Itô-Dupire's formula for $\mathbb{C}^{0,1}$ path-dependent functionals and approximate solutions of PPDEs

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Based on works with
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Initial motivation

☐ B. and Tan [4] : Solve a second order BSDE related to a (perfect) hedging problem under price impact

$$X_t = x_0 + \int_0^t \sigma_s(X, \mathfrak{g}_s) dW_s$$

$$Y_t = \Phi(X) - \int_t^T F_s(X, \mathfrak{g}_s) ds - \int_0^t Z_s dX_s \ \text{ and } \ Z_t = Z_0 + \int_0^t \mathfrak{g}_s dX_s - \mathfrak{B}_t.$$

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□ Derive a solution from a dual formulation of the form :

$$\mathrm{v}(t, \mathrm{x}) := \sup_\alpha \mathbb{E}\Big[\Phi\big(\bar{X}^{t,\mathrm{x},\alpha}\big) - \int_t^T G_s\big(\bar{X}^{t,\mathrm{x},\alpha}, \alpha_s\big) ds\Big],$$

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□ If v is Markovian : just need concave in space and decreasing in time or $C^{0,1}$ (cf. e.g. Gozzi and Russo [12], or Bandini and Russo [1]).

Dupire-Itô's formula : concave case

Definitions

- Notations
  - $\mathrm{x}_{t\wedge} := (\mathrm{x}_{t\wedge s})_{s\in[0,T]}$, (optional) stopped path.
  - $\mathrm{x} \oplus_t y := \mathrm{x} + y\mathbf{1}_{[t,T]}$ and $\mathrm{x} \boxplus_t y := \mathrm{x}\mathbf{1}_{[0,t)} + y\mathbf{1}_{[t,T]}$

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- If v is **Dupire-concave**, one can define its super-differential

$$\partial\mathrm{v}(t, \mathrm{x}) \;:=\; \left\{z : \mathrm{v}(t, \mathrm{x} \oplus_t y) \leq \mathrm{v}(t, \mathrm{x}) + z \cdot y, \;\forall\, y\right\}.$$

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in which $\{C^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}\}$ is a collection of non-decreasing processes and $H_s \in \partial \mathrm{v}(s, X^{s-})$ for all $s \in [0,T]$, $\mathcal{P}$-q.s, where

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$\Rightarrow$ **Solves our hedging problem with price impact.**

Application to robust super-hedging

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such that

$$\big|\Phi(\mathrm{x})\big| \ \le \ K\Big(1 + \mathrm{x}_T + \int_0^T \mathrm{x}_t |\mu|(dt)\Big), \quad \text{for all } \mathrm{x} \in D([0,T]),$$

and, for all $0 \le M_0 \le M_1$, $0 \le m_1 \le w_1 \wedge \varepsilon$ and $a_0, a_1 \in \mathbb{R}$,

$$\Big|\phi(M_1, m_1, a_1, w_1) - \phi(M_0, 0, a_0, 0)\Big| \le K\big(|a_1 - a_0| + w_1\big).$$

□ **Theorem :** Let $\mathcal{A}$ be the collection of all locally bounded $\mathbb{F}$-predictable processes H such that $\int_0^{\cdot} H_r dX_r$ is $\mathbb{Q}$-a.s. bounded from below by a $\mathbb{Q}$-martingale, for all $\mathbb{Q} \in \mathcal{M}^+(0, x)$.

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$$\mathrm{v}(0,\mathrm{x}) := \sup_{\mathbb{Q} \in \mathcal{M}^+(0,\mathrm{x})} \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big]$$

$$= \inf\Big\{ v \in \mathbb{R} \ : \exists H \in \mathcal{A} \text{ s.t. } v + \int_0^T H_r dX_r \geq \Phi(X), \ \mathcal{M}^+(0,x) - \mathrm{q.s.} \Big\}$$

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□ Remark : Similar to Guo, Tan and Touzi [13] but under more restrictive continuity assumptions + explicit representation of H. Comparing with other works, we allow for jumps without any dominating assumption (compare with Nutz [14]).

Dupire-Itô's formula : the $\mathbb{C}^{0,1}$-case

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- Horizontal derivative :

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- Vertical derivative :

$$\nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x}) \;:=\; \lim_{y \to 0} \frac{\mathrm{v}(t, \mathrm{x} \oplus_t y) - \mathrm{v}(t, \mathrm{x})}{y}.$$

□ Regularity

- $\mathrm{v} \in \mathbb{C}(\Theta)$ if is continuous.
- $\mathrm{v} \in \mathbb{C}_l(\Theta)$ if $\forall\ (t, \mathrm{x}) \in \Theta$ and $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$t' \leq t,\ |t - t'| + \|\mathrm{x}_{t\wedge} - \mathrm{x}'_{t'\wedge}\| \leq \delta \implies |\mathrm{v}(t, \mathrm{x}) - \mathrm{v}(t', \mathrm{x}')| \leq \varepsilon.$$

- $\mathbb{C}^{0,1}(\Theta)$: $\mathrm{v} \in \mathbb{C}(\Theta)$ and $\nabla_{\mathrm{x}}\mathrm{v} \in \mathbb{C}_l(\Theta)$.
- $\mathbb{C}^{1,2}(\Theta)$: $\mathrm{v} \in \mathbb{C}^{0,1}(\Theta)$, $\partial_t \mathrm{v}$ and $\nabla^2_{\mathrm{x}}\mathrm{v}$ belong to $\mathbb{C}_l(\Theta)$.

Reminder : Dupire-Itô's formula

□ Assume that $\mathrm{v} \in \mathbb{C}^{1,2}(\Theta)$ and that $X = M + A$ is a continuous semi-martingale, then

$$\mathrm{v}(t, X_t) \quad = \quad \mathrm{v}(0, X_0) + \int_0^t \nabla_{\mathrm{x}} \mathrm{v}(s, X_s) dM_s + \Gamma_t,$$

in which

$$\Gamma_t := \int_0^t \partial_t \mathrm{v}(s, X_s) ds + \int_0^t \nabla_{\mathrm{x}} \mathrm{v}(s, X_s) dA_s + \frac{1}{2} \int_0^t \nabla_{\mathrm{x}}^2 \mathrm{v}(s, X_s) d[X]_s.$$

See Dupire [9] and Cont and Fournié [7] + versions for càdlàg processes.

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□ Remark : See also Saporito [16] for a Meyer-Tanaka type formula assuming $\mathbb{C}^1$-regularity in time.

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□ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [12].

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Definitions :

- Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t \ := \ \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

whenever the limit exists in the sense of u.c.p.

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- X is a **weak Dirichlet process** if $X = X_0 + M + A$, where M is a local martingale and A is orthogonal such that $M_0 = A_0 = 0$.

Stability of Weak Dirichlet processes

Theorem [B., Loeper and Tan [3]] : Let $X = M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $\mathrm{v} \in \mathbb{C}^{0,1}$ such that v and $\nabla_{\mathrm{x}}\mathrm{v}$ are locally uniformly continuous and $[\cdots]$. Then,

$$\mathrm{v}(t, X) \;=\; \mathrm{v}(0, X) \;+\; \int_0^t \nabla_{\mathrm{x}}\mathrm{v}(s, X) dM_s \;+\; \Gamma_t, \quad t \in [0, T],$$

where Γ is a continuous orthogonal process, **if and only if**

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Remark : The condition (1) holds as soon as (for instance)

$$|\mathrm{v}(t, \mathrm{x}) - \mathrm{v}(t, \mathrm{x}')| \leq C \int_0^t |\mathrm{x}_s - \mathrm{x}'_s| d\mu_s$$

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□ Similar result for càdlàg processes (B. and Vallet [6]).

Application to robust super-hedging : toy model

- Let us consider a payoff function of the form

$$g(X) \;=\; g_\circ\Big(\int_0^T X_t \mu(dt)\Big),\; g_\circ \in C^{1+\alpha}(\mathbb{R}).$$

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where $\mathcal{P}(t,\mathrm{x}) := \big\{\mathbb{P}\;:\mathbb{P}[X_{t\wedge} = \mathrm{x}_{t\wedge}] = 1,\;$ and (2) holds on $[t, T]\big\}$.

- Ends up to showing that the PPDE

$$-\partial_t \mathrm{v} - \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \frac{\sigma^2}{2}\nabla_{\mathrm{x}}^2 \mathrm{v} = 0,\;\; \mathrm{v}(T,\cdot) = g$$

admits a $\mathbb{C}^{0,1}$-solution with $\nabla_{\mathrm{x}}\mathrm{v}$ locally uniformly continuous.

Approximate viscosity solutions of PPDEs

$$-\partial_t\varphi(t,\mathrm{x}) - F(t,\mathrm{x},\varphi(t,\mathrm{x}),\nabla_{\mathrm{x}}\varphi(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\varphi(t,\mathrm{x})) = 0,\ \varphi(T,\cdot) = g$$

B., Loeper and Tan [2].

Definition of solutions by approximation

□ Let $\pi = (\pi^n)_n$, with $\pi^n = (t_i^n)_{0\leq i\leq n}$, be an increasing sequence of time grids. Set

$$\bar{\mathrm{x}}^n := \sum_{i=0}^{n-1} \mathrm{x}_{t_i^n} \mathbf{1}_{[t_i^n, t_{i+\mathbf{1}}^n)} + \mathrm{x}_{t_n^n} \mathbf{1}_{\{T\}}$$

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□ We say that a continuous function v^n is a π^n-**viscosity solution** of

$$-\partial_t \varphi(t,\mathrm{x}) - F(t,\mathrm{x},\varphi(t,\mathrm{x}),\nabla_{\mathrm{x}}\varphi(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\varphi(t,\mathrm{x})) = 0 \ \forall \ t < T$$

if it is of the form

$$\sum_{i=0}^{n-1} \mathbf{1}_{[t_i^n, t_{i+1}^n)} v_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x)$$

in which each $v_i^n(\cdot, \bar{\mathrm{x}}^n_{\wedge t_i^n}, \cdot)$ is a viscosity solution on $\mathbb{R}^d \times [t_i^n, t_{i+1}^n)$ of

$$-\partial_t v_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x) - F(t,\bar{\mathrm{x}}^n_{\wedge t_i^n}, v_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x), Dv_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x), D^2 v_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x)) = 0$$

$$v_i^n(t_{i+1}^n-, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x) = v_{i+1}^n(t_{i+1}^n, \bar{\mathrm{x}}^n_{\wedge t_i^n} \boxplus_{t_{i+1}^n} x, x)$$

□ We say that v is a **π-approximate-viscosity solution** on $D([0,T])$ of

$$-\partial_t \mathrm{v}(t,\mathrm{x}) - F(t,\mathrm{x},\mathrm{v}(t,\mathrm{x}),\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\mathrm{v}(t,\mathrm{x})) = 0\ ,\ \ t < T$$

with terminal condition

$$\mathrm{v}(T,\cdot) = g$$

if $v^n(t,\mathrm{x},\mathrm{x}_t) \to \mathrm{v}(t,\mathrm{x})$ for all $(t,\mathrm{x}) \in [0,T] \times D([0,T])$ where $(v^n)_n$ is the sequence defined as above with

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□ Typical examples : Semi-linear PPDEs or HJB equations.
$\Rightarrow$ In both cases, amounts to replacing X by $\bar{X}^n$ in the coefficients and payoff.

But we also want to consider general non-linear parabolic PPDEs.

Existence, comparison, stability

- We focus on the case where

$$F(t, \mathrm{x}, r, p, q) = H(t, \mathrm{x}, r, p, q) + \rho(t, \mathrm{x})r + b(t, \mathrm{x})p + \frac{1}{2}\sigma^2(t, \mathrm{x})q$$

where all the coefficients are continuous and Lipschitz/uniformly continuous in space $[\cdots]$ + standard assumptions to have comparison and existence of a viscosity solution with linear growth in finite dimension (for the $F(\cdot, \bar{\mathrm{x}}^n_{\wedge t_i^n}, \cdot)$).

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Theorem : Let g be uniformly continuous, then $\exists$ a unique π-approximate viscosity solution v on $D([0,T])$.

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- If π' is another increasing sequence of time grids and if v' is the π'-approximate viscosity solution, then $\mathrm{v}' = \mathrm{v}$.

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Proposition : Comparison and stability holds in the class of solutions.

Remark : We have precise estimates on the approximation error $|v^n(t, \mathrm{x}, \mathrm{x}_t) - \mathrm{v}(t, \mathrm{x})|$ (depending on the regul. of x).

Regularity in the fully non-linear case

- For terminal conditions of the form (can be made more abstract)

$$g(\mathrm{x}) \;=\; g_\circ\Big(\int_0^T \mathrm{x}_t\mu(dt)\Big),$$

where $g_\circ \in C^{1+\alpha}(\mathbb{R})$ is bounded, and μ is a finite positive measure with at most finitely many atoms on $[0, T]$.

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- Two cases

(a) Either $\alpha \in (0,1)$ and $F(t,\mathrm{x},y,z,\gamma) = F_1(t)y + F_2(t)z + F_3(t,\gamma)$,

(b) Or $\alpha = 1$ and $F(t,\mathrm{x},y,z,\gamma) = F_1(t,y,\gamma) + F_2(t)z$ with $y \in \mathbb{R} \mapsto F_1(t,y,\gamma) \in C^1$ with bounded and Lipschitz first order derivative, uniformly in $\gamma \in \mathbb{R}$ and $t \leq T$.

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- In any case $\gamma \mapsto F(\cdot,\gamma)$ is concave or $d \le 2$.

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Theorem : $\nabla_{\mathrm{x}} \mathrm{v}$ is well-defined and locally uniformly continuous.

Regularity in the semi-linear case

□ Consider the case where

$$F(t, \mathrm{x}, r, p, q) = f(t, \mathrm{x}, r, p\sigma(t, \mathrm{x})) + b(t, \mathrm{x})p + \frac{1}{2}\sigma^2(t, \mathrm{x})q$$

with

- f, b and σ are Fréchet differentiable with $|\mu_f|$, $|\mu_b|$ and $|\mu_\sigma|$ dominated by a bounded non-negative measure μ.
- f is C^1_b in p, uniformly.
- g is Fréchet differentiable with $|\mu_g|$ dominated by μ .
- $(\mathrm{x}, r, p) \mapsto (\mu_g(\cdot; \mathrm{x}), \mu_b(\cdot; t, \mathrm{x}), \mu_\sigma(\cdot; t, \mathrm{x}), \mu_f(\cdot; t, \mathrm{x}, r, p))$ and well as $(\partial_r, \partial_p) f(t, \cdot)$ are uniformly α-Hölder, uniformly in $t \leq T$.

Regularity in the semi-linear case

☐ Consider the case where

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Theorem : Under the above conditions, $\nabla_{\mathrm{x}}\mathrm{v}$ is well-defined. It is locally uniformly α-Hölder in space (+ uniform continuity in time apparts from atoms of μ).

- In particular, the solution v satisfies

$$\begin{aligned} \mathrm{v}(t,X) =& g(X) + \int_t^T f(s,\mathrm{x},\mathrm{v}(s,X),\nabla_{\mathrm{x}}\mathrm{v}(s,X)\sigma(s,X))ds \\ & - \int_t^T \nabla_{\mathrm{x}}\mathrm{v}(s,X)\sigma(s,X)dW_s, \\ X_t =& X_0 + \int_0^t b(s,X)ds + \int_0^t \sigma(s,X)dW_s. \end{aligned}$$

Related works on solutions of PPDE

- Ekren, Touzi, Zhang [10] : test function in terms of optimal stopping and non-linear expectations, instead of tangent time-space points. Very weak notion. Essentially requires concavity or $d \leq 2$ + uniform ellipticity and continuity.

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- Cosso and Russo [8] : test functions in the more classical spirit of tangent points $\min\{(v - \varphi)(t, \mathrm{x}),\ (t, \mathrm{x}) \in [0; T] \times \Omega\}$ (or max). Introduce an approach based on Gauge functions for comparison, that still needs to be worked out.

Thank you !

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