$. But, by a continuity argument, it is satisfied if the point $\bar{Z}_{\phi_t}$ at which the diffusion coefficient is evaluated is not too far from the current position $\bar{Z}_t$, i.e. $\gamma(t,\phi_t)$ is small as well. See Lemma A.3 below.

Now we can define the main objects of this section: given $\ell \geq 0$, $\tau \in \mathcal{T}$, and an integer $p \geq 1$, we set

$$\begin{aligned}
\theta_\ell(\tau) &:= \inf\{t \geq \tau \ : \ P_t \leq \ell\}, \\
\theta_\ell^\pi(\tau) &:= \inf\{t \geq \tau \ : \ t \in \pi, \ P_t \leq \ell\}, \\
\Phi_\ell^p(\tau) &:= \mathbb{E}_\tau\left[(\theta_\ell(\tau) - \tau)^p\right]^{\frac{1}{p}}, \\
\Phi_\ell^{p,\pi}(\tau) &:= \mathbb{E}_\tau\left[(\theta_\ell^\pi(\tau) - \tau)^p\right]^{\frac{1}{p}}.
\end{aligned}$$

Our aim is to provide pointwise estimates on $\Phi_0^1(\tau)$ and $\Phi_0^{1,\pi}(\tau)$. Our arguments require an additional control on the first conditional moment of the times to exit.

Assumption (L) (Uniform bound on expectations of exit times). *One has that $\Phi_0^{1,\pi}(\tau) \leq L$ for all $\tau \in \mathcal{T}$.*

In Assumption **(L)** (similarly in Proposition 2.2 and Lemma 2.8 below) we keep in mind that $\theta_0(\tau) \leq \theta_0^\pi(\tau)$. Therefore one has $\Phi_0^1(\tau) \leq \Phi_0^{1,\pi}(\tau)$, so that $\Phi_0^{1,\pi}(\tau) \le L$ automatically implies $\Phi_0^1(\tau) \le L$. It should be emphasized that Assumption **(L)** concerns the given process $(Z_t)_{t\ge 0}$ and distance δ, and therefore the fixed distance process $(P_t)_{t\ge 0}$, and that the same constant $L \ge 1$ as before is taken for notational simplicity. We refer to [Fre85, Chapter III, Lemma 3.1] for sufficient conditions ensuring that the exit times of a stochastic differential equation have finite moments, that are bounded only in terms of the diameter of the domain, the bounds on the coefficients of the stochastic differential equation and a partial ellipticity condition.

In Lemma 2.8 below, we show that **(L)** implies that $\theta_0^\pi(\tau) - \tau$ has finite exponential moments, uniformly in $\tau \in \mathcal{T}$. We conclude this subsection with some equivalent variants of condition **(L)**. The proof is provided in the Appendix.

Proposition 2.2. *The condition* **(L)** *is equivalent to either of the following ones:*

(L') *There is a $L' \ge 1$ such that, for all $\tau \in \mathcal{T}$,*

$$\Phi_0^{1,\pi}(\tau) \le L' \text{ a.s. on } \{P_\tau > 0\}.$$

(L") *There exist $c > 0$ and $\alpha \in (0,1)$ such that, for all $\tau \in \mathcal{T}$,*

$$\mathbb{P}_\tau[\theta_0^\pi(\tau) \ge \tau + c] \le \alpha.$$

2.2 First moment control near the boundary

Now we are in a position to state the main results of this section. We will denote by $\mathcal{T}^\pi$ the set of stopping times with values in π. Remember that the following can be applied to situations where $\pi = \mathbb{R}_+$, in which case assumption **(Z)** is automatically satisfied and the extra term $|\pi|^{\frac{1}{2}}$ below vanishes.

Theorem 2.3. *Let the assumptions* **(Z)**, **(P)** *and* **(L)** *be satisfied.*

(a) *If $\tau \in \mathcal{T}^\pi$, then*

$$\Phi_0^1(\tau) \le c_{(2.3)} \Big[P_\tau + |\pi|\Big] \mathbf{1}_{\{P_\tau \ge 0\}},$$

where $c_{(2.3)} = c_{(2.3)}(r, L, d, \kappa) > 0$.

(b) *If $\tau \in \mathcal{T}$, then*

$$\Phi_0^1(\tau) \le d_{(2.3)} \Big[P_\tau + |\pi|^{\frac{1}{2}}\Big] \mathbf{1}_{\{P_\tau \ge 0\}},$$

where $d_{(2.3)} = d_{(2.3)}(r, L, d, \kappa) > 0$.

The proof of this theorem will be given in Section 2.5 below. Its counterpart for discrete exit times corresponds to the following statement when $\pi \neq \mathbb{R}_+$, and is proved in Section 2.6.

Theorem 2.4. *Let the assumptions* **(Z)**, **(P)** *and* **(L)** *be satisfied. Then there exists an* $\varepsilon_{(2.4)} = \varepsilon_{(2.4)}(r, L, d, \kappa) > 0$ *such that if* $|\pi| \leq \varepsilon_{(2.4)}$ *then one has*

$$\Phi_0^{1,\pi}(\tau) \leq d_{(2.4)} \left[|P_\tau| + |\pi|^{\frac{1}{2}} \right] \quad \text{for} \quad \tau \in \mathcal{T},$$

where $d_{(2.4)} = d_{(2.4)}(r, L, d, \kappa) > 0$.

Theorem 2.3 is similar to [GM07, Lemma 4.2], in which the time horizon is bounded and the counterpart of **(P-*ii*)** does not require $\gamma(\cdot, \phi) \leq r$. Our additional requirement yields to a weaker assumption and explains the presence of the additional $|\pi|$-terms in our result. We also refer to [Fre85, Chapter III, Section 3.3] who considers a Markovian setting for a uniformly fast exit of a diffusion from a domain.

Theorem 2.4 is of similar nature but is much more delicate to establish. An attempt to obtain such a result for the Euler scheme of stochastic differential equations on a finite time horizon can be found in [BM09] by a combination of their Lemmas 5.1, 5.2 and 5.3. However, they were only able to achieve a bound in $O_{|\pi| \to 0}(|\pi|^{\frac{1}{2}-\varepsilon})$ for all $0 < \varepsilon < 1/2$. We shall comment on this in Section 3 below. The absolute values on P_τ account for the case where Z_τ is outside $\mathcal{O}$ and $\tau \notin \mathcal{T}^\pi$ yielding a positive time to exit.

The proofs of the above theorems are divided in several steps and provided in the next subsections (see Sections 2.5 and 2.6 for the final arguments). Both start with arguments inspired by [Fre85] and that were already exploited in [BM09]. One important novelty is our set of assumptions where we do not use any Markovian hypothesis and where we only assume that the delay to exit is uniformly bounded in expectation with respect to the initial time. Furthermore, we also refine many important estimates of [BM09] and use a new final recursion argument which is presented in Section 2.6. This recursion is crucial in order to recover the bound $O_{|\pi| \to 0}(|\pi|^{\frac{1}{2}})$, in contrast to the bound $O_{|\pi| \to 0}(|\pi|^{\frac{1}{2}-\varepsilon})$ in [BM09].

Remark 2.5. Lemma 2.8 below implies the same estimates for $(\Phi_0^{p,\pi}(\tau))^p$ and $(\Phi_0^{p,\pi}(\tau))^p$, $p \geq 2$, as obtained for $\Phi_0^{1,\pi}(\tau)$ and $\Phi_0^{1,\pi}(\tau)$ in Theorems 2.3 and 2.4.

Remark 2.6. Theorems 2.3 and 2.4 extend to the case where $\mathcal{O}$ is the intersection of countable many $(\mathcal{O}_i)_{i \in I}$ satisfying the assumptions **(P)** and

(L) for some family of processes $(P^i)_{i\in I}$ with the same $L \geq 1$ and $r \in (0, L^{-3}/4)$. Indeed, denote by $\Phi_{0i}^{1,\pi}$ and Φ_{0i}^1 the counterparts of $\Phi_0^{1,\pi}$ and Φ_0^1 associated to $\mathcal{O}_i$, $i \in I$, then we have, a.s.,

$$\Phi_0^1(\tau) \leq \inf_{i\in I} \Phi_{0i}^1(\tau) \text{ and } \Phi_0^{1,\pi}(\tau) \leq \inf_{i\in I} \Phi_{0i}^{1,\pi}(\tau)$$

whenever $\mathcal{O} = \cap_{i\in I}\mathcal{O}_i$.

Remark 2.7. Take $d = 1$, $\mathcal{O} = (-\infty, 1) \subset \mathcal{Z} = \mathbb{R}$, $\pi = \mathbb{R}_+$, and let $Z = |W|^2 + z_0$ with $1/2 < z_0 < 1$. As distance function take an appropriate $\delta \in C^\infty(\mathbb{R})$ with δ constant outside $(0, 2)$ and $\delta(z) = 1 - z$ on $[1/2, 3/2]$. Then the conditions **(Z)**, **(P)** and **(L)** are satisfied and $\Phi_0^1(0) = \mathbb{E}[\theta_0(0)] = \mathbb{E}\left[|W_{\theta_0(0)}|^2\right] = 1 - z_0 = P_0$, which coincides with the upper-bound of Theorem 2.3 up to a multiplicative constant.

2.3 Freidlin type inequalities on moments of exit times

We start with a-priori estimates inspired by the proof of the exponential fast exit of Freidlin [Fre85, Lemma 3.3, Chapter 3]: a uniform bound on the conditional expected times to exit implies the existence of uniform conditional exponential moments for these exit times. We adapt Freidlin's arguments to our setting.

Lemma 2.8. *Let assumption* **(L)** *hold, $p \geq 1$ be an integer, $L^{(p)} := p!L^p$, and $\tau \in \mathcal{T}$. Then we have*

$$(\Phi_0^p(\tau))^p \leq c_{p,(2.8)}\Phi_0^1(\tau) \quad \textit{and} \quad (\Phi_0^{p,\pi}(\tau))^p \leq c_{p,(2.8)}\Phi_0^{1,\pi}(\tau)$$

with $c_{p,(2.8)} := pL^{(p-1)}$. Consequently,

$$\begin{aligned} (\Phi_0^{p,\pi}(\tau))^p &\leq L^{(p)}, \\ \mathbb{E}_\tau\left[e^{c(\theta_0^\pi(\tau)-\tau)}\right] &\leq (1 - cL)^{-1}, \end{aligned}$$

where $c \in [0, L^{-1})$.

Proof. 1. The estimates for $\Phi_0^p(\tau)$ and $\Phi_0^{p,\pi}(\tau)$ are obtained in the same way, we only detail the second one by an induction over p. The case $p = 1$ is an identity. Assume that the statement is proven for some $p \geq 1$. Observe that, on $\{\theta_0^\pi(\tau) > t \geq \tau\} = \{\forall s \in [\tau, t] \cap \pi : Z_s \in \mathcal{O}\}$, we have

$$\begin{aligned} \theta_0^\pi(\tau) = \inf\{s \geq \tau : s \in \pi, Z_s \notin \mathcal{O}\} &= \inf\{s \geq t \vee \tau : s \in \pi, Z_s \notin \mathcal{O}\} \\ &= \theta_0^\pi(t \vee \tau). \end{aligned}$$

Hence, for $A \in \mathcal{F}_\tau$ we can write

$$\begin{aligned}
& \frac{\mathbb{E}\left[(\Phi_0^{p+1,\pi}(\tau))^{p+1}\mathbf{1}_A\right]}{p+1} \\
= & \int_0^\infty \mathbb{E}\left[\mathbf{1}_A(\theta_0^\pi(\tau)-t)^p\mathbf{1}_{\{\theta_0^\pi(\tau)>t\geq\tau\}}\right]dt \\
= & \int_0^\infty \mathbb{E}\left[\mathbf{1}_A\mathbb{E}_{t\vee\tau}[(\theta_0^\pi(t\vee\tau)-t\vee\tau)^p]\mathbf{1}_{\{\theta_0^\pi(\tau)>t\geq\tau\}}\right]dt \\
\leq & \; p!L^{p-1}\int_0^\infty \mathbb{E}\left[\mathbf{1}_A\mathbb{E}_{t\vee\tau}[\theta_0^\pi(t\vee\tau)-t\vee\tau]\mathbf{1}_{\{\theta_0^\pi(\tau)>t\geq\tau\}}\right]dt \\
\leq & \; p!L^p\int_0^\infty \mathbb{E}\left[\mathbf{1}_A\mathbf{1}_{\{\theta_0^\pi(\tau)>t\geq\tau\}}\right]dt \\
\leq & \; L^{(p)}\mathbb{E}\left[\mathbf{1}_A\mathbb{E}_\tau\left[\theta_0^\pi(\tau)-\tau\right]\right],
\end{aligned}$$

so that the proof is complete because $A \in \mathcal{F}_\tau$ was arbitrary.

2. The consequently part is now obvious. □

2.4 An a-priori control in terms of the probability of strictly sub-harmonic paths

Now we provide a control on $\Phi_0^1(\tau)$ in terms of the conditional probability of

$$\mathcal{A}_0^\tau := \{2Pb+|a|^2 \geq L^{-2}/2 \text{ on } [\tau,\theta_0(\tau)]\}^c.$$

Intuitively we can say, the more non-degenerate the process P_t^2 from τ to $\theta_0(\tau)$ is, the smaller is the time of exit.

Lemma 2.9. *Let assumptions* **(L)** *and* **(P-*i*)** *be satisfied. Then there exists a constant* $c_{(2.9)} = c_{(2.9)}(L,d) > 0$ *such that, for all* $\tau \in \mathcal{T}$,

$$\Phi_0^1(\tau) \leq c_{(2.9)}\mathbb{P}_\tau\left[\mathcal{A}_0^\tau\right].$$

Proof. Let $E := \{P_\tau \geq 0\} \in \mathcal{F}_\tau$ so that $P_{\theta_0(\tau)} = 0$ on E and $\Phi_0^1(\tau) = 0$ on E^c. Moreover, on E we obtain that

$$\begin{aligned}
\theta_0(\tau)-\tau \;\leq\; & \mathbf{1}_{(\mathcal{A}_0^\tau)^c}2L^2\int_\tau^{\theta_0(\tau)}(2P_sb_s+|a_s|^2)ds+(\theta_0(\tau)-\tau)\mathbf{1}_{\mathcal{A}_0^\tau} \\
= \; & \mathbf{1}_{(\mathcal{A}_0^\tau)^c}2L^2(|P_{\theta_0(\tau)}|^2-|P_\tau|^2)-\mathbf{1}_{(\mathcal{A}_0^\tau)^c}2L^2\int_\tau^{\theta_0(\tau)}2P_sa_s^\top dW_s \\
& +(\theta_0(\tau)-\tau)\mathbf{1}_{\mathcal{A}_0^\tau} \\
\leq \; & -\mathbf{1}_{(\mathcal{A}_0^\tau)^c}4L^2\int_\tau^{\theta_0(\tau)}P_sa_s^\top dW_s+(\theta_0(\tau)-\tau)\mathbf{1}_{\mathcal{A}_0^\tau}.
\end{aligned}$$

Using the bound on $\Phi_0^1(\tau)$ from assumption **(L)** and the bounds from assumption **(P-*i*)**, we obtain $\mathbb{E}\int_0^\infty \mathbf{1}_{\{\tau<s\le\theta_0(\tau)\}}P_s^2|a_s|^2ds<\infty$ and, on E,

$$\begin{aligned}\mathbb{E}_\tau\left[-\mathbf{1}_{(\mathcal{A}_0^\tau)^c}\int_\tau^{\theta_0(\tau)} P_s a_s^\top dW_s\right] &= \mathbb{E}_\tau\left[\mathbf{1}_{\mathcal{A}_0^\tau}\int_\tau^{\theta_0(\tau)} P_s a_s^\top dW_s\right]\\ &\le L^2\sqrt{d}\,\mathbb{P}_\tau[\mathcal{A}_0^\tau]^{\frac12}(\Phi_0^1(\tau))^{\frac12}.\end{aligned}$$

On the other hand, Lemma 2.8 implies

$$\mathbb{E}_\tau\left[(\theta_0(\tau)-\tau)\mathbf{1}_{\mathcal{A}_0^\tau}\right]\le \Phi_0^2(\tau)\mathbb{P}_\tau[\mathcal{A}_0^\tau]^{\frac12}\le\left[c_{2,(2.8)}\Phi_0^1(\tau)\mathbb{P}_\tau[\mathcal{A}_0^\tau]\right]^{\frac12}.$$

Combining the above estimates and using the inequality $ab\le a^2+\frac14 b^2$ gives, on E,

$$\begin{aligned}\Phi_0^1(\tau) &\le 4L^4\sqrt{d}\,\mathbb{P}_\tau[\mathcal{A}_0^\tau]^{\frac12}(\Phi_0^1(\tau))^{\frac12}+\left[c_{2,(2.8)}\Phi_0^1(\tau)\mathbb{P}_\tau[\mathcal{A}_0^\tau]\right]^{\frac12}\\ &\le 16L^8 d\,\mathbb{P}_\tau[\mathcal{A}_0^\tau]+\frac14\Phi_0^1(\tau)+c_{2,(2.8)}\mathbb{P}_\tau[\mathcal{A}_0^\tau]+\frac14\Phi_0^1(\tau),\end{aligned}$$

which leads to the required result. □

2.5 Proof of Theorem 2.3

We start by two lemmas before we turn to the proof of Theorem 2.3.

Lemma 2.10. *Let $\Psi\in\{\Phi_0^1,\Phi_0^{1,\pi}\}$ and assume that there is a constant $c>0$ such that for all $\tau\in\mathcal{T}^\pi$ one has that*

$$\Psi(\tau)\le c\left[P_\tau+|\pi|^{\frac12}\right]\mathbf{1}_{\{0\le P_\tau\le r\}}+L\mathbf{1}_{\{r<P_\tau\}}.$$

Then for all $0<\tilde r<r$ there is a $d_{(2.10)}=d_{(2.10)}(r-\tilde r,L,d,c)>0$ such that for all $\tau\in\mathcal{T}$ one has that

$$\Psi(\tau)\le d_{(2.10)}\left[|P_\tau|+|\pi|^{\frac12}\right]\mathbf{1}_{\{|P_\tau|\le\tilde r\}}+L\mathbf{1}_{\{\tilde r<|P_\tau|\}}.$$

Proof. The case $\pi=\mathbb{R}_+$ is trivial because $\Psi(\tau)\le L$ so that we can assume that $\pi\ne\mathbb{R}^+$. Using

$$\Psi(\tau)\le\mathbb{E}_\tau\left[\Psi(\phi_\tau^+)+|\pi|\right]$$

and

$$\mathbb{E}_\tau\left[|P_{\phi_\tau^+}-P_\tau|\right]\le L[1+\sqrt{d}]|\pi|^{\frac12}=:A|\pi|^{\frac12},$$

we can conclude by

$$\begin{aligned}&\Psi(\tau)\\ \le\ &\mathbb{E}_\tau\left[\Psi(\phi_\tau^+)+|\pi|\right]\mathbf{1}_{\{|P_\tau|\le\tilde r\}}+L\mathbf{1}_{\{\tilde r<|P_\tau|\}}\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{E}_\tau \left[c \left[P_{\phi_\tau^+} + |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{0 \le P_{\phi_\tau^+} \le r\}} + L\mathbf{1}_{\{r < P_{\phi_\tau^+}\}} + |\pi| \right] \mathbf{1}_{\{|P_\tau| \le \tilde r\}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + L\mathbf{1}_{\{\tilde r < |P_\tau|\}} \\
&\le \left[c|P_\tau| + [c(1+A)+1]|\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{|P_\tau| \le \tilde r\}} \\
&\qquad + L\mathbb{P}_\tau \left[r < P_{\phi_\tau^+}, |P_\tau| \le \tilde r \right] \mathbf{1}_{\{|P_\tau| \le \tilde r\}} + L\mathbf{1}_{\{\tilde r < |P_\tau|\}} \\
&\le \left[c|P_\tau| + [c(1+A)+1]|\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{|P_\tau| \le \tilde r\}} \\
&\qquad + L\mathbb{P}_\tau \left[|P_{\phi_\tau^+} - P_\tau| \ge r - \tilde r \right] \mathbf{1}_{\{|P_\tau| \le \tilde r\}} + L\mathbf{1}_{\{\tilde r < |P_\tau|\}} \\
&\le \left[c|P_\tau| + \left[c(1+A) + 1 + \frac{LA}{r - \tilde r} \right] |\pi|^{\frac{1}{2}} \right] \mathbf{1}_{\{|P_\tau| \le \tilde r\}} + L\mathbf{1}_{\{\tilde r < |P_\tau|\}}.
\end{aligned}
$$

□

Next we control the quantity $\mathbb{P}_\tau[\mathcal{A}_0^\tau]$ to make Lemma 2.9 applicable:

Lemma 2.11. *Assume that* **(Z)** *and* **(P)** *hold. Then for all $c > 0$ there exists an $\eta(c) = \eta(c, r, L, d) > 0$ such that*

$$
\mathbb{P}_\tau[\mathcal{A}_0^\tau] \le \eta(c) P_\tau + c\,\Phi_0^1(\tau) + \kappa\left(\frac{2}{c}, r\right)|\pi| \quad \text{a.s. on } \{P_\tau \in [0, r]\}, \tag{5}
$$

where $\tau \in \mathcal{T}^\pi$ and $\mathcal{A}_0^\tau := \{2Pb + |a|^2 \ge L^{-2}/2 \text{ on } [\tau, \theta_0(\tau)]\}^c$.

Proof. Let $\tilde\theta_r(\tau) := \inf\{t \ge \tau : P_t = r\} \in [0, \infty]$. Assumption **(P-*ii*)** implies $2Pb + |a|^2 \ge L^{-2}/2$ $\mathbb{P}$-a.s. on $\{|P| \vee \gamma(\cdot, \phi) \le r\}$ for $r \le L^{-3}/4$. It follows from the restriction $\tau \in \mathcal{T}^\pi$ that on

$$
E := \{P_\tau \in [0, r]\}
$$

we have, $\mathbb{P}$-a.s., that

$$
\begin{aligned}
(\mathcal{A}_0^\tau)^c &\supseteq \left\{ \sup_{\tau \le t \le \theta_0(\tau)} |P_t| \le r \right\} \cap \left\{ \sup_{\tau \le t \le \theta_0(\tau)} \gamma(t, \phi_t \vee \tau) \le r \right\} \\
&\supseteq \{\theta_0(\tau) \le \tilde\theta_r(\tau)\} \cap \left\{ \sup_{\tau \le t \le \theta_0(\tau)} \gamma(t, \phi_t \vee \tau) \le r \right\}.
\end{aligned}
$$

Setting $\mathcal{B}_T := \{\sup_{\tau \le t \le \tau + T} \gamma(t, \phi_t \vee \tau) \le r\}$ for $T := 2c^{-1}$, we continue on E with

$$
\begin{aligned}
\mathbb{P}_\tau[\mathcal{A}_0^\tau] &\le \mathbb{P}_\tau[\mathcal{A}_0^\tau,\ \theta_0(\tau) \le \tau + T, \mathcal{B}_T] + \mathbb{P}_\tau[\theta_0(\tau) > \tau + T] + \mathbb{P}_\tau[\mathcal{B}_T^c] \\
&\le \mathbb{P}_\tau[\tilde\theta_r(\tau) < \theta_0(\tau) \le \tau + T, \mathcal{B}_T] + \mathbb{P}_\tau[\theta_0(\tau) > \tau + T] + \mathbb{P}_\tau[\mathcal{B}_T^c] \\
&\le \mathbb{P}_\tau[\tilde\theta_r(\tau) < \theta_0(\tau) \le \tau + T, \mathcal{B}_T] + \frac{c}{2}\,\Phi_0^1(\tau) + \kappa\left(\frac{2}{c}, r\right)|\pi|, \qquad (6)
\end{aligned}
$$

where the last inequality follows from Chebyshev's inequality and assumption **(Z)**. To treat the first term in (6) we set, for $T \geq 0$,

$$\theta_{0,r}^T := \theta_0(\tau) \wedge \tilde{\theta}_r(\tau) \wedge (\tau + T).$$

In view of assumption **(P)** we can define $\mathbb{Q} \sim \mathbb{P}$ by the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = H := \mathcal{E}\left(- \int_\tau^\cdot \lambda_s^\top dW_s \right)_{\theta_{0,r}^T},$$

where

$$\lambda := a\ [|a|^{-2} \wedge L^2]\ b\ \mathbf{1}_{[\tau, \theta_{0,r}^T]} \quad \text{so that} \quad |\lambda| \leq L^4 \sqrt{d} =: \lambda_\infty,$$

and deduce from Girsanov's Theorem (cf. [Bic10, p.163]) that

$$W^{\mathbb{Q}} := W + \mathbf{1}_{[\tau,\infty)} \int_\tau^{\theta_{0,r}^T \wedge \cdot} \lambda_s ds$$

is a Brownian motion associated to $\mathbb{Q}$. For any given $\ell > 1$ we obtain

$$\begin{aligned}
&\mathbb{P}_\tau[\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T] \\
&\leq \mathbb{P}_\tau[H^{-1} > \ell] + \mathbb{E}_\tau^{\mathbb{Q}}\left[H^{-1} \mathbf{1}_{\{H^{-1} \leq \ell\}} \mathbf{1}_{\{\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T\}} \mathbf{1}_{\mathcal{B}_T} \right] \\
&\leq \mathbb{P}_\tau[H^{-1} > \ell] + \ell\ \mathbb{Q}_\tau\left[\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T \right]. \qquad (7)
\end{aligned}$$

The first term above can be estimated, by using Chebyshev's inequality, the inequality $\theta_{0,r}^T \leq \theta_0(\tau)$, and Lemma 2.8:

$$\begin{aligned}
\mathbb{P}_\tau\left[H^{-1} > \ell\right] &\leq \frac{1}{|\log \ell|^2} 2\mathbb{E}_\tau\left[\frac{1}{4}\lambda_\infty^4 |\theta_{0,r}^T - \tau|^2 + \lambda_\infty^2 (\theta_{0,r}^T - \tau) \right] \\
&\leq \frac{(L\lambda_\infty^4 + 2\lambda_\infty^2)}{|\log \ell|^2} \Phi_0^1(\tau) \leq \frac{c}{2} \Phi_0^1(\tau), \qquad (8)
\end{aligned}$$

where the last inequality holds by taking the constant $\ell = \ell(c, L, d)$ large enough. To handle the second term in (7), set

$$M_t := \mathbb{E}_t^{\mathbb{Q}}[P_\tau] + \mathbf{1}_{[\tau,\infty)}(t) \int_\tau^t \mathbf{1}_{\{s < \theta_{0,r}^T\}} a_s^\top dW_s^{\mathbb{Q}} \quad \text{for} \quad t \geq 0$$

so that M is a u.i. $\mathbb{Q}$-martingale. Let $\theta_0^M(\tau)$ and $\theta_r^M(\tau)$ be the first hitting times after τ of levels 0 and r by M, and set $\theta_{0,r}^{M,T} := \theta_0^M(\tau) \wedge \theta_r^M(\tau) \wedge (\tau + T)$. Recalling assumption **(P)**, we see that

$$\mathbf{1}_{[\tau,\infty)}(t) \int_\tau^t \mathbf{1}_{\{s < \theta_{0,r}^T\}} a_s^\top \lambda_s ds = \mathbf{1}_{[\tau,\infty)}(t) \int_\tau^t \mathbf{1}_{\{s < \theta_{0,r}^T\}} b_s ds \quad \text{on } \mathcal{B}_T \cap E.$$

Hence, on $\mathcal{B}_T \cap E$ the processes M and P coincide on $[\tau, \theta^T_{0,r}]$. By the optional sampling theorem and the non-negativity of $(M_t)_{t\in[\tau,\theta^{M,T}_{0,r}]}$ a.s. on E, we then deduce

$$\begin{aligned} P_\tau \mathbf{1}_E = M_\tau \mathbf{1}_E = \mathbb{E}^{\mathbb{Q}}_\tau(\mathbf{1}_E M_{\theta^{M,T}_{0,r}}) &\geq \mathbf{1}_E r \mathbb{Q}_\tau\left(\theta^M_r(\tau) < \theta^M_0(\tau) \wedge (\tau+T)\right) \\ &\geq \mathbf{1}_E r \mathbb{Q}_\tau\left(\tilde{\theta}_r(\tau) < \theta_0(\tau) < \tau + T,\ \mathcal{B}_T\right). \end{aligned}$$

Plugging this inequality together with (8) into (7), gives on E that

$$\mathbb{P}_\tau\left[\tilde{\theta}_r(\tau) < \theta_0(\tau) \leq \tau + T, \mathcal{B}_T\right] \leq \frac{c}{2}\Phi^1_0(\tau) + \frac{\ell(c,L,d)}{r}P_\tau.$$

□

Proof of Theorem 2.3. Part (a): For $\tau \in \mathcal{T}^\pi$ and $c > 0$ Lemmas 2.9 and 2.11 imply

$$\Phi^1_0(\tau) \leq c_{(2.9)}\mathbb{P}_\tau\left[\mathcal{A}^\tau_0\right] \leq c_{(2.9)}\left[\eta(c)P_\tau + c\ \Phi^1_0(\tau) + \kappa\left(\frac{2}{c}, r\right)|\pi|\right]$$

a.s. on $\{P_\tau \in [0,r]\}$. Specializing to $c = 1/(2c_{(2.9)})$ leads to

$$\Phi^1_0(\tau) \leq 2c_{(2.9)}\left[\eta((2c_{(2.9)})^{-1})P_\tau + \kappa\left(4c_{(2.9)}, r\right)|\pi|\right]$$

on $\{P_\tau \in [0,r]\}$. On $\{P_\tau < 0\}$ we simply have $\Phi^1_0(\tau) = 0$, while $\Phi^1_0(\tau) \leq L$ on $\{P_\tau > r\}$ by assumption **(L)**. This implies

$$\Phi^1_0(\tau) \leq \bar{c}_{(2.3)}\left[P_\tau + |\pi|\right]\mathbf{1}_{\{0\leq P_\tau \leq r\}} + L\mathbf{1}_{\{r<P_\tau\}}, \tag{9}$$

where $\bar{c}_{(2.3)} = \bar{c}_{(2.3)}(r,L,d,\kappa) > 0$. By a change of the constant $\bar{c}_{(2.3)}$ the assertion follows.

Part (b): Combining (9) and Lemma 2.10, we derive

$$\Phi^1_0(\tau) \leq \bar{d}_{(2.3)}\left[|P_\tau| + |\pi|^{\frac{1}{2}}\right]\mathbf{1}_{\{|P_\tau|\leq\tilde{r}\}} + L\mathbf{1}_{\{\tilde{r}<|P_\tau|\}}$$

for $0 < \tilde{r} < r$ and $\tau \in \mathcal{T}$, where $\bar{d}_{(2.3)} = \bar{d}_{(2.3)}(r,\tilde{r},L,d,\kappa) > 0$. Observe that $\Phi^1_0(\tau) = 0$ for $P_\tau \leq 0$, thus the above r.h.s. needs to be specialized only for $P_\tau \geq 0$. Then, choosing $\tilde{r} = r/2$ and adapting $\bar{d}_{(2.3)}$, we obtain part (b). □

2.6 Proof of Theorem 2.4

Now we are in a position to conclude the proof of Theorem 2.4. It is based on a recursion argument. Namely, given $\tau \in \mathcal{T}$ such that $0 \leq P_\tau \leq r$, we wait until the next time ϑ in $\mathbb{R}_+$ such that Z hits the boundary. The time it takes, $\vartheta - \tau$, is controlled by Theorem 2.3. If $Z_{\phi^+_\vartheta} \notin \mathcal{O}$, then we stop: $\theta^\pi_0(\tau) - \tau \leq \vartheta - \tau + |\pi|$. If not, then we know from standard estimates (Lemma 2.12 below) that $P_{\phi^+_\vartheta} \in [0, r]$, up to some event with a probability controlled by $O(|\pi|^{\frac{1}{2}})$. In this case one can restart the above procedure from $\phi^+_\vartheta \in \pi$. Again, one waits for the next time in $\mathbb{R}_+$ such that Z reaches the boundary and stops if $Z \notin \mathcal{O}$ at the following time in π. One iterates this procedure. The key point is that the probability of the event set $\{Z_{\phi^+_\vartheta} \in \mathcal{O}\}$ is uniformly controlled by some $\alpha < 1$ (see Lemma 2.13 below).

Before we start with the proof of Theorem 2.4 we state two lemmas that are needed. The first one can be verified by Doob's maximal inequality and assumption **(P-*i*)**:

Lemma 2.12. *Under the assumption* **(P-*i*)** *one has, for all* $\tau \in \mathcal{T}$ *and* $\lambda > 0$,

$$\mathbb{P}_\tau\left[\max_{\tau \leq t \leq \phi^+_\tau} |P_t - P_\tau| \geq \lambda\right] \leq \frac{1}{\lambda}\mathbb{E}_\tau\left[\max_{\tau \leq t \leq \phi^+_\tau} |P_t - P_\tau|^2\right]^{\frac{1}{2}} \leq \frac{c_{(2.12)}}{\lambda}|\pi|^{\frac{1}{2}},$$

where $c_{(2.12)} := L + 2\sqrt{d}L$.

Lemma 2.13. *Let assumptions* **(Z)** *and* **(P)** *hold. Then there exists an* $0 < \alpha_{(2.13)} = \alpha_{(2.13)}(r, L, d, \kappa) < 1$ *such that, a.s.,*

$$\mathbb{P}_\tau\left[P_{\phi^+_\tau} > 0\right] \leq \alpha_{(2.13)} \quad \text{on} \quad \left\{\gamma(\tau, \phi_\tau) \leq \frac{r}{2}, P_\tau = 0\right\} \in \mathcal{F}_\tau$$

for all $\tau \in \mathcal{T}$ *and* $0 < |\pi| \leq \varepsilon_{(2.13)} = \varepsilon_{(2.13)}(r, L, d, \kappa)$.

Proof. It is sufficient to check for $B \in \mathcal{F}_\tau$ of positive measure with

$$B \subseteq \left\{\gamma(\tau, \phi_\tau) \leq \frac{r}{2}, P_\tau = 0\right\}$$

that

$$\mathbb{P}\left[P_{\phi^+_\tau} > 0, B\right] \leq \alpha_{(2.13)}\mathbb{P}\left[B\right].$$

Let

$$\mathcal{B} := \left\{\max_{\tau \leq t \leq \phi^+_\tau} (|P_t| \vee \gamma(t, \phi_t \vee \tau)) \leq \frac{r}{2}\right\} \cap B$$

so that

$$\mathcal{B} \subseteq \left\{\max_{\tau \leq t \leq \phi^+_\tau} (|P_t| \vee \gamma(t, \phi_t)) \leq r\right\}. \tag{10}$$

We use assumptions **(P)**, **(Z)** and Lemma 2.12 to continue with

$$\mathbb{P}\left[P_{\phi^+_\tau} > 0, B\right]$$

$$
\begin{aligned}
&= \mathbb{P}\left[\int_\tau^{\phi_\tau^+} b_s ds + \int_\tau^{\phi_\tau^+} a_s^\top dW_s > 0, B\right] \\
&\le \mathbb{P}\left[\int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B}\right] + \mathbb{P}\left[\mathcal{B}^c \cap B\right] \\
&\le \mathbb{P}\left[\int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B}\right] \\
&\quad + \left[\kappa\left(1, \frac{r}{2}\right)|\pi| + \frac{2c_{(2.12)}}{r}|\pi|^{\frac{1}{2}}\right]\mathbb{P}\left[B\right].
\end{aligned}
$$

Assuming that we are able to show that

$$
\mathbb{P}\left[\int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B}\right] \le \theta \mathbb{P}\left[B\right] \tag{11}
$$

for some $\theta = \theta(L, d) \in (0, 1)$, the proof would be complete as

$$
\mathbb{P}\left[P_{\phi_\tau^+} > 0, B\right] \le \left[\theta + \kappa\left(1, \frac{r}{2}\right)|\pi| + \frac{2c_{(2.12)}}{r}|\pi|^{\frac{1}{2}}\right]\mathbb{P}\left[B\right]
$$

and $\varepsilon_{(2.13)} = \varepsilon_{(2.13)}(r, L, d, \kappa) > 0$ can be taken small enough to guarantee

$$
\mathbb{P}\left[P_{\phi_\tau^+} > 0, B\right] \le \alpha \mathbb{P}\left[B\right]
$$

for some $\alpha = \alpha(r, L, d, \kappa) \in (0, 1)$. In order to check (11) we let

$$
M_t := e^\top W_{t\wedge\tau} + \int_\tau^{\tau\vee t} \bar{a}_s^\top dW_s
$$

where $\bar{a}_s := a_s \mathbf{1}_{\{s \le \phi_\tau^+\}} + e\mathbf{1}_{\{s > \phi_\tau^+\}}$ with $e = d^{-\frac{1}{2}}(1, ..., 1)^\top$. Define $\Lambda(s) := \inf\{t \ge 0 : \langle M \rangle_t > s\}$. Applying the Dambis-Schwarz Theorem [RY05, p. 181] yields that $B := M_\Lambda$ is a Brownian motion in the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ defined by $\mathcal{G} = \mathcal{F}_\Lambda$ and $M = B_{\langle M \rangle}$. One can also check that $\mathcal{F}_\tau \subseteq \mathcal{G}_{\langle M \rangle_\tau}$. Letting $\eta := \phi_\tau^+ - \tau$ (which is $\mathcal{F}_\tau$-measurable), observing $\eta \le \sqrt{\eta}$ and $\eta L^{-2} \le \langle M \rangle_{\tau+\eta} - \langle M \rangle_\tau \le \eta d L^2$ on $\mathcal{B}$ by assumption **(P)** and (10), and taking an auxiliary one-dimensional Brownian motion $\widetilde{B}$ defined on some $(\widetilde{\Omega}, \widetilde{\mathbb{P}})$, we conclude by

$$
\begin{aligned}
&\mathbb{P}\left[\int_\tau^{\phi_\tau^+} a_s^\top dW_s > -L(\phi_\tau^+ - \tau), \mathcal{B}\right] \\
&= \mathbb{P}\left[B_{\langle M \rangle_{\phi_\tau^+}} - B_{\langle M \rangle_\tau} > -L(\phi_\tau^+ - \tau), \mathcal{B}\right] \\
&\le \mathbb{P}\left[\sup_{t \in [\langle M \rangle_\tau + \eta L^{-2}, \langle M \rangle_\tau + \eta d L^2]} B_t - B_{\langle M \rangle_\tau} > -L\eta, \mathcal{B}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad \mathbb{P}\left[\sup_{t\in[\langle M\rangle_\tau+\eta L^{-2},\langle M\rangle_\tau+\eta dL^2]} B_t - B_{\langle M\rangle_\tau} > -L\eta, B\right] \\
&\leq \quad \widetilde{\mathbb{P}}\times\mathbb{P}\left[\sup_{\eta L^{-2}\leq u\leq \eta dL^2} \widetilde{B}_u > -L\sqrt{\eta}, B\right] \\
&\leq \quad \widetilde{\mathbb{P}}\times\mathbb{P}\left[\sup_{L^{-2}\leq u\leq dL^2} \widetilde{B}_u > -L, B\right] \\
&= \quad \widetilde{\mathbb{P}}\left[\sup_{L^{-2}\leq u\leq dL^2} \widetilde{B}_u > -L\right]\mathbb{P}\left[B\right] \\
&=: \quad \theta\mathbb{P}\left[B\right].
\end{aligned}
$$

$\square$

Proof of Theorem 2.4. (a) First we assume that $\tau \in \mathcal{T}^\pi$. For $i \geq 0$ we define

$$\vartheta_0 := \theta_0(\tau)\ , \quad \vartheta_{i+1} := \theta_0(\phi^+_{\vartheta_i})\ , \quad \vartheta^\pi_0 := \theta^\pi_0(\tau)\ , \quad \vartheta^\pi_{i+1} := \theta^\pi_0(\phi^+_{\vartheta_i}),$$

$$E_i := \{P_{\phi^+_{\vartheta_i}} > 0\}, \quad \text{and} \quad \mathcal{A}_i := \cap_{0\leq j\leq i} E_j \in \mathcal{F}_{\phi^+_{\vartheta_i}}.$$

1. From the definitions we obtain for $i \geq 0$:

a) $\vartheta_{i+1} \leq \vartheta^\pi_{i+1}$ (by definitions of the stopping times);

b) $\phi^+_{\vartheta_{i+1}} \leq \vartheta_{i+1} + |\pi|$ (by the definition of ϕ^+);

c) $\vartheta^\pi_{i+1} = \vartheta^\pi_{i+2}$ on $E_{i+1} = \{Z_{\phi^+_{\vartheta_{i+1}}} \in \mathcal{O}\}$ (since $\phi^+_{\vartheta_{i+1}} < \vartheta^\pi_{i+1}$ on E_{i+1});

d) $\vartheta^\pi_{i+1} \leq \vartheta_{i+1} + |\pi|$ on $(E_{i+1})^c = \{Z_{\phi^+_{\vartheta_{i+1}}} \notin \mathcal{O}\}$ (by definition of the stopping time ϑ^π_{i+1}).

Item c) leads to

$$\vartheta^\pi_{i+1} = \vartheta^\pi_{i+2}\mathbf{1}_{E_{i+1}} + \vartheta^\pi_{i+1}\mathbf{1}_{[E_{i+1}]^c},$$

$$\vartheta^\pi_{i+1} - \phi^+_{\vartheta_i} = (\vartheta^\pi_{i+2} - \phi^+_{\vartheta_{i+1}})\mathbf{1}_{E_{i+1}} + (\phi^+_{\vartheta_{i+1}} - \phi^+_{\vartheta_i})\mathbf{1}_{E_{i+1}} + (\vartheta^\pi_{i+1} - \phi^+_{\vartheta_i})\mathbf{1}_{[E_{i+1}]^c}.$$

With b) and d) we continue to

$$
\begin{aligned}
\vartheta^\pi_{i+1} - \phi^+_{\vartheta_i} \quad &\leq \quad (\vartheta^\pi_{i+2} - \phi^+_{\vartheta_{i+1}})\mathbf{1}_{E_{i+1}} + (\vartheta_{i+1} + |\pi| - \phi^+_{\vartheta_i})\mathbf{1}_{E_{i+1}} \\
&\quad\quad + (\vartheta_{i+1} + |\pi| - \phi^+_{\vartheta_i})\mathbf{1}_{[E_{i+1}]^c} \\
&= \quad (\vartheta^\pi_{i+2} - \phi^+_{\vartheta_{i+1}})\mathbf{1}_{E_{i+1}} + |\pi| + (\vartheta_{i+1} - \phi^+_{\vartheta_i})
\end{aligned}
$$

and

$$\mathbb{E}_\tau\left[(\vartheta^\pi_{i+1} - \phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right] \quad \leq \quad \mathbb{E}_\tau\left[(\vartheta^\pi_{i+2} - \phi^+_{\vartheta_{i+1}})\mathbf{1}_{\mathcal{A}_i}\mathbf{1}_{E_{i+1}}\right]$$

$$
\begin{aligned}
& +|\pi|\mathbb{P}_\tau[\mathcal{A}_i] + \mathbb{E}_\tau\left[(\vartheta_{i+1} - \phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right] \\
= \;& \mathbb{E}_\tau\left[(\vartheta^\pi_{i+2} - \phi^+_{\vartheta_{i+1}})\mathbf{1}_{\mathcal{A}_{i+1}}\right] \\
& +|\pi|\mathbb{P}_\tau[\mathcal{A}_i] + \mathbb{E}_\tau\left[(\vartheta_{i+1} - \phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right].
\end{aligned}
$$

Summing up the above inequalities from $i = 0$ to $i = n-1$ yields

$$
\begin{aligned}
\mathbb{E}_\tau\left[(\vartheta^\pi_1 - \phi^+_{\vartheta_0})\mathbf{1}_{\mathcal{A}_0}\right] \;\le\; & \mathbb{E}_\tau\left[(\vartheta^\pi_{n+1} - \phi^+_{\vartheta_n})\mathbf{1}_{\mathcal{A}_n}\right] \\
& + \sum_{i=0}^{n-1}\left(|\pi|\mathbb{P}_\tau[\mathcal{A}_i] + \mathbb{E}_\tau\left[(\vartheta_{i+1} - \phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right]\right). \quad (12)
\end{aligned}
$$

2. For $\sigma \in \mathcal{T}$ set $\mathcal{A}^\sigma := \{\gamma(\sigma, \phi_\sigma) \le r/2\} \in \mathcal{F}_\sigma$ so that, for $i \ge 1$,

$$
\begin{aligned}
\mathbb{P}_\tau[\mathcal{A}_i] \;=\; & \mathbb{E}_\tau\left[\mathbf{1}_{\mathcal{A}_{i-1}}\mathbf{1}_{\mathcal{A}^{\vartheta_i}}\mathbb{P}_{\vartheta_i}[E_i]\right] + \mathbb{E}_\tau\left[\mathbf{1}_{\mathcal{A}_{i-1}}\mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[[\mathcal{A}^{\vartheta_i}]^c \cap E_i\right]\right] \\
\le\; & \alpha_{(2.13)}\mathbb{P}_\tau[\mathcal{A}_{i-1}] + \mathbb{E}_\tau\left[\mathbf{1}_{\mathcal{A}_{i-1}}\mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[[\mathcal{A}^{\vartheta_i}]^c \cap E_i\right]\right],
\end{aligned}
$$

because of $\mathcal{F}_{\phi^+_{\vartheta_{i-1}}} \subseteq \mathcal{F}_{\vartheta_i}$, Lemma 2.13, and $P_{\vartheta_i} = 0$ on $\mathcal{A}_{i-1}$. To treat the second term we take a fixed $T > 0$ and use **(Z)** and **(L)** to get

$$
\begin{aligned}
\mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[[\mathcal{A}^{\vartheta_i}]^c \cap E_i\right] \;\le\; & \mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[\{\gamma(\vartheta_i, \phi_{\vartheta_i}) > r/2\} \cap \{\vartheta_i \le \phi^+_{\vartheta_{i-1}} + T\}\right] \\
& + \mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[\vartheta_i > \phi^+_{\vartheta_{i-1}} + T\right] \\
\le\; & \mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[\sup_{\phi^+_{\vartheta_{i-1}} \le t \le \phi^+_{\vartheta_{i-1}} + T} \gamma(t, \phi_t \vee \phi^+_{\vartheta_{i-1}}) > r/2\right] \\
& + \mathbb{P}_{\phi^+_{\vartheta_{i-1}}}\left[\vartheta_i > \phi^+_{\vartheta_{i-1}} + T\right] \\
\le\; & \kappa(T, r/2)|\pi| + \mathbb{E}_{\phi^+_{\vartheta_{i-1}}}[\theta_0(\phi^+_{\vartheta_{i-1}})]/T \\
\le\; & \kappa(T, r/2)|\pi| + L/T.
\end{aligned}
$$

By taking $T > 0$ large enough and then $\varepsilon_{(13)} \in (0, \varepsilon_{(2.13)}]$ small enough such that

$$
\alpha_{(2.13)} + \kappa(T, r/2)\varepsilon_{(13)} + \frac{L}{T} =: \alpha < 1
$$

and assuming that $|\pi| \le \varepsilon_{(13)}$, we obtain $\mathbb{P}_\tau[\mathcal{A}_i] \le \alpha\mathbb{P}_\tau[\mathcal{A}_{i-1}]$ and, by induction,

$$
\mathbb{P}_\tau[\mathcal{A}_j] \le \alpha^j \quad \text{for all } \; j \ge 0. \quad (13)
$$

3. Let us set $F_i := \{P_{\phi^+_{\vartheta_i}} > r\}$ for $i \ge 0$. Because of $\phi^+_{\vartheta_i} \in \mathcal{T}^\pi$, applying (9) from the proof of Theorem 2.3 and using the fact that $\mathcal{A}_i \in \mathcal{F}_{\phi^+_{\vartheta_i}}$ and assumption **(L)**, lead to

$$
\mathbb{E}_\tau\left[(\vartheta_{i+1} - \phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right]
$$

$$
\begin{aligned}
&= \mathbb{E}_\tau\left[\mathbb{E}_{\phi^+_{\vartheta_i}}\left[(\vartheta_{i+1}-\phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right]\right]\\
&\le \mathbb{E}_\tau\left[\bar{c}_{(2.3)}\big((P_{\phi^+_{\vartheta_i}})^+ + |\pi|\big)\mathbf{1}_{\mathcal{A}_{i-1}}\mathbf{1}_{E_i\cap[F_i]^c}\right] + L\mathbb{P}_\tau\left[\mathcal{A}_i\cap F_i\right]\\
&\le \mathbb{E}_\tau\left[\bar{c}_{(2.3)}\big((P_{\phi^+_{\vartheta_i}})^+ + |\pi|\big)\mathbf{1}_{\mathcal{A}_{i-1}}\right] + L\mathbb{E}_\tau\left[1_{\mathcal{A}_{i-1}}\mathbb{P}_{\vartheta_i}\left[F_i\right]\right],
\end{aligned}
$$

where $\mathcal{A}_{-1} := \Omega$. Because $P_{\vartheta_i}\le 0$, Lemma 2.12 implies

$$
\mathbb{E}_{\vartheta_i}\left[(P_{\phi^+_{\vartheta_i}})^+\right] \le c_{(2.12)}|\pi|^{\frac12} \quad\text{and}\quad \mathbb{P}_{\vartheta_i}\left[F_i\right] \le \frac{c_{(2.12)}}{r}|\pi|^{\frac12},
$$

and (13) yields

$$
\begin{aligned}
&\mathbb{E}_\tau\left[(\vartheta_{i+1}-\phi^+_{\vartheta_i})\mathbf{1}_{\mathcal{A}_i}\right]\\
\le\ &\mathbb{E}_\tau\left[\bar{c}_{(2.3)}\big(c_{(2.12)}|\pi|^{\frac12} + |\pi|^{\frac12}\big)\mathbf{1}_{\mathcal{A}_{i-1}}\right] + L\mathbb{P}_\tau\left[\mathcal{A}_{i-1}\right]\frac{c_{(2.12)}}{r}|\pi|^{\frac12}\\
\le\ &D|\pi|^{\frac12}\alpha^{(i-1)_+}
\end{aligned}
$$

with $D := \bar{c}_{(2.3)}c_{(2.12)} + \bar{c}_{(2.3)} + Lc_{(2.12)}/r$. If we insert the last estimate into (12) and let $n\to+\infty$, then we get

$$
\mathbb{E}_\tau\left[(\vartheta_1^\pi - \phi^+_{\vartheta_0})\mathbf{1}_{\mathcal{A}_0}\right] \le |\pi|^{\frac12}\frac{|\pi|^{\frac12} + (2-\alpha)D}{1-\alpha},
$$

where we exploit Lemma 2.8 to check

$$
\mathbb{E}_\tau\left[|\vartheta^\pi_{n+1} - \phi^+_{\vartheta_n}|\mathbf{1}_{\mathcal{A}_n}\right] \le \sqrt{L^{(2)}}\mathbb{P}_\tau\left[\mathcal{A}_n\right]^{\frac12}.
$$

Observe now that

$$
\theta_0^\pi(\tau) = \left[\phi^+_{\vartheta_0} + (\vartheta_1^\pi - \phi^+_{\vartheta_0})\right]\mathbf{1}_{\mathcal{A}_0} + \phi^+_{\vartheta_0}\mathbf{1}_{[\mathcal{A}_0]^c} \le |\pi| + \vartheta_0 + (\vartheta_1^\pi - \phi^+_{\vartheta_0})\mathbf{1}_{\mathcal{A}_0},
$$

so that by an application of the previous estimate, (9) and Assumption **(L)** we obtain

$$
\Phi_0^{1,\pi}(\tau) \le \bar{c}_{(2.4)}\left[P_\tau + |\pi|^{\frac12}\right]\mathbf{1}_{\{0\le P_\tau\le r\}} + L\mathbf{1}_{\{r<P_\tau\}}, \tag{14}
$$

for $\tau\in\mathcal{T}^\pi$ and $\bar{c}_{(2.4)} = \bar{c}_{(2.4)}(r,L,d,\kappa) > 0$.

(b) We now consider the general case $\tau\in\mathcal{T}$. Applying Lemma 2.10 to (14) we obtain for $0<\tilde{r}<r$ that

$$
\Phi_0^{1,\pi}(\tau) \le \bar{d}_{(2.4)}\left[|P_\tau| + |\pi|^{\frac12}\right]\mathbf{1}_{\{|P_\tau|\le\tilde{r}\}} + L\mathbf{1}_{\{\tilde{r}<|P_\tau|\}},
$$

where $\bar{d}_{(2.4)} = \bar{d}_{(2.4)}(r,\tilde{r},L,d,\kappa) > 0$. Taking $\tilde{r} = r/2$ and adapting $\bar{d}_{(2.4)}$, we obtain the statement of the theorem. □

3 General $\mathbf{L}_1$-error for exit time approximations

The main application we develop in this paper is the study of the error made by estimating the exit time θ of a diffusion X from a domain $\mathcal{O}$ by the discrete exit time $\bar{\theta}$ of an approximation process $\bar{X}$, which can be X itself or its Euler or Milstein scheme etc, computed on a grid $\bar{\pi}$. We only assume that the corresponding distance processes remain close, at least at the order $|\bar{\pi}|^{\frac{1}{2}}$ in $\mathbf{L}_1$. If X exits before $\bar{X}$, then our assumptions imply that $\bar{X}$ is close to the boundary as well. If we also know that the expectation of the time it takes to the approximation scheme $\bar{X}$ to exit the domain is proportional to its distance to the boundary up to an additional term $|\bar{\pi}|^{\frac{1}{2}}$, then we can conclude that $\mathbb{E}[|\bar{\theta}-\theta|\mathbf{1}_{\{\theta\le\bar{\theta}\}}]$ is controlled in $|\bar{\pi}|^{\frac{1}{2}}$. The same idea applies if $\bar{X}$ exits before X. In this section, we show how Theorems 2.3 and 2.4 are used to follow this idea. We start with an abstract statement and then specialize it to the case where X solves a stochastic differential equation and $\bar{X}$ is its Euler scheme.

3.1 Upper-bound in an abstract setting

We fix an open non-empty subset $\mathcal{O}$ of a metric space $(\mathcal{Z}, d_{\mathcal{Z}})$, satisfying the assumptions of Section 2.1, and two $\mathcal{Z}$-valued processes X and $\bar{X}$. We consider the first exit time $\theta_0 := \theta_0(0)$ of X on $\pi := \mathbb{R}_+$ and $\bar{\theta}_0^{\bar{\pi}} := \bar{\theta}_0^{\bar{\pi}}(0)$ of $\bar{X}$ on $\bar{\pi} \subsetneq \mathbb{R}_+$ (where $\bar{\pi}$ satisfies the conditions of Section 2.1), i.e.

$$\theta_0 := \inf\{t \ge 0 \ : \ X_t \notin \mathcal{O}\} \quad \text{and} \quad \bar{\theta}_0^{\bar{\pi}} := \inf\{t \ge 0 \ : \ t \in \bar{\pi} \text{ and } \bar{X}_t \notin \mathcal{O}\}.$$

We let $\bar{\phi}$ and $\bar{\phi}^+$ be the functions defined in (1) associated to $\bar{\pi}$.

We also fix a *distance function* $\delta : \mathcal{Z} \mapsto \mathbb{R}$ such that $\delta > 0$ on $\mathcal{O}$, $\delta = 0$ on $\partial\mathcal{O}$, and $\delta < 0$ on $\bar{\mathcal{O}}^c$, and set $P := \delta(X)$ and $\bar{P} := \delta(\bar{X})$.

Throughout this section we assume that the assumptions **(Z)**, **(P)** and **(L)** of Section 2.1 hold for (X, π, P), $(\bar{X}, \bar{\pi}, \bar{P})$, and δ with the same (r, L, κ). Obviously, the estimate contained in **(Z)** is trivial for (X, π, P) since $\pi = \mathbb{R}_+$ and ϕ is the identity.

Theorem 3.1. *Assume a stopping time $\upsilon : \Omega \to [0, \infty]$ and some $\rho > 0$ such that*

$$\mathbb{E}\left[|P_\vartheta - \bar{P}_\vartheta|\right] \le \rho|\bar{\pi}|^{\frac{1}{2}} \quad \textit{for all } \ \vartheta \in \mathcal{T} \ \textit{ with } \ \vartheta \le \theta_0 \wedge \upsilon. \tag{15}$$

Then for all integers $p \geq 1$ there exist $c_{(3.1)} = c_{(3.1)}(r, L, d, \kappa, p, \rho) > 0$ and $\varepsilon_{(3.1)} = \varepsilon_{(3.1)}(r, L, d, \kappa) > 0$ such that, for $|\bar{\pi}| \leq \varepsilon_{(3.1)}$,

$$\mathbb{E}\left[|[\theta_0 \wedge \upsilon] - [\bar{\theta}_0^{\bar{\pi}} \wedge \upsilon]|^p\right] \leq c_{(3.1)} \, |\bar{\pi}|^{\frac{1}{2}}.$$

Proof. Define $\upsilon_0 := \theta_0 \wedge \upsilon$ and $\bar{\upsilon}_0 := \bar{\theta}_0^{\bar{\pi}} \wedge \upsilon$. We observe that

$$\begin{aligned}
\mathbb{E}_{\bar{\upsilon}_0}\left[[\upsilon_0 - \bar{\upsilon}_0]^p\right] &\leq (\Phi_0^p(\bar{\upsilon}_0))^p \quad \text{on } \{\upsilon_0 \geq \bar{\upsilon}_0\}, \\
\mathbb{E}_{\upsilon_0}\left[[\bar{\upsilon}_0 - \upsilon_0]^p\right] &\leq (\Phi_0^{p,\pi}(\upsilon_0))^p \quad \text{on } \{\upsilon_0 < \bar{\upsilon}_0\}
\end{aligned}$$

and continue with Lemma 2.8 to get

$$\begin{aligned}
\mathbb{E}_{\bar{\upsilon}_0}\left[[\upsilon_0 - \bar{\upsilon}_0]^p\right] &\leq p! L^{p-1} \Phi_0^1(\bar{\upsilon}_0) \quad \text{on } \{\upsilon_0 \geq \bar{\upsilon}_0\}, \\
\mathbb{E}_{\upsilon_0}\left[[\bar{\upsilon}_0 - \upsilon_0]^p\right] &\leq p! L^{p-1} \Phi_0^{1,\pi}(\upsilon_0) \quad \text{on } \{\upsilon_0 < \bar{\upsilon}_0\}.
\end{aligned}$$

Applying Theorem 2.3 to (X, π, P) and $\tau = \bar{\upsilon}_0$ we get

$$\begin{aligned}
\mathbb{E}_{\bar{\upsilon}_0}\left[[\upsilon_0 - \bar{\upsilon}_0]^p\right] &\leq p! L^{p-1} c_{(2.3)} P_{\bar{\upsilon}_0} \mathbf{1}_{\{P_{\bar{\upsilon}_0} \geq 0\}} \\
&\leq p! L^{p-1} c_{(2.3)} |P_{\bar{\upsilon}_0} - \bar{P}_{\bar{\upsilon}_0}|
\end{aligned}$$

on $\{\upsilon_0 > \bar{\upsilon}_0\}$, where we use that on $\{\upsilon_0 > \bar{\upsilon}_0\}$ we have $\bar{\upsilon}_0 = \bar{\theta}_0^{\bar{\pi}}$ and therefore $\bar{P}_{\bar{\upsilon}_0} = \bar{P}_{\bar{\theta}_0^{\bar{\pi}}} \leq 0$. Consequently,

$$\mathbb{E}_{\bar{\upsilon}_0}\left[[\upsilon_0 - \bar{\upsilon}_0]^p\right] \leq p! L^{p-1} c_{(2.3)} |P_{\bar{\upsilon}_0} - \bar{P}_{\bar{\upsilon}_0}| \text{ on } \{\upsilon_0 \geq \bar{\upsilon}_0\}.$$

Applying Theorem 2.4 to $(\bar{X}, \bar{\pi}, \bar{P})$ and $\tau = \upsilon_0$ implies

$$\begin{aligned}
\mathbb{E}_{\upsilon_0}\left[[\bar{\upsilon}_0 - \upsilon_0]^p\right] &\leq p! L^{p-1} d_{(2.4)} [|\bar{P}_{\upsilon_0}| + |\bar{\pi}|^{\frac{1}{2}}] \\
&= p! L^{p-1} d_{(2.4)} [|\bar{P}_{\upsilon_0} - P_{\upsilon_0}| + |\bar{\pi}|^{\frac{1}{2}}]
\end{aligned}$$

on $\{\upsilon_0 < \bar{\upsilon}_0\}$, where (similarly as above) on this set $\upsilon_0 = \theta_0$ and therefore $P_{\upsilon_0} = P_{\theta_0} = 0$. Letting $\vartheta := \upsilon_0 \wedge \bar{\upsilon}_0$, the above inequalities imply

$$\mathbb{E}_{\vartheta}\left[|\upsilon_0 - \bar{\upsilon}_0|^p\right] \leq p! L^{p-1} [c_{(2.3)} \vee d_{(2.4)}] \left[|P_{\vartheta} - \bar{P}_{\vartheta}| + |\bar{\pi}|^{\frac{1}{2}}\right],$$

which, by Assumption (15), leads to the desired result. □

We conclude this section with sufficient conditions ensuring that (15) holds. In the following, $\|\cdot\|_q$ denotes the $\mathbf{L}_q$-norm for $q \geq 1$. The proof being standard, it is postponed to the Appendix.

Lemma 3.2. *Assume $\vartheta, \tau \in \mathcal{T}$ such that $0 \leq \vartheta \leq \tau$.*

(a) *We have that*

$$\left\| d_{\mathcal{Z}}\left(X_\vartheta, \bar{X}_\vartheta\right) \right\|_1 \leq \inf_{1 < q < \infty} \sum_{k=0}^{\infty} \mathbb{P}\left[\tau \geq k\right]^{\frac{q-1}{q}} \left\| \sup_{t \in [k, k+1)} d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right) \right\|_q .$$

(b) *Assume* $\alpha > 0$, $0 < \beta < \infty$, $1 < q < \infty$, *and* $Q(\cdot, q) : \{0, 1, 2, ...\} \to \mathbb{R}_+$ *such that*

   (i) $\mathbb{P}[\tau \geq k] \leq \alpha e^{-\beta k}$ *for* $k = 0, 1, 2, ...,$

   (ii) $\left\| \sup_{t \in [k,k+1)} d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right) \right\|_q \leq Q(k, q) |\bar{\pi}|^{\frac{1}{2}}$ *for* $k = 0, 1, 2, ...$

   (iii) $c := \sum_{k=0}^{\infty} e^{\beta\left(\frac{1}{q}-1\right)k} Q(k, q) < \infty.$

   Then one has $\left\| d_{\mathcal{Z}}\left(X_\vartheta, \bar{X}_\vartheta\right) \right\|_1 \leq \alpha^{1-\frac{1}{q}} c |\bar{\pi}|^{\frac{1}{2}} = O(|\bar{\pi}|^{\frac{1}{2}}).$

Here we have some kind of trade-off between the decay of $\mathbb{P}[\tau \geq k]$, measured by β, and the growth of $\left\| \sup_{t \in [k,k+1)} d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right) \right\|_q$ measured by $Q(\cdot, \cdot)$. In the product $e^{\beta\left(\frac{1}{q}-1\right)k} Q(k, q)$ the factor $Q(k, q)$ is thought to be increasing in q, but the factor $e^{\beta\left(\frac{1}{q}-1\right)k}$ decreases as β and q increase.

Combining Theorem 3.1 and Lemma 3.2, and using

$$|P_\theta - \bar{P}_\theta| = |\delta(X_\theta) - \delta(\bar{X}_\theta)| \leq L d_{\mathcal{Z}}\left(X_\theta, \bar{X}_\theta\right),$$

gives the following corollary.

Corollary 3.3. *Let* υ *be a stopping time. Assume that the conditions of Lemma 3.2(b) are satisfied with* $\tau = \theta_0 \wedge \upsilon$*, and let* $p \geq 1$ *be an integer. Then there is a* $c > 0$*, depending at most on* $(r, L, d, \kappa, p, \alpha, \beta, q, Q)$*, such that*

$$\mathbb{E}\left[|[\theta_0 \wedge \upsilon] - [\bar{\theta}_0^{\bar{\pi}} \wedge \upsilon]|^p \right] \leq c\, |\bar{\pi}|^{\frac{1}{2}} \quad \textit{whenever} \quad |\bar{\pi}| \leq \varepsilon_{(3.1)}$$

with $\varepsilon_{(3.1)} > 0$ *taken from Theorem 3.1.*

3.2 Application to the Euler scheme approximation of the first exit time of a SDE

Now we specialize the discussion to the case where $\mathcal{Z} = \mathbb{R}^d$ endowed with the usual Euclidean norm $|\cdot|$ and where X is the strong solution of the stochastic differential equation

$$X_t = x_0 + \int_0^t \mu(X_s) ds + \int_0^t \sigma(X_s) dW_s$$

for some fixed $x_0 \in \mathcal{O}$, where $(\mu, \sigma) : \mathbb{R}^d \to (\mathbb{R}^d, \mathbb{R}^{d \times d})$ satisfy

Assumption 3.4. *There exists* $0 < L_\mu, L_\sigma \leq L$ *such that, for all* $x, y \in \mathbb{R}^d$,

$$|\mu(x) - \mu(y)| \leq L_\mu |x - y|, \quad |\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|,$$

and $|\mu(x)| + |\sigma(x)| \leq L$.

Remark 3.5. As usual some rows or columns of σ can be equal to 0. In particular, the first component of X can be seen as the time component by setting the first entry of μ equal to 1 and the first row of σ equal to 0, i.e. $X_t = (t, X_t^\flat)$ where $X^\flat$ is a diffusion process in $\mathbb{R}^{d-1}$. This allows to consider time dependent coefficients (where one could investigate to what extent weaker assumptions on the first coordinate of μ and σ, like 1/2-Hölder continuity, would be sufficient for the purpose of this paper). This formalism allows also to consider time-dependent domains as in [GM10], i.e. $\mathcal{O} = \bigcup_{t\geq 0} \left(\{t\} \times \mathcal{O}_t^\flat\right)$ where $(\mathcal{O}_t^\flat)_{t\geq 0}$ is a family of domains in $\mathbb{R}^{d-1}$. Then the distance function $\delta((t, x^\flat))$ shall be the signed spatial distance to the boundary $\mathcal{O}_t^\flat$.

In the following we denote by $D\delta$ and $D^2\delta$ the gradient (considered as row vector) and the Hessian matrix of δ, respectively. To verify condition **(P)** we use the following sufficient assumption:

Assumption 3.6. *There exists a bounded C_b^2 function $\delta : \mathbb{R}^d \mapsto \mathbb{R}$ such that $\delta > 0$ on $\mathcal{O}$, $\delta = 0$ on $\partial\mathcal{O}$ and $\delta < 0$ on $\bar{\mathcal{O}}^c$, which satisfies $|D\delta| \leq 1$ and the non-characteristic boundary condition*

$$|D\delta\ \sigma| \geq 2L^{-1} \quad \text{on } \{|\delta| \leq r\}. \tag{16}$$

Note that this condition is usually satisfied if σ is uniformly elliptic and the domain has a C^2 compact boundary, see e.g. [GiTr01].

We let $\bar{X}$ be the Euler scheme based on the grid $\bar{\pi}$, i.e.

$$\bar{X}_t = x_0 + \int_0^t \mu(\bar{X}_{\bar{\phi}_s})ds + \int_0^t \sigma(\bar{X}_{\bar{\phi}_s})dW_s. \tag{17}$$

We are now in a position to state the main results of this section, whose proofs are postponed to the end of the section. Note that a sufficient condition for the assumption (18) below is given in Lemma A.4. See also [Fre85, Chapter 3].

Theorem 3.7. *Let the Assumptions 3.4 and 3.6 hold and assume that*

$$\mathbb{E}_\tau\left[|\bar{\theta}_0^{\bar{\pi}}(\tau) - \tau| + |\theta_0(\tau) - \tau|\right] \leq L \quad \text{for all} \quad \tau \in \mathcal{T}. \tag{18}$$

Let υ be a stopping time with values in $\mathbb{R}_+ \cup \{\infty\}$. Assume that there are $\rho > 0$, $4 \leq q < \infty$, and $\beta > \frac{qd}{q-1}(6L_\mu + 3qL_\sigma^2)$ such that

$$\mathbb{P}\left[\theta_0 \wedge \upsilon \geq k\right] \leq \rho e^{-\beta k} \quad \text{for all } k = 0, 1, 2, ...$$

Then there exist $c, \varepsilon > 0$ and, for any integer $p \geq 1$, a constant $c_p > 0$ such that, for $|\bar{\pi}| \leq \varepsilon$,

$$\mathbb{E}\left[|[\theta_0 \wedge \upsilon] - [\bar{\theta}_0^{\bar{\pi}} \wedge \upsilon]|^p\right] \leq c_p\ |\bar{\pi}|^{\frac{1}{2}} \quad \text{and} \quad \left(\mathbb{E}\left[|X_{\theta_0\wedge\upsilon} - \bar{X}_{\bar{\theta}_0^{\bar{\pi}}\wedge\upsilon}|^2\right]\right)^{\frac{1}{2}} \leq c\ |\bar{\pi}|^{\frac{1}{4}}.$$

Remark 3.8. Assuming (for example) $\upsilon \equiv \infty$, for the purpose of this paper the estimate $\mathbb{E}\left[|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p\right] \leq c_p|\bar{\pi}|^{\frac{1}{2}}$ is sufficient, as we know from [Gob00, GM10] that it can not be improved for $p = 1$. However, it would be of interest to find the optimal exponents $\alpha_p > 0$ such that $\mathbb{E}\left[|\theta_0 - \bar{\theta}_0^{\bar{\pi}}|^p\right] \leq c_p|\bar{\pi}|^{\alpha_p}$, in the case $p > 1$. This is left for future studies.

In the case where we are only interested in a finite horizon problem, then the integrability condition (18) is not necessary.

Theorem 3.9. *Let the Assumptions 3.4 and 3.6 hold. Fix $T > 0$. Then there exist $c, \varepsilon > 0$ and, for any integer $p \geq 1$, a constant $c_p > 0$ such that, for $|\bar{\pi}| \leq \varepsilon$,*

$$\mathbb{E}\left[|[\theta_0 \wedge T] - [\bar{\theta}_0^{\bar{\pi}} \wedge T]|^p\right] \leq c_p\, |\bar{\pi}|^{\frac{1}{2}} \quad \text{and} \quad \left(\mathbb{E}\left[\left|X_{\theta_0 \wedge T} - \bar{X}_{\bar{\theta}_0^{\bar{\pi}} \wedge T}\right|^2\right]\right)^{\frac{1}{2}} \leq c\, |\bar{\pi}|^{\frac{1}{4}}.$$

Remark 3.10. The main aim of [BM09] was to study the strong error made when approximating the solution of a BSDE whose terminal condition is of the form $g(X_{\theta_0 \wedge T})$, for some Lipschitz map g and $T > 0$, by a backward Euler scheme; see [BM09] for the corresponding definitions and references. Theorem 3.9 complements [BM09, Theorem 3.1] in which the upper-bound takes the form $O_{|\bar{\pi}|\to 0}(|\bar{\pi}|^{\frac{1}{2}-\varepsilon})$ for all $0 < \varepsilon < 1/2$. Moreover, the upper-bound of the second inequality of [BM09, Theorem 3.3] is of the form $O_{|\bar{\pi}|\to 0}(|\bar{\pi}|^{\frac{1}{4}-\varepsilon})$ for all $0 < \varepsilon < 1/4$. This comes from the control they obtained on the exit time of their Theorem 3.1. With Theorem 3.9 of this paper it can be reduced to $O_{|\bar{\pi}|\to 0}(|\bar{\pi}|^{\frac{1}{4}})$. Our results open the door to the study of backward Euler type approximations of BSDEs with a terminal condition of the form $g(X_{\theta_0})$, i.e. there is no finite time horizon $T > 0$. This will however require to study at first the regularity of the solution of the BSDE, which is beyond the scope of this paper.

Proof of Theorems 3.7 and 3.9. (a) Theorem 3.7 is an immediate consequence of Lemmas 3.2, A.1, A.2 and A.3 (see the Appendix below) and Corollary 3.3.

(b) To prove Theorem 3.9 we verify that condition (18) can be avoided when the time horizon is bounded. First we extend $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ equipped with the Euclidean metric and consider a function $\varrho \in C_b^2(\mathbb{R})$ such that

1. $\varrho \leq 0$ and $\varrho(0) = 0$,
2. ϱ is strictly increasing on $[-2L, 0]$ and strictly decreasing on $[0, 2L]$,
3. $\varrho \equiv -A$ on $[-2L, 2L]^c$ for some $A > L$,

4. $D\varrho = 1$ on $[-L-(r/2), -(r/2)]$ and $D\varrho = -1$ on $[(r/2), L+(r/2)]$.

Note that our assumptions $0 < r < 1/(4L^3)$ and $L \geq 1$ guarantee the existence of such a ϱ. Moreover, we can assume that $|\delta|_\infty \leq L$ as P and $\bar{P}$ take values in $[-L, L]$ only. We define the Lipschitz function $\delta^\# : \mathbb{R}^{d+1} \to \mathbb{R}$ by

$$\delta^\#(x,y) := \delta(x) + \varrho(y)$$

and extend the open set $\mathcal{O}$ to an open set

$$\mathcal{O}^\# := \{\delta^\# > 0\} \subseteq \mathbb{R}^{d+1}.$$

By our construction we have that

1. $\emptyset \neq \mathcal{O}^\# \subsetneq \overline{\mathcal{O}^\#} \subsetneq \mathbb{R}^{d+1}$,
2. $\mathcal{O}^\# \subseteq \mathbb{R}^d \times [-2L, 2L]$,
3. $\delta^\#$ is a distance function for $\mathcal{O}^\#$ in the sense of Assumption **(P)**.

Assume an auxiliary one-dimensional Brownian motion $B = (B_t)_{t\geq 0}$ on a complete probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and define $\bar{\Omega} := \Omega \times \Omega'$ equipped with the completion $\bar{\mathcal{F}}$ of $\mathcal{F} \otimes \mathcal{F}'$ with respect to $\bar{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}'$. We extend the processes W, B, X and $\bar{X}$ canonically to $\bar{\Omega}$ (where we keep the notation of the processes) and define the additional process Y by $Y_t := B_{t\vee T} - B_T$. The right-continuous augmentation of the natural filtration of the $(d+1)$-dimensional Brownian motion (W, B) is denoted by $(\bar{\mathcal{F}}_t)_{t\geq 0}$. Therefore, letting

$$X^\# := \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \bar{X}^\# := \begin{pmatrix} \bar{X} \\ Y \end{pmatrix}, \quad \text{and} \quad (P^\#, \bar{P}^\#) := (\delta^\#(X^\#), \delta^\#(\bar{X}^\#)),$$

we obtain a setting that fulfills the assumptions of this paper. Now we check that $(X^\#, \pi, P^\#)$ and $(\bar{X}^\#, \bar{\pi}, \bar{P}^\#)$ satisfy the conditions **(Z)**, **(P)** and **(L)** with possibly modified parameters (κ, L, r).

Assumption **(Z)**: For $(X^\#, \pi, P^\#)$ the assumption is trivial, the case $(\bar{X}^\#, \bar{\pi}, \bar{P}^\#)$ follows from the proof of Lemma A.1.

Assumption **(P)** for $(X^\#, \pi, P^\#)$: The process $P^\#$ admits an Itô decomposition

$$dP^\# = b^\# dt + a^{\#\top} d \begin{pmatrix} W \\ B \end{pmatrix},$$

where $b^\#$ is uniformly bounded and

$$a^{\#\top} := \begin{pmatrix} D\delta(X)\sigma(X) & 0 \\ (0,\ldots,0) & D\varrho(Y)\mathbf{1}_{[T,\infty)} \end{pmatrix}$$

is also bounded. The condition **(P-*ii*)** follows from the observation that $|\delta^{\#}| \leq r/2$ implies either $|\delta| \leq r$ or $r/2 \leq |\varrho| \leq r/2 + L$.

Assumption **(P)** for $(\bar{X}^{\#}, \bar{\pi}, \bar{P}^{\#})$: Similarly, using **(P-*ii*)** for $r/2$, implies that $|\bar{P}_t| \leq r$ and $|\bar{X}_t - \bar{X}_{\phi_t}| \leq r/2$, or $r/2 \leq |\varrho| \leq r/2 + L$.

Assumption **(L)**: It is sufficient to check the exit time of the process Y from $[-2L, 2L]$ computed on $\bar{\pi}$. This follows by the arguments of the proof of Lemma A.4.

Finally we observe that $P_{t\wedge T} = P^{\#}_{t\wedge T}$, $\bar{P}_{t\wedge T} = \bar{P}^{\#}_{t\wedge T}$, $\theta_0 \wedge T = \theta_0^{\#} \wedge T$, and $\bar{\theta}_0^{\bar{\pi}} \wedge T = \bar{\theta}_0^{\bar{\pi},\#} \wedge T$, where the quantities without $\#$ are taken with respect to $(X, \bar{X}, \mathcal{O})$ and the other ones for $(X^{\#}, \bar{X}^{\#}, \mathcal{O}^{\#})$. This implies that $\mathbb{E}\left[|[\theta_0 \wedge T] - [\bar{\theta}_0^{\bar{\pi}} \wedge T]|^p\right] \leq c_p \, |\bar{\pi}|^{\frac{1}{2}}$ and therefore $\left(\mathbb{E}\left[|X_{\theta_0\wedge T} - \bar{X}_{\bar{\theta}_0^{\bar{\pi}}\wedge T}|^2\right]\right)^{\frac{1}{2}} \leq c\, |\bar{\pi}|^{\frac{1}{4}}$. □

A Appendix

A.1 Proof of Proposition 2.2.

(L)$\Longleftrightarrow$(L'): The condition **(L)** obviously implies **(L')**. Conversely, since $\Phi_0^{1,\pi}(\tau) \leq \mathbb{E}_\tau\left[\Phi_0^{1,\pi}(\phi_\tau^+)\right] + |\pi|$, where $|\pi| \leq 1$, and $\Phi_0^{1,\pi}(\phi_\tau^+) = 0$ on $\{P_{\phi_\tau^+} \leq 0\}$, the assumption **(L')** implies that $\Phi_0^{1,\pi}(\tau) \leq L' + 1$ for all $\tau \in \mathcal{T}$.

(L)$\Longleftrightarrow$(L"): Indeed, **(L)** implies **(L")** by Markov's inequality applied to the level $c := L/\alpha$ for a given $\alpha \in (0,1)$. Conversely, the fact that $\theta_0^\pi(\tau + kc) = \theta_0^\pi(\tau)$ on $\{\theta_0^\pi(\tau) \geq \tau + kc\}$ implies that

$$\mathbb{P}_\tau[\theta_0^\pi(\tau) \geq \tau + (k+1)c] = \mathbb{E}_\tau\left[\mathbf{1}_{\{\theta_0^\pi(\tau)\geq\tau+kc\}}\mathbb{P}_{\tau+kc}[\theta_0^\pi(\tau+kc) \geq \tau + (k+1)c]\right].$$

Applying **(L")** inductively, allows us to conclude that the left-hand side above is controlled by α^{k+1}. It follows that

$$\begin{aligned}
\mathbb{E}_\tau[\theta_0^\pi(\tau) - \tau] &\leq c + c\sum_{k\geq 0}\mathbb{P}_\tau[\theta_0^\pi(\tau) \geq \tau + (k+1)c] \\
&\leq c + c\sum_{k\geq 0}\alpha^{k+1} = c/(1-\alpha) =: L.
\end{aligned}$$

This proves that **(L")** implies **(L)**. □

A.2 Proof of Lemma 3.2

(a) For $q > 1$ we simply observe that

$$\left\|d_{\mathcal{Z}}\left(X_\vartheta, \bar{X}_\vartheta\right)\right\|_1 \leq \sum_{k=0}^{\infty}\mathbb{E}\left[\sup_{t\in[k,k+1)} d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right)\mathbf{1}_{\vartheta\in[k,k+1)}\right]$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{\infty} \mathbb{P}\left[\vartheta \in [k, k+1)\right]^{\frac{q-1}{q}} \left\| \sup_{t\in[k,k+1)} d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right) \right\|_q \\
&\leq \sum_{k=0}^{\infty} \mathbb{P}\left[\tau \geq k\right]^{\frac{q-1}{q}} \left\| \sup_{t\in[k,k+1)} d_{\mathcal{Z}}\left(X_t, \bar{X}_t\right) \right\|_q .
\end{aligned}
$$

(b) follows immediately. □

A.3 Verification of the assumptions for the Euler scheme approximation

All over this section, we work under the framework of Section 3.2. We start with the condition **(Z)** that is - in a sense - independent from the set $\mathcal{O}$.

Lemma A.1. *Under the Assumption 3.4 the processes X and $\bar{X}$ satisfy condition* **(Z)***, where the function $\kappa : \mathbb{R}_+ \times (0, \infty) \to \mathbb{R}_+$ depends at most on (L, d).*

Proof. For the process X with the time-net $\pi = \mathbb{R}_+$, it is trivially satisfied. Let us fix $\tau \in \mathcal{T}$, $A \in \mathcal{F}_\tau$ of positive measure, $T > 0$ and set

$$Y_t := \bar{X}_{\tau+tT}\mathbf{1}_A \quad \text{for } t \in [0,1].$$

For $2 < p < \infty$ Assumption 3.4 implies that

$$\mathbb{E}\left[|Y_t - Y_s|^p\right] \leq c[T^p + T^{\frac{p}{2}}]|t-s|^{\frac{p}{2}}\mathbb{P}[A]$$

for some $c = c(L, d, p) > 0$ independent from the choice of $A \in \mathcal{F}_\tau$. Fix $\alpha \in (0, \frac{1}{2} - \frac{1}{p})$. Then it follows from the continuity of Y and (the proof of) Kolmogorov's theorem in [RY05, Theorem 2.1, p.26] that

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_A \sup_{\tau\leq t\leq\tau+T} |\bar{X}_t - \bar{X}_{\bar{\phi}_t\vee\tau}|^p\right] &\leq \mathbb{E}\left[\mathbf{1}_A \sup_{|t-s|\leq|\bar{\pi}|/T} |Y_t - Y_s|^p\right] \\
&\leq \frac{|\bar{\pi}|^{p\alpha}}{T^{p\alpha}}\mathbb{E}\left[\mathbf{1}_A \sup_{|t-s|\leq|\bar{\pi}|/T; s,t\in D} \frac{|Y_t - Y_s|^p}{(|\bar{\pi}|/T)^{p\alpha}}\right] \\
&\leq \frac{|\bar{\pi}|^{p\alpha}}{T^{p\alpha}}\mathbb{E}\left[\mathbf{1}_A \sup_{0\leq s<t\leq 1; s,t\in D} \frac{|Y_t - Y_s|^p}{|t-s|^{p\alpha}}\right] \\
&\leq \frac{|\bar{\pi}|^{p\alpha}}{T^{p\alpha}} c'[T^p + T^{\frac{p}{2}}]\mathbb{P}[A],
\end{aligned}
$$

where $c' = c'(c, p, \alpha) > 0$ and $D \subseteq [0,1]$ are the dyadic points. Choosing $p = 6$ and $\alpha = 1/6 \in (0, \frac{1}{2} - \frac{1}{p}) = (0, \frac{2}{6})$ gives

$$\mathbb{E}_\tau\left[\sup_{\tau\leq t\leq\tau+T} |\bar{X}_t - \bar{X}_{\bar{\phi}_t\vee\tau}|^p\right] \leq c'[T^5 + T^2]|\bar{\pi}|$$

and

$$\mathbb{P}_\tau\left[\sup_{\tau\le t\le\tau+T}|\bar X_t-\bar X_{\bar\phi_t\vee\tau}|\ge\rho\right]\le\frac{c'[T^5+T^2]}{\rho^p}|\bar\pi|\quad\text{for }\rho>0.$$

□

The next lemma is similar to [Avi07, Theorem A.1], which however involves a T^2 term in the exponent, while we need a linear term. It corresponds to the condition (b-ii) of Lemma 3.2.

Lemma A.2. *If Assumption 3.4 holds, then one has for all $4\le q<\infty$ that*

$$\left\|\sup_{t\in[0,T]}|X_t-\bar X_t|\right\|_q\le Q(T,q)|\bar\pi|^{\frac12},$$

where $Q(T,q):=cQ_q(T)e^{\alpha T}$ with $c>0$ depending at most on (q,L,L_μ,L_σ,d), a non-negative polynomial Q_q, and $\alpha:=d(6L_\mu+3qL_\sigma^2)$.

Proof. 1. Let $2\le v<\infty$ and set $\Delta:=X-\bar X$. It follows from Itô's Lemma that

$$\begin{aligned}|\Delta_s|^{2v}&=\int_0^s 2v|\Delta_u|^{2v-2}\Delta_u^\top d\Delta_u+\sum_{i=1}^d\int_0^s v|\Delta_u|^{2v-2}d\langle\Delta^i\rangle_u\\&\quad+\sum_{i,j=1}^d 2v(v-1)\int_0^s\Delta_u^i\Delta_u^j|\Delta_u|^{2v-4}d\langle\Delta^i,\Delta^j\rangle_u.\end{aligned}$$

Under Assumption 3.4 we obtain

$$\begin{aligned}&\mathbb{E}\left[|\Delta_s|^{2v}\right]\\\le&\int_0^s 2v\mathbb{E}\left[|\Delta_u|^{2v-1}|\mu(X_u)-\mu(\bar X_{\bar\phi_u})|\right]du\\&+\int_0^s v(1+2d(v-1))\mathbb{E}\left[|\Delta_u|^{2v-2}|\sigma(X_u)-\sigma(\bar X_{\bar\phi_u})|^2\right]du\\\le&\ A\int_0^s\mathbb{E}\left[|\Delta_u|^{2v-1}\left(|\Delta_u|+|\bar X_u-\bar X_{\bar\phi_u}|\right)\right]du\\&+B\int_0^s\mathbb{E}\left[|\Delta_u|^{2v-2}||\Delta_u|^2+|\bar X_u-\bar X_{\bar\phi_u}|^2|\right]du\\=&\ [A+B]\int_0^s\mathbb{E}\left[|\Delta_u|^{2v}\right]du+A\int_0^s\mathbb{E}\left[|\Delta_u|^{2v-1}|\bar X_u-\bar X_{\bar\phi_u}|\right]du\\&+B\int_0^s\mathbb{E}\left[|\Delta_u|^{2v-2}|\bar X_u-\bar X_{\bar\phi_u}|^2\right]du\end{aligned}$$

for $A:=2vL_\mu$ and $B:=2v(1+2d(v-1))L_\sigma^2\le 6dv^2L_\sigma^2$. Exploiting

$$|\Delta_u|^{2v-1}|\bar X_u-\bar X_{\bar\phi_u}|\le\frac{2v-1}{2v}|\Delta_u|^{2v}+\frac{1}{2v}|\bar X_u-\bar X_{\bar\phi_u}|^{2v}$$

and

$$|\Delta_u|^{2v-2}|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^2 \leq \frac{v-1}{v}|\Delta_u|^{2v} + \frac{1}{v}|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}$$

we arrive at

$$\begin{aligned}
& \mathbb{E}\left[|\Delta_s|^{2v}\right] \\
\leq\ & \left[A + B + A\frac{2v-1}{2v} + B\frac{v-1}{v}\right]\int_0^s \mathbb{E}\left[|\Delta_u|^{2v}\right] du \\
& + \left[\frac{A}{2v} + \frac{B}{v}\right]\int_0^s \mathbb{E}\left[|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}\right] du \\
\leq\ & 2\left[A+B\right]\int_0^s \mathbb{E}\left[|\Delta_u|^{2v}\right] du + \left[\frac{A}{2v} + \frac{B}{v}\right]\int_0^s \mathbb{E}\left[|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}\right] du \\
\leq\ & 12d[vL_\mu + v^2 L_\sigma^2]\int_0^s \mathbb{E}\left[|\Delta_u|^{2v}\right] du \\
& + 6d[L_\mu + vL_\sigma^2]\int_0^s \mathbb{E}\left[|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}\right] du.
\end{aligned}$$

Exploiting

$$\sup_{u\geq 0} \mathbb{E}\left[|\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}\right] \leq c_v\ |\bar{\pi}|^v$$

for some constant $c_v = c(v, L) > 0$, where we use the boundedness part of Assumption 3.4, we derive

$$\mathbb{E}\left[|\Delta_s|^{2v}\right] \leq 12d[vL_\mu + v^2 L_\sigma^2]\int_0^s \mathbb{E}\left[|\Delta_u|^{2v}\right] du + 6d[L_\mu + vL_\sigma^2] s c_v |\bar{\pi}|^v$$

and, by Gronwall's Lemma,

$$\mathbb{E}\left[|\Delta_s|^{2v}\right] \leq 6d[L_\mu + vL_\sigma^2] c_v s e^{s12d[vL_\mu + v^2 L_\sigma^2]}|\bar{\pi}|^v.$$

2. Using the Itô decomposition of $|\Delta|^2$ and the Burkholder-Davis-Gundy and Hölder inequalities, we obtain (for another constant $c'_v = c'(v, L_\mu, L_\sigma) > 0$) that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq T}|\Delta_s|^{2v}\right] &\leq c'_v\ [T^{v-1} + T^{v/2-1}]\int_0^T \mathbb{E}\left[|\Delta_u|^{2v} + |\bar{X}_u - \bar{X}_{\bar{\phi}_u}|^{2v}\right] du \\
&\leq c'_v\ [T^v + T^{v/2}]|\bar{\pi}|^v\left[c_v + 6d[L_\mu + vL_\sigma^2]c_v e^{12dT[vL_\mu + v^2L_\sigma^2]}\right] \\
&\leq c'_v c_v\ [T^v + T^{v/2}]\left[1 + 6d[L_\mu + vL_\sigma^2]\right]e^{12dT[vL_\mu + v^2L_\sigma^2]}|\bar{\pi}|^v.
\end{aligned}$$

Consequently, for $q \geq 4$,

$$\left\|\sup_{0\leq s\leq T}|\Delta_s|\right\|_q \leq C(q, L, L_\mu, L_\sigma, d) Q_q(T) e^{Td[6L_\mu + 3qL_\sigma^2]}|\bar{\pi}|^{\frac{1}{2}}.$$

□

Now, we verify assumption **(P)**:

Lemma A.3. *Let the Assumptions 3.4 and 3.6 hold. Then P and $\bar{P}$ satisfy the condition* **(P)** *for $r > 0$ small enough and $L \geq 1$ large enough, independently of $|\bar{\pi}|$.*

Proof. First we apply Itô's Lemma to obtain that $d\bar{P}_t = \bar{b}_t dt + \bar{a}_t^\top dW_t$ with

$$\bar{b}_t := D\delta(\bar{X}_t)\mu(\bar{X}_{\bar{\phi}_t}) + \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(\bar{X}_{\bar{\phi}_t})D^2\delta(\bar{X}_t)] \text{ and } \bar{a}_t^\top := D\delta(\bar{X}_t)\sigma(\bar{X}_{\bar{\phi}_t}).$$

Up to an increase of L in Assumption **(P)** (which potentially leads to a decrease of r to satisfy $0 < r < 1/(4L^3)$), condition (i) is satisfied because $\delta, \mu, \sigma, D\delta, D^2\delta$ are bounded. Since $D\delta$ is bounded by L and σ is L-Lipschitz,

$$\left|D\delta(\bar{X}_t)\sigma(\bar{X}_{\bar{\phi}_t}) - D\delta(\bar{X}_t)\sigma(\bar{X}_t)\right| \leq L^2|\bar{X}_{\bar{\phi}_t} - \bar{X}_t|.$$

Consequently,

$$\begin{aligned} |\bar{a}_t| &\geq |D\delta(\bar{X}_t)\sigma(\bar{X}_t)| - \left|D\delta(\bar{X}_t)\sigma(\bar{X}_{\bar{\phi}_t}) - D\delta(\bar{X}_t)\sigma(\bar{X}_t)\right| \\ &\geq |D\delta(\bar{X}_t)\sigma(\bar{X}_t)| - L^2|\bar{X}_t - \bar{X}_{\bar{\phi}_t}|. \end{aligned}$$

For $|\bar{P}_t| \leq r$, $|\bar{X}_t - \bar{X}_{\bar{\phi}_t}| \leq r$ and $0 < r < 1/(4L^3)$ this finally gives $|\bar{a}_t| \geq 1/L$ so that $\bar{P}$ satisfies **(P-*ii*)**. The argument for P is analogous. $\square$

We finally consider consider the Assumption **(L)**. Conditions of type (19) below can be found in [Fre85, Chapter 3].

Lemma A.4. *Let Assumption 3.4 be satisfied and assume an $R > 0$ and a non-increasing function $\varphi : [0,\infty) \to (0,\infty)$ with $\lim_{T\to\infty}\varphi(T) = 0$ such that*

$$\sup_{x\in\mathcal{O}} \mathbb{P}\left[\theta_0^x(R) \geq T\right] \leq \varphi(T) \quad \textit{for all} \quad T \geq 0, \tag{19}$$

where the open set $\mathcal{O}(R) := \mathcal{O} + B_R$ (B_R is the open ball centered at zero with radius $R > 0$) satifies $\mathcal{O}(R) \subsetneq \overline{\mathcal{O}(R)} \subsetneq \mathbb{R}^d$ and $\theta_0^x(R) := \inf\{t \geq 0 : X_t^x \notin \mathcal{O}(R)\}$ for $x \in \mathcal{O}$ with $(X_t^x)_{t\geq 0}$ being the diffusion started in $x \in \mathbb{R}^d$. Then there exist $\bar{\varepsilon} \in (0,1]$ and a constant $K > 0$ such that $|\bar{\pi}| \leq \bar{\varepsilon}$ implies

$$\mathbb{E}_\tau\left[\bar{\theta}_0^{\bar{\pi}}(\tau) - \tau\right] + \mathbb{E}_\tau\left[\theta_0(\tau) - \tau\right] \leq K \quad \textit{for all} \quad \tau \in \mathcal{T}.$$

Proof. We only consider the estimate which involves $\bar{\theta}_0^{\bar{\pi}}(\tau)$ (the other one follows directly from Proposition 2.2). By Proposition 2.2 the case $\bar{\theta}_0^{\bar{\pi}}(\tau)$ can be reduced to find $\alpha \in (0,1)$ and $c > 0$ with

$$\mathbb{P}_\tau\left[\bar{\theta}_0^{\bar{\pi}}(\tau) - \tau \geq c\right] \leq \alpha \ \text{ for all } \ \tau \in \mathcal{T}. \tag{20}$$

Because for $c > 1$ one has

$$\mathbb{P}_\tau\left[\bar{\theta}_0^{\bar{\pi}}(\tau) - \tau \geq c\right] \leq \mathbb{E}_\tau\left[\mathbb{P}_{\bar{\phi}_\tau^+}\left[\bar{\theta}_0^{\bar{\pi}}(\bar{\phi}_\tau^+) - \bar{\phi}_\tau^+ \geq c - 1\right]\right],$$

it is sufficient to check (20) for $\tau \in \mathcal{T}^{\bar\pi}$. Given $\tau \in \mathcal{T}^{\bar\pi}$, we let

$$\check X_t = x_0 + \int_0^t \check\mu_s ds + \int_0^t \check\sigma_s dW_s$$

with $\check\mu_t := \mathbf{1}_{(0,\tau]}(t)\mu(\check X_{\bar\phi_t}) + \mathbf{1}_{(\tau,\infty)}(t)\mu(\check X_t)$ and with the corresponding definition for $\check\sigma$. Let $\check\theta_0(\tau, R) := \inf\{t \geq \tau : \check X_t \notin \mathcal{O}(R)\}$. For $c \geq 2$ and $\tau \in \mathcal{T}^{\bar\pi}$ with $|\bar\pi| \leq \bar\varepsilon$, where $\bar\varepsilon \in (0, 1]$ is chosen at the end of the proof, we get from Lemma A.2, applied to $q = 4$ and some $T_0 > 0$, for a set $A \in \mathcal{F}_\tau$ of positive measure with $A \subseteq \{\tau = t\} \cap \{\bar X_t^{\bar\pi} \in \mathcal{O}\}$ (note that τ takes only countable many values) that

$$\begin{aligned}
& \mathbb{P}\left[A \cap \{\bar\theta_0^{\bar\pi}(\tau) - \tau \geq c\}\right] \\
= \; & \mathbb{P}\left[A \cap \{\bar\theta_0^{\bar\pi}(t) - t \geq c\}\right] \\
\leq \; & \mathbb{P}\left[A \cap \{\check\theta_0(t,R) - t \geq c/2\}\right] \\
& \qquad + \mathbb{P}\left[A \cap \{|\check X_{\check\theta_0(t,R)} - \bar X_{\check\theta_0(t,R)}| \geq R/2\}\right] \\
& \qquad + \mathbb{P}\left[A \cap \{|\bar X_{\bar\phi^+_{\check\theta_0(t,R)}} - \bar X_{\check\theta_0(t,R)}| \geq R/2\}\right] \\
\leq \; & \mathbb{P}[A]\left[\sup_{x\in\mathcal{O}} \mathbb{P}[\theta_0^x(R) \geq c/2]\right. \\
& \qquad \left. + \sup_{x\in\mathcal{O}} \sup_{|\tilde\pi|\leq\bar\varepsilon} \mathbb{P}\left[|X^x_{\theta_0^x(R)} - \bar X^{x,\tilde\pi}_{\theta_0^x(R)}| \geq R/2\right] + \frac{c'(L)}{R^2}|\bar\pi|\right] \\
\leq \; & \mathbb{P}[A]\left[\sup_{x\in\mathcal{O}} \mathbb{P}[\theta_0^x(R) \geq c/2] + \sup_{x\in\mathcal{O}} \mathbb{P}[\theta_0^x(R) \geq T_0]\right. \\
& \left. + \sup_{x\in\mathcal{O}} \sup_{|\tilde\pi|\leq\bar\varepsilon} \mathbb{P}\left[|X^x_{\theta_0^x(R)\wedge T_0} - \bar X^{x,\tilde\pi}_{\theta_0^x(R)\wedge T_0}| \geq R/2\right] + \frac{c'(L)}{R^2}|\bar\pi|\right] \\
\leq \; & \mathbb{P}[A]\left[\varphi\left(\frac{c}{2}\right) + \varphi(T_0) + \left(\frac{2}{R}Q_{(\mathrm{A.2})}(T_0, 4)\right)^4 \bar\varepsilon^2 + \frac{c'(L)}{R^2}\bar\varepsilon\right],
\end{aligned}$$

where $\bar X^{x,\tilde\pi}$ is the Euler scheme for X^x based on the net $\tilde\pi$. First we choose $c \geq 2$ and $T_0 > 0$ large enough, then $\bar\varepsilon$ small enough in order to arrange (20) for all $\tau \in \mathcal{T}^{\bar\pi}$ and some $\alpha \in (0, 1)$. □

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