Second order stochastic target problems with generalized market impact

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Abstract

We extend the study of [7, 19] to stochastic target problems with general market impacts. Namely, we consider a general abstract model which can be associated to a fully nonlinear parabolic equation. Unlike [7, 19], the equation is not concave and the regularization/verification approach of [7] can not be applied. We also relax the gamma constraint of [7]. In place, we need to generalize the *a priori* estimates of [19] and exhibit smooth solutions from the classical parabolic equations theory. Up to an additional approximating argument, this allows us to show that the super-hedging price solves the parabolic equation and that a perfect hedging strategy can be constructed when the coefficients are smooth enough. This representation leads to a general dual formulation. We finally provide an asymptotic expansion around a model without impact.

1 Introduction

Inspired by [1, 19], the authors in [6, 7] considered a financial market with permanent price impact (and possibly a resilience effect), in which the impact function behaves as a linear function (around the origin) in the number of purchased stocks. This class of models is dedicated to the pricing and hedging of derivatives in situations where the notional of the product hedged is such that the delta-hedging is non-negligible compared to the average daily volume traded on the underlying asset. As opposed to [6], the options considered in [7, 19] are

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covered, meaning that the buyer of the option delivers, at the inception, the required initial delta position, and accepts a mix of stocks (at their current market price) and cash as payment for the final claim. This is a common practice which eliminates the cost incurred by the initial and final hedge. In [19], the author considers a Black-Scholes type model, while the model of [7] is a local volatility one.

Motivated by these works, we consider in this paper a general abstract model of market impact in which the dynamics of the stocks X, the *wealth*[1] V and the number of stocks Y held in the portfolio follow dynamics of the form

$$X = x + \int_t^\cdot \mu(s, X_s, \gamma_s, b_s)ds + \int_t^\cdot \sigma(s, X_s, \gamma_s)dW_s \tag{1}$$

$$Y = y + \int_t^\cdot b_s ds + \int_t^\cdot \gamma_s dX_s \tag{2}$$

$$V = v + \int_t^\cdot F(s, X_s, \gamma_s)ds + \int_t^\cdot Y_s dX_s \tag{3}$$

where (y, b, γ) are the controls, and we consider the general super-hedging problem:

$$\mathrm{v}(t, x) := \inf\{v = c + yx \ : \ (c, y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}(t, x, v, y) \neq \emptyset\},$$

in which

$$\mathcal{G}(t, x, v, y) = \Big\{(b, \gamma) : V_T^{t,x,v,\phi} \geq g(X_T^{t,x,\phi}) \text{ for } \phi := (y, b, \gamma)\Big\},$$

and g is the payoff function associated to a European claim. Note that, in the above, the number of stocks in the portfolio is taken in the form of an Itô process controlled by (b, γ). The process γ is the *gamma* of the portfolio describing the change in the number of stocks held in the portfolio following a change of the stock's price. It will be later on identified to the *gamma* of the option to be hedged. This is a key quantity in all our analysis. The fact that the bounded variation part of Y is absolutely continuous is for technical reasons. The function F entering in the dynamics of the wealth models the liquidity costs. We refer to Example 2.1 below for a typical example.

Given the above dynamics, one can easily be convinced, by using formal computations based on the geometric dynamic programming principle of [22], see also the discussion just after Remark 3.1, that v should be a super-solution of the fully nonlinear parabolic equation

$$0 \leq -\partial_t \mathrm{v} - \bar{F}(\cdot, \cdot, \partial_x^2 \mathrm{v}) \quad \text{and} \quad (|F| + |\sigma|)(\cdot, \cdot, \partial_x^2 \mathrm{v}) < \infty.$$

in which

$$\bar{F}(t, x, z) := \frac{1}{2}\sigma^2(t, x, z)z - F(t, x, z).$$

[1] More precisely: the value of the cash plus the number of stocks in the portofolio times the current value of the stocks.

The right-hand side constraint in the previous inequalities is of importance. Indeed $(F, \sigma)(t, x, \cdot)$ can typically be singular and only finite on an interval of the form $(-\infty, \bar{\gamma}(t, x))$, as it is the case in [7]. Under this last assumption, one can actually expect that v is a viscosity solution of

$$\min\{-\partial_t \mathrm{v} - \bar{F}(\cdot, \cdot, \partial_x^2 \mathrm{v}) \ , \ \bar{\gamma} - \partial_x^2 \mathrm{v}\} = 0 \text{ on } [0, T) \times \mathbb{R}, \tag{4}$$

with T-terminal condition given by the smallest function $\hat{g} \geq g$ such that $\partial_x^2 \hat{g} \leq \bar{\gamma}(T, \cdot)$.

In [7], the authors impose a strong (uniform) constraint on the controls of the form $\gamma \leq \tilde{\gamma}(\cdot, X^{t,x,\phi})$ with $\tilde{\gamma}$ such that $F(\cdot, \cdot, \tilde{\gamma}) \leq C$ for some $C > 0$, and obtain that v is actually the unique viscosity solution of (4) with $\tilde{\gamma}$ in place of $\bar{\gamma}$, and terminal condition $\hat{g}$ (defined with $\tilde{\gamma}$ as well). Their proof of the super-solution property mimicks arguments of [11], and we can follow this approach. As for the sub-solution property, they could not prove the appropriate dynamic programming principle, and the standard direct arguments could not be used. Instead, they employed a regularization argument for viscosity solutions, inspired by [16], together with a verification procedure. In [7], the authors critically use the fact that $\bar{F}$ is convex.

Our setting here is different. First, as in [19], we do not impose a uniform constraint on our strategies. Our controls can take values arbitrarily close to the singularity $\bar{\gamma}(\cdot, X^{t,x,\phi})$ and the equation (4) is possibly degenerate. Even for $\bar{F}$ defined as in [7] our setting is more general in a sense. Second, $\bar{F}$ is not assumed to be convex.

For these reasons, we can not reproduce the smoothing/verification argument of [7] to deduce that v is actually a subsolution.

In this paper, we therefore proceed differently and generalise arguments used in [19] in the context of a Black-Scholes type model. Namely, we directly use the theory of parabolic equations to prove the existence of smooth solutions to (4) whenever $\hat{g}$ is smooth and satisfies a constraint of the form $\partial_x^2 \hat{g} \leq \bar{\gamma}(T, \cdot) - \varepsilon$, for some $\varepsilon > 0$. Our analysis heavily relies on new *a priori* estimates, see Proposition 3.10 below, thanks to which one can appeal to the continuity method in a rather classical way, see the proof of Theorem 3.11. We then let ε go to 0 to conclude that v indeed solves (4) in the viscosity solution sense, see Theorem 3.5 below.

We also discuss two important issues that were not considered in [7] but already studied in [19] in a Black-Scholes type model:

- The first one concerns the asymptotic expansion of the price around a model without market impact. As in [19], we show that a first order expansion can be established, see Proposition 4.3 below. But, we also prove that one can deduce from it a strategy that matches the terminal face-lifted payoff $\hat{g}$ at any prescribed level of precision in $\mathbb{L}^\infty$-norm, see Proposition 4.6.
- The second one concerns the existence of a dual formulation. It can be established when $\bar{F}$ is convex in its last argument, see Theorem 5.2. Applied to the

model discussed in [7], see Example 2.1 below, it takes the form

$$\begin{aligned}
\mathrm{v}(t,x) &= \sup_{\mathfrak{s}} \mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \frac{1}{2}\frac{(\mathfrak{s}_s - \sigma_\circ(t, X_s^{t,x,\mathfrak{s}}))^2}{f(X_s^{t,x,\mathfrak{s}})} ds\right] \\
&= \sup_{\mathfrak{s}} \mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}}) - \int_t^T \frac{1}{2}\frac{(\mathfrak{s}_s - \sigma_\circ(t, X_s^{t,x,\mathfrak{s}}))^2}{f(X_s^{t,x,\mathfrak{s}})} ds\right]
\end{aligned}$$

in which $X^{t,x,\mathfrak{s}} = x + \int_t^\cdot \mathfrak{s}_s dW_s$, $\sigma_\circ$ is the volatility surface in a the market without impact and $f > 0$ is the impact function, the limit case $f \equiv 0$ corresponding to the absence of impact. It can be interpreted as the formulation of the super-hedging price with volatility uncertainty. The difference being that the formula is penalized by the squared distance of the realized volatility term $\mathfrak{s}$ to the original local volatility $\sigma_\circ(\cdot, X^{t,x,\mathfrak{s}})$ associated to the model, weighted by the inverse of the impact function $f(X^{t,x,\mathfrak{s}})$. It can also be seen as a martingale optimal transport problem, see [19, Section 4.1] for details.

To conclude, let us refer to [4, 5, 3, 10, 11, 13, 18, 20, 21, 22], and the references therein. Also for related works, see [7] for a discussion.

The rest of this paper is organized as follows. The general abstract market model is described in Section 2 and the characterization of v as a solution of a parabolic equation is proved in Section 3. The asymptotic expansion and the dual formulation are provided and discussed in Sections 4 and 5.

General notations. Throughout this paper, Ω is the canonical space of continuous functions on $\mathbb{R}_+$ starting at 0, $\mathbb{P}$ is the Wiener measure, W is the canonical process, and $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is the augmentation of its raw filtration $\mathcal{F}^\circ = (\mathcal{F}_t^\circ)_{t\geq 0}$. All random variables are defined on $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$, the integer $n \geq 1$ is given by the context. Unless otherwise specified, inequalities involving random variables are taken in the $\mathbb{P}-$a.s. sense. We use the convention $x/0 = \mathrm{sign}(x) \times \infty$ with $\mathrm{sign}(0) = +$. We denote by $\partial_x^n \varphi$ the nth-order derivative of a function φ with respect to its x-component, whenever it is well-defined. For E, F, G, three subsets of $\mathbb{R}$, We denote by $C_b^{h,k}(E \times F)$ the set of continuous functions on $E \times F$ which have bounded partial derivatives of order from 1 to h with respect to the first variable and from 1 to k to the second variable. We denote by $C^{h,k,l}(E \times F \times G)$ the set of continuous functions on $E \times F \times G$ which have partial derivatives of order from 1 to h with respect to the first variable, from 1 to k to the second variable and from 1 to l to the third variable. We denote by $C_b^h(E \times F)$ the set of continuous functions on $E \times F$ which have bounded partial derivatives of order 1 to h. If in addition its h-th order derivatives are uniformly α-Hölder, with $\alpha \in (0,1)$, we say that it belongs to $C_b^{h+\alpha}(E \times F)$. We omit the spaces E, F, G if they are clearly given by the context.

2 Abstract market impact model

We first describe our abstract market with impact. It generalizes the model studied in [6, 7, 19]. We use the representation of the hedging strategies described in [7], which is necessary to obtain the supersolution characterization of the super-hedging price of Proposition 3.8 below. How to get to the market evolution (1)-(2)-(3) is explained briefly in Example 2.1.

Let us start with the definition of the class of admissible controls (b, γ) that enter into the Itô decomposition of Y in (2). Given $k \geq 1$, we denote by $\mathcal{A}_k^\circ$ the collection of continuous and $\mathbb{F}$-adapted processes (b, γ) such that

$$\gamma = \gamma_0 + \int_0^{\cdot} \beta_s ds + \int_0^{\cdot} \alpha_s dW_s$$

where (α, β) is continuous, $\mathbb{F}$-adapted, and $\zeta := (b, \gamma, \alpha, \beta)$ is essentially bounded by k and such that

$$\mathbb{E}\left[\sup\left\{|\zeta_{s'} - \zeta_s|,\ t \leq s \leq s' \leq s + \delta \leq T\right\} | \mathcal{F}_t^\circ\right] \leq k\delta$$

for all $0 \leq \delta \leq 1$ and $t \in [0, T - \delta]$. We then define

$$\mathcal{A}^\circ := \cup_k \mathcal{A}_k^\circ.$$

Let $F : [0, T] \times \mathbb{R}^2 \mapsto \mathbb{R} \cup \{\infty\}$ be a continuous map and let

$$\mathcal{D} := \{F < \infty\}$$

be its domain. We assume that there exists a map $(t, x) \to \bar{\gamma}(t, x) \in \mathbb{R} \cup \{+\infty\}$ such that

$$\mathcal{D} = \{(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} :\ z \in (-\infty, \bar{\gamma}(t, x))\}, \tag{5}$$

and that

$$\bar{\gamma} \text{ is either uniformly continuous, or identically equal to } +\infty. \tag{6}$$

We now let $\mu : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathcal{D} \to \mathbb{R}$ be two continuous maps such that, for all $\varepsilon > 0$,

$$\begin{aligned} &\mu \text{ is Lipschitz, with linear growth in its second variable, on } \mathcal{D}_{\varepsilon,\varepsilon^{-1}} \times \mathbb{R}, \\ &\sigma \text{ is Lipschitz, with linear growth in its second variable, on } \mathcal{D}_{\varepsilon,\varepsilon^{-1}}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathcal{D}_\varepsilon &:= \{(t, x, z) \in [0, T] \times \mathbb{R}^2 : F(t, x, z) \leq \varepsilon^{-1}\}, \\ \mathcal{D}_{\varepsilon,k} &:= \mathcal{D}_\varepsilon \cap ([0, T] \times \mathbb{R} \times [-k, k]) \text{ for } k \in (0, \infty). \end{aligned} \tag{8}$$

Then, given $(t,x,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ and $\phi = (y,b,\gamma) \in \mathbb{R} \times \mathcal{A}^{\circ}$, we define $(X^{t,x,\phi}, Y^{t,x,\phi}, V^{t,x,v,\phi})$ as the solution on $[t,T]$ of

$$X = x + \int_t^{\cdot} \mu(s, X_s, \gamma_s, b_s)ds + \int_t^{\cdot} \sigma(s, X_s, \gamma_s)dW_s \tag{9}$$

$$Y = y + \int_t^{\cdot} b_s ds + \int_t^{\cdot} \gamma_s dX_s \tag{10}$$

$$V = v + \int_t^{\cdot} F(s, X_s, \gamma_s)ds + \int_t^{\cdot} Y_s dX_s \tag{11}$$

satisfying $(X_t, Y_t, V_t) = (x, y, v)$, whenever $(\cdot, X, \gamma)$ takes values in $\mathcal{D}_{\varepsilon,k}$ on $[t,T]$, for some $\varepsilon, k > 0$. If this is the case, we say that ϕ belongs to $\mathcal{A}^{\varepsilon}_k$. For ease of notations, we set $\mathcal{A} := \cup_{\varepsilon,k>0}\mathcal{A}^{\varepsilon}_k$.
For a payoff function $g : \mathbb{R} \to \mathbb{R}$ the super-hedging price of the covered European claim associated to g is then defined as

$$\mathrm{v}(t,x) := \inf\{v = c + yx \ : \ (c,y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}(t,x,v,y) \neq \emptyset\}, \tag{12}$$

in which

$$\mathcal{G}(t,x,v,y) = \left\{\phi = (y,b,\gamma) \in \mathcal{A} : V_T^{t,x,v,\phi} \geq g(X_T^{t,x,\phi})\right\}$$

whenever this set is non-empty. Note that

$$\mathrm{v}(t,x) = \inf_{\varepsilon>0} \mathrm{v}^{\varepsilon}(t,x) \ \text{ where } \ \mathrm{v}^{\varepsilon}(t,x) := \inf_{k>0} \mathrm{v}^{\varepsilon}_k(t,x) \tag{13}$$

in which

$$\mathrm{v}^{\varepsilon}_k(t,x) := \inf\{v = c + yx \ : \ (c,y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}^{\varepsilon}_k(t,x,v,y) \neq \emptyset\}, \tag{14}$$

with

$$\mathcal{G}^{\varepsilon}_k(t,x,v,y) = \left\{\phi = (y,b,\gamma) \in \mathcal{A}^{\varepsilon}_k : V_T^{t,x,v,\phi} \geq g(X_T^{t,x,\phi})\right\}.$$

In the following, we assume as in [7] that

$$g \text{ is lower-semicontinuous, bounded from below, and } g^+ \text{ has linear growth.} \tag{15}$$

Example 2.1 (Example of derivation of the evolution equations)**.** *We close this section with an example of formal derivation of the above abstract dynamics. In the spirit of [1, 19], let us consider a linear market impact model in which an (infinitesimal) order to buy dY_t stocks at t leads to a immediate price move of $f(t, X_{t-}, \gamma_t)dY_t$ and is followed by an immediate relaxation (or resilience) so that permanent price move is $\bar{f}(t, X_{t-}, \gamma_t)dY_t$ for some $\bar{f} \leq f$. The average execution price will be $X_{t-} + \frac{1}{2}f(t, X_{t-}, \gamma_t)dY_t$. Then, following the computations* *done in [1, 19], see also the rigorous proof in [6] for details*[2], the portfolio value V *corresponding to the holding in cash plus the number of stocks in the portfolio evaluated at their current price* X *is given by*[3]

$$V = v + \int_t^{\cdot} Y_s dX_s - \int_t^{\cdot} (\frac{1}{2}f - \bar{f})(s, X_s, \gamma_s) d\langle Y \rangle_s.$$

The contribution $(\frac{1}{2}f - \bar{f})(s, X_s, \gamma_s)d\langle Y \rangle_s$ *is due to the spread between the execution price of the trade and the final price after market impact. It can be either positive or negative. The fact that* f *and* $\bar{f}$ *can depend on* γ *is discussed in [19]. Let us now assume that* X *evolves according to* $dX_t = \sigma_\circ(t, X_t)dW_t + \mu_\circ(t, X_t)dt$ *in the absence of trade. Then, arguing again as in [6], we obtain the modified dynamics*

$$dX_t = \sigma_\circ(t, X_t)dW_t + \mu_\circ(t, X_t)dt + \bar{f}(t, X_t, \gamma_t)dY_t + \partial_x \bar{f}(t, X_t, \gamma_t)\gamma_t \sigma_\circ^2(t, X_t)dt.$$

Combining this with (10), and formally solving in dX, *we obtain that*

$$\sigma(t, X_t, \gamma_t) = \frac{\sigma_\circ(t, X_t)}{1 - \bar{f}(t, X_t, \gamma_t)\gamma_t},$$

so that the dynamics of V *can be written as*

$$V = v + \int_t^{\cdot} Y_s dX_s - \int_t^{\cdot} (\frac{1}{2}f - \bar{f})(s, X_s, \gamma_s) \left(\frac{\sigma_\circ(s, X_s)\gamma_s}{1 - \bar{f}(s, X_s, \gamma_s)\gamma_s} \right)^2 ds.$$

Note that, as observed in [6], the drift $\mu_\circ$ *is also affected by the market impact, but that this does not affect the pricing equation. It is therefore not taken into account in our abstract model. The model studied in [6, 7] corresponds to* $f = f(x)$ *(no dependency in* t, γ*) and* $\bar{f} = f$ *(no immediate resilience). In this particular case, the functions* σ *and* F *are given by*

$$\sigma(t, x, z) = \frac{\sigma_\circ(t,x)}{1 - \bar{f}(x)z}, \ \bar{\gamma} = 1/\bar{f}$$
$$F(t, x, z) = \frac{1}{2} \left(\frac{\sigma_\circ(t,x)z}{1 - \bar{f}(x)z} \right)^2 \bar{f}(x) \mathbf{I}_{\{\bar{f}(x)z < 1\}} + \infty \mathbf{I}_{\{\bar{f}(x)z \geq 1\}}.$$

Remark 2.2. *As in [7, Section 4], a (non-immediate) resilience effect could be added in our model. This would take the form of a drift term in the dynamic of* X, *that depends on past orders. As explained in [7, Section 4], it would not play any role in this setting of covered options. Note also that the above setting allows to consider market impact functions that are not globally linear, but only "linear around* 0*", see [7, Remark 2.3] for a precise discussion.*

[2] The continuous time version is obtained by considering the limit dynamics of a discrete time trading model, as the speed of trading goes to infinity.

[3] Obviously, this is only a theoretical value, the liquidation value of the portfolio being different.

3 PDE characterization

The parabolic equation associated to v can be formally derived as follows. Assume that v is smooth and that a perfect hedging strategy $\phi = (y, b, \gamma)$ can be found when starting at t from $v = \mathrm{v}(t, x)$ if the stock price is x at t. Then, we expect to have $V^{t,x,v,\phi} = \mathrm{v}(\cdot, X^{t,x,\phi})$ which, by Itô's lemma combined with (9)-(11), implies that

$$\begin{aligned} & F(s, X_s^{t,x,\phi}, \gamma_s)ds + Y_s^{t,x,\phi} dX_s^{t,x,\phi} \\ & = (\partial_t \mathrm{v} + \frac{1}{2}\sigma^2(\cdot,\cdot,\gamma_s)\partial_x^2 \mathrm{v})(s, X_s^{t,x,\phi})ds + \partial_x \mathrm{v}(s, X_s^{t,x,\phi})dX_s^{t,x,\phi} \end{aligned}$$

for $s \in [t, T]$. By identifying the different terms, we obtain

$$F(s, X_s^{t,x,\phi}, \gamma_s) = (\partial_t \mathrm{v} + \frac{1}{2}\sigma^2(\cdot,\cdot,\gamma_s)\partial_x^2 \mathrm{v})(s, X_s^{t,x,\phi}) \text{ and } Y_s^{t,x,\phi} = \partial_x \mathrm{v}(s, X_s^{t,x,\phi}).$$

Another application of Itô's lemma to the second equation then leads to

$$\gamma_s = \partial_x^2 \mathrm{v}(s, X_s^{t,x,\phi}),$$

recall (10). The combination of the above reads

$$0 = -(\partial_t \mathrm{v} + \bar{F}(\cdot,\cdot,\partial_x^2 \mathrm{v}))(s, X_s^{t,x,\phi}) \text{ and } (|F| + |\sigma|)(\cdot,\cdot,\partial_x^2 \mathrm{v})(s, X_s^{t,x,\phi}) < \infty,$$

in which

$$\bar{F}(t,x,z) := \frac{1}{2}\sigma^2(t,x,z)z - F(t,x,z), \quad \text{for } (t,x,z) \in \mathcal{D}. \tag{16}$$

Remark 3.1. *The model discussed in [7] corresponds to*

$$\bar{F}(t,x,z) = \frac{1}{2}\frac{\sigma_\circ^2(t,x)z}{1 - f(x)z}\mathbf{I}_{\{f(x)z<1\}} + \infty \mathbf{I}_{\{f(x)z \geq 1\}}.$$

As usual, perfect equality can not be ensured because of the gamma constraint induced by the above. We therefore only expect to have

$$0 \leq -(\partial_t \mathrm{v} + \bar{F}(\cdot,\cdot,\partial_x^2 \mathrm{v}))(s, X_s^{t,x,\phi}) \text{ and } (|F| + |\sigma|)(\cdot,\cdot,\partial_x^2 \mathrm{v})(s, X_s^{t,x,\phi}) < \infty.$$

Recalling (5), this leads to the fact that v should be a super-solution of the parabolic equation

$$\min\{-\partial_t \varphi - \bar{F}(\cdot,\cdot,\partial_x^2 \varphi)\ ,\ \bar{\gamma} - \partial_x^2 \varphi\} = 0 \text{ on } [0,T) \times \mathbb{R}. \tag{17}$$

By minimality, it should indeed be a solution. Moreover, as usual, the gamma constraint $\partial_x^2 \varphi \leq \bar{\gamma}$ needs to propagate up to the boundary, so that we can only expect that v satisfies the terminal condition

$$\lim_{(t',x') \to (T,x)} \varphi(t', x') = \hat{g}(x) \text{ for } x \in \mathbb{R}, \tag{18}$$

where $\hat{g}$ is the face-lift of g, i.e.[4]

$$\hat{g} = \inf\{h \in C^2(\mathbb{R}) : h \geq g \text{ and } \partial_x^2 h \leq \bar{\gamma}(T, \cdot)\}.$$

See Remark 3.7 below for ease of comparison with [7].

Remark 3.2. *When $\bar{\gamma} \equiv +\infty$, the above reads*

$$-\partial_t \varphi - \bar{F}(\cdot, \cdot, \partial_x^2 \varphi) = 0 \text{ on } [0, T) \times \mathbb{R} \text{ and } \lim_{(t',x') \to (T,x)} \varphi(t', x') = g(x) \text{ on } \mathbb{R}.$$

In order to prove that v is actually a continuous viscosity solution of the above, we need some additional assumptions. First, we assume that $\bar{F}$ is smooth enough,

$$\bar{F} \in C^1(\mathcal{D}) \text{ and } \bar{F} \in C_b^{1,3,3}(\mathcal{D}_{\varepsilon,\varepsilon^{-1}}), \ \varepsilon \in (0, \varepsilon_\circ], \tag{19}$$

$$\bar{F} \text{ is uniformly continuous on } \mathcal{D}_\varepsilon, \ \varepsilon \in (0, \varepsilon_\circ], \tag{20}$$

where $\varepsilon_\circ > 0$, and that

$$F(\cdot, \cdot, 0) = 0, . \tag{21}$$

For later use, note that the above implies

$$\bar{F}(\cdot, \cdot, 0) = 0. \tag{22}$$

We also assume that there exists $L_\circ, M > 0$ such that, on $\mathcal{D}$ and for all $\varepsilon \in (0, \varepsilon_\circ]$,

$$|\partial_t \bar{F} / \bar{F}| \leq L_\circ, \text{ and } |\partial_x^2 \bar{F}(\cdot, \cdot, z)| \leq M|z| \text{ for all } z \in (-\infty, 0], \tag{23}$$

that

$$\partial_z \bar{F} > 0 \text{ on } \mathcal{D}_\varepsilon \text{ and } \sup_{\{(t,x,z) \in \mathcal{D}_{\varepsilon,\varepsilon^{-1}}\}} (|\partial_z \bar{F}| + |1/\partial_z \bar{F}|) < \infty, \tag{24}$$

$$\inf_{\mathcal{D}_{\varepsilon,\varepsilon^{-1}}} \sigma > 0. \tag{25}$$

$$F \text{ is uniformly continuous on } \mathcal{D}_\varepsilon, \tag{26}$$

$$\sup_{\mathcal{D}_\varepsilon} |F| < \infty, \tag{27}$$

and that, for all $\varepsilon \in (0, \varepsilon_\circ]$, there exists a uniformly continuous map $\bar{\gamma}_\varepsilon$ such that

$$\mathcal{D}_\varepsilon = \{(t, x, z) \in [0, T] \times \mathbb{R}^2 : z \leq \bar{\gamma}_\varepsilon(t, x)\}. \tag{28}$$

Moreover,

$$\inf(\bar{\gamma}_{\varepsilon'} - \bar{\gamma}_\varepsilon) > 0, \quad \text{for all } 0 < \varepsilon' < \varepsilon < \varepsilon_\circ. \tag{29}$$

[4]Here and in the definition of $\hat{g}^\varepsilon$ below, the inf is taken with respect to the point-wise ordering on the set of real valued maps. We shall see in Remark 3.7 below that it is actually continuous .

Remark 3.3. *All these conditions are satisfied in the model of [7].*

Finally, we assume that

$$\hat{g}^{\varepsilon} := \inf\{h \geq g : h \in C^2(\mathbb{R}), F(T, \cdot, \partial_x^2 h) \leq \varepsilon^{-1}\} \tag{30}$$

satisfies

$$\begin{aligned}&\text{the maps } (\hat{g}^{\varepsilon})_{\varepsilon>0} \text{ are uniformly continuous, uniformly in } \varepsilon > 0,\\ &\text{bounded from below, have uniform linear growth,}\\ &\text{and converge uniformly towards } \hat{g}.\end{aligned} \tag{31}$$

and that there exists $k_\circ \geq 1$ such that, recall (14),

$$[\mathrm{v}_k^{\varepsilon}]^+ \text{ has linear growth, uniformly in } k \geq k_\circ, \tag{32}$$

for all $0 < \varepsilon \leq \varepsilon_\circ$, in which we use the convention $1/0 = \infty$ and identify $\hat{g}$ with $\hat{g}^0$.

Remark 3.4. *Note that* (31) *implies*

$$\hat{g} \textit{ is uniformly continuous, bounded from below, and has linear growth.} \tag{33}$$

In the case where $\bar{\gamma} = +\infty$, $\hat{g} = g$, and therefore, in this case, we assume indeed that g is uniformly continuous.

Under the above conditions, we can state the main result of this section.

Theorem 3.5. *The function* v *is a continuous viscosity solution of* (17) *such that* $\lim_{t'\uparrow T, x'\to x} \mathrm{v}(t', x') = \hat{g}(x)$ *for all $x \in \mathbb{R}$. If moreover there exists $\alpha \in (0, 1)$ such that $\hat{g} \in C_b^{4+\alpha}$, $|\partial_x^2 \hat{g}| \leq \varepsilon^{-1}$ and $(T, \cdot, \partial_x^2 \hat{g}) \in \mathcal{D}_\varepsilon$ for some $\varepsilon > 0$, then, for each $(t, x) \in [0, T) \times \mathbb{R}$, we can find $\phi \in \mathcal{A}$ such that $V_T^{t,x,v,\phi} = \hat{g}(X_T^{t,x,\phi})$ with $v = \mathrm{v}(t, x)$.*

In [7], the authors also provide a viscosity solution characterization of v, but in their case

(i) admissible strategies should satisfy $\gamma \leq \tilde{\gamma}(\cdot, X^{t,x,\phi})$ for some given function $\tilde{\gamma} < \bar{\gamma}$ (uniformly on $[0, T] \times \mathbb{R}$),

(ii) $\bar{F}(\cdot, \cdot, \tilde{\gamma}) < \infty$,

(iii) $\bar{F}(t, x, \cdot)$ is convex on $(-\infty, \tilde{\gamma}(t, x)]$ for all $(t, x) \in [0, T] \times \mathbb{R}$.

None of these assumptions are imposed here, and we also consider the case $\bar{\gamma} \equiv +\infty$.

Still, the supersolution property can essentially be proved by mimicking the arguments of [11, Section 5], up to considering a weak formulation of our stochastic target problem. To be more precise, this will provide a supersolution $\underline{\mathrm{v}}^{\varepsilon}$ of (17)

that will serve as a lower bound for v^ε, see Proposition 3.8 for a precise statement. In [7], the subsolution property could not be proved directly as in [11]. The reason is that the feedback effect of the controled state dynamics (X, Y, V) prevented them to establish the required geometric dynamic programming principle. Instead, they used a smoothing argument in the spirit of [16]. This however requires $\bar{F}$ to be convex, which, again, is not the case in our generalized setting. We will instead rely on the theory of parabolic equations. We shall show that (17) admits smooth solutions for terminal conditions Φ satisfying a uniform gamma constraint $(T, \cdot, \partial_x^2 \Phi) \in \mathcal{D}_\varepsilon$ for some $\varepsilon > 0$, see Corollary 3.12 below. In this case, a simple verification argument shows that the solution $\hat{u}$ is greater than the super-hedging price of the payoff Φ taken as a terminal condition. On the other hand, a comparison principle implies that it is smaller than any supersolution of the same equation, see Proposition 3.9 for a precise statement. In particular, if the terminal condition Φ is $\hat{g}$, then $\mathrm{v}^\varepsilon \geq \underline{\mathrm{v}}^\varepsilon \geq \hat{u} \geq \mathrm{v}$ and therefore $\mathrm{v} = \hat{u}$, by sending ε to 0. If the terminal condition $\hat{g}$ does not satisfy the required constraints on its second order derivative, then one can approximate it by functions Φ^ε and Φ_ε satisfying the above mentioned constraints and such that $\Phi^\varepsilon \geq \hat{g} \geq \Phi_\varepsilon$, for $\varepsilon > 0$. The corresponding solutions u^ε and u_ε to (17) will satisfy $u^\varepsilon \geq \mathrm{v}$ while $\mathrm{v} \geq u_\varepsilon$, because u^ε and u_ε are both the super-hedging and perfect-hedging prices of Φ^ε and Φ_ε. Again, a comparison argument will show that $u^\varepsilon - u_\varepsilon$ goes to 0 as $\varepsilon \to 0$. By stability, their common limit is a viscosity solution of (17) with terminal condition $\hat{g}$, for suitable choices of $\Phi^\varepsilon, \Phi_\varepsilon \to \hat{g}$, see Section 3.3. And therefore so is v. We detail this is the subsequent subsections.
From a general methodological perspective, note that the smoothing approach of [7], as initiated by [8], and the version we use here provide an alternative to the dynamic programming approach when part of it can not be proved. The approach of [7, 8] is relatively easy to implement when the PDE operator is concave, while the one we use here does not require concavity but much more technical work on the PDE itself. It can certainly be used in various contexts.

Remark 3.6. *Note that there is in general no hope to prove the existence of a classical solution solution to* (17)*. In particular, because the second order derivative of the boundary condition can be on the boundary of the domain of* $\bar{F}$*. Also note that we do not provide uniqueness in Theorem 3.5. We only provide a partial comparison result for this type of equations in Proposition 3.9. It is enough for our main result and we leave the study of a more general comparison result to future research. Note however that our scheme of proof induces that* v *is actually the biggest subsolution of* (17) *in the class of functions with linear growth, see Remark 3.14 below in which we also suggest a numerical procedure for approximating* v.

We conclude this section with a remark on our definition of the face-lift of g.

Remark 3.7. *In [7], the face-lift is defined as the smallest function above* g *that is a viscosity supersolution of the equation* $\bar{\gamma} - \partial_x^2 \varphi = 0$*. It is obtained by* *considering any twice continuously differentiable function* $\bar{\Gamma}$ *such that* $\partial_x^2 \bar{\Gamma} = \bar{\gamma}$, *and then setting*

$$\bar{g} := (g - \bar{\Gamma})^{conc} + \bar{\Gamma},$$

in which the superscript conc means concave envelope, cf. [23, Lemma 3.1]. This actually corresponds to our definition. The fact that $\hat{g} \geq \bar{g}$ *is trivialy deduced from the supersolution property in the definition of* $\bar{g}$. *Let us prove the converse inequality. Fix* $\varepsilon \in (0, \varepsilon_\circ]$, *and define* $\bar{g}_\varepsilon$ *as* $\bar{g}$ *but with* $\bar{\gamma} - \varepsilon$ *in place of* $\bar{\gamma}$. *Fix* $\psi \in C_b^\infty$ *with compact support, such that* $\int \psi(y)dy = 1$ *and* $\psi \geq 0$, *and define* $\bar{g}_n^\varepsilon(x) := \int \bar{g}_\varepsilon(y) n\psi(n(y-x))dy$ *for* $n \geq 1$. *Since* $\bar{g}_\varepsilon$ *is the sum of a concave function and a* C^2 *function, one can consider the measure* m_ε *associated to its second derivative and it satisfies* $m_\varepsilon(dy) \leq (\bar{\gamma}(y) - \varepsilon)dy$. *Then,* $\partial_x^2 \bar{g}_n^\varepsilon(x) = \int \bar{g}_\varepsilon(y) n^2 \partial_x^2 \psi(n(y-x))dy = \int n\psi(n(y-x))dm_\varepsilon(y) \leq \int n\psi(n(y-x))(\bar{\gamma}(y) - \varepsilon)dy$. *Now, note that* $\bar{g}$ *is continuous and therefore uniformly continuous on compact sets. Then, up to using the approximation from above argument of [7, Lemma 3.2], we can assume that it is uniformly continuous. Since* $\bar{\gamma}$ *is also uniformly continuous, see* (6), *one can find* $\kappa, \varepsilon > 0$ *such that* $\bar{g}_n^{\varepsilon,\kappa} : x \in \mathbb{R} \mapsto \bar{g}_n^\varepsilon(x) + \kappa$ *is* C^2, $\partial_x^2 \bar{g}_n^{\varepsilon,\kappa} \leq \bar{\gamma}$ *and* $\bar{g}_n^{\varepsilon,\kappa} \geq g$. *By definition, it follows that* $\bar{g}_n^{\varepsilon,\kappa} \geq \hat{g}$. *Clearly,* $(\bar{g}_n^{\varepsilon,\kappa})_{\varepsilon,\kappa>0,n\geq 1}$ *converges pointwise to* $\bar{g}$ *as* $n \to \infty$ *and* $(\varepsilon, \kappa) \to 0$ *in a suitable way. This shows that* $\bar{g} \geq \hat{g}$.

3.1 Supersolution property of a lower bound and partial comparison

In this section, we produce a supersolution of a version of (17) that is associated to v^ε, recall (13), and that is a lower bound for v^ε. We also prove a partial comparison result on this version that will be of important use later on. Recall the definition of $\hat{g}^\varepsilon$ in (30).

Proposition 3.8. *For each* $\varepsilon \in (0, \varepsilon_\circ]$ *small enough, there exists a continuous function* $\underline{\mathrm{v}}^\varepsilon \leq \mathrm{v}^\varepsilon$ *that has linear growth, is bounded from below, is a viscosity super-solution of*

$$\min\{-\partial_t \varphi - \bar{F}(\cdot, \cdot, \partial_x^2 \varphi) \,,\ \varepsilon^{-1} - F(\cdot, \cdot, \partial_x^2 \varphi)\} = 0 \ \text{on } [0, T) \times \mathbb{R} \qquad (\mathrm{Eq}_\varepsilon)$$

and satisfies $\liminf_{t' \uparrow T, x' \to x} \underline{\mathrm{v}}^\varepsilon(t', x') \geq \hat{g}^\varepsilon(x)$ *for all* $x \in \mathbb{R}$.

Proof. This follows from exactly the same arguments as in [7, Section 3.1]. We only explain the differences. As in [7, Section 3.2], we first introduce a sequence of weak formulations. On $(C(\mathbb{R}_+))^5$, let us denote by $(\tilde{\zeta} := (\tilde{\gamma}, \tilde{b}, \tilde{\alpha}, \tilde{\beta}), \tilde{W})$ the coordinate process and let $\tilde{\mathbb{F}}^\circ = (\tilde{\mathcal{F}}_s^\circ)_{s \leq T}$ be its raw filtration. We say that a probability measure $\tilde{\mathbb{P}}$ belongs to $\tilde{\mathcal{A}}_k$ if $\tilde{W}$ is a $\tilde{\mathbb{P}}$-Brownian motion and if for all $0 \leq \delta \leq 1$ and $r \geq 0$ it holds $\tilde{\mathbb{P}}$-a.s. that

$$\tilde{\gamma} = \tilde{\gamma}_0 + \int_0^\cdot \tilde{\beta}_s ds + \int_0^\cdot \tilde{\alpha}_s d\tilde{W}_s \ \text{ for some } \tilde{\gamma}_0 \in \mathbb{R}, \qquad (34)$$

$$\sup_{\mathbb{R}_+} |\tilde{\zeta}| \le k \ , \tag{35}$$

and

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sup\left\{|\tilde{\zeta}_{s'} - \tilde{\zeta}_s|,\ r \le s \le s' \le s + \delta\right\} |\tilde{\mathcal{F}}^\circ_r\right] \le k\delta. \tag{36}$$

For $\tilde{\phi} := (y, \tilde{\gamma}, \tilde{b})$, $y \in \mathbb{R}$, we define $(\tilde{X}^{x,\tilde{\phi}}, \tilde{Y}^{\tilde{\phi}}, \tilde{V}^{x,v,\tilde{\phi}})$ as in (9)-(10)-(11) associated to the control $(\tilde{\gamma}, \tilde{b})$ with time-t initial condition (x, y, v), and with $\tilde{W}$ in place of W. For $t \le T$ and $k \ge 1$, we say that $\tilde{\mathbb{P}} \in \tilde{\mathcal{G}}_{k,\varepsilon}(t, x, v, y)$ if

$$\left[\tilde{V}^{x,v,\tilde{\phi}}_T \ge \hat{g}^\varepsilon(\tilde{X}^{x,\tilde{\phi}}_T),\ F(\cdot, \tilde{X}^{x,\tilde{\phi}}, \tilde{\gamma}) \le \varepsilon^{-1} \text{ and } \tilde{\gamma} \in [-k, k] \text{ on } \mathbb{R}_+\right] \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{37}$$

We finally define

$$\tilde{\mathrm{v}}^\varepsilon_k(t, x) := \inf\{v = c + yx \ : \ (c, y) \in \mathbb{R} \times [-k, k] \text{ s.t. } \tilde{\mathcal{A}}_k \cap \tilde{\mathcal{G}}_{k,\varepsilon}(t, x, v, y) \neq \emptyset\}.$$

Step 1. We first provide bounds for $\tilde{\mathrm{v}}^\varepsilon_k$. Note that $\tilde{\mathrm{v}}^\varepsilon_k \le \mathrm{v}^\varepsilon_k$, so that (32) implies that $[\tilde{\mathrm{v}}^\varepsilon_k]^+$ has linear growth, uniformly in $k \ge k_\circ$. Moreover, note that the fact that σ is Lipschitz with linear growth in its second variable, uniformly on $\mathcal{D}_{\varepsilon,k} \times \mathbb{R}$ (see (7)), implies that $\tilde{X}^{t,x,\tilde{\phi}}$ is a square integrable martingale under $\tilde{\mathbb{P}}$ for any $\tilde{\phi} := (y, \tilde{\gamma}, \tilde{b})$, and that the same holds for $\int_t^\cdot \tilde{Y}^{t,\tilde{\phi}}_s d\tilde{X}^{t,x,\tilde{\phi}}_s$. Then, the inequality

$$v + \int_t^T F(s, \tilde{X}^{t,x,\tilde{\phi}}_s, \tilde{\gamma}_s) ds + \int_t^T \tilde{Y}^{t,\tilde{\phi}}_s d\tilde{X}^{t,x,\tilde{\phi}}_s \ge \hat{g}^\varepsilon(\tilde{X}^{t,x,\tilde{\phi}}_T)$$

combined with (27) and (15) implies that $v \ge -\sup|g^-| - T \sup_{\mathcal{D}_\varepsilon} F > -\infty$. This shows that $\tilde{\mathrm{v}}^\varepsilon_k$ is bounded from below, uniformly in $k \ge k_0$.
Step 2. We claim that

$$\underline{\mathrm{v}}^\varepsilon(t, x) := \liminf_{\substack{(k,t',x') \to (\infty,t,x) \\ (t',x') \in [0,T) \times \mathbb{R}}} \tilde{\mathrm{v}}^\varepsilon_k(t', x'), \quad (t, x) \in [0, T] \times \mathbb{R},$$

is a viscosity supersolution of (Eq_ε). To prove this, it suffices to show that it holds for each $\tilde{\mathrm{v}}^\varepsilon_k$, with $k \ge k_\circ$, and then to apply standard stability results, see e.g. [2]. By the same arguments as in [7, Proposition 3.15], each $\tilde{\mathrm{v}}^\varepsilon_k$ is lower-semicontinuous[5]. Given a C^∞_b test function φ and $(t_0, x_0) \in [0, T) \times \mathbb{R}$ such that (t_0, x_0) achieves a strict minimum of $\tilde{\mathrm{v}}^\varepsilon_k - \varphi$,

$$(\text{strict}) \min_{[0,T) \times \mathbb{R}} (\tilde{\mathrm{v}}^\varepsilon_k - \varphi) = (\tilde{\mathrm{v}}^\varepsilon_k - \varphi)(t_0, x_0) = 0,$$

[5]The use of the weak formulation is exactly motivated by the fact that it ensures this lower-semicontinuity property, which is required in the arguments we will appeal to for the derivation of super-solution property. Unfortunately, no alternative argument seems available so far without lower-semicontinuity for stochastic target problems involving a second order constraint.

we first use (25) and the arguments of [7, Step 1-2, proof of Theorem 3.16] to obtain that there exists $\tilde{\gamma}_0$ such that

$$\partial_x^2\varphi(t_0,x_0) \le \tilde{\gamma}_0 \text{ and } F(t_0,x_0,\tilde{\gamma}_0) \le \varepsilon^{-1}.$$

Then, the same arguments as in [7, Proof of Theorem 3.16] combined with (16) and (24) lead to

$$\begin{aligned} 0 \le& F(t_0,x_0,\tilde{\gamma}_0) - \partial_t\varphi(t_0,x_0) - \frac{1}{2}\sigma^2(t_0,x_0,\tilde{\gamma}_0)\partial_x^2\varphi(t_0,x_0) \\ &- \frac{1}{2}\left(\tilde{\gamma}_0 - \partial_x^2\varphi(t_0,x_0)\right)\sigma^2(t_0,x_0,\tilde{\gamma}_0) \\ =& -\partial_t\varphi(t_0,x_0) - \bar{F}(t_0,x_0,\tilde{\gamma}_0) \\ \le& -\partial_t\varphi(t_0,x_0) - \bar{F}(t_0,x_0,\partial_x^2\varphi(t_0,x_0)). \end{aligned}$$

Finally, the T-boundary condition is obtained as in [7, Proof of Theorem 3.16], recall our assumption (15), as well as Remark 3.7. □

We now provide a partial comparison result that will be used later on. Note that a full comparison result could be proved as in [7, Theorem 3.11] when $\bar{F}$ is convex, by mimicking their arguments. It is however not the case in general. Given the strategy of our proof, it is not required in this paper. In the following, we interpret (Eq$_\varepsilon$) by using the convention $0^{-1} = \infty$ in the case $\varepsilon = 0$.

Proposition 3.9. *Let U be an upper semicontinuous viscosity subsolution of* (Eq$_\varepsilon$) *for $\varepsilon \in [0,\varepsilon_\circ]$. Let V be a lower semicontinuous viscosity supersolution of* (Eq$_{\varepsilon'}$) *for some $\varepsilon' \in (\varepsilon,\varepsilon_\circ]$. Assume that U and V have linear growth and that $U \le V$ on $\{T\}\times\mathbb{R}$, then $U \le V$ on $[0,T]\times\mathbb{R}$.*

Proof. Set $\hat{U}(t,x) := e^{\rho t}U(t,x)$, $\hat{V}(t,x) := e^{\rho t}V(t,x)$ for some $\rho > 0$. Then, $\hat{U}$ is a subsolution of

$$\min\left\{\rho\varphi - \partial_t\varphi - e^{\rho\cdot}\bar{F}(\cdot,\cdot,e^{-\rho\cdot}\partial_x^2\varphi),\ \varepsilon^{-1} - F(\cdot,\cdot,e^{-\rho\cdot}\partial_x^2\varphi)\right\} = 0 \qquad (38)$$

and $\hat{V}$ is a supersolution of

$$\min\left\{\rho\varphi - \partial_t\varphi - e^{\rho\cdot}\bar{F}(\cdot,\cdot,e^{-\rho\cdot}\partial_x^2\varphi),\ (\varepsilon')^{-1} - F(\cdot,\cdot,e^{-\rho\cdot}\partial_x^2\varphi)\right\} = 0 \qquad (39)$$

on $[0,T)\times\mathbb{R}$.
Assume that $\sup_{[0,T]\times\mathbb{R}}(\hat{U}-\hat{V}) > 0$. Then, there exists $\eta > 0$ such that, for all $n > 0$ and all $\lambda > 0$ small enough,

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}^2}\left[\hat{U}(t,x) - \hat{V}(t,y) - \frac{\lambda}{2}|x|^2 - \frac{n}{2}|x-y|^2\right] \ge \eta > 0. \qquad (40)$$

Denote by (t_n,x_n,y_n) the point at which this supremum is achieved. Since $\hat{V}(T,\cdot) \ge \hat{U}(T,\cdot)$, we have $t_n < T$. Moreover, standard arguments, see e.g., [12, Proposition 3.7], lead to

$$\lim_{n\to\infty} n|x_n - y_n|^2 = 0. \qquad (41)$$

We now apply Ishii's lemma, see e.g. [12, Theorem 8.3], to obtain the existence of $(a_n, M_n, N_n) \in \mathbb{R}^3$ such that

$$(a_n, n(x_n - y_n) + \lambda x_n, M_n) \in \bar{\mathcal{P}}^{2,+}\hat{U}(t_n, x_n)$$
$$(a_n, -n(x_n - y_n), N_n) \in \bar{\mathcal{P}}^{2,-}\hat{V}(t_n, y_n),$$

in which $\bar{\mathcal{P}}^{2,+}$ and $\bar{\mathcal{P}}^{2,-}$ denote as usual the *closed* parabolic super- and subjets, see [12], and

$$\begin{pmatrix} M_n & 0 \\ 0 & -N_n \end{pmatrix} \le 3n \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 3\lambda + \frac{\lambda^2}{n} & -\lambda \\ -\lambda & 0 \end{pmatrix}.$$

In particular, $M_n \le N_n + 2\lambda + \lambda^2/n$. Since $\hat{V}$ is a supersolution of (39) and $\varepsilon < \varepsilon'$, (26) and (41) imply that $F(t_n, x_n, e^{-\rho t_n} M_n) < \varepsilon^{-1}$ for $\lambda > 0$ small enough and n large enough. Hence,

$$\rho\hat{U}(t_n, x_n) - a_n - e^{\rho t_n}\bar{F}(t_n, x_n, e^{-\rho t_n} M_n) \le 0.$$

On the other hand, the supersolution property of $\hat{V}$ combined with (20) and (24) implies that

$$\begin{aligned} 0 \le & \rho\hat{V}(t_n, y_n) - a_n - e^{\rho t_n}\bar{F}(t_n, y_n, e^{-\rho t_n} N_n) \\ \le & \rho\hat{V}(t_n, y_n) - a_n - e^{\rho t_n}\bar{F}(t_n, y_n, e^{-\rho t_n} M_n) + e^{\rho t_n}\delta(e^{-\rho t_n}(2\lambda + \lambda^2/n)) \end{aligned}$$

in which $\delta(z) \to 0$ as $z \to 0$. Hence,

$$\begin{aligned} \rho(\hat{U}(t_n, x_n) - \hat{V}(t_n, y_n)) \le & \, e^{\rho t_n}\left(\bar{F}(t_n, x_n, e^{-\rho t_n} M_n) - \bar{F}(t_n, y_n, e^{-\rho t_n} M_n)\right) \\ & + e^{\rho t_n}\delta(e^{-\rho t_n}(\lambda + \lambda^2/n)). \end{aligned}$$

Recalling (41) and (20), we obtain a contradiction to (40) by sending $n \to \infty$ and then $\lambda \to 0$. $\square$

3.2 Regularity of solutions to $(\mathrm{Eq}_\varepsilon)$

To complete the characterization of Proposition 3.8, we now study the regularity of solutions to $(\mathrm{Eq}_\varepsilon)$. We shall indeed show that $(\mathrm{Eq}_\varepsilon)$ admits a smooth solution u^ε such that $(\cdot, \cdot, \partial_x^2 u) \in \mathcal{D}_\varepsilon$ on $[0, T] \times \mathbb{R}$, for $\varepsilon > 0$ small enough and for a certain class of terminal conditions. A simple verification argument will then show that u^ε dominates the super-hedging price v if the terminal data Φ^ε associated to u^ε dominates $\hat{g}$. A lower bound u_ε for v can also be constructed by considering a terminal condition $\Phi_\varepsilon \le \hat{g}$ and using our comparison result of Proposition 3.9 combined with Proposition 3.8. Then, letting $\Phi_\varepsilon, \Phi^\varepsilon \to \hat{g}$ in a suitable way will be enough to show that v is actually a solution of (Eq_0), i.e. to conclude the proof of Theorem 3.5.
The strategy we employ consists in establishing *a priori* estimates for the second derivative of the solution to $(\mathrm{Eq}_\varepsilon)$. Once established, the equation becomes uniformly parabolic, and higher regularity follows by standard parabolic regularity

(see [17]). Then, the continuity method (see [15]) allows us to actually construct the solution to (Eq_ε).
Let us start with uniform estimates for solutions to (Eq_ε) such that $(\cdot,\cdot,\partial_x^2 u) \in \mathcal{D}_{\varepsilon'}$ for some $\varepsilon' > 0$, in the case where the terminal condition Φ is smooth and satisfies a similar constraint.

Proposition 3.10. *Let u and Φ be two continuous functions such that*

(i) $\Phi \in C^2(\mathbb{R})$ *with* $|\partial_x^2 \Phi| \le K_\Phi$ *for some* $K_\Phi > 0$,

(ii) $(T,\cdot,\partial_x^2\Phi) \in \mathcal{D}_{\varepsilon_\Phi}$ *for some* $\varepsilon_\Phi > 0$,

(iii) $u \in C^{1,4}([0,T)\times\mathbb{R}) \cap C^{0,2}([0,T]\times\mathbb{R})$ *with* $|\partial_x^2 u| \le K'$ *for some* $K' > 0$,

(iv) $(\cdot,\cdot,\partial_x^2 u) \in \mathcal{D}_{\varepsilon'}$ *for some* $\varepsilon' > 0$.

Assume that u solves

$$\partial_t u + \bar{F}(\cdot,\cdot,\partial_x^2 u) = 0 \quad \textit{on } [0,T)\times\mathbb{R}, \tag{Eq_0}$$
$$u(T,\cdot) = \Phi \quad \textit{on } \mathbb{R}. \tag{42}$$

Then,

a. $(\cdot,\cdot,\partial_x^2 u) \in \mathcal{D}_\varepsilon$ *on* $[0,T]\times\mathbb{R}$, *for some* $\varepsilon > 0$ *that depends only on* ε_Φ *and* $L_\circ$,

b. $|\partial_x^2 u| \le K$ *on* $[0,T]\times\mathbb{R}$ *where* K *depends only on* K_Φ.

c. If Φ is globally Lipschitz, then u is also globally Lipschitz with Lipschitz constant controlled by the one of Φ.

d. u is the unique $C^{1,2}([0,T)\times\mathbb{R}) \cap C^0([0,T]\times\mathbb{R})$ solution of (Eq_0)-(42) *such that $(\cdot,\cdot,\partial_x^2 u) \in \mathcal{D}_{\varepsilon''}$ for some $\varepsilon'' > 0$.*

e. For some $\alpha \in (0,1)$ depending on K_Φ and the assumptions on $\bar{F}$, $u \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T)\times\mathbb{R})$. Moreover, for any compact subset $C' \subset [0,T)\times\mathbb{R}$, there is a constant $C(C',K_\Phi,\bar{F})$ such that

$$\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}(C')} \le C(C',K_\Phi,\bar{F}).$$

f. If moreover $\Phi \in C^{2+\alpha}$, $u \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\mathbb{R})$.

Proof. a. Let $V := \bar{F}(\cdot,\cdot,\partial_x^2 u)$. Then, on $[0,T)\times\mathbb{R}$,

$$\partial_t V = \partial_t \bar{F}(\cdot,\cdot,\partial_x^2 u) + \partial_z \bar{F}(\cdot,\cdot,\partial_x^2 u)\partial_t\partial_x^2 u$$

in which, by (Eq_0), $\partial_t\partial_x^2 u + \partial_x^2 V = 0$. Hence,

$$\partial_t V + \partial_z \bar{F}(\cdot,\cdot,\partial_x^2 u)\partial_x^2 V = \partial_t \bar{F}(\cdot,\cdot,\partial_x^2 u) = \frac{\partial_t \bar{F}(\cdot,\cdot,\partial_x^2 u)}{\bar{F}(\cdot,\cdot,\partial_x^2 u)} V, \tag{43}$$

recall (23). For $(t,x) \in [0,T] \times \mathbb{R}$, let $\bar{X}^{t,x}$ be the solution of

$$\bar{X} = x + \int_t^{\cdot} (2\partial_z \bar{F}(\cdot,\cdot,\partial_x^2 u)(s,\bar{X}_s))^{\frac{1}{2}} dW_s.$$

By (iv), (19) and (24), it is well-defined. Combining Itô's Lemma and a standard localizing argument using (19) and (23), we obtain

$$V(t,x) = \mathbb{E}[V(T,\bar{X}_T^{t,x})e^{-\int_t^T (\partial_t \bar{F}(\cdot,\cdot,\partial_x^2 u)/\bar{F}(\cdot,\cdot,\partial_x^2 u))(s,\bar{X}_s^{t,x})ds}]. \tag{44}$$

By definition of V and the fact that $\partial_x^2 u(T,\cdot) = \partial_x^2 \Phi$ by (iii), this shows that $(\cdot,\cdot,\partial_x^2 u) \in \mathcal{D}_\varepsilon$ on $[0,T] \times \mathbb{R}$, for some $\varepsilon > 0$ that depends only on $L_\circ$ and ε_Φ.
b. To obtain the bound on $\partial_x^2 u$, we first differentiate twice (Eq$_0$) with respect to x, recall (19) and (iii). Letting $Z(t,x) = \partial_x^2 u(t,x)$, this yields

$$\partial_t Z + 2\partial_x \partial_z \bar{F} \partial_x Z + \partial_z \bar{F} \partial_x^2 Z + \partial_z^2 \bar{F}(\partial_x Z)^2 = -\partial_x^2 \bar{F}.$$

We now consider

$$(t,x) \mapsto \underline{Z}(t,x) := \min\{0, \inf Z(T,\cdot)\} e^{M(T-t)},$$

in which M is given in (23). Then,

$$\partial_t \underline{Z} + 2\partial_x \partial_z \bar{F} \partial_x \underline{Z} + \partial_z \bar{F} \partial_x^2 \underline{Z} + \partial_z^2 \bar{F}(\partial_x \underline{Z})^2 \quad = \quad -M\underline{Z} \geq -\partial_x^2 \bar{F}(t,x,\underline{Z}).$$

Under the current assumptions, Z is uniformly bounded on $[0,T] \times \mathbb{R}$. Moreover, from assumption (19), $\partial_x^2 \bar{F}$ is uniformly continuous on $\mathcal{D}_{\varepsilon,\varepsilon^{-1}}$, for all $\varepsilon > 0$ small enough, hence, by (22) and [12, Proof of comparison, Theorem 5.1], the comparison principle holds between Z and $\underline{Z}$, and yields that $\underline{Z} \leq Z$ globally on $[0,T] \times \mathbb{R}$. The upper bound is obtained in the exact same way.
c. The assertion about the Lipschitz regularity also follows from the linearised equation satisfied by $\kappa = \partial_x u$:

$$\partial_t \kappa + \partial_z \bar{F}(\cdot,\cdot,\partial_x^2 u)\partial_x^2 \kappa + \partial_x \bar{F}(\cdot,\cdot,\partial_x \kappa) = 0, \ \kappa(T,\cdot) = \partial_x \Phi.$$

Under the assumptions (24), (19), and (22), this implies that

$$\kappa(t,x) = \mathbb{E}[\partial_x \Phi(\tilde{X}_T^{t,x})]$$

where

$$\tilde{X}^{t,x} = x + \int_t^{\cdot} \left(2\partial_z \bar{F}(\cdot,\cdot,\partial_x^2 u)\right)^{\frac{1}{2}} (s,\tilde{X}_s^{t,x}) dW_s + \int_t^{\cdot} \frac{\partial_x \bar{F}(\cdot,\cdot,\partial_x^2 u)}{\partial_x^2 u}(s,\tilde{X}_s^{t,x}) ds,$$

and the result follows. (Note that, since $\bar{F}(\cdot,\cdot,0) = 0$ and $\bar{F} \in C_b^{1,3,3}(\mathcal{D}_{\varepsilon,\varepsilon^{-1}})$, the map $z \mapsto \frac{\partial_x \bar{F}(\cdot,\cdot,z)}{z}$ is bounded and Lipschitz - after extending it to $\partial_z \partial_x \bar{F}(\cdot,\cdot,0)$ at 0.)
d. Consider another solution u'. Then, b. implies that u and u' have at most a quadratic growth. Moreover, a. allows one to consider a uniformly parabolic equation. Then, the fact that $u = u'$ follows from standard arguments.

e. Differentiating the equation (Eq_0) with respect to x we have that $w = \partial_x u$ satisfies

$$\partial_t w + \partial_x(\bar{F}(t, x, \partial_x w)) = 0.$$

We then apply the result of [17, Theorem 12.1] to conclude that $\partial_x w \in C^{\alpha/2,\alpha}_{loc}$. This estimate provides the local in space Hölder continuity. To obtain the estimate in the time variable, we use the original equation (Eq_0). Since $\bar{F}(t, x, u(t, x))$ is Hölder continuous and $\partial_t u(t, x) = -\bar{F}(t, x, \partial_x^2 u(t, x))$, this implies the regularity in time as well. The stated local estimate follows from [17, Theorem 12.1, (12.4)].

f. If $\Phi \in C^{2+\alpha}$, then by following the arguments of [17, Theorem 5.13], and invoking the argument used to prove point e, we obtain that $u \in C^{1+\frac{\alpha}{2},2+\alpha}$ globally up to time T.

□

We are now in position to construct a smooth solution to (Eq_0).

Theorem 3.11. *Let Φ be a continuous map such that $|\partial_x^2 \Phi| \le \varepsilon^{-1}$ and $(T, \cdot, \partial_x^2 \Phi) \in \mathcal{D}_\varepsilon$ for some $\varepsilon > 0$. Then, there exists a solution u of* (Eq_0)*-*(42) *that belongs to $C([0, T] \times \mathbb{R}) \cap C^{1,4}_{loc}([0, T) \times \mathbb{R})$, such that $|\partial_x^2 u| \le (\varepsilon_{\Phi,L_\circ})^{-1}$ and $(\cdot, \cdot, \partial_x^2 u) \in \mathcal{D}_{\varepsilon_{\Phi,L_\circ}}$ on $[0, T] \times \mathbb{R}$, for some $\varepsilon_{\Phi,L_\circ} > 0$ that only depends on Φ and $L_\circ$. If Φ is globally Lipschitz, then u is also globally Lipschitz with Lipschitz constant controlled by the one of Φ. If moreover there exists $\alpha \in (0, 1)$ such that $\Phi \in C^{4+\alpha}_b$ then $u \in C^{1,4}_b$.*

Proof. This follows by using the continuity method (cf. [15, Chap. 17.2]). We first mollify Φ into a function Φ_n so that $\partial_x^5 \Phi_n$ is bounded, and at the same time $\bar{F}$ into a function $\bar{F}_n$ such that $\bar{F}_n(\cdot, \cdot, z)$ is C^∞ on each $\{(t, x) \in [0, T] \times \mathbb{R} : (t, x, z) \in \mathcal{D}_{\varepsilon'}\}$, $\varepsilon' > 0$, for all $z \in \mathbb{R}$, with derivatives' bounds on $\{(t, x) \in [0, T] \times \mathbb{R} : (t, x, z) \in \mathcal{D}_{\varepsilon'}\}$, $\varepsilon' > 0$, that are locally uniform with respect to z. This is possible, since $\bar{\gamma}$ and $\bar{F}$ are uniformly continuous (recall (6) and (20)), by taking a compactly supported smoothing kernel $\psi \in C^\infty(\mathbb{R})$ and considering

$$\begin{aligned}
\Phi_n &= n \int_{\mathbb{R}} \Phi(y)\psi(n(y - \cdot))dy, \\
\bar{F}_n(\cdot, \cdot, z) &= n^2 \int_{[0,T]\times\mathbb{R}} \bar{F}(s, y, z)\psi(n(s - \cdot))\psi(n(y - \cdot))dsdy,
\end{aligned}$$

and taking n large enough with respect to ε'. For later use, note that $\bar{F}_n(T, \cdot, \partial_x^2 \Phi_n) \le 2\varepsilon^{-1}$, for n large enough. Set

$$G_n(\varphi, \theta) := [\partial_t \varphi + \bar{F}_n(\cdot, \partial_x^2 \varphi)]\mathbf{I}_{[0,T)} + \mathbf{I}_{\{T\}}(\varphi - \theta \Phi_n) \;\text{ for } \varphi \in C^{1,4}_b,$$

and let $E_n \subset [0, 1]$ be the set of real number $\theta \in [0, 1]$ for which a $C^{1,4}_b$ solution u^n_θ to $G_n(u^n_\theta, \theta) = 0$ exists such that it satisfies the condition (iii)-(iv) of Proposition

3.10. By (22), $u_0 \equiv 0$ solves $G_n(u_0, 0) = 0$ so that $0 \in E_n$. Hence, E_n is non empty. Moreover, for every $\theta \in E_n$, the linearised operator associated to G_n is

$$(\tilde{u}, \tilde{\theta}) \in C^{1,2} \times E_n \mapsto L_n(\tilde{u}, \tilde{\theta}) := [\partial_t \tilde{u} + \partial_z \bar{F}_n(t, x, \partial_x^2 u)\partial_x^2 \tilde{u}]\mathbf{I}_{[0,T)} + \mathbf{I}_{\{T\}}(\tilde{u} - \tilde{\theta}\Phi_n).$$

It is uniformly parabolic (recall (24)) with coefficients in C^∞. For $\tilde{\theta}$ fixed, the equation $L_n(\tilde{u}, \tilde{\theta}) = 0$ is therefore a linear, uniformly parabolic equation, with smooth coefficients. The terminal data is smooth, has linear growth and bounded derivatives of order 1 up to 5. Standard parabolic regularity theory (see [14]) yields that the linearised equation with respect to $\tilde{u}$ is solvable in $C_b^{1,4}$. By the implicit function theorem, see e.g. [15, Theorem 17.6], E_n is open in $[0, 1]$. By the *a priori* estimates of Proposition 3.10, E_n is also closed for n large enough. In particular, we have a uniform (with respect to θ) *a priori* estimate in $C^{1+\frac{\alpha}{2}, 2+\alpha}$. This, given our assumptions on $\Phi_n, \bar{F}_n$ and from standard parabolic regularity, implies that the corresponding solution is uniformly (with respect to θ) bounded in $C^{1,4}$.
Therefore, $E_n = [0, 1]$ and u_1^n is well defined, and uniformly bounded in $C^{1+\frac{\alpha}{2}, 2+\alpha}$. Note that, by a. of Proposition 3.10, $\varepsilon' > 0$ can be chosen such that $(\cdot, \cdot, \partial_x^2 u_1^n) \in \mathcal{D}_{\varepsilon', \varepsilon'^{-1}}$ on $[0, T] \times \mathbb{R}$, for all n large enough. Since $(\bar{F}_n)_{n \geq 1}$ is uniformly parabolic, uniformly in n, and given our initial smoothness assumptions on $\bar{F}$, see assumption (19), u_1^n is uniformly bounded in $C_{loc}^{1,4}([0, T) \times \mathbb{R})$. If moreover $(\Phi_n)_{n \geq 1}$ is bounded in $C_b^{4+\alpha}$ uniformly in n, then $(u_1^n)_{n \geq 1}$ is $C_b^{1,4}$ uniformly in n (see again [17, Theorem 5.13] applied to $\partial_x^2 u_1^n$). It remains to send $n \to \infty$ to deduce the required result. □

3.3 Full chacterization of the super-hedging price and perfect hedging in the smooth case

We are now about to conclude the proof of Theorem 3.5. Let $\hat{u}$ be the function constructed in Theorem 3.11 for $\Phi = \hat{g}$, assuming that $\hat{g}$ satisfies the required constraints. We first establish that $\hat{u}$ permits to apply a perfect hedging strategy of the face-lifted payoff whenever it is smooth enough, and that it coincides with the super-hedging price.

Corollary 3.12. *Assume that there exists* $\alpha \in (0, 1)$ *such that* $\hat{g} \in C_b^{4+\alpha}$, *that* $|\partial_x^2 \hat{g}| \leq \varepsilon^{-1}$ *and* $(T, \cdot, \partial_x^2 \hat{g}) \in \mathcal{D}_\varepsilon$ *for some* $\varepsilon > 0$. *Let* $\hat{u}$ *be the function constructed in Theorem 3.11 for* $\Phi = \hat{g}$. *Then,* $\mathrm{v} = \hat{u}$ *and, for each* $(t, x) \in [0, T] \times \mathbb{R}$, *we can find* $\phi \in \mathcal{A}$ *such that* $V_T^{t,x,v,\phi} = \hat{g}(X_T^{t,x,\phi})$.

Proof. It follows from Theorem 3.11, Itô's lemma and (16) that $\hat{u}$ induces an exact replication strategy:

$$\begin{aligned}\hat{g}(X_T^{t,x,\phi}) =&\hat{u}(t,x) + \int_t^T \left[\partial_t \hat{u} + \frac{1}{2}\sigma^2(\cdot,\cdot,\partial_x^2\hat{u})\partial_x^2\hat{u}\right](s, X_s^{t,x,\phi})ds \\ &+ \int_t^T \partial_x \hat{u}(s, X_s^{t,x,\phi})dX_s^{t,x,\phi} \\ =&\hat{u}(t,x) + \int_t^T F(s, X_s^{t,x,\phi}, \gamma_s)ds + \int_t^T Y_s^{t,x,\phi}dX_s^{t,x,\phi}\end{aligned}$$

in which $\phi = (y, b, \gamma)$ with

$$y = \partial_x \hat{u}(t,x),\ b = ([\partial_t + \frac{1}{2}\sigma^2(\cdot,\cdot,\gamma)\partial_x^2]\partial_x\hat{u})(\cdot, X_\cdot^{t,x,\phi}),\ \gamma = \partial_x^2\hat{u}(\cdot, X_\cdot^{t,x,\phi}).$$

Hence, $\hat{u} \geq \mathrm{v}$. Moreover, $\hat{u}$ is a viscosity subsolution of ($\mathrm{Eq}_{\varepsilon'}$) for all $\varepsilon' \geq 0$ small enough. Since $\hat{g}$ is globally Lipschitz, $\hat{u}$ is also globally Lipschitz (Theorem 3.11), and therefore has linear growth. By Proposition 3.8, $\mathrm{v}^\varepsilon \geq \underline{\mathrm{v}}^\varepsilon$ that is a supersolution of (Eq_ε) and satisfies $\liminf_{t'\uparrow T, x'\to x} \underline{\mathrm{v}}^\varepsilon(t',x') \geq \hat{g}^\varepsilon(x) \geq \hat{g}(x) = \hat{u}(T,x)$ for all $x \in \mathbb{R}$. Then, Proposition 3.9 implies that $\mathrm{v}^\varepsilon \geq \hat{u}$. Taking the inf over $\varepsilon > 0$ leads to $\mathrm{v} \geq \hat{u}$. □

We can now conclude the proof of Theorem 3.5.
Proof of Theorem 3.5. We begin the proof with the following approximation lemma, whose proof is deferred after the end of the Theorem's proof.

Lemma 3.13. *For all $\varepsilon > 0$, there exists $\Phi_\varepsilon, \Phi^\varepsilon \in C^2$ such that, for $\Psi \in \{\Phi_\varepsilon, \Phi^\varepsilon\}$,*

$$\Psi \in C_b^5(\mathbb{R}),\ |\partial_x^2\Psi| \leq \varepsilon^{-1},\ (T, \cdot, \partial_x^2\Psi) \in \mathcal{D}_\varepsilon,$$

and

$$\Phi_\varepsilon \leq \hat{g} \leq \Phi^\varepsilon,\ \Phi^\varepsilon - \Phi_\varepsilon \leq \delta(\varepsilon),$$

in which $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$.

Let u^ε and u_ε be the (smooth) solutions to (Eq_0) associated to Φ^ε and Φ_ε respectively, as in Theorem 3.11. By applying Corollay 3.12 to Φ^ε in place of $\hat{g}$, we deduce that u^ε is the super-hedging price of $\Phi^\varepsilon \geq \hat{g}$ so that $u^\varepsilon \geq \mathrm{v}$. Similarly $u_\varepsilon \leq \mathrm{v}$, and therefore $u_\varepsilon \leq \mathrm{v} \leq u^\varepsilon$.
By the comparison principle, we also have

$$0 \leq u^\varepsilon - u_\varepsilon \leq \sup\{\Phi^\varepsilon - \Phi_\varepsilon\} \leq \delta(\varepsilon).$$

It follows that v is the uniform limit of a sequence of continuous functions, and is therefore continuous. Each of the functions u_ε solves (17), recall (5). Standard stability results, see e.g. [2], imply that v is a viscosity solution to (17)-(18).
The other assertions in Theorem 3.5 are immediate consequences of Corollary 3.12. □

Remark 3.14. *By Proposition 3.9, the function u^ε defined in the proof of Theorem 3.5 is the unique solution of* (Eq$_0$) *with terminal condition Φ^ε, in the class of functions with linear growth. This opens the door to the construction of a numerical scheme for the computation of u^ε, and therefore of* v *by passing to the limit $\varepsilon \to 0$. Moreover, if v is a upper-semicontinuous subsolution of* (17)*, with linear growth, such that $v(T,\cdot) \le \hat{g}$, then the comparison result of Proposition 3.9 implies that $v \le u^\varepsilon$. Since $u^\varepsilon \to \mathrm{v}$, this proves that* v *is the biggest subsolution of* (17)*, with linear growth, associated to the boundary condition $\hat{g}$.*

Proof of Lemma 3.13. The proof uses standard approximation arguments, and we only state the main ingredients. Consider $\hat{g}^\varepsilon$ as in (30), see also Remark 3.7. Then, it follows from (28) that its second derivative measure satisfies $\partial_x^2 \hat{g}^\varepsilon(dx) \le \bar{\gamma}_\varepsilon(x)dx$. Consider a smooth mollification $\hat{g}_n^\varepsilon$ of $\hat{g}^\varepsilon$ as in Remark 3.7. By (31), it converges uniformly to $\hat{g}$ as $n \to \infty$ and $\varepsilon \to 0$. By (29), for n large enough with respect to ε, $\partial_x^2 \hat{g}_n^\varepsilon < \bar{\gamma}_{\varepsilon/2} \le \bar{\gamma}_{\varepsilon'}$, and $|\partial_x^2 \hat{g}_n^\varepsilon| \le 1/\varepsilon'$, for some $0 < \varepsilon' \le \varepsilon/2$, see Remark 3.7. Since the convergence is uniform, we can add to (resp. substract from) $\hat{g}_n^\varepsilon$ a constant $k_n^\varepsilon \ge 0$ to ensure that $\hat{g}_n^\varepsilon + k_n^\varepsilon \ge \hat{g}$ (resp. $\hat{g}_n^\varepsilon - k_n^\varepsilon \le \hat{g}$) such that k_n^ε goes to 0 as $n \to \infty$ and $\varepsilon \to 0$. □

4 Asymptotic analysis

We now consider the case where the impact of the γ process in the dynamics of (X, V) is small. Our aim is to obtain an asymptotic expansion around an impact free model. More precisely, we consider the dynamics

$$
\begin{aligned}
X^{\epsilon,t,x,\phi} &= x + \int_t^\cdot \mu(s, X_s^{\epsilon,t,x,\phi}, \epsilon\gamma_s, \epsilon b_s)ds + \int_t^\cdot \sigma(s, X_s^{\epsilon,t,x,\phi}, \epsilon\gamma_s)dW_s \\
V^{\epsilon,t,x,v,\phi} &= v + \int_t^\cdot \epsilon^{-1} F(s, X_s^{\epsilon,t,x,\phi}, \epsilon\gamma_s)ds + \int_t^\cdot Y_s^{\epsilon,t,x,\phi} dX_s^{\epsilon,t,x,\phi},\ \epsilon > 0,
\end{aligned}
$$

and denote by v^ϵ the corresponding super-hedging price.
We place ourself in the context of Corollary 3.12 for the coefficients $\mu(\cdot,\cdot,\epsilon\cdot,\epsilon\cdot)$, $\sigma(\cdot,\cdot,\epsilon\cdot)$ and $\epsilon^{-1}F(\cdot,\cdot,\epsilon\cdot)$. In particular, we assume that $\hat{g} \in C^2$ is such that $\epsilon^{-1}\bar{F}(T,\cdot,\epsilon\partial_x^2\hat{g})$ is bounded on $\mathbb{R}$, for $\epsilon > 0$ small enough.

In the following, we use the notation

$$
(\bar{F}_0, \partial_z^n \bar{F}_0) := (\bar{F}(\cdot,\cdot,0), \partial_z^n \bar{F}(\cdot,\cdot,0)), \quad \text{for } n = 1, 2.
$$

Remark 4.1. *Note that the model of [7] corresponds to*

$$
\sigma(t,x,\epsilon z) = \frac{\sigma_\circ(t,x)}{1 - \epsilon f(x) z}\ ,\ \epsilon^{-1}F(t,x,\epsilon z) = \frac{1}{2}\left(\frac{\sigma_\circ(t,x)z}{1-\epsilon f(x)z}\right)^2 \epsilon f(x).
$$

Our scaling therefore amounts to consider a small impact function $x \mapsto \epsilon f(x)$. In order to interpret the result of Proposition 4.3 below, also observe that

$$
(2\partial_z \bar{F}_0(t,x))^{\frac{1}{2}} = \sigma_\circ(t,x) \ \ \textit{and}\ \ \partial_z^2 \bar{F}_0(t,x) = \sigma_\circ^2(t,x) f(x).
$$

Our expansion is performed around the solution v^0 of

$$\partial_t \mathrm{v}^0 + \partial_z \bar{F}_0 \partial_x^2 \mathrm{v}^0 = 0 \text{ on } [0,T) \times \mathbb{R} \text{ and } \mathrm{v}^0(T,\cdot) = \hat{g} \text{ on } \mathbb{R}. \tag{45}$$

Remark 4.2. *Let the conditions of Corollary 3.12 hold and assume that* $\bar{F} \in C^{1,3,1}_{loc}(\mathcal{D})$ *with*

$$|\partial_x \partial_z \bar{F}_0| + |\partial_x^2 \partial_z \bar{F}_0| \textit{ uniformly bounded.} \tag{46}$$

Then, v^0 *is the unique solution in* $C_b^{1,2}([0,T] \times \mathbb{R}) \cap C^{1,3}([0,T) \times \mathbb{R}])$ *of* (45). *This follows from* (24) *and standard estimates.*

The following expansion requires some additional regularity on $\hat{g}$ that will in general not be satisfied in applications. However, one can reduce to it up to a slight approximation argument (i.e. by smoothing $\hat{g}$ if needed in practice).

Proposition 4.3. *Assume that the conditions of Corollary 3.12 hold with* $\bar{F}^\epsilon := \epsilon^{-1}\bar{F}(\cdot,\cdot,\epsilon\cdot)$ *in place of* $\bar{F}$*, uniformly in* $\epsilon \in (0, \epsilon_\circ]$*, for some* $\epsilon_\circ > 0$*. Assume further that* $\bar{F} \in C^{1,2,3}_{loc}(\mathcal{D})$*, that* (46) *and*

$$\sup_{\mathcal{D}_\epsilon} \left(|\partial_z^2 \bar{F}_0| + |\partial_z^3 \bar{F}_0| + |\partial_x \partial_z^2 \bar{F}_0| + |\partial_x^2 \partial_z^2 \bar{F}_0| \right) < \infty \tag{47}$$

hold. Then, there exists some $o(\epsilon)$*, which does not depend on* x*, such that*

$$\begin{aligned} \mathrm{v}^\epsilon(0,x) =& \mathrm{v}^0(0,x) + \frac{\epsilon}{2} \mathbb{E}\left[\int_0^T \left[\partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2 \right](s, \tilde{X}_s^0) ds \right] + o(\epsilon) \\ =& \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\, \mathbb{E}\left[\partial_x \hat{g}(T, \tilde{X}_T^0) \tilde{Y}_T \right] + o(\epsilon) \end{aligned}$$

where, for $z \in \mathbb{R}$*,* $\tilde{X}^z$ *is the solution on* $[0,T]$ *of*

$$\tilde{X}^z = x + \int_t^\cdot (2\partial_z \bar{F}(\cdot, z\partial_x^2 \mathrm{v}^0(\cdot)))^{\frac{1}{2}}(s, \tilde{X}_s^z) dW_s, \tag{48}$$

and $\tilde{Y} := \partial_z \tilde{X}^z|_{z=0}$*, solves*

$$\tilde{Y} = \frac{1}{\sqrt{2}} \int_t^\cdot \frac{\partial_x \partial_z \bar{F}_0(s, \tilde{X}_s^0)\tilde{Y}_s + \partial_z^2 \bar{F}_0 \partial_x^2 \mathrm{v}^0(s, \tilde{X}_s^0)}{\sqrt{\partial_z \bar{F}_0(s, \tilde{X}_s^0)}} dW_s.$$

Proof. By Corollary 3.12, each v^ϵ associated to $\epsilon \in (0, \epsilon_\circ]$ solves

$$\partial_t \mathrm{v}^\epsilon + \epsilon^{-1} \bar{F}(\cdot,\cdot,\epsilon \partial_x^2 \mathrm{v}^\epsilon) = 0.$$

Moreover, it follows from our assumptions and Corollary 3.12 that $(\cdot,\cdot,\mathrm{v}^\epsilon) \in \mathcal{D}_\epsilon$ for all $\epsilon \in (0,\epsilon_\circ]$. Then, the fact that $\bar{F}(\cdot,\cdot,0) = 0$ implies that

$$\partial_t \mathrm{v}^\epsilon + \partial_z \bar{F}_0 \partial_x^2 \mathrm{v}^\epsilon + \frac{1}{2}\epsilon \partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^\epsilon|^2 = O(\epsilon^2),$$

in which the $O(\epsilon^2)$ is uniform since $|\partial_z^3 \bar{F}_0|$ is uniformly bounded on $\mathcal{D}_\varepsilon$ by assumption. Let $\Delta v^\epsilon := (\mathrm{v}^\epsilon - \mathrm{v}^0)/\epsilon$. By the above, (45) and Remark 4.2, it solves

$$\begin{aligned} O(\epsilon) =& \partial_t \Delta v^\epsilon + \partial_z \bar{F}_0 \partial_x^2 \Delta v^\epsilon + \frac{1}{2}\partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2 \\ &+ \frac{1}{2}\epsilon^2 \partial_z^2 \bar{F}_0 |\partial_x^2 \Delta v^\epsilon|^2 + \epsilon \partial_z^2 \bar{F}_0 \partial_x^2 \Delta v^\epsilon \partial_x^2 \mathrm{v}^0, \end{aligned}$$

in which $O(\epsilon)$ is uniform on $[0, T) \times \mathbb{R}$. By Theorem 3.11, Remark 4.2, and the same arguments as in this remark, $(\partial_x^2 \Delta v^\epsilon,\ \partial_z^2 \bar{F}_0,\ \partial_x^2 \mathrm{v}^0)_{0<\epsilon\le\epsilon_\circ}$ is locally bounded. Since $\Delta v^\epsilon(T, \cdot) = 0$, it follows that

$$\Delta v^\epsilon(0, x) = \mathbb{E}\left[\frac{1}{2}\int_0^T [\partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2](s, \tilde{X}_s^0) ds\right] + O(\epsilon).$$

Hence, $\Delta v := \lim_{\epsilon\to 0} \Delta v^\epsilon$ is given by

$$\Delta v(0, x) = \mathbb{E}\left[\frac{1}{2}\int_0^T [\partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2](s, \tilde{X}_s^0) ds\right]. \tag{49}$$

Moreover, $\partial_x \mathrm{v}^0$ satisfies

$$\partial_t(\partial_x \mathrm{v}^0) + \partial_x \partial_z \bar{F}_0 \partial_x^2 \mathrm{v}^0 + \partial_z \bar{F}_0 \partial_x^2 (\partial_x \mathrm{v}^0) = 0, \tag{50}$$

recall Remark 4.2.
Applying Itô's lemma to $\partial_x \mathrm{v}^0(t, \tilde{X}_t^0)\tilde{Y}_t$, we obtain

$$\begin{aligned} &d(\partial_x \mathrm{v}^0(t, \tilde{X}_t^0)\tilde{Y}_t) = \partial_t \partial_x \mathrm{v}^0(t, \tilde{X}_t^0)\tilde{Y}_t dt + \partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0)\tilde{Y}_t d\tilde{X}_t^0 + \partial_x \mathrm{v}^0(t, \tilde{X}_t^0) d\tilde{Y}_t \\ &+ \partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0) d\langle \tilde{Y}, \tilde{X}^0\rangle_t + \frac{1}{2}\partial_x^2(\partial_x \mathrm{v}^0(t, \tilde{X}_t^0))\tilde{Y}_t d\langle \tilde{X}^0\rangle_t \\ &= \Big(\partial_t \partial_x \mathrm{v}^0(t, \tilde{X}_t^0) + \partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0)\partial_x \partial_z \bar{F}_0(t, \tilde{X}_t^0) + \partial_x^2(\partial_x \mathrm{v}^0(t, \tilde{X}_t^0))\partial_z \bar{F}_0(t, \tilde{X}_t^0)\Big)\tilde{Y}_t dt \\ &+ \partial_z^2 \bar{F}_0(t, \tilde{X}_t^0)(\partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0))^2 dt + \partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0)\tilde{Y}_t d\tilde{X}_t^0 + \partial_x \mathrm{v}^0(t, \tilde{X}_t^0) d\tilde{Y}_t \\ &= \partial_z^2 \bar{F}_0(t, \tilde{X}_t^0)(\partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0))^2 dt + \partial_x^2 \mathrm{v}^0(t, \tilde{X}_t^0)\tilde{Y}_t d\tilde{X}_t^0 + \partial_x \mathrm{v}^0(t, \tilde{X}_t^0) d\tilde{Y}_t \end{aligned}$$

where we use (50) to get the last equality.
Therefore, taking expectation on both sides, we have

$$\mathbb{E}\left[\partial_x \mathrm{v}^0(T, \tilde{X}_T^0)\tilde{Y}_T\right] = \mathbb{E}\left[\int_0^T [\partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2](s, \tilde{X}_s^0) ds\right],$$

which leads to

$$\Delta v(0, x) = \frac{1}{2}\mathbb{E}\left[\partial_x \mathrm{v}^0(T, \tilde{X}_T^0)\tilde{Y}_T\right] = \frac{1}{2}\mathbb{E}\left[\partial_x \hat{g}(T, \tilde{X}_T^0)\tilde{Y}_T\right].$$

$\square$

Remark 4.4. *For later use, note that the above proof implies that* Δv *defined in* (49) *satisfies*

$$\partial_t \Delta v + \partial_z \bar{F}_0 \partial_x^2 \Delta v + \frac{1}{2} \partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2 = 0 \quad \text{on } [0,T) \times \mathbb{R}.$$

Remark 4.5. *A more tractable formulation can be obtained in the particular case where* $(\partial_z \bar{F}_0, \partial_z^2 \bar{F}_0) = (\lambda_1, \lambda_2)$ *is constant and* $\partial_x \partial_z \bar{F}_0 = 0$. *This is the case in the model of [7], see Example 2.1, whenever* $\sigma_\circ$ *and* f *are constant, see e.g. Remark 4.1. Then,* $\partial_x \mathrm{v}^0(\cdot, \tilde{X}^0) = \partial_x \mathrm{v}^0(0,x) + \int_0^\cdot \sqrt{2\lambda_1} \partial_x^2 \mathrm{v}^0(s, \tilde{X}_s^0) dW_s$ *by* (50)*, so that*

$$\begin{aligned}
\frac{\epsilon}{2} \mathbb{E}\left[\int_0^T [\partial_z^2 \bar{F}_0 |\partial_x^2 \mathrm{v}^0|^2](s, \tilde{X}_s^0) ds\right] &= \frac{\epsilon \lambda_2}{4\lambda_1} \mathbb{E}\left[\int_0^T \left[\sqrt{2\lambda_1} \partial_x^2 \mathrm{v}^0(s, \tilde{X}_s^0)\right]^2 ds\right] \\
&= \frac{\epsilon \lambda_2}{4\lambda_1} \mathbb{E}\left[\left(\partial_x \hat{g}(\tilde{X}_T^0) - \partial_x \mathrm{v}^0(0,x)\right)^2\right] \\
&= \frac{\epsilon \lambda_2}{4\lambda_1} \mathbb{E}\left[\left(\partial_x \hat{g}(\tilde{X}_T^0) - \mathbb{E}[\partial_x \hat{g}(\tilde{X}_T^0)]\right)^2\right] \\
&= \frac{\epsilon \lambda_2}{4\lambda_1} \mathrm{Var}\left[\partial_x \hat{g}(\tilde{X}_T^0)\right]
\end{aligned}$$

and the computation of the gamma $\partial_x^2 \mathrm{v}^0$ *is not required. Such a formulation does not seem available in general.*

The expansion of Proposition 4.3 leads to a natural approximate hedging strategy. The result is stated in terms of the function Δv introduced in the proof of Proposition 4.3, see (49).

Proposition 4.6. *Assume that the conditions of Proposition 4.3 hold and that*

(i) $\partial_z^2 \bar{F}_0 \in C_b^{1,4}([0,T] \times \mathbb{R})$,

(ii) $(t,x,z) \in [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto \frac{1}{2\epsilon} \sigma^2(t,x,\epsilon z)$ *is bounded and uniformly Lipschitz in its two last components, uniformly in* $\epsilon \in (0, \epsilon_0]$.

Then, there exists a constant $C > 0$ *such that, for each* $\epsilon \in (0, \epsilon_0]$ *and* $x \in \mathbb{R}$,

$$|V_T^{\epsilon,0,x,v^\epsilon,\phi^\epsilon} - \hat{g}(X_T^{\epsilon,0,x,\phi^\epsilon})| \le C\epsilon^2$$

in which

$$v^\epsilon := \mathrm{v}^0(0,x) + \epsilon \Delta v(0,x)$$

and $\phi^\epsilon = (y^\epsilon, b^\epsilon, \gamma^\epsilon) \in \mathcal{A}$ *with*

$$\begin{aligned}
y^\epsilon &= \partial_x(\mathrm{v}^0 + \epsilon \Delta v)(0,x), \\
b^\epsilon &= \left[\partial_t + \frac{1}{2\epsilon} \sigma^2(\cdot, \cdot, \epsilon \partial_x^2(\mathrm{v}^0 + \epsilon \Delta v)) \partial_x^2\right] \partial_x(\mathrm{v}^0 + \epsilon \Delta v)(\cdot, X^{\epsilon,0,x,\phi^\epsilon}), \\
\gamma^\epsilon &= \partial_x^2(\mathrm{v}^0 + \epsilon \Delta v)(\cdot, X^{\epsilon,0,x,\phi^\epsilon}).
\end{aligned}$$

Proof. For ease of notations, we write σ_ϵ for $\epsilon^{-\frac{1}{2}}\sigma(\cdot,\epsilon\cdot)$. We let $Y^\epsilon = \partial_x(\mathrm{v}^0 + \epsilon\Delta v)(\cdot, X^{\epsilon,0,x,\phi^\epsilon})$, and only write X^ϵ for $X^{\epsilon,0,x,\phi^\epsilon}$ in the following. Note that (48), (49), (i) and (24) imply that $\Delta v \in C_b^{1,4}([0,T]\times\mathbb{R})$.Then, the dynamics are well-defined thanks to Remark 4.2, and $\phi^\epsilon \in \mathcal{A}$. Set $F_\epsilon := F(\cdot,\cdot,\epsilon\,\cdot)/\epsilon$. By applying Itô's Lemma, using Remark 4.2, Remark 4.4 and the definition of $\bar{F}_\epsilon$ together with (22), we obtain

$$
\begin{aligned}
&\hat{g}(X_T^\epsilon) - v^\epsilon - \int_0^T Y_t^\epsilon dX_t^\epsilon - \int_0^T F_\epsilon(t, X_t^\epsilon, \gamma_t^\epsilon)dt \\
=&\mathrm{v}^0(T, X_T^\epsilon) + \epsilon\Delta v(T, X_T^\epsilon) - \mathrm{v}^0(0,x) - \epsilon\Delta v(0,x) - \int_0^T Y_t^\epsilon dX_t^\epsilon \\
&- \int_0^T F_\epsilon(\cdot, \partial_x^2(\mathrm{v}^0 + \epsilon\Delta v))(t, X_t^\epsilon)dt \\
=&\int_0^T \left[\bar{F}_\epsilon(\cdot, \partial_x^2(\mathrm{v}^0+\epsilon\Delta v)) - \partial_z\bar{F}_0\partial_x^2(\mathrm{v}^0+\epsilon\Delta v) - \frac{\epsilon}{2}\partial_z^2\bar{F}_0|\partial_x^2\mathrm{v}^0|^2\right](t, X_t^\epsilon)dt.
\end{aligned}
$$

Recalling that (19) is assumed to hold for $\bar{F}_\epsilon$, uniformly in $\epsilon \in (0,\epsilon_\circ]$, that $\partial_x^2\mathrm{v}^0$ and $\partial_x^2\Delta v$ are bounded, as well as (22), a second order Taylor expansion implies

$$
\bar{F}_\epsilon(\cdot, \partial_x^2(\mathrm{v}^0+\epsilon\Delta v)) - \partial_z\bar{F}_0\partial_x^2(\mathrm{v}^0+\epsilon\Delta v) - \frac{\epsilon}{2}\partial_z^2\bar{F}_0|\partial_x^2\mathrm{v}^0|^2 = O(\epsilon^2),
$$

in which $O(\epsilon^2)$ is uniform on $[0,T]\times\mathbb{R}$. □

5 Dual representation formula in the convex case

In this last section, we assume that

$$
z \in \mathbb{R} \mapsto \bar{F}(t,x,z) \text{ is convex and bounded from below,} \tag{51}
$$

$$
\lim_{z\to\bar{\gamma}(t,x)} \partial_z\bar{F}(t,x,z) = \infty \quad \text{for all } (t,x)\in[0,T]\times\mathbb{R}. \tag{52}
$$

Note that the second assumption is automatically satisfied if $\bar{\gamma} < \infty$, since in this case $\lim_{z\to\bar{\gamma}(t,x)} \bar{F}(t,x,z) = \infty$. Both are satisfied is the model studied in [7], see Remark 3.1.

Whenever $\bar{\gamma} < \infty$, let us now use the extension $\bar{F}(\cdot,\cdot,z) := \infty$ for $z \in [\bar{\gamma},\infty)$ and define the Fenchel-Moreau transform

$$
\bar{F}^*(\cdot,\cdot,\mathfrak{v}) := \sup_{z\in\mathbb{R}}\left(\frac{1}{2}\mathfrak{v}z - \bar{F}(\cdot,\cdot,z)\right),\ \mathfrak{v}\in\mathbb{R}.
$$

The conditions (51) and (52) ensure that $\bar{F}^*(t,x,\cdot)$ is finite on $\mathbb{R}_+$ and takes the value $+\infty$ on $\mathbb{R}_-\setminus\{0\}$. The function $\bar{F}$ being lower-semicontinuous on $\mathbb{R}_+$, convex and proper in its last argument, it follows that

$$
\bar{F}(\cdot,\cdot,z) = \sup_{\mathrm{s}\in\mathbb{R}_+}\left(\frac{1}{2}\mathrm{s}^2 z - \bar{F}^*(\cdot,\cdot,\mathrm{s}^2)\right). \tag{53}
$$

$$
\bar{F}^*(\cdot,\cdot,2\partial_z\bar{F}(\cdot,\cdot,z)) = \partial_z\bar{F}(\cdot,\cdot,z)z - \bar{F}(\cdot,\cdot,z), \quad \text{for } z<\bar{\gamma}. \tag{54}
$$

Remark 5.1. *It follows from* (53) *that a function* V *is a viscosity supersolution (resp. subsolution) on* $[0,T)\times\mathbb{R}$ *of*

$$\min\{-\partial_t\varphi - \bar{F}(\cdot,\cdot,\partial_x^2\varphi)\ ,\ \bar{\gamma} - \partial_x^2\varphi\} = 0$$

if and only if it is a viscosity supersolution (resp. subsolution) on $[0,T)\times\mathbb{R}$ *of*

$$\inf_{\mathrm{s}\in\mathbb{R}_+}\left(\bar{F}^*(\cdot,\cdot,\mathrm{s}^2) - \partial_t\varphi - \frac{1}{2}\mathrm{s}^2\partial_x^2\varphi\right) = 0. \tag{55}$$

This suggests, in the spirit of [24], that v admits a dual formulation in terms of an optimal control problem.

Theorem 5.2. *Assume that* (51) *and* (52) *hold. Let* S *denote the collection of non-negative bounded adapted processes. Then, for all* $(t,x)\in[0,T)\times\mathbb{R}$,

$$\begin{aligned}\mathrm{v}(t,x) &= \sup_{\mathfrak{s}\in\mathrm{S}}\mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s,X_s^{t,x,\mathfrak{s}},\mathfrak{s}_s^2)ds\right] \qquad (56)\\ &= \sup_{\mathfrak{s}\in\mathrm{S}}\mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s,X_s^{t,x,\mathfrak{s}},\mathfrak{s}_s^2)ds\right]\end{aligned}$$

in which

$$X^{t,x,\mathfrak{s}} = x + \int_t^{\cdot}\mathfrak{s}_s dW_s,\quad \mathfrak{s}\in\mathrm{S}.$$

If moreover the conditions of Corollary 3.12 hold, then the optimum is achieved by the Markovian control

$$\hat{\mathfrak{s}}_{t,x} := \left(2\partial_z\bar{F}(\cdot,\cdot,\partial_x^2\mathrm{v})(\cdot,X^{t,x,\hat{\mathfrak{s}}_{t,x}})\right)^{\frac{1}{2}}.$$

Remark 5.3. *The model studied in [7] corresponds to*

$$\bar{F}^*(t,x,s^2) = \frac{1}{2}\frac{(s-\sigma_\circ(t,x))^2}{f(x)},\quad \textit{for } s\geq 0.$$

See Remark 3.1. The result of Theorem 5.2 above can then be formally interpreted as follows. The larger the impact function f*, the more the optimal control can deviate from the volatility associated to the model without market impact. When* f *tends to* 0*, the optimal control needs to converge to the volatility of the impact free model* $\sigma_\circ$*, and one recovers the usual pricing rule at the limit.*

Proof of Theorem 5.2. 1. We first prove the first equality in (56) in the case where the conditions of Corollary 3.12 hold. Let v denote the right-hand side of (56). Recalling from Remark 5.1, Corollary 3.12 and Theorem 3.11 that v is a smooth supersolution of (55), we deduce that $\mathrm{v}\geq v$ by a simple verification argument. Let now $\hat{X}$ be the solution of

$$\hat{X} = x + \int_t^{\cdot}(2\partial_z\bar{F}(\cdot,\cdot,\partial_x^2\mathrm{v})(s,\hat{X}_s))^{\frac{1}{2}}dW_s.$$

It is well defined, recall Corollary 3.12, Theorem 3.11, (24) and (19), and corresponds to $X^{t,x,\hat{\mathfrak{s}}}$ with

$$\hat{\mathfrak{s}} := (2\partial_z \bar{F}(\cdot,\cdot,\partial_x^2 \mathrm{v})(\cdot,\hat{X}))^{\frac{1}{2}},$$

which is bounded. Moreover, (54) implies that

$$\mathrm{v}(t,x) = \mathbb{E}\Big[\hat{g}(\hat{X}_T) - \int_t^T \bar{F}^*(s,\hat{X}_s,\hat{\mathfrak{s}}_s^2)ds\Big],$$

which shows that $\mathrm{v} \le v$ since $\hat{\mathfrak{s}}$ is bounded.
2. We now extend the first equality in (56) to the general case. Let $\{\Phi_\varepsilon,\Phi^\varepsilon\}$ be as in the proof of Theorem 3.5 at the end of Section 3, and let u^ε and u_ε be the (smooth) solutions to (Eq_0) associated to Φ^ε and Φ_ε respectively, as in Theorem 3.5. Then $\Phi_\varepsilon \le \hat{g} \le \Phi^\varepsilon$, $u_\varepsilon \le \mathrm{v} \le u^\varepsilon$ and $(u^\varepsilon - u_\varepsilon, \Phi^\varepsilon - \Phi_\varepsilon)_{\varepsilon>0}$ converges uniformly to 0 as $\varepsilon \to 0$. Define v_ε and v^ε as v but with Φ_ε and Φ^ε in place of $\hat{g}$. Then, $v_\varepsilon \le v \le v^\varepsilon$ and $(v^\varepsilon - v_\varepsilon)_{\varepsilon>0}$ converges uniformly to 0 as $\varepsilon \to 0$. Since, by 1., $(v_\varepsilon, v^\varepsilon) = (u_\varepsilon, u^\varepsilon)$, the required result follows.
3. It remains to prove the second equality in (56). Define

$$\tilde{v}(t,x) := \sup_{\mathfrak{s}\in \mathrm{S}} \mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s,X_s^{t,x,\mathfrak{s}},\mathfrak{s}_s^2)ds\right],\ (t,x)\in[0,T)\times\mathbb{R}.$$

In view of 2., we know that $\tilde{v}$ is bounded from above by v. Since $\bar{F}^*(\cdot,0)^+$ and g^- are bounded, see (51) and (15), it is also bounded from below, by a constant. Then, it follows from [9] that the lower-semicontinuous enveloppe $\tilde{v}_*$ of $\tilde{v}$ is a viscosity supersolution of (55) such that $\tilde{v}_*(T,\cdot) \ge g$, recall (15). It is in particular a supersolution of $\bar{\gamma} - \partial_x^2\varphi \ge 0$ on $[0,T)\times\mathbb{R}$, by Remark 5.1. Then, the same arguments as in [7, Step 3.b., proof of Theorem 3.16] imply that $\tilde{v}_*(T,\cdot) \ge \hat{g}$. By [9] again, we also have that

$$\tilde{v}(t,x) \ge \mathbb{E}\left[\tilde{v}_*(T,X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s,X_s^{t,x,\mathfrak{s}},\mathfrak{s}_s^2)ds\right],\ \text{ for any } \mathfrak{s}\in \mathrm{S}.$$

Hence,

$$\tilde{v}(t,x) \ge \sup_{\mathfrak{s}\in \mathrm{S}} \mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s,X_s^{t,x,\mathfrak{s}},\mathfrak{s}_s^2)ds\right].$$

□

We conclude this section with a result showing that any optimal control control $\hat{\mathfrak{s}}$ should be such that $\hat{g}(X_T^{t,x,\hat{\mathfrak{s}}}) = g(X_T^{t,x,\hat{\mathfrak{s}}})$.

Proposition 5.4. *Let the condition of Theorem 5.2 hold and assume that* $\bar{F}(\cdot,\cdot,\kappa)$ *is uniformly bounded on* $[0,T]\times\mathbb{R}$ *for some* $\kappa > 0$. *Fix* $(t,x)\in[0,T)\times\mathbb{R}$

and let $(\mathfrak{s}^n)_{n\geq 1}$ *be such that*

$$\mathrm{v}(t,x) = \lim_{n\uparrow\infty} \mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}^n}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}^n}, (\mathfrak{s}_s^n)^2)ds\right].$$

Then, $(X_T^{t,x,\mathfrak{s}^n})_{n\geq 1}$ *is tight, and any limiting law* ν *associated to a subsequence satisfies* $\nu(\hat{g} > g) = 0$.

Proof. We only write X^n for $X^{t,x,\mathfrak{s}^n}$ and let

$$J_n := \mathbb{E}\left[g(X_T^n) - \int_t^T \bar{F}^*(s, X_s^n, (\mathfrak{s}_s^n)^2)ds\right],$$

$n \geq 1$. Then, (15) and (51) imply that one can find $C > 0$ such that

$$\begin{aligned} -C &\leq \mathbb{E}[C + \frac{\kappa}{4}|X_T^n|^2 - \int_t^T \frac{\kappa}{2}(\mathfrak{s}_s^n)^2 ds + T \sup \bar{F}(\cdot,\cdot,\kappa)] \\ &\leq \mathbb{E}[C - \int_t^T \frac{\kappa}{4}(\mathfrak{s}_s^n)^2 ds + T \sup \bar{F}(\cdot,\cdot,\kappa)]. \end{aligned}$$

Hence, $\sup_{n\geq 1} \mathbb{E}[\int_t^T (\mathfrak{s}_s^n)^2 ds] < \infty$. Let ν_n be the law associated to X_T^n. The above shows that $(\nu_n)_{n\geq 1}$ is tight. Let us consider a subsequence $(\nu_{n_k})_{k\geq 1}$ that converges to some law ν. If $\nu(\hat{g} > g) > 0$, then one can find $\delta > 0$ such that $\mathbb{E}[\hat{g}(X_T^{n_k})] \geq \mathbb{E}[g(X_T^{n_k})] + \delta$ for all $k \geq 1$ large enough, which would imply that

$$\lim_{k\to\infty} \mathbb{E}\left[\hat{g}(X_T^{n_k}) - \int_t^T \bar{F}^*(s, X_s^{n_k}, (\mathfrak{s}_s^{n_k})^2)ds\right] \geq \lim_{k\to\infty} J_{n_k} + \delta,$$

a contradiction to Theorem 5.2. □

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