# Simple bounds for transaction costs

B. Bouchard

Ceremade - Univ. Paris-Dauphine, PSL University

Joint work with J. Muhle-Karbe (Carnegie Mellon University)

# Problem formulation

- $S$ : $d$-dimensional continuous semimartingale.
- Frictionless market

$$X^\theta := X_0 + \int_0^\cdot \theta_s^\top dS_s.$$

# Problem formulation

- $S$ : $d$-dimensional continuous semimartingale.
- Frictionless market

$$X^{\theta} := X_0 + \int_0^{\cdot} \theta_s^{\top} dS_s.$$

- Market with transaction costs (on volumes for the moment)

$$X^{\vartheta,\varepsilon} := X_0 + \int_0^{\cdot} \vartheta_s^{\top} dS_s - \varepsilon \int_0^{\cdot} d|\vartheta|_s - \mathbf{1}_{\{T\}} \varepsilon |\vartheta_T|.$$

# Problem formulation

□ $S$ : $d$-dimensional continuous semimartingale.

□ Frictionless market

$$X^{\theta} := X_0 + \int_0^{\cdot} \theta_s^{\top} dS_s.$$

□ Market with transaction costs (on volumes for the moment)

$$X^{\vartheta,\varepsilon} := X_0 + \int_0^{\cdot} \vartheta_s^{\top} dS_s - \varepsilon \int_0^{\cdot} d|\vartheta|_s - \mathbf{1}_{\{T\}}\varepsilon|\vartheta_T|.$$

□ Compare $X^{\theta}$ and $X^{\vartheta,\varepsilon}$ in terms of an $\mathbf{L}_p$ norm or in terms of expected utility. In particular, compare

$$\sup_{\theta} \mathbb{E}[U(X_T^{\theta})] \quad \text{and} \quad \sup_{\vartheta} \mathbb{E}[U(X_T^{\vartheta,\varepsilon})]$$

# Expected results

- Balance between deviating from $\theta$ and paying transaction costs (local time control).

# Expected results

□ Balance between deviating from $\theta$ and paying transaction costs (local time control). If we stay at distance $\delta$, the local time total variation should be of order $1/\delta$ (by scaling of the Brownian motion).

# Expected results

- Balance between deviating from $\theta$ and paying transaction costs (local time control). If we stay at distance $\delta$, the local time total variation should be of order $1/\delta$ (by scaling of the Brownian motion).

- **$\mathbf{L}_p$-bounds**

$$\|X_t^{\theta} - X_t^{\vartheta,\varepsilon}\|_{\mathbf{L}_p} \leq C\,\varepsilon^{\frac{1}{2}}, \quad \text{for } \delta \sim \varepsilon^{\frac{1}{2}}.$$

# Expected results

- Balance between deviating from $\theta$ and paying transaction costs (local time control). If we stay at distance $\delta$, the local time total variation should be of order $1/\delta$ (by scaling of the Brownian motion).

- **$\mathbf{L}_p$-bounds**

$$\|X_t^\theta - X_t^{\vartheta,\varepsilon}\|_{\mathbf{L}_p} \leq C\,\varepsilon^{\frac{1}{2}}, \quad \text{for } \delta \sim \varepsilon^{\frac{1}{2}}.$$

- **Expected utility bounds**

$$|\sup_\theta \mathbb{E}[U(X_T^\theta)] - \sup_\vartheta \mathbb{E}[U(X_T^{\vartheta,\varepsilon})]| \leq C\,\varepsilon^{\frac{2}{3}}, \quad \text{for } \delta \sim \varepsilon^{\frac{1}{3}}.$$

# Litterature (part of...)

□ PDE approach : Goes back to Shreve and Soner (94), Whalley and Wilmott (97) - utility based pricing -, Jaceneck and Shreve (04) and Rogers (04) - ideas -.

# Litterature (part of...)

□ PDE approach : Goes back to Shreve and Soner (94), Whalley and Wilmott (97) - utility based pricing -, Jaceneck and Shreve (04) and Rogers (04) - ideas -.

Further developped by Bichuch (14) - power utility as well.

# Litterature (part of...)

□ PDE approach : Goes back to Shreve and Soner (94), Whalley and Wilmott (97) - utility based pricing -, Jaceneck and Shreve (04) and Rogers (04) - ideas -.

Further developped by Bichuch (14) - power utility as well.

□ Homogenization approach : Soner and Touzi (13), and Soner, Touzi and Possamai (15), ... Systematic but need to know that the order is $O(\varepsilon^{\frac{2}{3}})$ + regularity/bounds on the Merton problem.

# Litterature (part of...)

□ PDE approach : Goes back to Shreve and Soner (94), Whalley and Wilmott (97) - utility based pricing -, Jaceneck and Shreve (04) and Rogers (04) - ideas -.

Further developped by Bichuch (14) - power utility as well.

□ Homogenization approach : Soner and Touzi (13), and Soner, Touzi and Possamai (15), ... Systematic but need to know that the order is $O(\varepsilon^{\frac{2}{3}})$ + regularity/bounds on the Merton problem.

□ Probabilistic approach (Shadow price and duality) : Kallsen and Muhle-Karbe (15) - ideas -, Kallsen and Li (13) (exponential utility), Gerhold et al. (14) (long run power utility).

# Litterature (part of...)

□ PDE approach : Goes back to Shreve and Soner (94), Whalley and Wilmott (97) - utility based pricing -, Jaceneck and Shreve (04) and Rogers (04) - ideas -.

Further developped by Bichuch (14) - power utility as well.

□ Homogenization approach : Soner and Touzi (13), and Soner, Touzi and Possamai (15), ... Systematic but need to know that the order is $O(\varepsilon^{\frac{2}{3}})$ + regularity/bounds on the Merton problem.

□ Probabilistic approach (Shadow price and duality) : Kallsen and Muhle-Karbe (15) - ideas -, Kallsen and Li (13) (exponential utility), Gerhold et al. (14) (long run power utility).

In all cases, complex arguments, quite heavy conditions and special care on the design of the transaction region (hand-made).

# Litterature (part of...)

☐ PDE approach : Goes back to Shreve and Soner (94), Whalley and Wilmott (97) - utility based pricing -, Jaceneck and Shreve (04) and Rogers (04) - ideas -.

Further developped by Bichuch (14) - power utility as well.

☐ Homogenization approach : Soner and Touzi (13), and Soner, Touzi and Possamai (15), ... Systematic but need to know that the order is $O(\varepsilon^{\frac{2}{3}})$ + regularity/bounds on the Merton problem.

☐ Probabilistic approach (Shadow price and duality) : Kallsen and Muhle-Karbe (15) - ideas -, Kallsen and Li (13) (exponential utility), Gerhold et al. (14) (long run power utility).

In all cases, complex arguments, quite heavy conditions and special care on the design of the transaction region (hand-made).

See also Cai, Rosenbaum and Tankov (17) for tracking errors (general asymptotic lower bounds in probability).

# General idea

□ Balance between deviating from $\widehat{\theta}$ and paying transaction costs (local time control).

# General idea

□ Balance between deviating from $\widehat{\theta}$ and paying transaction costs (local time control).

□ We just write down these ideas using the very simplest transaction region + use a trick from B., Elie and Moreau (12) : leads to a very simple proof of the bounds of order $\epsilon^{\frac{1}{2}}$ and $\epsilon^{\frac{2}{3}}$.

# General idea

□ Balance between deviating from $\widehat{\theta}$ and paying transaction costs (local time control).

□ We just write down these ideas using the very simplest transaction region + use a trick from B., Elie and Moreau (12) : leads to a very simple proof of the bounds of order $\epsilon^{\frac{1}{2}}$ and $\epsilon^{\frac{2}{3}}$.

□ Based on mild moment conditions, that can be checked by using Malliavin calculus in complete Itô semimartingale frameworks.

# General idea

□ Balance between deviating from $\widehat{\theta}$ and paying transaction costs (local time control).

□ We just write down these ideas using the very simplest transaction region + use a trick from B., Elie and Moreau (12) : leads to a very simple proof of the bounds of order $\epsilon^{\frac{1}{2}}$ and $\epsilon^{\frac{2}{3}}$.

□ Based on mild moment conditions, that can be checked by using Malliavin calculus in complete Itô semimartingale frameworks.

□ We restrict to bounded risk aversion but it can be made more general.

# General idea

□ Balance between deviating from $\widehat{\theta}$ and paying transaction costs (local time control).

□ We just write down these ideas using the very simplest transaction region + use a trick from B., Elie and Moreau (12) : leads to a very simple proof of the bounds of order $\epsilon^{\frac{1}{2}}$ and $\epsilon^{\frac{2}{3}}$.

□ Based on mild moment conditions, that can be checked by using Malliavin calculus in complete Itô semimartingale frameworks.

□ We restrict to bounded risk aversion but it can be made more general.

□ Can be complemented by the approach of Soner and Touzi to derive explicit expansion.

# Elementary $L_p$-bounds

□ The simplest possible transaction region : $\vartheta$ solves the Skhorohod problem

$$\left\{ \begin{array}{c} \theta - \vartheta \in [-\delta, \delta]^d \text{ on } [0, T], \\ \sum_{i=1}^d \left( \int_0^T \mathbf{1}_{\{\theta_t^i - \vartheta_t^i = \delta\}} d\vartheta_t^{i+} + \int_0^T \mathbf{1}_{\{\theta_t^i - \vartheta_t^i = -\delta\}} d\vartheta_t^{i-} \right) = 0. \end{array} \right.$$

# Elementary $L_p$-bounds

□ The simplest possible transaction region : $\vartheta$ solves the Skhorohod problem

$$\left\{ \begin{array}{c} \theta - \vartheta \in [-\delta, \delta]^d \text{ on } [0, T], \\ \sum_{i=1}^d \left( \int_0^T \mathbf{1}_{\{\theta^i_t - \vartheta^i_t = \delta\}} d\vartheta^{i+}_t + \int_0^T \mathbf{1}_{\{\theta^i_t - \vartheta^i_t = -\delta\}} d\vartheta^{i-}_t \right) = 0. \end{array} \right.$$

□ Take $\varphi$ such that $-\varphi'(-1) = \varphi'(1) = 1$, $|\varphi| \vee |\varphi'| \vee |\varphi''| \leq 1$ and set $Z := (\theta - \vartheta)/\delta \in [-1, 1]$. Then ($d = 1$ case),

# Elementary $L_p$-bounds

□ The simplest possible transaction region : $\vartheta$ solves the Skhorohod problem

$$\begin{cases} \theta - \vartheta \in [-\delta, \delta]^d \text{ on } [0, T], \\ \sum_{i=1}^d \left( \int_0^T \mathbf{1}_{\{\theta_t^i - \vartheta_t^i = \delta\}} d\vartheta_t^{i+} + \int_0^T \mathbf{1}_{\{\theta_t^i - \vartheta_t^i = -\delta\}} d\vartheta_t^{i-} \right) = 0. \end{cases}$$

□ Take $\varphi$ such that $-\varphi'(-1) = \varphi'(1) = 1$, $|\varphi| \vee |\varphi'| \vee |\varphi''| \leq 1$ and set $Z := (\theta - \vartheta)/\delta \in [-1, 1]$. Then ($d = 1$ case),

$$\varphi(Z_t) = \varphi(Z_0) + \frac{1}{\delta} \left( \int_0^t \varphi'(Z_s) d(\theta - \vartheta)_s + \frac{1}{2\delta} \int_0^t \varphi''(Z_s) d\langle \theta \rangle_s \right)$$

# Elementary $L_p$-bounds

□ The simplest possible transaction region : $\vartheta$ solves the Skhorohod problem

$$\begin{cases} \theta - \vartheta \in [-\delta, \delta]^d \text{ on } [0, T], \\ \sum_{i=1}^d \left( \int_0^T \mathbf{1}_{\{\theta^i_t - \vartheta^i_t = \delta\}} d\vartheta^{i+}_t + \int_0^T \mathbf{1}_{\{\theta^i_t - \vartheta^i_t = -\delta\}} d\vartheta^{i-}_t \right) = 0. \end{cases}$$

□ Take $\varphi$ such that $-\varphi'(-1) = \varphi'(1) = 1$, $|\varphi| \vee |\varphi'| \vee |\varphi''| \leq 1$ and set $Z := (\theta - \vartheta)/\delta \in [-1, 1]$. Then ($d = 1$ case),

$$\begin{aligned} \varphi(Z_t) &= \varphi(Z_0) + \frac{1}{\delta} \left( \int_0^t \varphi'(Z_s) d(\theta - \vartheta)_s + \frac{1}{2\delta} \int_0^t \varphi''(Z_s) d\langle \theta \rangle_s \right) \\ &= \varphi(Z_0) + \frac{1}{\delta} \left( \int_0^t \varphi'(Z_s) d\theta_s - |\vartheta|_t + \frac{1}{2\delta} \int_0^t \varphi''(Z_s) d\langle \theta \rangle_s \right). \end{aligned}$$

# Elementary $L_p$-bounds

□ The simplest possible transaction region : $\vartheta$ solves the Skhorohod problem

$$\begin{cases} \theta - \vartheta \in [-\delta, \delta]^d \text{ on } [0, T], \\ \sum_{i=1}^d \left( \int_0^T \mathbf{1}_{\{\theta^i_t - \vartheta^i_t = \delta\}} d\vartheta^{i+}_t + \int_0^T \mathbf{1}_{\{\theta^i_t - \vartheta^i_t = -\delta\}} d\vartheta^{i-}_t \right) = 0. \end{cases}$$

□ Take $\varphi$ such that $-\varphi'(-1) = \varphi'(1) = 1$, $|\varphi| \vee |\varphi'| \vee |\varphi''| \leq 1$ and set $Z := (\theta - \vartheta)/\delta \in [-1, 1]$. Then ($d = 1$ case),

$$\begin{aligned} \varphi(Z_t) &= \varphi(Z_0) + \frac{1}{\delta} \left( \int_0^t \varphi'(Z_s) d(\theta - \vartheta)_s + \frac{1}{2\delta} \int_0^t \varphi''(Z_s) d\langle \theta \rangle_s \right) \\ &= \varphi(Z_0) + \frac{1}{\delta} \left( \int_0^t \varphi'(Z_s) d\theta_s - |\vartheta|_t + \frac{1}{2\delta} \int_0^t \varphi''(Z_s) d\langle \theta \rangle_s \right). \end{aligned}$$

Thus, there exists $\xi \in \mathcal{B}_1$ (i.e. $\|\xi\| \leq 1$) s.t.

$$|\vartheta| \leq 2d\delta + \int_0^\cdot \xi_s^\top d\theta_s + \frac{1}{2\delta} \langle \theta \rangle.$$

Recall :

$$|\vartheta| \leq 2d\delta + \int_0^{\cdot} \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle.$$

□ **Assumption :** For some $p \geq 1$ and $\mathbb{Q} \sim \mathbb{P}$,

$$\sup_{\xi \in \mathcal{B}_1} \Big|\Big| \int_0^T \xi_s^\top d\theta_s \Big|\Big|_{\mathbf{L}_p(\mathbb{Q})} + \|\langle\theta\rangle_T\|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p).$$

Recall :

$$|\vartheta| \leq 2d\delta + \int_0^{\cdot} \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle.$$

□ **Assumption :** For some $p \geq 1$ and $\mathbb{Q} \sim \mathbb{P}$,

$$\sup_{\xi \in \mathcal{B}_1} \left|\left| \int_0^T \xi_s^\top d\theta_s \right|\right|_{\mathbf{L}_p(\mathbb{Q})} + \|\langle\theta\rangle_T\|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p).$$

□ **Proposition :** Fix $\delta \in (0,1)$, then

$$\| \, |\vartheta|_T \, \|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p)\left(1 + \frac{1}{\delta}\right).$$

Recall :

$$|\vartheta| \le 2d\delta + \int_0^{\cdot} \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle.$$

□ **Assumption :** For some $p \ge 1$ and $\mathbb{Q} \sim \mathbb{P}$,

$$\sup_{\xi\in\mathcal{B}_1} \Big\| \int_0^T \xi_s^\top d\theta_s \Big\|_{\mathbf{L}_p(\mathbb{Q})} + \|\langle\theta\rangle_T\|_{\mathbf{L}_p(\mathbb{Q})} \le C(p).$$

□ **Proposition :** Fix $\delta \in (0,1)$, then

$$\| \, |\vartheta|_T \, \|_{\mathbf{L}_p(\mathbb{Q})} \le C(p)\left(1 + \frac{1}{\delta}\right).$$

□ Remark : Suppose that $\theta$ is a $\mathbb{Q}$-Brownian motion and choose $\varphi(z) = z^2/2$ for $z \in [-1,1]$, then

$$\mathbb{E}^{\mathbb{Q}}\left[|\vartheta|_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\delta(\varphi(Z_0) - \varphi(Z_t)) + \frac{d}{2\delta}t\right] \ge -\frac{1}{2} + \frac{d}{2\delta}t.$$

Generally speaking : this is just the Brownian motion scalling propety...

□ As a consequence :

$$\left|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\right| = \left|\int_0^t (\vartheta_s - \theta_s)^\top dS_s - \varepsilon|\vartheta|_t - \mathbf{1}_{\{T\}}\varepsilon|\vartheta_T|\right|$$
$$\leq \delta\left|\int_0^t \tilde{\xi}_s^\top dS_s\right| + 2\varepsilon\left(2d\delta + \int_0^t \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle_t\right)$$

where $\xi, \tilde{\xi} \in \mathcal{B}_1$.

□ As a consequence :

$$\left|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\right| = \left|\int_0^t (\vartheta_s - \theta_s)^\top dS_s - \varepsilon|\vartheta|_t - \mathbf{1}_{\{T\}}\varepsilon|\vartheta_T|\right|$$
$$\leq \delta\left|\int_0^t \tilde{\xi}_s^\top dS_s\right| + 2\varepsilon\left(2d\delta + \int_0^t \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle_t\right)$$

where $\xi, \tilde{\xi} \in \mathcal{B}_1$.

□ **Assumption :** $\sup_{\tilde{\xi}\in\mathcal{B}_1} \left|\left| \int_0^T \tilde{\xi}_t^\top dS_t \right|\right|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p)$.

□ As a consequence :

$$\left|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\right| = \left|\int_0^t (\vartheta_s - \theta_s)^\top dS_s - \varepsilon|\vartheta|_t - \mathbf{1}_{\{T\}}\varepsilon|\vartheta_T|\right|$$
$$\leq \delta \left|\int_0^t \tilde{\xi}_s^\top dS_s\right| + 2\varepsilon\left(2d\delta + \int_0^t \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle_t\right)$$

where $\xi, \tilde{\xi} \in \mathcal{B}_1$.

□ **Assumption :** $\sup_{\tilde{\xi}\in\mathcal{B}_1} \left|\left| \int_0^T \tilde{\xi}_t^\top dS_t \right|\right|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p)$.

□ **Proposition :**

$$\|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\|_{\mathbf{L}_p(\mathbb{Q})} \leq \delta\ C(p) + \ 2\varepsilon\ C(p)\left(1 + \frac{1}{\delta}\right).$$

□ As a consequence :

$$\left|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\right| = \left|\int_0^t (\vartheta_s - \theta_s)^\top dS_s - \varepsilon|\vartheta|_t - \mathbf{1}_{\{T\}}\varepsilon|\vartheta_T|\right|$$
$$\leq \delta\left|\int_0^t \tilde{\xi}_s^\top dS_s\right| + 2\varepsilon\left(2d\delta + \int_0^t \xi_s^\top d\theta_s + \frac{1}{2\delta}\langle\theta\rangle_t\right)$$

where $\xi, \tilde{\xi} \in \mathcal{B}_1$.

□ **Assumption :** $\sup_{\tilde{\xi}\in\mathcal{B}_1} \left\|\int_0^T \tilde{\xi}_t^\top dS_t\right\|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p)$.

□ **Proposition :**

$$\|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\|_{\mathbf{L}_p(\mathbb{Q})} \leq \delta\ C(p) + \ 2\varepsilon\ C(p)\left(1 + \frac{1}{\delta}\right).$$

For $\delta = \varepsilon^{1/2} \in (0,1)$,

$$\|X_t^{\vartheta,\varepsilon} - X_t^{\theta}\|_{\mathbf{L}_p(\mathbb{Q})} \leq C(p)\ \varepsilon^{1/2}.$$

□ Remark : if $\theta$ is a Brownian motion and $S$ an Itô semi-martingale :

$$\begin{aligned}\delta\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \xi_s^\top \mu_s^S ds\right] - c\varepsilon\left(1+\frac{1}{\delta}\right) &\le \mathbb{E}^{\mathbb{Q}}\left[X_t^{\vartheta,\varepsilon} - X_t^{\theta}\right] \\ &\le \delta\mathbb{E}^{\mathbb{Q}}\left[\int_0^t \xi_s^\top \mu_s^S ds\right] - c'\varepsilon\left(1+\frac{1}{\delta}\right).\end{aligned}$$

Cannot do better in general... unless $S$ is a $\mathbb{Q}$-martingale (as in the utility maximization problem).

# Elementary bounds for utility maximization

- **Assumption :** $U$ has a bounded risk aversion, namely

$$0 < r < -\frac{U''(x)}{U'(x)} < R < \infty, \quad \text{for constants } r,\ R \text{ and all } x \in \mathbb{R}.$$

# Elementary bounds for utility maximization

□ **Assumption :** $U$ has a bounded risk aversion, namely

$$0 < r < -\frac{U''(x)}{U'(x)} < R < \infty, \quad \text{for constants } r,\ R \text{ and all } x \in \mathbb{R}.$$

□ **Admissibility** $\mathcal{A}$ (resp. $\mathcal{A}^{\epsilon}$) : $X^{\theta}$ (resp. $X^{\vartheta,\epsilon}$) is a supermartingale under all absolutely continuous martingale measures with finite entropy.

# Elementary bounds for utility maximization

□ **Assumption :** $U$ has a bounded risk aversion, namely

$$0 < r < -\frac{U''(x)}{U'(x)} < R < \infty, \quad \text{for constants } r,\ R \text{ and all } x \in \mathbb{R}.$$

□ **Admissibility** $\mathcal{A}$ (resp. $\mathcal{A}^{\epsilon}$) : $X^{\theta}$ (resp. $X^{\vartheta,\epsilon}$) is a supermartingale under all absolutely continuous martingale measures with finite entropy.

□ There exists an optimizer $\widehat{\theta} \in \mathcal{A}$ and a dual optimizer $\widehat{\mathbb{Q}} \sim \mathbb{P}$ s.t.

$$\frac{U'(X_T^{\widehat{\theta}})}{\mathbb{E}[U'(X_T^{\widehat{\theta}})]} = \frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}.$$

Set $\Delta_T^\varepsilon := U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})$. For some $\zeta^\varepsilon = \lambda X_T^{\vartheta,\varepsilon} + (1-\lambda) X_T^{\widehat{\theta}}$,

$$\mathbb{E}[\Delta_T^\varepsilon] = \mathbb{E}\left[ U'(X_T^{\widehat{\theta}}) \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right) + \frac{1}{2} U''(\zeta^\varepsilon) \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^2 \right]$$

Set $\Delta_T^\varepsilon := U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})$. For some $\zeta^\varepsilon = \lambda X_T^{\vartheta,\varepsilon} + (1-\lambda) X_T^{\widehat{\theta}}$,

$$\begin{aligned}\mathbb{E}[\Delta_T^\varepsilon] &= \mathbb{E}\left[U'(X_T^{\widehat{\theta}})\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}U''(\zeta^\varepsilon)\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right] \\ &\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right],\end{aligned}$$

with $\alpha := \mathbb{E}[U'(X_T^{\widehat{\theta}})]$.

Set $\Delta_T^\varepsilon := U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})$. For some $\zeta^\varepsilon = \lambda X_T^{\vartheta,\varepsilon} + (1-\lambda) X_T^{\widehat{\theta}}$,

$$\begin{aligned}\mathbb{E}[\Delta_T^\varepsilon] &= \mathbb{E}\left[U'(X_T^{\widehat{\theta}})\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}U''(\zeta^\varepsilon)\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right] \\ &\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right],\end{aligned}$$

with $\alpha := \mathbb{E}[U'(X_T^{\widehat{\theta}})]$. Because risk aversion is bounded from above by $R$,

$$\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})} = \frac{U''(\zeta^\varepsilon)}{U'(\zeta^\varepsilon)}\frac{U'(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})} \geq -Re^{R|\zeta^\varepsilon - X_T^{\widehat{\theta}}|}.$$

Set $\Delta_T^\varepsilon := U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})$. For some $\zeta^\varepsilon = \lambda X_T^{\vartheta,\varepsilon} + (1-\lambda) X_T^{\widehat{\theta}}$,

$$\begin{aligned}
\mathbb{E}[\Delta_T^\varepsilon] &= \mathbb{E}\left[U'(X_T^{\widehat{\theta}})\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}U''(\zeta^\varepsilon)\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right] \\
&\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right],
\end{aligned}$$

with $\alpha := \mathbb{E}[U'(X_T^{\widehat{\theta}})]$. Because risk aversion is bounded from above by $R$,

$$\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})} = \frac{U''(\zeta^\varepsilon)}{U'(\zeta^\varepsilon)}\frac{U'(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})} \geq -Re^{R|\zeta^\varepsilon - X_T^{\widehat{\theta}}|}.$$

Thus,

$$\mathbb{E}[\Delta_T^\varepsilon] \geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) - \frac{R}{2}e^{R|\zeta^\varepsilon - X_T^{\widehat{\theta}}|}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right]$$

Set $\Delta_T^\varepsilon := U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})$. For some $\zeta^\varepsilon = \lambda X_T^{\vartheta,\varepsilon} + (1-\lambda) X_T^{\widehat{\theta}}$,

$$\begin{aligned}\mathbb{E}[\Delta_T^\varepsilon] &= \mathbb{E}\left[U'(X_T^{\widehat{\theta}})\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}U''(\zeta^\varepsilon)\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right] \\ &\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) + \frac{1}{2}\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right],\end{aligned}$$

with $\alpha := \mathbb{E}[U'(X_T^{\widehat{\theta}})]$. Because risk aversion is bounded from above by $R$,

$$\frac{U''(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})} = \frac{U''(\zeta^\varepsilon)}{U'(\zeta^\varepsilon)}\frac{U'(\zeta^\varepsilon)}{U'(X_T^{\widehat{\theta}})} \geq -Re^{R|\zeta^\varepsilon - X_T^{\widehat{\theta}}|}.$$

Thus,

$$\begin{aligned}\mathbb{E}[\Delta_T^\varepsilon] &\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) - \frac{R}{2}e^{R|\zeta^\varepsilon - X_T^{\widehat{\theta}}|}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right] \\ &= \alpha \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right) - \frac{R}{2}e^{R\lambda|X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}|}\left(X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}\right)^2\right].\end{aligned}$$

□ **Assumption :** There exists $\iota > 0$ such that

$$\sup_{\xi \in \mathcal{B}_1} \left\{ \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \int_0^T \xi_t^\top d\widehat{\theta}_t}] \right\} + \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \langle \widehat{\theta} \rangle_T} + e^{\iota \langle S \rangle_T}] \leq C.$$

□ **Assumption :** There exists $\iota > 0$ such that

$$\sup_{\xi \in \mathcal{B}_1} \left\{ \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \int_0^T \xi_t^\top d\widehat{\theta}_t}] \right\} + \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \langle \widehat{\theta} \rangle_T} + e^{\iota \langle S \rangle_T}] \leq C.$$

Then,

$$\mathbb{E}[\Delta_T^\varepsilon] \geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right) - \frac{R}{2} e^{R\lambda |X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}|} \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^2 \right]$$

□ **Assumption :** There exists $\iota > 0$ such that

$$\sup_{\xi \in \mathcal{B}_1} \left\{ \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \int_0^T \xi_t^\top d\widehat{\theta}_t}] \right\} + \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \langle \widehat{\theta} \rangle_T} + e^{\iota \langle S \rangle_T}] \leq C.$$

Then,

$$\begin{aligned} \mathbb{E}[\Delta_T^\varepsilon] &\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right) - \frac{R}{2} e^{R\lambda |X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}|} \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^2 \right] \\ &\geq -2\alpha\varepsilon \mathbb{E}^{\widehat{\mathbb{Q}}}[\mathcal{R}_\delta(\xi)_t] - C \mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^{2\eta} \right]^{\frac{1}{\eta}} \end{aligned}$$

with $\eta > 1$ and $\mathcal{R}_\delta(\xi)_t := 2d\delta + \int_0^t \xi_s^\top d\widehat{\theta}_s + \frac{1}{2\delta} \langle \widehat{\theta} \rangle_t$.

□ **Assumption :** There exists $\iota > 0$ such that

$$\sup_{\xi \in \mathcal{B}_1} \left\{ \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \int_0^T \xi_t^\top d\widehat{\theta}_t}] \right\} + \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \langle \widehat{\theta} \rangle_T} + e^{\iota \langle S \rangle_T}] \le C.$$

Then,

$$\begin{aligned} \mathbb{E}[\Delta_T^\varepsilon] &\ge \alpha \mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right) - \frac{R}{2} e^{R\lambda |X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}|} \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^2 \right] \\ &\ge -2\alpha\varepsilon \mathbb{E}^{\widehat{\mathbb{Q}}}[\mathcal{R}_\delta(\xi)_t] - C\mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^{2\eta} \right]^{\frac{1}{\eta}} \end{aligned}$$

with $\eta > 1$ and $\mathcal{R}_\delta(\xi)_t := 2d\delta + \int_0^t \xi_s^\top d\widehat{\theta}_s + \frac{1}{2\delta} \langle \widehat{\theta} \rangle_t$.

□ **Theorem :** $\mathbb{E}[U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})] \ge -C\, \varepsilon^{\frac{2}{3}}$, for $\delta = \varepsilon^{\frac{1}{3}}$.

□ **Assumption :** There exists $\iota > 0$ such that

$$\sup_{\xi \in \mathcal{B}_1} \left\{ \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \int_0^T \xi_t^\top d\widehat{\theta}_t}] \right\} + \mathbb{E}^{\widehat{\mathbb{Q}}}[e^{\iota \langle \widehat{\theta} \rangle_T} + e^{\iota \langle S \rangle_T}] \leq C.$$

Then,

$$\begin{aligned}
\mathbb{E}[\Delta_T^\varepsilon] &\geq \alpha \mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right) - \frac{R}{2} e^{R\lambda |X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}}|} \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^2 \right] \\
&\geq -2\alpha\varepsilon \mathbb{E}^{\widehat{\mathbb{Q}}}[\mathcal{R}_\delta(\xi)_t] - C \mathbb{E}^{\widehat{\mathbb{Q}}} \left[ \left( X_T^{\vartheta,\varepsilon} - X_T^{\widehat{\theta}} \right)^{2\eta} \right]^{\frac{1}{\eta}}
\end{aligned}$$

with $\eta > 1$ and $\mathcal{R}_\delta(\xi)_t := 2d\delta + \int_0^t \xi_s^\top d\widehat{\theta}_s + \frac{1}{2\delta} \langle \widehat{\theta} \rangle_t$.

□ **Theorem :** $\mathbb{E}[U(X_T^{\vartheta,\varepsilon}) - U(X_T^{\widehat{\theta}})] \geq -C\, \varepsilon^{\frac{2}{3}}$, for $\delta = \varepsilon^{\frac{1}{3}}$.
In particular,

$$\left| \sup_{\vartheta} \mathbb{E}[U(X_T^{\vartheta,\varepsilon})] - \sup_{\theta} \mathbb{E}[U(X_T^{\theta})] \right| \leq C\, \varepsilon^{\frac{2}{3}}.$$

# Complete Itô diffusion case

- We assume that

$$S = S_0 + \int_0^{\cdot} \mu(S_t)dt + \int_0^{\cdot} \sigma(S_t)dW_t.$$

# Complete Itô diffusion case

☐ We assume that

$$S = S_0 + \int_0^{\cdot} \mu(S_t)dt + \int_0^{\cdot} \sigma(S_t)dW_t.$$

☐ **Assumption :** $\lambda := \sigma^{-1}\mu$, $\sigma$ and $\sigma^{-1}$ are $C_b^2 \cap C_b^0$. $U \in C^3(\mathbb{R})$, and $U'''/U''$ is bounded. (can easily relax to be more general...)

# Complete Itô diffusion case

- We assume that

$$S = S_0 + \int_0^{\cdot} \mu(S_t)dt + \int_0^{\cdot} \sigma(S_t)dW_t.$$

- **Assumption :** $\lambda := \sigma^{-1}\mu$, $\sigma$ and $\sigma^{-1}$ are $C_b^2 \cap C_b^0$. $U \in C^3(\mathbb{R})$, and $U'''/U''$ is bounded. (can easily relax to be more general...)

Simply write that

$$X_T^{\widehat{\theta}} = (U')^{-1}\left(c\ d\widehat{\mathbb{Q}}/d\mathbb{P}\right) \quad \text{and} \quad \widehat{\theta}_t^{\top}\sigma(S_t) = \mathbb{E}^{\widehat{\mathbb{Q}}}[D_t X_T^{\widehat{\theta}}|\mathcal{F}_t]$$

and use standard estimates (need second order Malliavin derivatives to control the martingale part of $\theta$).

# Complete Itô diffusion case

- We assume that

$$S = S_0 + \int_0^{\cdot} \mu(S_t)dt + \int_0^{\cdot} \sigma(S_t)dW_t.$$

- **Assumption :** $\lambda := \sigma^{-1}\mu$, $\sigma$ and $\sigma^{-1}$ are $C_b^2 \cap C_b^0$. $U \in C^3(\mathbb{R})$, and $U'''/U''$ is bounded. (can easily relax to be more general...)

Simply write that

$$X_T^{\widehat{\theta}} = (U')^{-1}\left(c\ d\widehat{\mathbb{Q}}/d\mathbb{P}\right) \quad \text{and} \quad \widehat{\theta}_t^{\top}\sigma(S_t) = \mathbb{E}^{\widehat{\mathbb{Q}}}[D_t X_T^{\widehat{\theta}} | \mathcal{F}_t]$$

and use standard estimates (need second order Malliavin derivatives to control the martingale part of $\theta$).

- **Proposition :** $\theta$ is bounded and is of the form

$$\theta = \theta_0 + \int_0^{\cdot} \alpha_t dt + \int_0^{\cdot} \gamma_t dW_t^{\widehat{\mathbb{Q}}},$$

where $\theta_0 \in \mathbb{R}$ and $\alpha$, $\gamma$ are bounded adapted processes.

# Costs on transacted amounts

- We now write the frictionless wealth process as

$$X_t^{\theta} := X_0 + \int_0^t (\theta_s / S_s)^\top dS_s.$$

# Costs on transacted amounts

- We now write the frictionless wealth process as

$$X_t^\theta := X_0 + \int_0^t (\theta_s / S_s)^\top dS_s.$$

- The frictional wealth process is

$$X_t^{\vartheta,\varepsilon} := X_0 + \int_0^t \left(Y_s^\vartheta / S_s\right)^\top dS_s - \varepsilon \int_0^t d|\vartheta|_s - \mathbf{1}_{\{T\}} \varepsilon |Y_T^\vartheta|,$$

where

$$Y_t^\vartheta := \int_0^t \left(Y_s^\vartheta / S_s\right)^\top dS_s + \vartheta_t.$$

# Costs on transacted amounts

- We now write the frictionless wealth process as

$$X_t^\theta := X_0 + \int_0^t (\theta_s/S_s)^\top dS_s.$$

- The frictional wealth process is

$$X_t^{\vartheta,\varepsilon} := X_0 + \int_0^t \left(Y_s^\vartheta/S_s\right)^\top dS_s - \varepsilon \int_0^t d|\vartheta|_s - \mathbf{1}_{\{T\}}\varepsilon|Y_T^\vartheta|,$$

where

$$Y_t^\vartheta := \int_0^t \left(Y_s^\vartheta/S_s\right)^\top dS_s + \vartheta_t.$$

- The Skorokhod problem becomes

$$\left\{ \begin{array}{c} \theta - Y^\vartheta \in [-\delta,\delta]^d \text{ on } [0,T], \\ \sum_{i=1}^d \left( \int_0^T \mathbf{1}_{\{\theta_t^i - Y_t^{\vartheta,i} = \delta\}} d\vartheta_t^{i+} + \int_0^T \mathbf{1}_{\{\theta_t^i - Y_t^{\vartheta,i} = -\delta\}} d\vartheta_t^{i-} \right) = 0. \end{array} \right.$$

# Costs on transacted amounts

- We now write the frictionless wealth process as

$$X_t^\theta := X_0 + \int_0^t (\theta_s/S_s)^\top dS_s.$$

- The frictional wealth process is

$$X_t^{\vartheta,\varepsilon} := X_0 + \int_0^t \left(Y_s^\vartheta/S_s\right)^\top dS_s - \varepsilon \int_0^t d|\vartheta|_s - \mathbf{1}_{\{T\}}\varepsilon|Y_T^\vartheta|,$$

where

$$Y_t^\vartheta := \int_0^t \left(Y_s^\vartheta/S_s\right)^\top dS_s + \vartheta_t.$$

- The Skorokhod problem becomes

$$\begin{cases} \theta - Y^\vartheta \in [-\delta,\delta]^d \text{ on } [0,T], \\ \sum_{i=1}^d \left(\int_0^T \mathbf{1}_{\{\theta_t^i - Y_t^{\vartheta,i}=\delta\}} d\vartheta_t^{i+} + \int_0^T \mathbf{1}_{\{\theta_t^i - Y_t^{\vartheta,i}=-\delta\}} d\vartheta_t^{i-}\right) = 0. \end{cases}$$

But the analysis is very similar.... (simply a bit more painful to write).

# Main references


M. Bichuch.
Pricing a contingent claim liability with transaction costs using asymptotic analysis for optimal investment.
*Finance Stoch.* 18(3) : 651–694, 2014.

J. Cai, M. Rosenbaum, and P. Tankov.
Asymptotic lower bounds for optimal tracking : a linear programming approach.
*Ann. Appl. Probab.* 27(4) : 2455–2514, 2017.

K. Janecek, K. and S.E. Shreve.
Asymptotic analysis for optimal investment and consumption with transaction costs.
*Finance Stoch.* 8(2) : 181–206, 2004.

J. Kallsen and J. Muhle-Karbe.
The general structure of optimal investment and consumption with small transaction costs.
*Math. Finance* 27(3) : 659–703, 2017.

L.C.G. Rogers.
Why is the effect of proportional transaction costs $O(2/3)$ ?
In G. Yin and Q. Zhang, editors, *Mathematics of Finance*, pages 303–308. Amer. Math. Soc., Providence, RI, 2004.

H.M. Soner and N. Touzi.
Homogenization and asymptotics for small transaction costs.
*SIAM J. Control Optim.*, 51(4), 2893–2921, 2013.

A.E. Whalley and P. Wilmott.
An asymptotic analysis of an optimal hedging model for option pricing with transaction costs.
*Math. Finance*, 7(3) :307–324, 1997.