A $\mathbb{C}^{0,1}$-functional Itô's formula and its applications in mathematical finance

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Abstract

Using Dupire's notion of vertical derivative, we provide a functional (path-dependent) extension of the Itô's formula of Gozzi and Russo (2006) that applies to $C^{0,1}$-functions of continuous weak Dirichlet processes. It is motivated and illustrated by its applications to the hedging or superhedging problems of path-dependent options in mathematical finance, in particular in the case of model uncertainty. In this context, we also prove a new regularity result for the vertical derivative of candidate solutions to a class of path-depend PDEs, using an approximation argument which seems to be original and of own interest.

1 Introduction

Let X be a $\mathbb{R}^d$–valued continuous semimartingale with (unique) decomposition $X = X_0 + M + A$, where M is a continuous martingale and A is a finite variation process such that $M_0 = A_0 = 0$. Let $f : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $C^{1,2}$-function, then, Itô's Lemma says that

$$f(t, X_t) \quad = \quad f(0, X_0) + \int_0^t \nabla_x f(s, X_s) dM_s + \Gamma_t^f, \text{ a.s.}, \tag{1.1}$$

in which $\nabla_x f$ is the gradient in space of f, viewed as a line vector, and Γ^f is a continuous process with finite variation, given by

$$\Gamma_t^f := \int_0^t \partial_t f(s, X_s) ds + \sum_{1 \le i \le d} \int_0^t \nabla_{x^i} f(s, X_s) dA_s + \frac{1}{2} \sum_{1 \le i,j \le d} \int_0^t \nabla^2_{x^i x^j} f(s, X_s) d[X^i, X^j]_s.$$

If we assume in addition that X and $f(\cdot, X)$ are both local martingales, then $\Gamma^f \equiv 0$, a.s., so that the formula does not involve the partial derivatives $\partial_t f$ and $(\nabla^2_{x^i x^j} f)_{i,j \le d}$ any more. In this case, one might expect that the above formula still holds even if f is only $C^{0,1}$. This was in fact achieved by using the stochastic calculus via regularization theory that was developed in [25, 26, 27, 19, 1, 7].

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In this theory, notions of orthogonal (or zero energy) and weak Dirichlet processes have been introduced (see below for a precise definition), which generalize respectively the notions of finite variation processes and of semimartingales. It is proved that, for a $C^{0,1}$ functions f and a continuous weak Dirichlet process X with finite quadratic variation, the decomposition (1.1) still holds true for some orthogonal (or zero energy) process Γ^f. In particular, if X and $f(\cdot, X)$ are both continuous local martingales, the orthogonal process Γ^f must vanish, so that (1.1) reduces to

$$f(t, X_t) \;=\; f(0, X_0) + \int_0^t \nabla_x f(s, X_s) dX_s, \text{ a.s.} \tag{1.2}$$

This is typically the case in mathematical finance under the so-called no free lunch with vanishing risk property, see e.g. [10]. Such a formula is obviously very useful in many situations where $C^{1,2}$ regularity is difficult to prove, or not true at all. In particular, we refer to [19] for an application to a verification argument in a stochastic control problem.

In this paper, our first main objective is to provide an extension of (1.1) and (1.2) to the functional (path-dependent) case. For $\mathbb{C}^{1,2}$-functionals, the functional Itô's formula for continuous semimartingales has been investigated in [6, 8], using the notion of Dupire's [12] derivatives. For less regular functionals, a step forward in this direction was made in [29, 3, 4]. The results in [3, 4] were motivated, respectively, by a verification argument for the replication of path-dependent options in a model with market impact and by an optional decomposition theorem for supermartingales, which in turn was applied to derive original results in the field of robust hedging in mathematical finance. In the above papers, the functional does not even need to be differentiable in space but is assumed to be concave in space and non-increasing in time (in a sense that matches the notion of Dupire's derivative), up to a smooth function. These assumptions, which perfectly match the cases of application motivating [3, 4], allows one to show that Γ^f is non-increasing without complex analysis. It is restricted to càdlàg semimartingales in [4] and to continuous semimartingales in [3, Appendix]. The main objective of [29] is to establish a path-dependent Meyer-Tanaka's formula. It has the advantage over [3, 4] to provide an explicit expression of the non-decreasing process entering the decomposition in terms of local times, but it requires much more regularity.

Also notice that a weaker notion of differentiability of path-dependent functionals has been used in [13, 14] to define the viscosity solutions of path-dependent PDEs (see also [24] for an overview).

In this paper, we show that the arguments of [19] can be used to easily provide a functional version (1.1)-(1.2) using Dupire's notion of derivatives for functionals F defined on the space of paths. Unlike [1], we voluntarily restrict ourselves to the case where X has continuous paths for tractability, see Remark 2.9. Since we will use the stochastic calculus by regularization developed by Russo and Vallois, and their co-authors, we naturally provide a version for weak Dirichlet processes that extends [19] to the path-dependent case. In general, it requires additional conditions involving both the path-regularity of the underlying process X and of the path-dependent functional F, that are satisfied when X is a continuous semimartingale and F is smooth, or under other typical structure conditions on F, in particular if F is Fréchet differentiable in space.

Our main motivation comes from mathematical finance. In models without frictions, the prices of financial assets turn out to be semimartingales and even martingales under a suitable probability measure. The $\mathbb{C}^{0,1}$–functional Itô's formula allows one to understand the structure/relation between the martingale parts of different financial assets, which is the core problem for the hedging of risks. More concretely, we provide a new result on the super-hedging of path-dependent options, under model uncertainty, where the gradient of the value function provides the optimal super-hedging strategy. Unlike in [4], the situation we consider does not correspond to that of a concave functional, so that the results of [3, 4] can not be exploited. In particular, in this application, we prove an original regularity result for the vertical derivative of the super-hedging price. For this, we use a PDE approximation argument which seems to be original and which may open the door to a wide range of applications in the PPDE literature.

The rest of the paper is organized as follows. We first provide our version of the path-dependent Itô's formula for $\mathbb{C}^{0,1}$-functionals and continuous weak Dirichlet processes in Section 2. Applications in finance are then provided in Section 3.

2 Path-dependent Itô's formula for $\mathbb{C}^{0,1}$-functionals

In this section, we fix a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. The abbreviation u.c.p. denotes the uniform convergence in probability.

2.1 Preliminaries

We start with preliminaries on the stochastic calculus via regularization and the notion of Dupire's derivatives of path-dependent functions.

2.1.1 Itô's calculus via regularization and weak Dirichlet processes

Let us recall here some definitions and facts on the Itô calculus via regularization developped by Russo and Vallois [25, 26, 28]. See also Bandini and Russo [1] (and [27, 19]) for a version of the C^1–Itô's formula.

Definition 2.1. (i) *Let X be a real valued càdlàg process, and H be a process with paths in $L^1([0,T])$ a.s. The forward integral of H w.r.t. X is defined by*

$$\int_0^t H_s\, d^- X_s \ :=\ \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\int_0^t H_s\big(X_{(s+\varepsilon)\wedge t} - X_s\big)ds, \quad t\ge 0,$$

whenever the limit exists in the sense of u.c.p.

(ii) *Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X,Y]$ is defined by*

$$[X,Y]_t \ :=\ \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon}\int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s)ds, \quad t\ge 0,$$

whenever the limit exists in the sense of u.c.p.

(iii) *We say that a real valued càdlàg process X has finite quadratic variation, if its quadratic variation, defined by $[X] := [X, X]$, exists and is finite a.s.*

Remark 2.2. *When X is a (càdlàg) semimartingale and H is a càdlàg adapted process, $\int_0^t H_s \, d^- X_s$ coincides with the usual Itô's integral $\int_0^t H_s dX_s$. When X and Y are two semimartingales, $[X, Y]$ coincides with the usual bracket.*

Definition 2.3. (i) *We say that an adapted process A is orthogonal if $[A, N] = 0$ for any real valued continuous local martingale N.*

(ii) *An adapted process X is called a weak Dirichlet process if it has a decomposition of the form $X = X_0 + M + A$, where M is a local martingale and A is orthogonal such that $M_0 = A_0 = 0$.*

Remark 2.4. (i) *An adapted process with finite variation is orthogonal. Consequently, a semimartingale is in particular a continuous weak Dirichlet process.*

(ii) *An orthogonal process has not necessarily finite variation. For example, any deterministic process (with possibly infinite variation) is orthogonal.*

(iii) *The decomposition $X = X_0 + M + A$ for a continuous weak Dirichlet process X is unique, and both processes M and A in the decomposition are continuous.*

2.1.2 Dupire's derivatives of path-dependent functions

Let us denote by $C([0,T])$ the space of all $\mathbb{R}^d$–valued continuous paths on $[0,T]$, and by $D([0,T])$ the space of all $\mathbb{R}^d$–valued càdlàg paths on $[0,T]$, which are endowed with the uniform convergence topology induced by the norm $\|\mathrm{x}\| := \sup_{s\in[0,T]} |\mathrm{x}_s|$. Let $\Theta := [0,T] \times D([0,T])$. For $(t, \mathrm{x}) \in \Theta$, let us define the (optional) stopped path $\mathrm{x}_{t\wedge} := (\mathrm{x}_{t\wedge s})_{s\in[0,T]}$.

A function $F : \Theta \longrightarrow \mathbb{R}$ is said to be non-anticipative if $F(t, \mathrm{x}) = F(t, \mathrm{x}_{t\wedge})$ for all $(t, \mathrm{x}) \in \Theta$. A non-anticipative function $F : \Theta \longrightarrow \mathbb{R}$ is said to be continuous if, for all $(t, \mathrm{x}) \in \Theta$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t - t'| + \|\mathrm{x}_{t\wedge} - \mathrm{x}'_{t'\wedge}\| \le \delta \implies |F(t, \mathrm{x}) - F(t', \mathrm{x}')| \le \varepsilon.$$

Let $\mathbb{C}(\Theta)$ denote the class of all non-anticipative continuous functions. A non-anticipative function F is said to be left-continuous if, for all $(t, \mathrm{x}) \in \Theta$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t' \le t, \ |t - t'| + \|\mathrm{x}_{t\wedge} - \mathrm{x}'_{t'\wedge}\| \le \delta \implies |F(t, \mathrm{x}) - F(t', \mathrm{x}')| \le \varepsilon.$$

We denote by $\mathbb{C}_l(\Theta)$ the class of all non-anticipative left-continuous functions.

Let $F : \Theta \longrightarrow \mathbb{R}$ be a non-anticipative function, we follow Dupire [12] to define the Dupire's derivatives: F is said to be horizontally differentiable if, for all $(t, \mathrm{x}) \in [0, T) \times \Theta$, its horizontal derivative

$$\partial_t F(t, \mathrm{x}) \ := \ \lim_{h \searrow 0} \frac{F(t + h, \mathrm{x}_{t\wedge}) - F(t, \mathrm{x}_{t\wedge})}{h}$$

is well-defined ; F is said to be vertically differentiable if, for all $(t, \mathrm{x}) \in \Theta$, the function

$$y \longmapsto F(t, \mathrm{x} \oplus_t y) \text{ is differentiable at } 0, \text{ with } \mathrm{x} \oplus_t y \; := \; \mathrm{x}\mathbf{1}_{[0,t)} + (\mathrm{x}_t + y)\mathbf{1}_{[t,T]},$$

whose derivative at $y = 0$ is called the vertical derivative of F at (t, x), denoted by $\nabla_{\mathrm{x}} F(t, \mathrm{x})$. One can then similarly define the second-order derivative $\nabla^2_{\mathrm{x}} F$. Given

$$\mathbb{C}^{0,1}(\Theta) := \big\{ F \in \mathbb{C}(\Theta) \; : \nabla_{\mathrm{x}} F \text{ is well defined and } \nabla_{\mathrm{x}} F \in \mathbb{C}_l(\Theta) \big\},$$

we let $\mathbb{C}^{1,2}(\Theta)$ denote the class of all functions $F \in \mathbb{C}^{0,1}(\Theta)$ such that both $\partial_t F$ and $\nabla^2_{\mathrm{x}} F$ are well defined and belong to $\mathbb{C}_l(\Theta)$.

A functional $F : \Theta \longrightarrow \mathbb{R}$ is said to be locally bounded if, for all $K > 0$,

$$\sup_{t \in [0,T], \; \|\mathrm{x}\| \le K} \big| F(t, \mathrm{x}) \big| \; < \; \infty. \tag{2.1}$$

Further, F is said to be locally uniformly continuous if, for each $K > 0$, there exists a modulus of continuity[1] δ_K such that, for all $t \in [0, T]$, $h \in [0, T - t]$, $\|\mathrm{x}\| \le K$, $|y| \le K$,

$$\big| F(t, \mathrm{x}) - F(t + h, \mathrm{x}_{t\wedge}) \big| + \big| F(t, \mathrm{x}) - F(t, \mathrm{x} \oplus_t y) \big| \; \le \; \delta_K(h + |y|). \tag{2.2}$$

Let us denote by $\mathbb{C}^{u,b}_{\mathrm{loc}}(\Theta)$ the class of all locally bounded and locally uniformly continuous functions $F : \Theta \longrightarrow \mathbb{R}$. Notice that a continuous function defined on $[0, T] \times \mathbb{R}^d$ is automatically locally bounded and locally uniformly continuous, while it may not be true for a continuous function defined on Θ. This is the reason for introducing the class $\mathbb{C}^{u,b}_{\mathrm{loc}}(\Theta)$.

In the following, given a non-anticipative function $F : \Theta \longrightarrow \mathbb{R}$, we shall often write $F_t(\mathrm{x})$ in place of $F(t, \mathrm{x})$ for ease of notations.

2.2 Functional Itô's formula for $\mathbb{C}^{0,1}(\Theta)$-functions

We first provide a functional Itô's formula for continuous weak Dirichlet processes. More precisely, let $F \in \mathbb{C}^{0,1}$ and $X = M + A$ be a continuous weak Dirichlet process, we give a necessary and sufficient condition for the following decomposition:

$$F(t, X) \; = \; F(0, X) \; + \; \int_0^t \nabla_{\mathrm{x}} F_s(X) dM_s \; + \; \Gamma^F_t, \quad t \in [0, T], \tag{2.3}$$

where Γ^F is a continuous orthogonal process.

Theorem 2.5. *Let $X = X_0 + M + A$ be a continuous weak Dirichlet process with finite quadratic variation, where M is a (continuous) local martingale and A is an orthogonal process. Let $F \in \mathbb{C}^{0,1}(\Theta)$ be such that both F and $\nabla_{\mathrm{x}} F$ belong to $\mathbb{C}^{u,b}_{\mathrm{loc}}(\Theta)$, and assume that $s \mapsto \nabla_{\mathrm{x}} F_s(X)$ admits right-limits a.s. Then, $F(\cdot, X)$ is a continuous weak Dirichlet process with decomposition* (2.3) *if and only if, for all continuous martingale N,*

$$\frac{1}{\varepsilon} \int_0^{\cdot} \big(F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)) \big) \big(N_{s+\varepsilon} - N_s \big) ds \longrightarrow 0, \; u.c.p., \; as \; \varepsilon \longrightarrow 0. \tag{2.4}$$

[1] A non-negative function that is continuous at 0 and vanishes at 0.

The proof of Theorem 2.5 is postponed to the end of this section. Notice that, when $F(t,\mathrm{x}) = F_\circ(t,\mathrm{x}_t)$ for some $F_\circ : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$, it is clear that $F_{s+\varepsilon}(X) = F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s))$ so that (2.4) holds always true. Let us also provide a sufficient condition to ensure (2.4).

Proposition 2.6. *Assume that*

$$E^\varepsilon := \int_0^T \frac{1}{\varepsilon}\Big(F_{s+\varepsilon}(X) - F_{s+\varepsilon}\big(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)\big)\Big)^2 ds \longrightarrow 0, \text{ in probability} \tag{2.5}$$

as $\varepsilon \longrightarrow 0$. Then, condition (2.4) holds true.

Proof. Using Cauchy-Schwarz inequality, it follows that, for all continuous martingale N,

$$\begin{aligned}&\Big|\int_0^\cdot \frac{(F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)))}{\sqrt{\varepsilon}}\ \frac{N_{s+\varepsilon} - N_s}{\sqrt{\varepsilon}} ds\Big| \\ \le\ & \Big(\int_0^\cdot \frac{\big(F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s))\big)^2}{\varepsilon} ds\Big)^{1/2}\Big(\int_0^\cdot \frac{(N_{s+\varepsilon} - N_s)^2}{\varepsilon} ds\Big)^{1/2},\end{aligned}$$

which converges to 0 in the sense of u.c.p. by (2.5), together with the fact that N has finite quadratic variation. □

Remark 2.7. *The sufficient condition (2.5) is still quite abstract, we will provide more discussions on it in Section 2.3. In particular it is satisfied when X is a continuous semi-martingale and $F \in \mathbb{C}^{1,2}(\Theta)$, so that the result in Theorem 2.5 is consistent with that in [6]. Let us also notice that, to prove (2.3), it is indeed enough to check that for any sequence $(\varepsilon_n)_{n\ge1}$, such that $\varepsilon_n \longrightarrow 0$, there exists a subsequence $(\varepsilon_{n_k})_{k\ge1}$ along which the convergence in (2.4) holds true.*

We next provide a direct consequence of Theorems 2.5, by combining it with the Doob-Meyer decomposition, in the case where X is a continuous martingale and $F(\cdot, X)$ is a supermartingale. Notice that, in the following context, our result is more precise than the classical Doob-Meyer decomposition for supermartingales.

Corollary 2.8. *Let $F : \Theta \longrightarrow \mathbb{R}$ satisfy the conditions in Theorem 2.5. Assume in addition that X is a continuous local martingale, and $F(\cdot, X)$ is a supermartingale. Then*

$$F(t,X) \ =\ F(0,X)\ +\int_0^t \nabla_{\mathrm{x}} F_s(X) dX_s\ +\ A_t, \quad \textit{for all } t\in[0,T],$$

where A is a predictable non-increasing process.

Proof. It follows from Theorem 2.5 that the continuous supermartingale $F(\cdot, X)$ has the decomposition

$$F(t,X) \ =\ F(0,X) + \int_0^t \nabla_{\mathrm{x}} F_s(X) dX_s + \Gamma^F_t, \tag{2.6}$$

where Γ^F is a continuous (predictable) orthogonal process. At the same time, Γ^F should be a supermartingale, since $F(\cdot, X)$ is a supermartingale and $\int_0^\cdot \nabla_{\mathrm{x}} F_s(X) dX_s$ is a local martingale. Then, Γ^F has finite variation, and hence (2.6) coincides with the Doob-Meyer decomposition of $F(\cdot, X)$. As a conclusion, $\Gamma^F = A$ for some predictable non-increasing process A. $\square$

Proof of Theorem 2.5. Notice that $F \in \mathbb{C}(\Theta)$, $\nabla_{\mathrm{x}} F \in \mathbb{C}_l(\Theta)$ and X is a continuous process. Then the process $t \mapsto F_t(X)$ has a.s. continuous paths, $t \mapsto \nabla_{\mathrm{x}} F_t(X)$ has a.s. left-continuous paths (see e.g. [6, Lemma 2.6]). We now follow the arguments of [1, Theorem 5.15] to show that (2.4) is a necessary and sufficient condition for the decomposition (2.3).

(i) Let us define the process Γ^F by

$$\Gamma^F_\cdot \;:=\; F_\cdot(X) - \int_0^\cdot \nabla_{\mathrm{x}} F_s(X) dM_s.$$

We need to show that the condition (2.4) is necessary and sufficient to ensure that Γ^F is an orthogonal process (Definition 2.3), that is, for any continuous local martingales N,

$$\big[\Gamma^F, N\big] \;=\; \Big[F_\cdot(X) - \int_0^\cdot \nabla_{\mathrm{x}} F_s(X) dM_s \;,\; N\Big] \;=\; 0.$$

We first notice that, by [1, Proposition 2.8],

$$\Big[\int_0^\cdot \nabla_{\mathrm{x}} F_s(X) dM_s, N\Big] \;=\; \int_0^\cdot \nabla_{\mathrm{x}} F_s(X) d[M,N]_s \;=\; \int_0^\cdot \nabla_{\mathrm{x}} F_s(X) d[M,N]_s.$$

Then, to prove the decomposition (2.3), it is equivalent to show that, for any continuous local martingale N,

$$I^\varepsilon_\cdot := \frac{1}{\varepsilon}\int_0^\cdot \big(F_{s+\varepsilon}(X) - F_s(X)\big)\big(N_{s+\varepsilon} - N_s\big) ds \longrightarrow \int_0^\cdot \nabla_{\mathrm{x}} F_s(X) d[M,N]_s, \text{ as } \varepsilon \searrow 0, \text{ u.c.p.,} \tag{2.7}$$

Let us write $I^\varepsilon = I^{1,\varepsilon} + I^{2,\varepsilon}$, with

$$I^{1,\varepsilon}_t \;:=\; \frac{1}{\varepsilon}\int_0^t \big(F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)) - F_s(X)\big)\big(N_{s+\varepsilon} - N_s\big) ds,$$

and

$$I^{2,\varepsilon}_t \;:=\; \frac{1}{\varepsilon}\int_0^t \big(F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s))\big)\big(N_{s+\varepsilon} - N_s\big) ds.$$

(ii) Let us first consider $I^{1,\varepsilon}$ and write it as $I^{1,\varepsilon} = I^{11,\varepsilon} + I^{12,\varepsilon} + I^{13,\varepsilon} + I^{14,\varepsilon}$, where

$$I^{11,\varepsilon}_t := \frac{1}{\varepsilon}\int_0^t \big(F_{s+\varepsilon}(X_{s\wedge}) - F_s(X)\big)\big(N_{s+\varepsilon} - N_s\big) ds,$$

$$I^{12,\varepsilon}_t := \frac{1}{\varepsilon}\int_0^t \Delta^\varepsilon_s \cdot \big(X_{s+\varepsilon} - X_s\big)\big(N_{s+\varepsilon} - N_s\big) ds,$$

with

$$\Delta^\varepsilon_s := \int_0^1 \Big(\nabla_{\mathrm{x}} F_{s+\varepsilon}\big(X_{s\wedge} \oplus_{s+\varepsilon} \lambda(X_{s+\varepsilon} - X_s)\big) - \nabla_{\mathrm{x}} F_{s+\varepsilon}(X_{s\wedge})\Big) d\lambda.$$

and

$$I^{13,\varepsilon}_t := \frac{1}{\varepsilon}\int_0^t \big(\nabla_{\mathrm{x}} F_{s+\varepsilon}(X_{s\wedge}) - \nabla_{\mathrm{x}} F_s(X_{s\wedge})\big) \cdot \big(X_{s+\varepsilon} - X_s\big)\big(N_{s+\varepsilon} - N_s\big) ds,$$

$$I_t^{14,\varepsilon} := \frac{1}{\varepsilon}\int_0^t \nabla_{\mathrm{x}} F_s(X_{s\wedge})\cdot\big(X_{s+\varepsilon}-X_s\big)\big(N_{s+\varepsilon}-N_s\big)ds.$$

For the term $I^{11,\varepsilon}$, one has, by the integration by parts formula,

$$I_t^{11,\varepsilon} \;=\; \frac{1}{\varepsilon}\int_0^t \bigg(\big(F_{s+\varepsilon}(X_{s\wedge})-F_s(X)\big)\int_s^{s+\varepsilon} dN_u\bigg)\,ds \;=\; \int_0^{t+\varepsilon}\theta_u^{11,\varepsilon}dN_u,$$

where, by the uniform continuity condition (2.2) on F,

$$\theta_u^{11,\varepsilon} \;:=\; \frac{1}{\varepsilon}\int_{(u-\varepsilon)\vee 0}^{u}\big(F_{s+\varepsilon}(X_{s\wedge})-F_s(X)\big)ds \;\longrightarrow\; 0, \text{ for all } u\in[0,T], \text{ a.s.}$$

Then, by e.g. [21, Theorem I.4.31],

$$I^{11,\varepsilon}\longrightarrow 0, \text{ u.c.p. as } \varepsilon\longrightarrow 0.$$

For the terms $I^{12,\varepsilon}$ and $I^{13,\varepsilon}$, we notice that

$$\sup_{t\in[0,T]}\big(\big|I_t^{12,\varepsilon}\big|+\big|I_t^{13,\varepsilon}\big|\big) \;\le\; \delta_\varepsilon\,\Big(\frac{1}{\varepsilon}\int_0^T\big|X_{s+\varepsilon}-X_s\big|^2ds\Big)\Big(\frac{1}{\varepsilon}\int_0^T\big(N_{s+\varepsilon}-N_s\big)^2ds\Big),$$

where

$$\delta_\varepsilon := \sup_{0\le s\le T-\varepsilon}\Big(|\Delta_s^\varepsilon|+\big|\nabla_{\mathrm{x}}F_{s+\varepsilon}(X_{s\wedge})-\nabla_{\mathrm{x}}F_s(X_{s\wedge})\big|\Big)\longrightarrow 0, \text{ a.s. as } \varepsilon\searrow 0,$$

by the uniformly continuity condition (2.2) on $\nabla_{\mathrm{x}}F$. Since

$$\Big(\frac{1}{\varepsilon}\int_0^T\big(X_{s+\varepsilon}-X_s\big)^2ds\Big)\Big(\frac{1}{\varepsilon}\int_0^T\big(N_{s+\varepsilon}-N_s\big)^2ds\Big) \;\longrightarrow\; [X]_T[N]_T, \text{ u.c.p. as } \varepsilon\longrightarrow 0,$$

it follows that $I^{13,\varepsilon}\longrightarrow 0$ and $I^{13,\varepsilon}\longrightarrow 0$, u.c.p.

Finally, for $I^{14,\varepsilon}$, we apply [1, Corollary A.4 and Proposition A.6] to obtain that

$$I_\cdot^{14,\varepsilon}\longrightarrow\int_0^\cdot\nabla_{\mathrm{x}}F_s(X)d[M,N]_s, \text{ u.c.p., as } \varepsilon\longrightarrow 0,$$

so that

$$I_\cdot^{1,\varepsilon}\longrightarrow\int_0^\cdot\nabla_{\mathrm{x}}F_s(X)d[M,N]_s, \text{ u.c.p., as } \varepsilon\longrightarrow 0.$$

(iii) To conclude, we observe that (2.7) holds true (or equivalently (2.3) holds) if and only if $I^{2,\varepsilon}\longrightarrow 0$, u.c.p. (or equivalently (2.4) holds), for any continuous local martingale N. This concludes the proof. □

Remark 2.9. *The results and proof of Theorem 2.5 remain valid even if X is a càdlàg weak Dirichlet process with bounded quadratic variation, up to the fact that $\nabla_{\mathrm{x}}F_s(X)$ must be replaced by $\nabla_{\mathrm{x}}F_s(X\mathbf{1}_{[0,s)}+X_{s-}\mathbf{1}_{[s,T]})$ in (2.3). However, in this case, the decomposition of the weak Dirichlet process $F_\cdot(X)$ may not be unique. To see this, recall that any purely discontinuous martingale is orthogonal to a continuous martingale N, then one can always* *move a purely discontinuous martingale from the martingale part of $F.(X)$ to the orthogonal part of $F.(X)$ and the decomposition of the weak Dirichlet process $F.(X)$ stays valid.*

To ensure the uniqueness of the decomposition of $F.(X)$, one needs to use the notion of special weak Dirichlet process in [1], where the orthogonal part Γ^F entering (2.3) *is required to be predictable. Then it is possible to mimic the smoothing procedure of [1] to obtain such a decomposition for $F.(X)$. However, smoothing a $\mathbb{C}^{0,1}(\Theta)$-function into a $\mathbb{C}^{1,2}(\Theta)$-function requires various and heavy technical assumptions, see [29], which may be difficult to check in the applications we have in mind.*

2.3 Discussions on the condition (2.5)

The sufficient technical condition (2.5) used to ensure the decomposition result in Theorem 2.5 is still too abstract. Let us provide some more explicit sufficient conditions for (2.5). We first show that (2.5) holds true when X is a continuous semimartingale and $F \in \mathbb{C}^{1,2}(\Theta)$, which makes our result consistent with [6]. We will then provide some examples of sufficient conditions for (2.5) when F is not in $\mathbb{C}^{1,2}(\Theta)$. Also recall that (2.5) trivially holds when F is Markovian, i.e. $F(t, \mathrm{x}) = F_\circ(t, \mathrm{x}_t)$ for all $(t, \mathrm{x}) \in [0, T] \times D([0, T])$, for some $F_\circ : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$.

2.3.1 The case where $F \in \mathbb{C}^{1,2}(\Theta)$ and X is a continuous semimartingale

When X is a continuous semimartingale and $F \in \mathbb{C}^{1,2}(\Theta)$ with (local) bounded and uniformly continuous derivatives, one can check that (2.5) holds true by simply applying the functional Itô's formula of [6].

Proposition 2.10. *Let X be a continuous semimartingale and $F \in \mathbb{C}^{1,2}(\Theta)$ be such that $\partial_t F$ and $\nabla^2_{\mathrm{x}} F$ are locally bounded, and $\nabla_{\mathrm{x}} F \in \mathbb{C}^{u,b}_{\mathrm{loc}}$. Then, condition* (2.5) *holds true.*

Proof. For simplification of the notations, let us consider the one-dimensional case. First, for every fixed (s, ε), we apply the functional Itô's formula in [6, Theorem 4.1] on $F(X)$ to obtain that

$$\begin{aligned} F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge}) \;&=\; \big(F_{s+\varepsilon}(X) - F_s(X)\big) + \big(F_s(X) - F_{s+\varepsilon}(X_{s\wedge})\big) \\ = \int_s^{s+\varepsilon} \big(\partial_t F_r(X) - \partial_t F_r(X_{s\wedge})\big) dr &+ \int_s^{s+\varepsilon} \nabla_{\mathrm{x}} F_r(X) dX_r + \frac{1}{2}\int_s^{s+\varepsilon} \nabla^2_{\mathrm{x}} F_r(X) d[X]_r. \end{aligned}$$

Further, one can also apply the classical Itô's formula to $\phi(X_r) := F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_r - X_s))$ to obtain that

$$\begin{aligned} F_{s+\varepsilon}\big(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)\big) - F_{s+\varepsilon}\big(X_{s\wedge}\big) &= \int_s^{s+\varepsilon} \nabla_{\mathrm{x}} F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_r - X_s)) dX_r \\ &+ \frac{1}{2}\int_s^{s+\varepsilon} \nabla^2_{\mathrm{x}} F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_r - X_s)) d[X]_r. \end{aligned}$$

Then, it follows that

$$F_{s+\varepsilon}(X) - F_{s+\varepsilon}\big(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)\big) = \int_s^{s+\varepsilon} W^{1,\varepsilon}_{s,r} dr + \int_s^{s+\varepsilon} W^{2,\varepsilon}_{s,r} dX_r + \frac{1}{2}\int_s^{s+\varepsilon} W^{3,\varepsilon}_{s,r} d[X]_r, \tag{2.8}$$

where

$$W^{1,\varepsilon}_{s,r} := \partial_t F_r(X) - \partial_t F_r(X_{s\wedge}), \quad W^{2,\varepsilon}_{s,r} := \nabla_{\mathrm{x}} F_r(X) - \nabla_{\mathrm{x}} F_{s+\varepsilon}\big(X_{s\wedge} \oplus_{s+\varepsilon} (X_r - X_s)\big),$$

and

$$W^{3,\varepsilon}_{s,r} := \nabla^2_{\mathrm{x}} F_r(X) - \nabla^2_{\mathrm{x}} F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_r - X_s)).$$

By the local boundedness of $\partial_t F$, $\nabla_{\mathrm{x}} F$ and $\nabla^2_{\mathrm{x}} F$, it follows that

$$\sup_{0\le s\le r\le T,\ \varepsilon>0} \Big(\big|W^{1,\varepsilon}_{s,r}\big| + \big|W^{2,\varepsilon}_{s,r}\big| + \big|W^{3,\varepsilon}_{s,r}\big| \Big) < \infty, \text{ a.s.}$$

Further, since $\nabla_{\mathrm{x}} F$ satisfies the (locally) uniform continuity condition (2.2), for every fixed $r \in [0,T]$, one has

$$\big|W^{2,\varepsilon}_{s,r}\big| \le \big|\nabla_{\mathrm{x}} F_r(X) - \nabla_{\mathrm{x}} F_s(X)\big| + \delta_K(\varepsilon + |X_r - X_s|), \quad \text{whenever } \|X\| \le K.$$

As $s \mapsto \nabla_{\mathrm{x}} F_s(X)$ is left-continuous, then for every fixed $r \in [0,T]$,

$$\sup_{s\in[(r-\varepsilon)\vee 0, r]} \big|W^{2,\varepsilon}_{s,r}\big| \longrightarrow 0, \text{ a.s. as } \varepsilon \longrightarrow 0. \tag{2.9}$$

Let $X = X_0 + M + A$ where M is a continuous martingale and A is a finite variation process, and denote by $(|A|_t)_{t\in[0,T]}$ the total variation process of A. The two non-decreasing processes $|A|$ and $[M]$ are continuous, so that they are uniformly continuous on $[0,T]$, a.s. Recall the definition of E^ε in (2.5). It follows from (2.8) that

$$E^\varepsilon \ \le\ 4E^\varepsilon_1 + 4E^\varepsilon_2 + 4E^\varepsilon_3 + 4E^\varepsilon_4,$$

where

$$\begin{aligned}
E^\varepsilon_1 &:= \frac{1}{\varepsilon}\int_0^T \Big(\int_s^{s+\varepsilon} W^{1,\varepsilon}_{s,r} dr \Big)^2 ds \le \int_0^T \int_s^{s+\varepsilon} \big|W^{1,\varepsilon}_{s,r}\big|^2 dr ds \ \longrightarrow\ 0, \text{ a.s.,} \\
E^\varepsilon_2 &:= \frac{1}{\varepsilon}\int_0^T \Big(\int_s^{s+\varepsilon} W^{2,\varepsilon}_{s,r} dA_r \Big)^2 ds \ \le\ \int_0^T \big(|A|_{s+\varepsilon} - |A|_s\big) \frac{1}{\varepsilon} \int_s^{s+\varepsilon} \big|W^{2,\varepsilon}_{s,r}\big|^2 d|A|_r ds \\
&= \int_0^T \Big(\frac{1}{\varepsilon} \int_{(r-\varepsilon)\vee 0}^r \big(|A|_{s+\varepsilon} - |A|_s\big) \big|W^{2,\varepsilon}_{s,r}\big|^2 ds \Big) d|A|_r \ \longrightarrow\ 0, \text{ a.s.,} \\
E^\varepsilon_3 &:= \frac{1}{4\varepsilon}\int_0^T \Big(\int_s^{s+\varepsilon} W^{3,\varepsilon}_{s,r} d[X]_r \Big)^2 ds \\
&\le \int_0^T \Big(\frac{1}{\varepsilon} \int_{(r-\varepsilon)\vee 0}^r \big([X]_{s+\varepsilon} - [X]_s\big) \big|W^{3,\varepsilon}_{s,r}\big|^2 ds \Big) d[X]_r \ \longrightarrow\ 0, \text{ a.s.,}
\end{aligned}$$

and

$$E^\varepsilon_4 := \frac{1}{\varepsilon}\int_0^T \Big(\int_s^{s+\varepsilon} W^{2,\varepsilon}_{s,r} dM_r \Big)^2 ds.$$

To study the limit of E^ε_4, one can assume w.l.o.g. that $W^{2,\varepsilon}$ and $[M]_T$ are uniformly bounded by using localization techniques. Then, by (2.9),

$$\mathbb{E}\big[\big|E^\varepsilon_4\big|\big] = \frac{1}{\varepsilon}\mathbb{E}\Big[\int_0^T \int_s^{s+\varepsilon} \big|W^{2,\varepsilon}_{s,r}\big|^2 d[M]_r ds \Big] = \mathbb{E}\Big[\int_0^T \Big(\frac{1}{\varepsilon} \int_{(r-\varepsilon)\vee 0}^r \big|W^{2,\varepsilon}_{s,r}\big|^2 ds \Big) d[M]_r \Big] \longrightarrow 0.$$

It follows that $E^\varepsilon \longrightarrow 0$ in probability, and therefore that (2.5) holds true. □

2.3.2 Examples of sufficient conditions for (2.5)

We now provide examples of sufficient conditions for (2.5). The general idea behind them is to exploit Item (ii) of Definition 2.1 to control the terms in (2.5) by some quadratic variations, possibly up to an additional vanishing element. In the following, we let BV_+ denote the collection of all non-decreasing paths on $[0,T]$.

Proposition 2.11. *Assume that, for all* $\mathrm{x} \in D([0,T])$, $s \in [0,T]$ *and* $\varepsilon \in [0, T-s]$,

$$\big|F_{s+\varepsilon}(\mathrm{x}) - F_{s+\varepsilon}(\mathrm{x}_{s\wedge} \oplus_{s+\varepsilon} (\mathrm{x}_{s+\varepsilon} - \mathrm{x}_s))\big| \;\le\; \int_{(s,s+\varepsilon)} \phi\big(\mathrm{x}, |\mathrm{x}_u - \mathrm{x}_s|\big) db_u(\mathrm{x}),$$

where $\phi : C([0,T]) \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *satisfies* $\sup_{|y|\le K} \phi(\mathrm{x}, y) < \infty$, $\lim_{y\searrow 0} \phi(\mathrm{x}, y) = \phi(\mathrm{x}, 0) = 0$ *for all* $\mathrm{x} \in C([0,T])$ *and* $K > 0$, *and* $b : C([0,T]) \longrightarrow \mathrm{BV}_+$. *Then,* (2.5) *holds for any continuous process* X.

Proof. We first notice that, since X is a continuous process, then, for all $u \in [0,T]$,

$$\frac{1}{\varepsilon}\int_{(u-\varepsilon)\vee 0}^{u} \phi\big(X, |X_u - X_s|\big)^2 ds \longrightarrow \phi\big(X, 0\big)^2 = 0, \text{ a.s., as } \varepsilon \longrightarrow 0.$$

Recall the definition of E^ε in (2.5), and define the process B with finite variations by $B := b(X)$. Then, Minkowski's integral inequality implies that

$$\sqrt{E^\varepsilon} \;\le\; \int_0^T \Big(\frac{1}{\varepsilon}\int_{(u-\varepsilon)\vee 0}^{u} \phi\big(X, |X_u - X_s|\big)^2 ds\Big)^{1/2} dB_u \longrightarrow 0, \text{ a.s.,}$$

which concludes the proof. □

Example 2.12. *Assume that there exists a family of signed measures* $(\mu(\cdot; t, \mathrm{x}),\ (t, \mathrm{x}) \in [0,T] \times D([0,T]))$, *which is dominated by a non-negative finite measure* $\hat{\mu}$, *and a locally bounded map* $m : [0,T] \times D([0,T]) \times D([0,T]) \mapsto \mathbb{R}$ *such that, for all* $t \le T$ *and* $\mathrm{x}, \mathrm{x}' \in D([0,T])$ *satisfying* $\mathrm{x}_t = \mathrm{x}'_t$,

$$F(t,\mathrm{x}) - F(t,\mathrm{x}') = \int_{[0,t)} \big(\mathrm{x}_s - \mathrm{x}'_s\big)\mu(ds; t, \mathrm{x}) + o\big(\|\mathrm{x}_{t\wedge} - \mathrm{x}'_{t\wedge}\|\big) m(t, \mathrm{x}, \mathrm{x}'). \tag{2.10}$$

Then, one has $\partial_\lambda F_t(\mathrm{x} + \lambda(\mathrm{x}' - \mathrm{x})) = \int_0^t (\mathrm{x}'_s - \mathrm{x}_s)\mu(ds; t, \mathrm{x} + \lambda(\mathrm{x}' - \mathrm{x}))$. *It follows that*

$$\begin{aligned} F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)) &= \int_0^1 \partial_\lambda F_{s+\varepsilon}(X^{\varepsilon,\lambda}) d\lambda \\ &= \int_0^1 \int_{(s,s+\varepsilon)} (X_s - X_u)\mu(du; s+\varepsilon, X^{\varepsilon,\lambda}) d\lambda, \end{aligned}$$

with $X^\varepsilon := X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s)$ *and* $X^{\varepsilon,\lambda} := X^\varepsilon + \lambda(X - X^\varepsilon)$. *As* $(\mu(\cdot; t, \mathrm{x}))_{t,\mathrm{x}}$ *is dominated by* $\hat{\mu}$, *letting* $\hat{b}_u := \hat{\mu}([0,u])$, *one has*

$$\big|F_{s+\varepsilon}(X) - F_{s+\varepsilon}(X_{s\wedge} \oplus_{s+\varepsilon} (X_{s+\varepsilon} - X_s))\big| \;\le\; \int_{(s,s+\varepsilon)} |X_s - X_u| d\hat{b}_u.$$

Then, (2.5) *holds true when* X *has continuous paths, by Proposition 2.11.*

Notice that a Fréchet differentiable function in the sense of Clark [5] satisfies (2.10). *The difference is that we only need to check* (2.10) *for paths* x *such that* $\mathrm{x}_t = \mathrm{x}'_t$.

When X is a semimartingale, we can also exploit its semimartigale property to obtain sufficient conditions for (2.5).

Proposition 2.13. *Assume that, for all* $\mathrm{x} \in D([0,T])$, $s \in [0,T]$ *and* $\varepsilon \in [0, T-s]$,

$$\big|F_{s+\varepsilon}(\mathrm{x}) - F_{s+\varepsilon}(\mathrm{x}_{s\wedge} \oplus_{s+\varepsilon} (\mathrm{x}_{s+\varepsilon} - \mathrm{x}_s))\big| \;\le\; \phi\big(\mathrm{x}, \|\mathrm{x}_{(s+\varepsilon)\wedge} - \mathrm{x}_{s\wedge}\|, \varepsilon\big),$$

where $\phi : C([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ *satisfies*

$$\lim_{\varepsilon\searrow 0,\ y\searrow 0} \sup_{\|\mathrm{x}\|\le K} |\phi(\mathrm{x}, y, \varepsilon)|/y = 0, \text{ for all } K \ge 0.$$

Assume in addition that X is a continuous semimartingale. Then (2.5) *holds true.*

Proof. We first notice that

$$\int_0^T \frac{\phi(X, \|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\|, \varepsilon)^2}{\|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\|^2} \frac{\|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\|^2}{\varepsilon} ds \;\le\; C_\varepsilon^2 \int_0^T \frac{1}{\varepsilon}\|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\|^2 ds,$$

where

$$C_\varepsilon := \sup_{s\in[0,T]} \frac{\phi(X, \|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\|, \varepsilon)}{\|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\|} \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0.$$

Further, up to adding additional components, one can assume that each component of X is a martingale or a non-decreasing process. We therefore assume this and it suffices to consider the one dimensional case. Since X is a martingale or a non-decreasing process, there exists $C > 0$ such that

$$\mathbb{E}\Big[\int_0^T \frac{1}{\varepsilon}\big\|X_{(s+\varepsilon)\wedge} - X_{s\wedge}\big\|^2 ds\Big] \le C\mathbb{E}\Big[\int_0^T \frac{1}{\varepsilon}\big|X_{s+\varepsilon} - X_s\big|^2 ds\Big] \longrightarrow C\mathbb{E}\big[X\big]_T, \text{ in probability.}$$

This is enough to prove that $E^\varepsilon \longrightarrow 0$ in probability, so that (2.5) holds. □

By combining the conditions in Propositions 2.11 and 2.13, one obtains immediately new sufficient conditions for (2.5).

Corollary 2.14. *Assume that* $X = (X^1, X^2)$, *where* X^1 *is a continuous process,* X^2 *is a continuous semimartingale, and, for all* $0 \le s \le s+\varepsilon \le T$ *and* $\mathrm{x} = (\mathrm{x}^1, \mathrm{x}^2) \in D([0,T])$,

$$\big|F_{s+\varepsilon}(\mathrm{x}) - F_{s+\varepsilon}(\mathrm{x}_{s\wedge} \oplus_{s+\varepsilon} (\mathrm{x}_{s+\varepsilon} - \mathrm{x}_s))\big| \le \int_{[s,s+\varepsilon]} \phi_1\big(\mathrm{x}, |\mathrm{x}_u - \mathrm{x}_s|\big) db_u(\mathrm{x}) + \phi_2\big(\mathrm{x}, \|\mathrm{x}^2_{(s+\varepsilon)\wedge} - \mathrm{x}^2_{s\wedge}\|, \varepsilon\big),$$

where (ϕ_1, b) *satisfies the conditions in Proposition 2.11, and* ϕ_2 *satisfies the conditions in Proposition 2.13. Then* (2.5) *holds true.*

3 Applications in mathematical finance

In frictionless financial models, under the no-arbitrage (in the sense of no free lunch with vanishing risk) assumption, the prices of tradable financial assets need to be semimartingales, see e.g. [10]. If the pricing function of a financial derivative is $\mathbb{C}^{0,1}(\Theta)$, then one can apply the Itô's formula in Theorems 2.5 to characterize the martingale part of the derivative's price process, and therefore identify the hedging strategy. Below we provide some examples of such applications in finance.

3.1 General formulations under $\mathbb{C}^{0,1}(\Theta)$-regularity condition

3.1.1 Replication of path-dependent options

Let us consider a continuous martingale $X = (X_t)_{0\le t\le T}$, which represents the discounted price of some risky asset, and a path-dependent derivative with payoff $g(X)$ such that $\mathbb{E}[|g(X)|] < \infty$. Define

$$V(t,\mathrm{x}) := \mathbb{E}\big[g(X)\big|X_{t\wedge} = \mathrm{x}_{t\wedge}\big],\ (t,\mathrm{x}) \in [0,T]\times D([0,T]). \tag{3.1}$$

Proposition 3.1. *Assume that V belongs to $\mathbb{C}^{0,1}(\Theta)$ and satisfies all the conditions of Theorem 2.5. Then*

$$g(X) = \mathbb{E}\big[g(X)\big] + \int_0^T \nabla_{\mathrm{x}} V(t,X)dX_t.$$

Proof. Since $V(t,X)$ and $-V(t,X)$ are both supermartingales, the result follows from Corollary 2.8. □

Remark 3.2. (i) *The above result can be compared to [6, Theorem 5.2] but we require less regularity conditions ($\mathbb{C}^{0,1}(\Theta)$ and* (2.4) *rather than $\mathbb{C}^{1,2}_b(\Theta)$, which implies* (2.4) *by Propositions 2.6 and 2.10).*

(ii) *Let X be a diffusion process with dynamics*

$$X_t = X_0 + \int_0^t \mu(s,X)ds + \int_0^t \sigma(s,X)dW_t,$$

in which W is a Brownian motion and (μ,σ) are continuous, non-anticipative and Lipschitz in space. When $V \in \mathbb{C}^{1,2}(\Theta)$ in the sense of [6, Theorem 4.1], it is easy to deduce from their functional Itô's formula that V is a classical solution of the path-dependent PDE

$$\partial_t V + \mu\cdot\nabla_{\mathrm{x}} V + \frac{1}{2}\sigma\sigma^\top\cdot\nabla^2_{\mathrm{x}} V = 0.$$

Without the $\mathbb{C}^{1,2}(\Theta)$–regularity condition, one can still prove that V is a viscosity solution of the path-dependent PDE in the sense of [14], for which numerical algorithms can be found in [23, 30].

Remark 3.3. *As already mentioned in [12] and [6], the result of Proposition 3.1 is consistent with the classical Clark-Haussmann-Ocone formula. Indeed, let X be a continuous martingale with independent increments, and g be Fréchet differentiable with derivative λ_g, then by the Clark-Haussmann-Ocone formula (see e.g. Haussmann [20]),*

$$g(X) = \mathbb{E}[g(X)] + \int_0^T \mathbb{E}\big[\lambda_g(X;[t,T])\big|\mathcal{F}_t\big]dX_t.$$

On the other hand, for the value function V in (3.1)*, one can also compute the vertical derivative $\nabla_{\mathrm{x}} V$ from its definition to obtain that*

$$\nabla_{\mathrm{x}} V(t,\mathrm{x}) = \mathbb{E}\big[\lambda_g(X;[t,T])\big|X_{t\wedge} = \mathrm{x}_{t\wedge}\big].$$

3.1.2 Super-replication under model uncertainty

Let us now denote by $\Omega^\circ := D([0,T],\mathbb{R}^d)$ the canonical space of $\mathbb{R}^d$-valued càdlàg paths on $[0,T]$, let X be the canonical process, and $\mathbb{F}^\circ = (\mathcal{F}_t^\circ)_{t\in[0,T]}$ the canonical filtration. Let us denote by $\mathfrak{B}(\Omega^\circ)$ the space of all Borel probability measures on Ω°. We consider a subset $\mathcal{P} \subset \mathfrak{B}(\Omega^\circ)$, such that X is a $\mathbb{P}$–continuous local martingale satisfying $\mathbb{P}[X_0 = x_0] = 1$ for all $\mathbb{P} \in \mathcal{P}$, for some $x_0 \in \mathbb{R}^d$. Recall that, given a probability measure $\mathbb{P}$ on $(\Omega^\circ, \mathcal{F}_T^\circ)$ and a $\mathbb{F}^\circ$–stopping time τ taking values in $[0,T]$, a r.c.p.d. (regular conditional probability distribution) of $\mathbb{P}$ conditional to $\mathcal{F}_\tau^\circ$ is a family $(\mathbb{P}_\omega)_{\omega\in\Omega}$ of probability measures on $(\Omega^\circ, \mathcal{F}_T^\circ)$, such that $\omega \mapsto \mathbb{P}_\omega$ is $\mathcal{F}_\tau^\circ$–measurable, $\mathbb{P}_\omega[X_s = \omega_s,\ s \le \tau(\omega)] = 1$ for all $\omega \in \Omega$, and $\mathbb{E}^{\mathbb{P}}[\mathbf{1}_A|\mathcal{F}_\tau^\circ](\omega) = \mathbb{E}^{\mathbb{P}_\omega}[\mathbf{1}_A]$ for $\mathbb{P}$–a.e. $\omega \in \Omega^\circ$ for all $A \in \mathcal{F}_T^\circ$. Recall also that a subset A of a Polish space E is called an analytic set if there exists another Polish space E' together with a Borel subset $B \subset E \times E'$ such that $A = \{x \in E\ : (x,x') \in B\}$.

We further make the following assumptions.

Assumption 3.4. *One has $\mathcal{P} = \cup_{\omega\in\Omega^\circ}\mathcal{P}(0,\omega)$, for a collection of families of probability measures $\big(\mathcal{P}(t,\omega)\big)_{(t,\omega)\in[0,T]\times\Omega^\circ}$ on Ω°. Moreover, for every $(t,\omega) \in [0,T]\times\Omega^\circ$:*

1. *$\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{t\wedge})$, $\mathbb{P}[X_{t\wedge} = \omega_{t\wedge}] = 1$ for all $\mathbb{P} \in \mathcal{P}(t,\omega)$ and the graph set*
$$[[\mathcal{P}]] \ :=\ \big\{(t,\omega,\mathbb{P})\ : \mathbb{P} \in \mathcal{P}(t,\omega)\big\}$$
is an analytic subset of $[0,T]\times\Omega^\circ\times\mathfrak{B}(\Omega^\circ)$.

2. *Let $\mathbb{P} \in \mathcal{P}(t,\omega)$, $s \ge t$ and $(\mathbb{P}_\omega)_{\omega\in\Omega^\circ}$ be a family of regular conditional probability of $\mathbb{P}$ knowing $\mathcal{F}_s^\circ$, then $\mathbb{P}_\omega \in \mathcal{P}(s,\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega^\circ$.*

3. *Let $\mathbb{P} \in \mathcal{P}(t,\omega)$, $s \ge t$ and $(\mathbb{Q}_\omega)_{\omega\in\Omega^\circ}$ be a family such that $\omega \mapsto \mathbb{Q}_\omega$ is $\mathcal{F}_s^\circ$–measurable and $\mathbb{Q}_\omega \in \mathcal{P}(s,\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega^\circ$, then*
$$\mathbb{P}\otimes_s\mathbb{Q}_\cdot \in \mathcal{P}(t,\omega),$$
where $\mathbb{P}\otimes_s\mathbb{Q}_\cdot$ is defined by
$$\mathbb{E}^{\mathbb{P}\otimes_s\mathbb{Q}_\cdot}[\xi] := \int_{\Omega^\circ}\int_{\Omega^\circ}\xi(\omega')\mathbb{Q}_\omega(d\omega')\mathbb{P}(d\omega),\ \textit{for all bounded r.v. } \xi : \Omega^\circ \longrightarrow \mathbb{R}.$$

Let $g : \Omega^\circ \longrightarrow \mathbb{R}$ be such that $\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\big[\big|g(X)\big|\big] < \infty$, let us define
$$V(t,\omega) \ :=\ \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}^{\mathbb{P}}\big[g(X)\big],\ (t,\omega) \in [0,T]\times D([0,T]).$$

We simply write $V(0,x_0)$ for $V(0,\cdot)$. We also denote by $\mathcal{H}$ the collection of all $\mathbb{R}^d$–valued $\mathbb{F}^\circ$–predictable processes H such that $\int_0^T H_t^\top d\langle X\rangle_t H_t < \infty$, $\mathbb{P}$–a.s. and $\int_0^\cdot H_s dX_s$ is a $\mathbb{P}$–supermartingale, for all $\mathbb{P} \in \mathcal{P}$.

Proposition 3.5. *Let Assumption 3.4 hold true, and suppose in addition that $(V,X,\mathbb{P})$ satisfies the conditions of Theorem 2.5 for each $\mathbb{P} \in \mathcal{P}$. Then $V(\cdot,X)$ is a $\mathbb{P}$–supermartingale for every $\mathbb{P} \in \mathcal{P}$ and*
$$V(0,x_0) = \inf\Big\{x\ : x + \int_0^T H_t dX_t \ge g(X),\ H \in \mathcal{H},\ \mathbb{P}\text{–a.s. for all } \mathbb{P} \in \mathcal{P}\Big\}. \qquad (3.2)$$
Moreover, the superhedging problem at the r.h.s. of (3.2) is achieved by $H^* := \nabla_{\mathrm{x}}V(\cdot,X)$.

Proof. First, it is clear that one has the weak duality

$$V(0,x_0) \;\leq\; \inf\Big\{x\in\mathbb{R}\;: x+\int_0^T H_t dX_t \geq g(X),\; H\in\mathcal{H},\; \mathbb{P}\text{–a.s. for all } \mathbb{P}\in\mathcal{P}\Big\}.$$

Next, our stability conditions under conditioning and concatenation of Assumption 3.4 imply the dynamic programming principle.

$$V(t,\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[V(t+h,X)\big], \tag{3.3}$$

see e.g. [17, 18]. Together with the fact that $V\in\mathbb{C}^{0,1}$, this implies that $V(\cdot,X)$ is a $\mathbb{P}$–continuous supermartingale for every $\mathbb{P}\in\mathcal{P}$. By Corollary 2.8, one has

$$V(0,x_0)+\int_0^T \nabla_{\mathrm{x}} V(t,X)dX_t \geq g(X),\; \mathbb{P}\text{–a.s. for all } \mathbb{P}\in\mathcal{P}.$$

This implies the duality result (3.2) as well as the fact that $H^* := \nabla_{\mathrm{x}} V(\cdot,X)$ is the optimal superhedging strategy. □

Remark 3.6. *The duality result* (3.2) *in the model independent setting has been much investigated, see e.g. [11, 22]. In most cases, one obtains the existence of an optimal strategy H^* but without an explicit expression. In [22], the duality is obtained for just measurable payoff functions g, but they require $\mathcal{P}$ to be made of extremal martingale measures. Our duality result of Proposition 3.5 does not requires $\mathbb{P}\in\mathcal{P}$ to be extremal, but requires regularity conditions on the value function V. This in turn allows us to characterize the optimal superhedging strategy explicitly as the Dupire's vertical derivative of the pricing function, which also justify the initial motivation of Dupire [12] to introduce this notion of derivative.*

Remark 3.7. *The main idea in Propositions 3.1 and 3.5 is to show that the replication or super-replication prices of the options are supermartingales, so that one can apply the Doob-Meyer decomposition result of Corollary 2.8. We can also apply the same technique to other situations, such as the hedging of American options, the superhedging problems under constraints, etc., in which the option price process has a natural supermartingale structure (see e.g. [2]).*

3.2 Verification of the $\mathbb{C}^{0,1}(\Theta)$-regularity in a model with (bounded) uncertain volatility

Let us consider a more concrete superhedging problem in the context of an uncertain volatility model. Let $d=1$, $x_0\in\mathbb{R}$, $0\leq\underline{\sigma}<\overline{\sigma}$ be fixed, we denote by $\mathcal{P}_0$ the collection of all probability measures $\mathbb{P}$ such that $\mathbb{P}[X_0=x_0]=1$ and

$$dX_s = \sigma_s dW_s^{\mathbb{P}},\; \sigma_s\in[\underline{\sigma},\overline{\sigma}],\; s\in[0,T],\; \mathbb{P}\text{-a.s.} \tag{3.4}$$

for some $\mathbb{P}$–Brownian motion $W^{\mathbb{P}}$. We then consider a derivative option with payoff function $g: D([0,T])\longrightarrow\mathbb{R}$ satisfying the following conditions.

Assumption 3.8. (i) *The function g is bounded, and there exist $\alpha \in (0,1]$ and a finite positive measure μ on $[0,T]$ with at most finitely many atoms such that, for all $\mathrm{x}, \mathrm{x}' \in D([0,T])$, $B = [s,t) \subset [0,T]$ and $\delta \in \mathbb{R}$,*

$$|g(\mathrm{x}) - g(\mathrm{x}')| \le \int_0^T |\mathrm{x}_s - \mathrm{x}'_s| \mu(ds), \tag{3.5}$$

$\delta' \in \mathbb{R} \mapsto g(\mathrm{x} + \delta' \mathbf{1}_B)$ is differentiable and

$$\Big| \frac{dg(\mathrm{x} + \delta \mathbf{1}_B + \mathrm{x}')}{d\delta} - \frac{dg(\mathrm{x} + \delta \mathbf{1}_B)}{d\delta} \Big| \le \Big(\int_0^T \big|\mathrm{x}'_s\big| \mu(ds) \Big)^\alpha \mu(B). \tag{3.6}$$

(ii) *for any increasing sequence $0 = t_0 < t_1 < \cdots < t_n = T$ with $\max_{i<n} |t_{i+1} - t_i|$ small enough, for all $1 \le i < j < n$, there exists $p^{i,j} \neq 0$ such that, for all $\delta \in \mathbb{R}$, and $(x_\ell)_{0 \le \ell \le n-1} \subset \mathbb{R}^n$,*

$$\begin{aligned} & g\left(\sum_{\ell=0}^{n-1} (x_\ell + \delta \mathbf{1}_{\{\ell = i\}}) \mathbf{1}_{[t_\ell, t_{\ell+1})} + \mathbf{1}_{\{T\}} x_{n-1} \right) \\ & = g\left(\sum_{\ell=0}^{n-1} (x_\ell + p^{i,j} \delta \mathbf{1}_{\{\ell \ge j\}}) \mathbf{1}_{[t_\ell, t_{\ell+1})} + \mathbf{1}_{\{T\}} (x_{n-1} + p^{i,j} \delta) \right). \end{aligned} \tag{3.7}$$

Remark 3.9. *Let*

$$g(X) \;=\; g_\circ\Big(\int_0^T X_t \mu_0(dt) \Big),$$

where $g_\circ \in C^{1+\alpha}(\mathbb{R})$ is bounded, and μ_0 is a finite positive measure with at most finitely many atoms on $[0,T]$ satisfying $\mu_0([T-h,T]) \neq 0$ for all $h > 0$ small enough. Then it satisfies Assumption 3.8.

For each $(t, \mathrm{x}) \in [0,T] \times D([0,T])$, we define

$$\mathcal{P}(t,\mathrm{x}) := \big\{ \mathbb{P} \in \mathfrak{B}(\Omega^\circ) \;: \mathbb{P}[X_{t\wedge} = \mathrm{x}_{t\wedge}] = 1, \text{ and (3.4) holds on } [t,T] \big\}, \tag{3.8}$$

and

$$V(t,\mathrm{x}) \;:=\; \sup_{\mathbb{P} \in \mathcal{P}(t,\mathrm{x})} \mathbb{E}^{\mathbb{P}}\big[g(X)\big].$$

Proposition 3.10. *Let $\mathcal{P}_0$ and g be given as above. Then, V is vertically differentiable and the duality result (3.2) holds true with the optimal superhedging strategy $H^* := \nabla_{\mathrm{x}} V(\cdot, X)$.*

Proof. First, by rewriting $\mathbb{P} \in \mathcal{P}(t,\mathrm{x})$ as solution of a controlled martingale problem, it is easy to check that the graph set $[[\mathcal{P}_0]]$ is a closed set, and satisfies the stability conditions under conditioning and concatenation (see e.g. [18, Section 4]), so that Assumption 3.4 holds true. As in Proposition 3.5, one has the dynamic programming principle (3.3), and consequently, $V(\cdot, X)$ is a $\mathbb{P}$–supermartingale for every $\mathbb{P} \in \mathcal{P}_0$.

Next, we assume that μ has possible atoms on $\{0 = T_0 < T_1 < \cdots < T_n = T\}$. Then by Propositions 3.12 and 3.13 below, together with Propositions 2.6 and 2.11, it follows that V satisfies (2.4) and the $\mathbb{C}^{0,1}$–regularity as well as other conditions required in Theorem 2.5 on each interval $[T_k + \varepsilon, T_{k+1}]$, for all $k = 0, \cdots, n-1$, and all $\varepsilon > 0$ small enough.

Recalling that $V(\cdot, X)$ is a supermartingale under each $\mathbb{P} \in \mathcal{P}_0$, it follows by Corollary 2.8 that

$$V(T_{k+1}, X) - V(T_k + \varepsilon, X) \;\le\; \int_{T_k+\varepsilon}^{T_{k+1}} H_t^* dX_t, \text{ with } H_t^* := \nabla_{\rm x} V(t, X).$$

Taking the sum on $k = 0, \cdots, n-1$ and then letting $\varepsilon \longrightarrow 0$, we can then conclude as in Proposition 3.5 to obtain the duality result (3.2) and that H^* is the optimal strategy. $\square$

Remark 3.11. *The regularity property of V is given in Propositions 3.12 and 3.13 below, which seems to be original in the literature. Moreover, it can be naturally extended to payoff functions of the form $g\big(\int_0^T {\rm x}_t \rho_1(dt), \cdots, \int_0^T {\rm x}_t \rho_m(dt)\big)$, for finitely many measures $\rho_1, \cdots, \rho_m$. We nevertheless restrict to the one measure case to make the presentation more accessible.*

Proposition 3.12. *Let Assumption 3.8 hold true. Then for all $(t, {\rm x}, {\rm x}', h) \in [0, T] \times D([0,T]) \times D([0,T]) \times \mathbb{R}_+$ with $t + h \le T$, one has*

$$|V(t+h, {\rm x}_{t\wedge}) - V(t, {\rm x})| \le \overline{\sigma} h^{\frac{1}{2}} \mu([t, T]) \text{ and } |V(t, {\rm x}') - V(t, {\rm x})| \le \int_0^t |{\rm x}'_s - {\rm x}_s| \mu(ds). \quad (3.9)$$

Proof. It suffices to observe that (3.5) implies that

$$\begin{aligned} \big|V(t+h, {\rm x}_{t\wedge}) - V(t, {\rm x})\big| &\le \sup_{\mathbb{P} \in \mathcal{P}(t,{\rm x})} \mathbb{E}^{\mathbb{P}}\Big[\int_t^T \Big|{\rm x}_t + \int_{t+h}^{s \vee (t+h)} \sigma_r dW_r^{\mathbb{P}} - {\rm x}_t - \int_t^s \sigma_r dW_r^{\mathbb{P}}\Big| \mu(ds)\Big] \\ &\le \sup_{\mathbb{P} \in \mathcal{P}(t,{\rm x})} \mathbb{E}^{\mathbb{P}}\Big[\int_t^T \Big|\int_t^{s \wedge (t+h)} \sigma_r dW_r^{\mathbb{P}}\Big| \mu(ds)\Big]. \end{aligned}$$

The second estimate is also an immediate consequence of (3.5). $\square$

We can now state the main result of this section. It can be viewed as a first result on the regularity of solutions of path-dependent PDEs, see Remark 3.14 below.

Proposition 3.13. *Let Assumption 3.8 hold true. Then the vertical derivative $\nabla_{\rm x} V(t, {\rm x})$ is well defined for all $(t, {\rm x}) \in [0, T] \times D([0, T])$, and there exists $C > 0$ such that $|\nabla_{\rm x} V(t, {\rm x})| \le C$,*

$$|\nabla_{\rm x} V(t, {\rm x}') - \nabla_{\rm x} V(t, {\rm x})| \;\le\; C\Big(\Big|\int_0^t |{\rm x}'_s - {\rm x}_s| \mu(ds)\Big|^\alpha + |{\rm x}'_t - {\rm x}_t|^\alpha\Big), \quad (3.10)$$

and

$$|\nabla_{\rm x} V(t', {\rm x}_{t\wedge}) - \nabla_{\rm x} V(t, {\rm x})| \;\le\; C\big(|t' - t|^{\frac{\alpha}{2+2\alpha}} + \mu([t, t'))\big), \quad (3.11)$$

for all $t \le t' \le T$ and ${\rm x}, {\rm x}' \in D([0, T])$.

Proof. Without loss of generality, we restrict to the collection $D_0([0, T])$ of càdlàg paths x with initial condition ${\rm x}_0 = x_0$, where $x_0 \in \mathbb{R}$ is the constant introduced above (3.4).

1. Let us consider a sequence $(\pi^n)_{n\ge1}$ of discrete time grids, dense in $[0, T]$, such that $\pi^n = (t_i^n)_{0 \le i \le n} \subset [0, T]$ and $\{0, T\} \subset \pi^n \subset \pi^{n+1}$ for all $n \ge 1$, and $\max_{0\le i\le n-1} |t_{i+1}^n - t_i^n| \longrightarrow 0$ as $n \longrightarrow \infty$. Remembering that μ has at most finitely many atoms on $[0, T]$, one can choose

$(\pi^n)_{n\geq 1}$ such that $\{t\in[0,T]:\mu(\{t\})>0\}\subset\cup_{n\geq 1}\pi^n$. Next, let us define, for all $n\geq 1$ and $(t,\mathrm{x},x)\in[0,T]\times D_0([0,T])\times\mathbb{R}$,

$$V^n(t,\mathrm{x},x) := \sup_{\mathbb{P}\in\mathcal{P}(t,\mathrm{x}\oplus_t(x-\mathrm{x}_t))}\mathbb{E}^{\mathbb{P}}\big[g\big(\Pi^n[\mathrm{x}\mathbf{1}_{[0,t^n_{i+1})}+X\mathbf{1}_{[t^n_{i+1},T]}]\big)\big] \;\text{ if } t\in[t^n_i,t^n_{i+1}),\; i\leq n-1,$$

where

$$\Pi^n[\mathrm{x}] := \sum_{i=0}^{n-1}\mathrm{x}_{t^n_i}\mathbf{1}_{[t^n_i,t^n_{i+1})}+\mathrm{x}_{t^n_{n-1}}\mathbf{1}_{\{T\}}.$$

Notice that, for $t\in[t^n_i,t^n_{i+1})$, $V^n(t,\mathrm{x},x)$ depends on (x,x) only through $(\mathrm{x}_{t^n_1},\cdots,\mathrm{x}_{t^n_i},x)$. This motivates us to introduce $\Pi^{n,i}_t:\mathbb{R}^{i+1}\to D_0([0,T])$ defined for $i<n$ by

$$\Pi^{n,i}_t(y_1,\cdots,y_i,x) \;:=\; \sum_{j=0}^{i-1}y_j\mathbf{1}_{[t^n_j,t^n_{j+1})}+y_i\mathbf{1}_{[t^n_i,t)}+x\mathbf{1}_{[t,T]},\; t\in(t^n_i,t^n_{i+1}]$$

as well as

$$g^n(y_1,\ldots,y_{n-2},x) \;:=\; g\Big(\Pi^{n,n-2}_{t^n_{n-1}}(x_0,y_1,\cdots,y_{n-2},x)\Big),$$

and

$$v^n(t,y_1,\ldots,y_i,x) \;:=\; V^n\Big(t,\Pi^{n,i}_{t^n_{i+1}}(y_1,\cdots,y_i),x\Big),\; t\in[t^n_i,t^n_{i+1}).$$

Notice that, for all $t\in[t^n_i,t^n_{i+1})$, $\mathrm{x}\in D_0([0,T])$, $x\in\mathbb{R}$,

$$V^n(t,\mathrm{x},x) \;=\; V^n(t,\bar{\mathrm{x}}^n,x) \;=\; v^n(t,\mathrm{x}_{t^n_1},\cdots,\mathrm{x}_{t^n_i},x),\quad\text{with } \bar{\mathrm{x}}^n:=\Pi^n[\mathrm{x}].$$

We further observe from (3.5) that, for all $(t,\mathrm{x})\in[0,T]\times C([0,T])$ with $t\in[t^n_{i_\circ-1},t^n_{i_\circ})$ for some $1<i_\circ\leq n$,

$$\begin{aligned}
\big|V^n(t,\bar{\mathrm{x}}^n,\mathrm{x}_t)-V(t,\mathrm{x})\big| &\leq \int_0^{t^n_{i_\circ}}|\bar{\mathrm{x}}^n_s-\mathrm{x}_s|\mu(ds)+\sum_{i=i_\circ}^{n-1}\int_{t^n_i}^{t^n_{i+1}}\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_{t^n_i}^s\sigma_r dW^{\mathbb{P}}_r\Big|\Big]\mu(ds)\\
&\leq \int_0^{t^n_{i_\circ}}|\bar{\mathrm{x}}^n_s-\mathrm{x}_s|\mu(ds)+\sum_{i=i_\circ}^{n-1}\int_{t^n_i}^{t^n_{i+1}}\overline{\sigma}(s-t^n_i)^{\frac12}\mu(ds)\\
&\leq \int_0^{t^n_{i_\circ}}|\bar{\mathrm{x}}^n_s-\mathrm{x}_s|\mu(ds)+\overline{\sigma}\Big(\max_{0\leq i\leq n-1}|t^n_{i+1}-t^n_i|^{\frac12}\Big)\mu([0,T]).
\end{aligned}$$

As μ is a finite measure on $[0,T]$, it follows that

$$\big|V^n(t,\mathrm{x},\mathrm{x}_t)-V(t,\mathrm{x})\big| = \big|V^n(t,\bar{\mathrm{x}}^n,\mathrm{x}_t)-V(t,\mathrm{x})\big|\longrightarrow 0,\quad\text{as } n\longrightarrow\infty. \tag{3.12}$$

2. Let us set

$$F:\gamma\in\mathbb{R}\;\longmapsto\;\max_{a\in[\underline{\sigma}^2,\overline{\sigma}^2]}\frac12 a\gamma \;=\; \frac12\overline{\sigma}^2\gamma^+-\frac12\underline{\sigma}^2\gamma^-. \tag{3.13}$$

Then for each $i\leq n-2$, v^n is a continuous viscosity solution of

$$\partial_t v^n(t,z)+F(D^2v^n(t,z))=0 \;\text{ for } (t,z)\in[t^n_i,t^n_{i+1})\times\mathbb{R}^{i+1}, \tag{3.14}$$

$$\lim_{t\uparrow t^n_{i+1},(y',x')\longrightarrow(y,x)}v^n(t,y',x')=v^n(t^n_{i+1},y,x,x),\;\text{ for } (y,x)\in\mathbb{R}^i\times\mathbb{R}, \tag{3.15}$$

with terminal condition

$$v^n(t^n_{n-1},\cdot) = g^n. \tag{3.16}$$

In the above, Dv^n and D^2v^n denote for the first and second order derivative with respect to the last argument of v^n. The operator F being Lipschitz, it follows from standard arguments that this system satisfies a comparison principle among (semi-continuous) bounded viscosity solutions.

Let us denote by D_ig^n the partial derivative of g^n w.r.t. the i-th argument, then by (3.5)-(3.6), for all $z, z' \in \mathbb{R}^{n-1}$,

$$\big|D_ig^n(z)\big| \le \mu\big([t^n_i, t^n_{i+1})\big), \quad \big|D_ig^n(z+z') - D_ig^n(z)\big| \le \Big(\int_0^T \big|\Pi^{n,n-2}_{t^n_{n-1}}[z']_t\big|\mu(dt)\Big)^\alpha \mu\big([t^n_i, t^n_{i+1})\big).$$

We will next regularize (g^n, F). Let $\rho_k : \mathbb{R}^{n-1} \to \mathbb{R}_+$ be a C^∞ density function with compact support, and $g^n_k := g^n * \rho_k$ be the regularized function obtained by convolution. Then it is clear that g^n_k still satisfies

$$\big|D_ig^n_k(z)\big| \le \mu\big([t^n_i, t^n_{i+1})\big), \quad \big|D_ig^n_k(z+z') - D_ig^n_k(z)\big| \le \Big(\int_0^T \big|\Pi^{n,n-2}_{t^n_{n-1}}[z']_t\big|\mu(dt)\Big)^\alpha \mu\big([t^n_i, t^n_{i+1})\big). \tag{3.17}$$

Moreover, it follows from (3.7) that for each $i < j$, there exists $p^{i,j} \neq 0$ such that, for all $z \in \mathbb{R}^{n-1}$, $\delta \in \mathbb{R}$,

$$g^n_k\big(z + \delta e^{n-1}_i\big) \;=\; g^n_k\Big(z + p^{i,j}\delta \sum_{\ell=j}^{n-1} e^{n-1}_j\Big), \tag{3.18}$$

where e^{n-1}_i denotes the i-th standard unit vector in $\mathbb{R}^{n-1}$.

Since F is a convex function, one can approximate it by a C^∞ convex function F_k, $k \ge 1$, such that

$$F_k(\gamma) = \begin{cases} \frac{1}{2}\overline{\sigma}^2\gamma, & \text{for } \gamma \ge 1, \\ \frac{1}{2}\underline{\sigma}_k^2\gamma, & \text{for } \gamma \le -1, \end{cases} \quad \text{with } \underline{\sigma}_k^2 := \underline{\sigma}^2 \vee k^{-1}.$$

Let $F_k^*(a) = \sup_{\gamma\in\mathbb{R}}(a\gamma - F_k(\gamma))$ be the Fenchel transformation of F_k, so that

$$F_k(\gamma) \;=\; \sup_{a\in[\frac{1}{2}\underline{\sigma}_k^2, \frac{1}{2}\overline{\sigma}^2]} \big(a\gamma - F_k^*(a)\big).$$

Let v^n_k be the corresponding solutions of (3.14)-(3.15)-(3.16) with parameters (F_k, g^n_k) such that $(F_k, g^n_k) \to (F, g^n)$ as $k \to \infty$. Then $v^n_k \in C^{1,3}_b$ and

$$v^n_k \longrightarrow v^n, \text{ pointwise as } k \to \infty. \tag{3.19}$$

Moreover, as F_k is convex, the associated equations on v^n_k is still a HJB equation, so that v^n_k can be considered as the value function of a control problem:

$$v^n_k(t,y,x) = \sup_{\alpha\in\mathcal{A}_k} \mathbb{E}\Big[g^n_k\big(y_1,\cdots,y_i, X^{t,x,\alpha}_{t^n_{i+1}},\cdots, X^{t,x,\alpha}_{t^n_{n-1}}\big) - \int_t^T F_k^*(\frac{1}{2}\alpha_s^2)ds\Big], \quad t \in [t^n_i, t^n_{i+1}), \tag{3.20}$$

where $X^{t,x,\alpha}_s := x + \int_t^s \alpha_r dW_r,\ s \ge t$, and $\mathcal{A}_k$ is the collection of all progressively measurable process α taking value in $[\underline{\sigma}_k, \overline{\sigma}]$ on some filtered probability space equipped with a Brownian motion W.

3. For $i \le n-2$, $t \in [t^n_i, t^n_{i+1})$, and $(\mathrm{x}, x) \in D_0([0,T]) \times \mathbb{R}$, let us set

$$V^n_k(t, \mathrm{x}, x) \ := \ v^n_k(t, \mathrm{x}_{t^n_1}, \dots, \mathrm{x}_{t^n_i}, x).$$

We claim that there exists a constant $C > 0$ such that, for all $k, n \ge 1$, $t \in [t^n_i, t^n_{i+1})$, $h \in (0, T-t]$, $\mathrm{x}, \mathrm{x}' \in D_0([0,T])$,

$$\left|DV^n_k(t, \mathrm{x}, x)\right| \ \le \ C, \tag{3.21}$$

$$\big|DV^n_k(t, \mathrm{x}', \mathrm{x}'_t) - DV^n_k(t, \mathrm{x}, \mathrm{x}_t)\big| \ \le \ C\Big(\big|\int_0^t |\bar{\mathrm{x}}'^n_s - \bar{\mathrm{x}}^n_s|\mu(ds)\big|^\alpha + \big|\mathrm{x}'_t - \mathrm{x}_t\big|^\alpha\Big), \tag{3.22}$$

$$\big|DV^n_k(t+h, \mathrm{x}_{t\wedge\cdot}, \mathrm{x}_t) - DV^n_k(t, \mathrm{x}, \mathrm{x}_t)\big| \ \le \ C\big(h^{\frac{\alpha}{2+2\alpha}} + \mu([t, t+h))\big), \tag{3.23}$$

where DV^n_k denote the derivative of V^n_k with respect to its last argument. Then, for each $t \le t' \in [0,T]$, $\mathrm{x}, \mathrm{x}' \in D_0([0,T])$ and $x, x' \in \mathbb{R}$,

$$\big|V^n_k(t, \mathrm{x}, x) - V^n_k(t, \mathrm{x}, x') - DV^n_k(t, \mathrm{x}, x)(x - x')\big| \ \le \ C\big|x - x'\big|^{1+\alpha}. \tag{3.24}$$

Next, let $\mathbb{T} := \cup_{n\ge 1}\pi^n$ and $\mathbb{Q}$ be the set of all rational numbers, so that $\mathbb{T}\times\mathbb{Q}$ is a countable dense subset of $[0,T]\times\mathbb{R}$. We then define a countable subset Q_T of $[0,T]\times D_0([0,T])\times\mathbb{R}$ by

$$Q_T \ := \ \cup_{n\ge1}\big\{(t, \mathrm{x}, x)\ : t \in \mathbb{T},\ \mathrm{x} = \Pi^n[\mathrm{x}],\ x \in \mathbb{Q},\ \mathrm{x}_s \in \mathbb{Q} \text{ for } s \in [0,T]\big\}.$$

In view of (3.21) and the convergence results (3.12) and (3.19), one can extract a subsequence $(n_\ell, k_\ell)_{\ell\ge1}$, such that, for all $(t, \mathrm{x}, x) \in Q_T$,

$$\big(V^{n_\ell}_{k_\ell}(t, \mathrm{x}, \mathrm{x}_t), DV^{n_\ell}_{k_\ell}(t, \mathrm{x}, x)\big) \ \longrightarrow \ \big(V(t, \mathrm{x}), DV(t, \mathrm{x}, x)\big), \text{ as } \ell \longrightarrow \infty,$$

for some function $DV : Q_T \longrightarrow \mathbb{R}$. Moreover, by (3.21)-(3.22)-(3.23) and (3.24), DV satisfies

$$\big|V(t, \mathrm{x}\oplus_t x) - V(t, \mathrm{x}\oplus_t x') - DV(t, \mathrm{x}, x)(x - x')\big| \ \le \ C|x - x'|^{1+\alpha},$$

and

$$\begin{aligned} &\big|DV(t, \mathrm{x}, x) - DV(t', \mathrm{x}', x')\big| \\ \le \ & C\Big(\Big|\int_{[0,t')}\big|\mathrm{x}_{t\wedge s} - \mathrm{x}'_s\big|\mu(ds)\Big|^\alpha + \big|x - x'\big|^\alpha + |t - t'|^{\frac{\alpha}{2+2\alpha}} + \mu([t, t'))\Big), \end{aligned} \tag{3.25}$$

for all $(t, \mathrm{x}, x), (t', \mathrm{x}', x') \in Q_T$ such that $t \le t'$. Notice that under the distance

$$\rho\big((t, \mathrm{x}, x), (t', \mathrm{x}', x')\big) := \int_{[0,t')}\big|\mathrm{x}_{t\wedge s} - \mathrm{x}'_s\big|\mu(ds) + \big|x - x'\big| + |t - t'| + \mu([t, t')), \text{ when } t \le t',$$

Q_T is a dense subset of $[0,T]\times D_0([0,T])\times\mathbb{R}$. Then, by continuity of V and DV, recall (3.9) and (3.25), one can extend the definition of DV to $[0,T]\times D_0([0,T])\times\mathbb{R}$ in such a way that $DV(t, \mathrm{x}, \mathrm{x}_t) = \nabla_{\mathrm{x}} V(t, \mathrm{x})$ for all $(t, \mathrm{x}) \in [0, T] \times D_0([0, T])$, and $\nabla_{\mathrm{x}} V$ is uniformly bounded and satisfies (3.10)-(3.11).

4. It remains to prove (3.21)-(3.22)-(3.23).

a. We start by proving (3.21). Recall that $v_k^n \in C_b^{1,3}$. Let us denote by $\phi_k^{n,j}$ the derivative of v_k^n in its j-th space argument. For all $i \le n-2$ and $j \le i+1$, it solves

$$\partial_t \phi_k^{n,j}(t, z) + F_k'(D^2 v_k^n(t, z)) D^2 \phi_k^{n,j}(t, z) = 0, \quad (t, z) \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^{i+1}, \tag{3.26}$$

with the boundary condition

$$\lim_{t' \uparrow t_{i+1}^n} \phi_k^{n,j}(t, y, x) = \phi_k^{n,j}(t_{i+1}^n, y, x, x) + \mathbf{1}_{\{j=i+1\}} \phi_k^{n,i+2}(t_{i+1}^n, y, x, x), \quad (y, x) \in \mathbb{R}^i \times \mathbb{R}, \tag{3.27}$$

where

$$\phi_k^{n,j}(t_{n-1}^n, \cdot) = D_j g_k^n, \quad \text{for } j \le n-1. \tag{3.28}$$

As $x \in \mathbb{R} \mapsto F_k'(D^2 v_k^n(t, y, x))$ is Lipschitz, it follows from the Feynman-Kac formula that, for all $t \in [t_i^n, t_{i+1}^n)$, $x \in \mathbb{R}^{i+1}$ and $j \le i+1$,

$$\phi_k^{n,j}(t, z) \;=\; \mathbb{E}\Big[\Big(D_j g_k^n + \mathbf{1}_{\{j=i+1\}} \sum_{j'=i+2}^{n-1} D_{j'} g_k^n\Big)\big(\Pi^n(Y^{t,z})\big)\Big],$$

for some process $Y^{t,z}$. Then the first inequality in (3.17) implies that $|\phi_k^{n,j}| \le \mu([0, T])$.

b. We now prove (3.22). Let us fix $i \le n-2$, $j, \ell \le i+1$, $z' \in \mathbb{R}^\ell$ and then define, for $t \in [t_i^n, t_{i+1}^n)$, $z \in \mathbb{R}^{i+1}$,

$$\psi_k^{n,j,[z']_\ell^{i+1}}(t, z) \;:=\; \Big(\phi_k^{n,j}\big(t, z + [z']_\ell^{i+1}\big) - \phi_k^{n,j}(t, z)\Big),$$

where $[z']_\ell^{i+1} = (z_1', \cdots, z_\ell', 0, \cdots 0) \in \mathbb{R}^{i+1}$. Using (3.26), one obtains that,

$$\begin{aligned} 0 \;=\; & \partial_t \psi_k^{n,j,[z']_\ell^{i+1}}(t, z) + F_k'(D^2 v_k^n(t, z + [z']_\ell^{i+1})) D^2 \psi_k^{n,j,[z']_\ell^{i+1}}(t, z) \\ & + \; F_k''\big(A^{[z']_\ell^{i+1}}(t, z)\big) D^2 \phi_k^{n,j}(t, z) D\psi_k^{n,i+1,[z']_\ell^{i+1}}(t, z), \quad (t, z) \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^{i+1}, \end{aligned}$$

in which $A^{[z']_\ell^{i+1}}$ is a continuous function. We now observe that (3.18)-(3.20) imply that

$$\phi_k^{n,j}(t, \cdot) = \phi_k^{n,i+1}(t, \cdot) p^{j,i+1}$$

for some $p^{j,i+1} \neq 0$. Hence, $\psi_k^{n,j,[z']_\ell^{i+1}}$ satisfies the PDE

$$\begin{aligned} 0 \;=\; & \partial_t \psi_k^{n,j,[z']_\ell^{i+1}}(t, z) + \; F_k'\big(D^2 v_k^n(t, z + [z']_\ell^{i+1})\big) D^2 \psi_k^{n,j,[z']_\ell^{i+1}}(t, z) \\ & + \; \frac{1}{p^{j,i+1}} \Big[F_k''(A^{[z']_\ell^{i+1}}(t, z)) D^2 \phi_k^{n,j}(t, z)\Big] D\psi_k^{n,j,[z']_\ell^{i+1}}(t, z), \quad (t, z) \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^{i+1}, \end{aligned}$$

and, by (3.27)-(3.28),

$$\begin{aligned} \lim_{t' \uparrow t_{i+1}^n} \psi_k^{n,j,[z']_\ell^{i+1}}(t, y, x) = & \mathbf{1}_{\{\ell < i+1\}} \Big(\psi_k^{n,j,[z']_\ell^{i+2}} + \mathbf{1}_{\{j=i+1\}} \psi_k^{n,i+2,[z']_\ell^{i+2}}\Big)(t_{i+1}^n, y, x, x) \\ & + \mathbf{1}_{\{\ell=i+1\}} \Big(\psi_k^{n,j,[[z']]_{\ell+1}} + \mathbf{1}_{\{j=i+1\}} \psi_k^{n,i+2,[[z']]_{\ell+1}}\Big)(t_{i+1}^n, y, x, x), \end{aligned}$$

for $(y,x)\in\mathbb{R}^i\times\mathbb{R}$, and

$$\psi_k^{n,j,z''}(t_{n-1}^n,\cdot) \;=\; \Delta_{z''}D_jg_k^n \;:=\; \big(D_jg_k^n(\cdot+z'')-D_jg_k^n\big),\;\text{ for all } z''\in\mathbb{R}^{n-1},$$

in which

$$[[z']]_{\ell+p} \;:=\; (z'_1,\cdots,z'_\ell,z'_\ell,\cdots,z'_\ell)\in\mathbb{R}^{\ell+p},\; p\geq 1.$$

Then one can apply the Feynman-Kac formula to find a process $\tilde Y^{t,z}$ such that

$$\begin{aligned}\psi_k^{n,j,[z']_\ell^{i+1}}(t,z) =&\mathbf{1}_{\{\ell<i+1\}}\mathbb{E}\Big[\Big(\Delta_{[z']_\ell^{n-1}}D_jg_k^n+\mathbf{1}_{\{j=i+1\}}\sum_{j'=i+2}^{n-1}\Delta_{[z']_\ell^{n-1}}D_{j'}g_k^n\Big)\big(\Pi^n[\tilde Y^{t,z}]\big)\Big]\\ &+\mathbf{1}_{\{\ell=i+1\}}\mathbb{E}\Big[\Big(\Delta_{[[z']]_{\ell+n_\ell}}D_jg_k^n+\mathbf{1}_{\{j=i+1\}}\sum_{j'=i+2}^{n-1}\Delta_{[[z']]_{\ell+n_\ell}}D_{j'}g_k^n\Big)\big(\Pi^n[\tilde Y^{t,z}]\big)\Big],\end{aligned}$$

with $n_\ell:=n-1-\ell$. In view of the second inequality in (3.17), this concludes the proof of (3.22).

c. We finally prove (3.23). In view of the representation of v_k^n as the value function of an optimal control problem in (3.20), one can apply exactly the same arguments as in Proposition 3.12, together with (3.17), to obtain that, for all $t\leq t'\in[0,T]$ and $\mathrm{x},\mathrm{x}'\in D_0([0,T])$,

$$\big|V_k^n(t',\mathrm{x}_{t\wedge},\mathrm{x}_t)-V_k^n(t,\mathrm{x},\mathrm{x}_t)\big| \;\leq\; \overline{\sigma}|t'-t|^{\frac12}\mu([t,T]),$$

and

$$\big|V_k^n(t,\mathrm{x}',\mathrm{x}_t)-V_k^n(t,\mathrm{x},\mathrm{x}_t)\big| \;\leq\; \int_0^t\big|\bar{\mathrm{x}}_s^{\prime n}-\bar{\mathrm{x}}_s^n\big|\mu(ds).$$

Let us set $f:=DV_k^n$. It follows from the above estimations, together with (3.22), that,

$$\begin{aligned}&\big|f(t+h^{2+2\alpha},\mathrm{x}_{t\wedge},\mathrm{x}_t)-f(t,\mathrm{x},\mathrm{x}_t)\big|\\ &\leq h^{-1}\big|V_k^n(t+h^{2+2\alpha},\mathrm{x}_{t\wedge},\mathrm{x}_t+h)-V_k^n(t+h^{2+2\alpha},\mathrm{x}_{t\wedge},\mathrm{x}_t)-V_k^n(t,\mathrm{x},\mathrm{x}_t+h)+V_k^n(t,\mathrm{x},\mathrm{x}_t)\big|\\ &\quad+2Ch^\alpha\\ &\leq h^{-1}\big|V_k^n(t+h^{2+2\alpha},(\mathrm{x}\oplus_t h)_{t\wedge},\mathrm{x}_t+h)-V_k^n(t,\mathrm{x},\mathrm{x}_t+h)\big|\\ &\quad+h^{-1}\big|V_k^n(t,\mathrm{x},\mathrm{x}_t)-V_k^n(t+h^{2+2\alpha},\mathrm{x}_{t\wedge},\mathrm{x}_t)\big|\\ &\quad+h^{-1}\big|V_k^n(t+h^{2+2\alpha},\mathrm{x}_{t\wedge},\mathrm{x}_t+h)-V_k^n(t+h^{2+2\alpha},(\mathrm{x}\oplus_t h)_{t\wedge},\mathrm{x}_t+h)\big|\\ &\quad+2Ch^\alpha\\ &\leq 2Ch^\alpha+2\overline{\sigma}h^\alpha\mu([0,T])+\mu([t,t+h^{2+2\alpha})),\end{aligned}$$

for all $\mathrm{x}\in C_{x_0}([0,T])$ and $t<t+h\leq T$. This concludes the proof of (3.23). □

Remark 3.14. *Note that the same proof would go through if F, recall (3.13), was affine in place of assuming (3.7). In this case, the term F_k'' in Step 4.b. of the proof of Proposition 3.13 would simply be zero. This corresponds to the case where $\underline{\sigma}=\overline{\sigma}$. If we assume (3.7), then one can replace F in the proof by any convex function, say growing linearly at infinity. The specific definition of F does not play any role. Then, Proposition 3.13 can be seen as a first result on the regularity of path-dependent PDEs of the form*

$$\partial_t\mathrm{v}+F(\nabla_{\mathrm{x}}^2\mathrm{v})=0,$$

in the case where F is convex. Although, we do not define here the notion of solution explicitly, such PDEs are associated to optimal control problems for which one can, for instance, appeal to the notions of solution of [9]-[15]-[16], see also the references therein. We hope that this first step will open the door to the study of more general equations.

References

[1] Elena Bandini and Francesco Russo. Weak dirichlet processes with jumps. *Stochastic Processes and their Applications*, 127(12):4139–4189, 2017.

[2] Bruno Bouchard and Jean-François Chassagneux. *Fundamentals and advanced techniques in derivatives hedging.* Springer, 2016.

[3] Bruno Bouchard and Xiaolu Tan. Understanding the dual formulation for the hedging of path-dependent options with price impact. *arXiv preprint arXiv:1912.03946*, 2019.

[4] Bruno Bouchard and Xiaolu Tan. A quasi-sure optional decomposition and super-hedging result on the skorokhod space. *arXiv preprint arXiv:2004.11105*, 2020.

[5] John MC Clark. The representation of functionals of brownian motion by stochastic integrals. *The Annals of Mathematical Statistics*, pages 1282–1295, 1970.

[6] Rama Cont and David-Antoine Fournié. Functional itô calculus and stochastic integral representation of martingales. *The Annals of Probability*, 41(1):109–133, 2013.

[7] François Coquet, Adam Jakubowski, Jean Mémin, and Leszek Słominski. Natural decomposition of processes and weak dirichlet processes. In *In Memoriam Paul-André Meyer*, pages 81–116. Springer, 2006.

[8] Andrea Cosso and Francesco Russo. A regularization approach to functional itô calculus and strong-viscosity solutions to path-dependent pdes. *arXiv preprint arXiv:1401.5034*, 2014.

[9] Andrea Cosso and Francesco Russo. Crandall-lions viscosity solutions for path-dependent pdes: The case of heat equation. *arXiv preprint arXiv:1911.13095*, 2019.

[10] Freddy Delbaen and Walter Schachermayer. *The mathematics of arbitrage.* Springer Science & Business Media, 2006.

[11] Laurent Denis and Claude Martini. A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. *The Annals of Applied Probability*, 16(2):827–852, 2006.

[12] Bruno Dupire. Functional itô calculus. *Portfolio Research Paper*, 04, 2009.

[13] Ibrahim Ekren, Christian Keller, Nizar Touzi, Jianfeng Zhang, et al. On viscosity solutions of path dependent pdes. *The Annals of Probability*, 42(1):204–236, 2014.

[14] Ibrahim Ekren, Nizar Touzi, Jianfeng Zhang, et al. Viscosity solutions of fully nonlinear parabolic path dependent pdes: Part i. *The Annals of Probability*, 44(2):1212–1253, 2016.

[15] Ibrahim Ekren, Nizar Touzi, Jianfeng Zhang, et al. Viscosity solutions of fully nonlinear parabolic path dependent pdes: Part i. *Annals of Probability*, 44(2):1212–1253, 2016.

[16] Ibrahim Ekren, Nizar Touzi, Jianfeng Zhang, et al. Viscosity solutions of fully nonlinear parabolic path dependent pdes: Part ii. *Annals of Probability*, 44(4):2507–2553, 2016.

[17] Nicole El Karoui and Xiaolu Tan. Capacities, measurable selection and dynamic programming part i: abstract framework. *arXiv preprint arXiv:1310.3363*, 2013.

[18] Nicole El Karoui and Xiaolu Tan. Capacities, measurable selection and dynamic programming part ii: application in stochastic control problems. *arXiv preprint arXiv:1310.3364*, 2013.

[19] Fausto Gozzi and Francesco Russo. Weak dirichlet processes with a stochastic control perspective. *Stochastic Processes and their Applications*, 116(11):1563–1583, 2006.

[20] Ulrich G. Haussmann. Functionals of ito processes as stochastic integrals. *SIAM Journal on Control and Optimization*, 16(2):252–269, 1978.

[21] Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer Science & Business Media, 2013.

[22] Ariel Neufeld and Marcel Nutz. Superreplication under volatility uncertainty for measurable claims. *Electronic journal of probability*, 18, 2013.

[23] Zhenjie Ren and Xiaolu Tan. On the convergence of monotone schemes for path-dependent pdes. *Stochastic Processes and their Applications*, 127(6):1738–1762, 2017.

[24] Zhenjie Ren, Nizar Touzi, and Jianfeng Zhang. An overview of viscosity solutions of path-dependent pdes. In *Stochastic Analysis and Applications 2014*, pages 397–453. Springer, 2014.

[25] Francesco Russo and Pierre Vallois. Forward, backward and symmetric stochastic integration. *Probability theory and related fields*, 97(3):403–421, 1993.

[26] Francesco Russo and Pierre Vallois. The generalized covariation process and itô formula. *Stochastic Processes and their applications*, 59(1):81–104, 1995.

[27] Francesco Russo and Pierre Vallois. Ito formula for $c1$-functions of semimartingales. *Probability theory and related fields*, 104(1):27–41, 1996.

[28] Francesco Russo and Pierre Vallois. Elements of stochastic calculus via regularization. In *Séminaire de Probabilités XL*, pages 147–185. Springer, 2007.

[29] Yuri F. Saporito. The functional meyer–tanaka formula. *Stochastics and Dynamics*, 18(04):1850030, 2018.

[30] Jianfeng Zhang and Jia Zhuo. Monotone schemes for fully nonlinear parabolic path dependent pdes. *Journal of Financial Engineering*, 1(01):1450005, 2014.