A quasi-sure optional decomposition and super-hedging result on the Skorokhod space

Bruno Bouchard [*] Xiaolu Tan [†]

November 23, 2020

Abstract

We prove a robust super-hedging duality result for path-dependent options on assets with jumps, in a continuous time setting. It requires that the collection of martingale measures is rich enough and that the payoff function satisfies some continuity property. It is a by-product of a quasi-sure version of the optional decomposition theorem, which can also be viewed as a functional version of Itô's Lemma, that applies to non-smooth functionals (of càdlàg processes) which are concave in space and nonincreasing in time, in the sense of Dupire.

1 Introduction

A key element in the proof of the super-hedging duality is the optional decomposition theorem. Let X be a stochastic process on some filtered probability space and consider the class of all equivalent martingale measures under which X is a local martingale. Let V be a supermartingale under all these equivalent martingale measures. Then, the classical optional decomposition theorem states that there exists a predictable process H and a non-decreasing process C such that $V = V_0 + \int_0^\cdot H_r \cdot dX_r - C$, almost surely. Initially introduced by El Karoui and Quenez [10] in the case where X has continuous paths, it was then extended to the càdlàg paths case in Kramkov [18], Föllmer and Kabanov [12], Delbaen and Schachermayer [7], Föllmer and Kramkov [13].

The robust optional decomposition theorem has been recently studied, as a key step to prove (and also motivated by) the robust super-hedging duality. In this context, one considers a family $\mathcal{P}$ of (singular) martingale measures, under each of which the process V is a supermartingale. Applying the classical decomposition theorem, one obtains a family $(H^{\mathbb{P}}, C^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ such that $V = V_0 + \int_0^\cdot H^{\mathbb{P}}_r \cdot dX_r - C^{\mathbb{P}}$, $\mathbb{P}$–a.s. for all $\mathbb{P} \in \mathcal{P}$. Then the robust optional decomposition theorem consists essentially in aggregating the family $(H^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ into a universal process H, independent of $\mathbb{P}$.

When X is a continuous martingale, one can in fact express $H^{\mathbb{P}}$ as the ratio of the intensity of the (co-)quadratic variation terms $\langle V, X\rangle$ and $\langle X\rangle$ under each reference probability measure $\mathbb{P}$. Since the (co-)quadratic variation $\langle V, X\rangle$ and $\langle X\rangle$ can be defined universally and independently of $\mathbb{P}$, one can aggregate $(H^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ into a universal process H. This technique has been explored by Neufeld and Nutz [20], Possamaï, Royer and Touzi [23] and Biagini, Bouchard, Kardaras

[*] CEREMADE, Université Paris-Dauphine, PSL, CNRS. bouchard@ceremade.dauphine.fr.
[†] Department of Mathematics, The Chinese University of Hong Kong. xiaolu.tan@cuhk.edu.hk. The research of Xiaolu Tan is supported by CUHK Faculty of Science Direct Grant 2019-2020.

and Nutz [1]. The same is true when X is a càdlàg martingale whose jump part is "dominated" by its diffusion part (see Nutz [22] for a precise definition). Based on this observation, Nutz [22] established a robust version of the optional decomposition theorem for càdlàg processes. In all of this literature, almost no regularity condition is imposed on the payoff function.

The open question is how to aggregate the family $(H^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ when X is a càdlàg martingale, without the "domination" condition of Nutz [22]. In this paper, we investigate a functional Itô's calculus approach, with the objective to establish a robust optional decomposition theorem, as well as a robust super-hedging duality result. Although it requires a minimum of regularity on V, see below, it provides a partial answer that can turn out to be useful in many applications.

This idea was indeed a motivation for Dupire [9] to introduce the functional Itô's calculus. Clearly, when V is a $C^{1,2}$ smooth functional of (t, X), in the sense of Dupire, one can apply the functional Itô's formula rigorously established by Cont and Fournié [5, 6] to identify that $H^{\mathbb{P}} = \nabla_\omega V(\cdot, X)$, in which $\nabla_\omega V$ denotes the vertical derivative of V. When X is a one-dimensional continuous semimartingale, and V is a concave functional of X (in the sense of Dupire) satisfying some additional regularity conditions, Saporito [24] obtained a functional Meyer-Tanaka's formula that also provides a decomposition formula on V that is very close to the one we are looking for in the context of the optional decomposition. However, the regularity conditions they impose are usually too strong to be checked (or even not true especially in the case of [9]). Moreover, their Itô's formulas are given on functionals defined on the space of all $\mathbb{R}^d$-valued paths. For the applications in finance, one may for instance consider only positive valued paths, so that the value function is defined on the canonical space of $\mathbb{R}^d_+$–valued paths. But it is not trivial to extend such a functional to the whole space of $\mathbb{R}^d$–valued paths and at the same time keep the pathwise regularity/concavity properties.

In this paper, we consider the setting with multivariate càdlàg paths and build on concepts and ideas of [9, 5, 6], and in particular on the regularization technique of Saporito [24], to establish a robust optional decomposition formula, under some mild continuity, concavity and monotonicity conditions on $(t, \omega) \mapsto V(t, \omega)$. In the robust super-hedging problem, the supermartingale V is obtained as the supremum of the expectation of the payoff over a family of martingale measures. We then show that the conditions imposed on V are satisfied as soon as the family of martingale measures is rich enough, and the payoff function enjoys some continuity conditions. In terms of required regularity, our setting is obviously not as general as [22] but we do not require the "domination" conditions in [22] and we are able to provide an explicit expression of the optimal super-hedging strategy H as an element of the super-differential of V. This paper should thus be considered as a complement to earlier works.

As a by-product, we prove that any locally-bounded path-dependent Dupire-concave function of a $\mathbb{R}^d$-valued semi-martingale remains a semi-martingale, thus generalizing the result of Meyer [19, Chapter VI] (see also Carlen and Protter [4]).

The rest of this paper is organized as follow. We first introduce some notations that will be used all over this paper. We state our version of the robust optional decomposition theorem in Section 2. In Section 3, we provide a pretty general version of the robust super-hedging duality for continuous payoffs, including two typical examples in which the components of X are restricted to remain non-negative.

Notations. (i). Let $E \subseteq \mathbb{R}^d$ be a closed convex set, we denote by $\Omega = D([0, T], E)$ be the space of all càdlàg E–valued paths on $[0, T]$, with canonical filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ and canonical process $X(\omega) := \omega$. We endow Ω with the uniform convergence topology induced by the norm $\|\omega\| := \sup_{t\in[0,T]} |\omega_t|$ for $\omega \in \Omega$. For $(t, \omega) \in \Theta := [0, T] \times \Omega$, we consider the (optional) stopped path $\omega_{t\wedge\cdot} := (\omega_{t\wedge s})_{s\in[0,T]}$, and (predictable) stopped path $\omega^{t-} := (\omega^{t-}_s)_{s\in[0,T]}$ defined by $\omega^{t-}_s := \omega_s \mathbf{1}_{\{s\in[0,t)\}} + \omega_{t-}\mathbf{1}_{\{s\in[t,T]\}}$. A function $\varphi : \Theta \longrightarrow \mathbb{R}$ is said to be non-anticipative if $\varphi(t,\omega) = \varphi(t,\omega_{t\wedge\cdot})$ for all $(t,\omega)\in\Theta$.

(ii). For a function $\varphi : \Theta \longrightarrow \mathbb{R}$, we follow Dupire [9] (see also Cont and Fournié [6]) to introduce the Dupire derivatives as follows: φ is said to be horizontally differentiable if, for all $(t,\omega)\in[0,T)\times\Omega$, its horizontal derivative

$$\partial_t\varphi(t,\omega) \;:=\; \lim_{h\searrow 0}\frac{\varphi(t+h,\omega_{t\wedge\cdot})-\varphi(t,\omega_{t\wedge\cdot})}{h}$$

is well-defined ; φ is said to be vertically differentiable if, for all $(t,\omega)\in\Theta$, the function

$$y\in E\longmapsto \varphi(t,\omega\boxplus_t y)\in\mathbb{R} \text{ is differentiable, with } \omega\boxplus_t y := \omega\mathbf{1}_{[0,t)} + y\mathbf{1}_{[t,T]},$$

in which case its derivative at $y=\omega_t$ is defined as the vertical derivative $\nabla_\omega\varphi(t,\omega)$ of φ at (t,ω).

(iii). A non-anticipative function $\varphi : \Theta \longrightarrow \mathbb{R}$ is called right-continuous if, for all $(t,\omega)\in\Theta$, $\varepsilon>0$, there exists $\delta>0$ such that

$$t'\ge t,\ |t'-t| + \|\omega'_{t'\wedge\cdot}-\omega_{t\wedge\cdot}\|\le\delta \implies \big|\varphi(t',\omega')-\varphi(t,\omega)\big|\le\varepsilon.$$

Let us denote by $\mathbb{C}_{\mathrm{r}}(\Theta)$ the class of all right-continuous non-anticipative functions. We say that $\varphi\in\mathbb{C}^{0,1}_{\mathrm{r}}(\Theta)$ if both φ and $\nabla_\omega\varphi$ are well defined and belong to $\mathbb{C}_{\mathrm{r}}(\Theta)$. Similar to the terminology in Saporito [24], a non-anticipative functions $\varphi:\Theta\to\mathbb{R}$ is called right equi-continuous, which we write as $\varphi\in\mathbb{C}^{\mathrm{e}}_{\mathrm{r}}(\Theta)$, if for each $(t,\omega)\in\Theta$, for all $\varepsilon>0$ there exists $\delta>0$ such that

$$t'\ge t,\ |t'-t|+\|\omega'_{t'\wedge\cdot}-\omega_{t\wedge\cdot}\|\le\delta \implies \big|\varphi(t',\omega'\boxplus_{t'} y)-\varphi(t,\omega\boxplus_t y)\big|\le\varepsilon,\ \forall y\in B_1(\omega_t)\cap E,$$

where $B_1(\omega_t):=\{y\in\mathbb{R}^d\ :|\omega_t-y|\le 1\}$. It is clear that $\mathbb{C}^{\mathrm{e}}_{\mathrm{r}}(\Theta)\subset\mathbb{C}_{\mathrm{r}}(\Theta)$.

(iv). A non-anticipative map $\varphi:\Theta\longrightarrow\mathbb{R}$ is said to be Dupire-concave if, for all $t\in[0,T]$, $\omega^1,\omega^2\in\Omega$, such that $\omega^1=\omega^2$ on $[0,t)$, and $\theta\in[0,1]$,

$$\varphi(t,\theta\omega^1+(1-\theta)\omega^2)\ \ge\ \theta\varphi(t,\omega^1)+(1-\theta)\varphi(t,\omega^2). \tag{1}$$

Notice that the above definition of Dupire-concavity is the same as that in Saporito [24] or Köpfer and Rüschendorf [17]. For a Dupire-concave function φ, one can define the Dupire super-differential (set)

$$\partial\varphi(t,\omega)\ :=\ \big\{z\in\mathbb{R}^d:\varphi(t,\omega\boxplus_t y)\le\varphi(t,\omega)+z\cdot(y-\omega_t),\ \forall\, y\in E\big\}.$$

The map φ is said to be Dupire-nonincreasing in time if

$$\varphi(t+h,\omega_{t\wedge\cdot})\le\varphi(t,\omega_{t\wedge\cdot}),\quad\text{for all } (t,\omega)\in\Theta \text{ and } h\in[0,T-t].$$

The map φ is said to be locally equi-nonincreasing in time if, for all $K>0$, there is a non-decreasing function $\mathrm{r}_K:[0,T]\to\mathbb{R}$ and a module continuity[1] $\rho_K:\mathbb{R}_+\to\mathbb{R}_+$ such that, for all $0\le t\le t+h\le T$, $\|\omega\|\le K$, and $y\in B_1(\omega_t)\cap E$,

$$\varphi(t+h,\omega_{t\wedge\cdot}\boxplus_{t+h} y)\ \le\ \varphi(t,\omega\boxplus_t y)\ +\ \rho_K(|y-\omega_t|)\big(\mathrm{r}_K(t+h)-\mathrm{r}_K(t)\big). \tag{2}$$

Taking $y=\omega_t$ in (2), we see that a functional φ which is locally equi-nonincreasing in time is in particular Dupire-nonincreasing in time. Notice also that a map $\varphi:\Omega\longrightarrow\mathbb{R}$ can be also associated to the Dupire-nonincreasing map $t\mapsto\varphi(\omega_{t\wedge\cdot})$.

[1] That is ρ_K is non-negative, continuous around 0 and vanishes at 0.

(v). Given a locally bounded predictable process H, and a (càdlàg) semimartingale X, we write (as usual) $\int_s^t H_r \cdot dX_r$ for the stochastic integral $\int_{(s,t]} H_r \cdot dX_r$. In the case that the law of X depends on the reference probability measure $\mathbb{P}$, the integral $\int_s^t H_r \cdot dX_r$ depends also on $\mathbb{P}$, which is usually omitted whenever it is obviously given by the context.

2 Optional decomposition of Dupire concave functionals

We provide immediately our version of the optional decomposition theorem, which is the key ingredient for proving the super-hedging duality of Theorem 3.6 below. It can also be seen as a functional version of Itô's or Meyer-Tanaka's formula, as it generalizes both up to the fact that the bounded variation part entering our decomposition is not explicitly characterized. As opposed to the classical versions of the optional decomposition theorem mentioned in the introduction, it is a functional one as our starting point is not that $t \mapsto V(t,\omega)$ is a supermartingale under martingale measures (although one can easily check that our assumptions imply this).

Recall that $\Omega := D([0,T],E)$ denotes the canonical space of all E-value càdlàd paths on $[0,T]$. Throughout the paper, we assume that $E \subseteq \mathbb{R}^d$ is closed convex set, with non-empty interior, and moreover that there exists a compactly supported smooth density function $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\text{the map } y \longmapsto \phi(y-x) \text{ is supported in } E, \text{ for all } x \in E. \tag{3}$$

Remark 2.1 *Examples of sets E satisfying* (3) *could be $E = \mathbb{R}^d$, or $E = \mathbb{R}_+^d$, with $\mathbb{R}_+ := [0,\infty)$, or any cone of $\mathbb{R}^d$ (with non-empty interior).*

Let $\mathcal{P}$ denote the collection of all Borel probability measures on Ω, under which the canonical process X is a semimartingale, that is, a (càdlàg) process which can be decomposed as the sum of a local martingale and an adapted finite variation process. For $s \in [0,T]$, denote also $X_r^{s-}(\omega) := \omega_r^{s-}$, or equivalently, $X_r^{s-} := X_{s\wedge r} - \Delta X_s \mathbf{1}_{\{r \geq s\}}$ for all $r \in [0,T]$.

Theorem 2.2 *Let $V \in \mathbb{C}_{\mathrm{r}}^{\mathrm{e}}(\Theta)$ be Dupire-concave, locally equi-nonincreasing in time, and such that*

$$\sup \big\{ |V(t,\omega)| + |z| \;\; : (t,\omega) \in \Theta,\ \|\omega\| \leq K,\ z \in \partial V(t,\omega) \big\} < \infty, \quad \textit{for all } K > 0. \tag{4}$$

Then, there exists a $\mathbb{F}$–predictable locally bounded process $H : \Theta \longrightarrow \mathbb{R}^d$, together with a collection of non-decreasing processes $\{C^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}\}$, satisfying

$$V(t,X) = V(0,X) + \int_0^t H_s \cdot dX_s - C_t^{\mathbb{P}},\ t \in [0,T],\ \mathbb{P}\text{–a.s.} \ \forall\, \mathbb{P} \in \mathcal{P}. \tag{5}$$

Moreover, $H_s \in \partial V(s, X^{s-})$ for all $s \in [0,T]$, $\mathcal{P}$-q.s.

Let us make some remarks before proving this result.

Remark 2.3 *An explicit formula for H is given in Remark 2.7 as an element of the super-differential of $\partial V(\cdot, X^{\cdot-})$.*

(i) *From this point of view, it can be considered as a version of the functional Meyer-Tanaka's formula, except that $C^{\mathbb{P}}$ is not identified to be associated to local time processes. In particular, when $E = \mathbb{R}^d$ and $V(t,\omega) = f(\omega_t)$ for some convex function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, it satisfies clearly all the conditions in Theorem 2.2, and the decomposition result* (5) *implies the result of Meyer*

[19, Chapter VI] (see also Carlen and Protter [4]) which states that a convex function of a semimartingale is still a semimartingale. Our result provides a path-depend version of this (apply it to $V(t,\omega) := f(\omega_{t\wedge\cdot})$ with $f : \Omega \longrightarrow \mathbb{R}$). In the functional (path-dependent) case, when X is a one-dimensional process with continuous paths, such a decomposition has been derived in Saporito [24] with an explicit expression of $C^{\mathbb{P}}$ in terms of the local times of X, but under additional smoothness conditions (see also Bouchard and Tan [3] for a version when V is only in $\mathbb{C}^{0,1}_{\mathrm{r}}(\Theta)$).

(ii) *It is clear that one may have different versions of the process H, which depends on the kernel ϕ used in part* (ii) *of the proof of Theorem 2.2.*

(iii) *Such an explicit formula is not available in the approach of Nutz [22] because it is based on the aggregation argument mentioned in the introduction (and does not assume any continuity).*

Remark 2.4 *When $E = \mathbb{R}^d$, if we assume in addition that $V \in \mathbb{C}^{1,2}(\Theta)$ in the sense of Dupire, or that X is a one-dimensional continuous process and that V is differentiable in t, together with some other technical conditions, one can apply the functional Itô's formula in Cont and Fournié [6] or in Saporito [24] to deduce immediately the result of Theorem 2.2. However, in practice, V is usually obtained as the value function of an optimal control problem, and such regularity conditions are pretty difficult to check, especially in the path-dependent context.*

When $E \subsetneq \mathbb{R}^d$, one needs (at least formally) to extend the definition of V from the space of E–valued paths to the space of all $\mathbb{R}^d$–valued paths in order to apply the functional Itô's formulas of [6, 24]. In the path-dependent case, such an extension seems not trivial if it is required to keep the same concavity and regularity properties of V.

Remark 2.5 *In Theorem 2.2, the stochastic integrals $(\int_0^t H_s \cdot dX_s)_{t\le T}$ depend on the reference measure $\mathbb{P}$, the fundamental point being that H does not. However, following Nutz [21], these stochastic integrals could be aggregated into a single $\mathbb{F}^*$-optional process, with $\mathbb{F}^*$ defined as the universally augmented filtration. In this case, the corresponding nonincreasing processes $\{C^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}\}$ can also be aggregated into a process independent of $\mathbb{P}$. But, this requires to work under the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) plus the Continuum Hypothesis, as well as to assume the existence of a uniform dominating measure for the characteristics of X (see [21, Assumption 2.1]).*

Remark 2.6 *When $E = \mathbb{R}^d$, one has*

$$\sup\big\{|z| \ : z \in \partial\varphi(t,\omega)\big\} \ \le \ \sup_{|y|\le 1} \big|\varphi(t,\omega \boxplus_t y) - \varphi(t,\omega)\big|,$$

so that Condition (4) *is equivalent to assuming that $V : \Theta \longrightarrow \mathbb{R}$ is a locally bounded function.*

Proof. (of Theorem 2.2) (i) Let us first provide a proof under the conditions that $V \in \mathbb{C}^{0,1}_{\mathrm{r}}(\Theta)$ is Dupire-concave, Dupire-nonincreasing in time and (4) holds. In this case, $\nabla_\omega V \in \mathbb{C}_{\mathrm{r}}(\Theta)$ and, for each $(t,\omega) \in \Theta$, $\nabla_\omega V(t,\omega)$ is the unique element in $\partial V(t,\omega)$, or equivalently, in the super-differential of the map $y \mapsto V(r, \omega \boxplus_r y)$ at $y = \omega_r$.

(a) Let us fix $s < t$ and consider a sequence of deterministic discrete time grids $(\pi_n)_{n\ge1}$, where $\pi_n = \{t^n_k\}_{0\le k\le n}$ satisfies

$$s = t^n_0 < t^n_1 < \cdots < t^n_n = t \ \text{ and } \ |\pi_n| := \max_{k=1,\cdots,n} (t^n_k - t^n_{k-1}) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Next, with fixed $\delta > 0$, we define the sequences of $\mathbb{F}$–stopping times $(\tau^n_k)_{k\ge1}$, $n \ge 1$, by

$$\tau^n_0 \equiv s, \ \text{ and } \ \tau^n_{k+1} := \inf\big\{r{>}\tau^n_k \ : |\Delta X_r| \ge \delta \text{ or } r \in \pi_n\big\}, \ k \ge 0.$$

Observe that the random number $m_n := \max\{k \geq 0 : \tau_k^n \leq t\}$ is finite, but is not uniformly bounded in general.

For each $n \geq 1$ and $u \in [s,t]$, let the processes X^n and $X^{n,u-}$ be defined by

$$X_r^n := \sum_{k=0}^{m_n-1} X_{\tau_k^n} \mathbf{1}_{\{r \in [\tau_k^n, \tau_{k+1}^n)\}} + X_t \mathbf{1}_{\{r=t\}}, \quad X_r^{n,u-} := X_{u\wedge r}^n - \Delta X_u \mathbf{1}_{\{r \geq u\}}, \ r \in [s,t],$$

so that

$$X_{\tau_k^n}^n = X_{\tau_k^n} \ \text{ and } \ X_{\tau_{k+1}^n}^{n,\tau_{k+1}^n-} = X_{\tau_{k+1}^n -}, \quad \text{for each } k \geq 0.$$

Recall that V is non-anticipative, Dupire-concave and Dupire-nonincreasing in time, so that $V(\tau_k^n, X^n) \geq V(\tau_{k+1}^n, X_{\tau_k^n \wedge \cdot}^n)$. It follows that

$$V(\tau_{k+1}^n, X^n) - V(\tau_k^n, X^n) \ \leq \ \nabla_\omega V(\tau_{k+1}^n, X_{\tau_k^n\wedge\cdot}^n) \cdot (X_{\tau_{k+1}^n} - X_{\tau_k^n}), \quad \text{if } \tau_{k+1}^n \in \pi_n, \tag{6}$$

and

$$\begin{aligned} & V(\tau_{k+1}^n, X^n) - V(\tau_k^n, X^n) \\ \leq \ & V(\tau_{k+1}^n, X^n) - V(\tau_{k+1}^n, X^{n,\tau_{k+1}^n-}) + V(\tau_{k+1}^n, X^{n,\tau_{k+1}^n-}) - V(\tau_{k+1}^n, X_{\tau_k^n\wedge\cdot}^n) \\ \leq \ & \nabla_\omega V(\tau_{k+1}^n, X^{n,\tau_{k+1}^n-}) \cdot \Delta X_{\tau_{k+1}^n} + \nabla_\omega V(\tau_{k+1}^n, X_{\tau_k^n\wedge\cdot}^n) \cdot (X_{\tau_{k+1}^n-} - X_{\tau_k^n}), \quad \text{if } \tau_{k+1}^n \notin \pi_n. \end{aligned} \tag{7}$$

By summing up the two sides of (6) and (7) for $k = 0, \cdots, m_n - 1$, it follows that

$$V(t, X^n) - V(s, X^n) \leq I_n^\delta \tag{8}$$

with

$$\begin{aligned} I_n^\delta \ := \ & \sum_{k=0}^{m_n-1} \Big(\nabla_\omega V(\tau_{k+1}^n, X_{\tau_k^n\wedge\cdot}^n) \cdot (X_{\tau_{k+1}^n} - X_{\tau_k^n}) \Big) \mathbf{1}_{\{\tau_{k+1}^n \in \pi_n\}} \\ & + \sum_{k=0}^{m_n-1} \Big(\nabla_\omega V(\tau_{k+1}^n, X^{n,\tau_{k+1}^n-}) \cdot \Delta X_{\tau_{k+1}^n} + \nabla_\omega V(\tau_{k+1}^n, X_{\tau_k^n\wedge\cdot}^n) \cdot (X_{\tau_{k+1}^n-} - X_{\tau_k^n}) \Big) \mathbf{1}_{\{\tau_{k+1}^n \notin \pi_n\}}, \end{aligned}$$

where we add the superscript δ on I_n^δ to emphasis the dependence of the random number m_n and the stopping times $(\tau_k^n)_{1\leq k\leq m_n}$ on $\delta > 0$. The term I_n^δ can be written as an integral w.r.t. X, but the integrand may not be adapted to the filtration $\mathbb{F}$. This motivates us to introduce

$$I_n \ := \ \sum_{k=0}^{n-1} \nabla_\omega V(t_{k+1}^n, X_{t_k^n\wedge\cdot}^n) \cdot \big(X_{t_{k+1}^n} - X_{t_k^n} \big) \ = \ \int_s^t H_r^n dX_r,$$

where H^n is the $\mathbb{F}$-predictable process defined by

$$H^n := \sum_{k=0}^{n-1} \nabla_\omega V(t_{k+1}^n, X_{t_k^n\wedge\cdot}^n) \mathbf{1}_{\big\{(t_k^n, t_{k+1}^n]\big\}}.$$

Notice that, for all fixed $\delta > 0$ and $\omega \in \Omega$, there exists only a finite number of τ_{k+1}^n. Further, by (4), the terms $\nabla_\omega V(\tau_{k+1}^n, X_{\tau_k^n\wedge\cdot}^n)$ and $\nabla_\omega V(\tau_{k+1}^n, X^{n,\tau_{k+1}^n-})$ are uniformly bounded for every fixed $\omega \in \Omega$. Then by the continuity of $\nabla_\omega V$ and the fact that X has càdlàg paths, it is easy to see that, for every fixed $\omega \in \Omega$,

$$\lim_{n\longrightarrow\infty} \big| I_n^\delta(\omega) - I_n(\omega) \big| \ = \ 0. \tag{9}$$

(b) Let us now assume that, for some $\delta > 0$, $\mathbb{P}$ belongs to the collection of probability measures

$$\mathcal{P}_\delta \;:=\; \big\{\mathbb{P}' \in \mathcal{P} \;:\mathbb{P}'\big[|\Delta X_r| \in \{0\} \cup [\delta, \infty),\; \forall r \in [0,T]\big] = 1\big\},$$

i.e. X has only big jumps (with jump size bigger than δ) under $\mathbb{P}$. As $\nabla_\omega V \in \mathbb{C}_{\mathrm{r}}(\Theta)$, then

$$H^n_r \longrightarrow \nabla_\omega V^-(r, X), \text{ for all } r \in [s,t], \; \mathbb{P}\text{–a.s. for each } \mathbb{P} \in \mathcal{P}_\delta,$$

in which $(\nabla_\omega V^-(r,X))_{r\ge 0} := (\nabla_\omega V(r, X^{r-}))_{r\ge 0}$ is $\mathbb{F}$-predictable. Further, notice that one can localize the sequence of processes $(X_0 \mathbf{1}_{\{0\}} + \sum_{k=0}^{n-1} X_{t_k^n} \mathbf{1}_{(t_k^n, t_{k+1}^n]})_{n\ge 1}$ uniformly by using the sequence of $\mathbb{F}$–stopping times $\tau_m := \inf\{t \;: |X_t| \ge m\}$, $m \ge 1$. Then, by (4), the sequence $(|H^n|)_{n\ge 1}$ can be uniformly bounded by a locally bounded predictable process. By Jacod and Shiryaev [15, Theorem I.4.31], and after possibly passing to a subsequence, it follows that

$$\int_s^t H^n_r \cdot dX_r \longrightarrow \int_s^t \nabla_\omega V^-(r,X) \cdot dX_r \;\text{ and }\; V(t, X^n) \longrightarrow V(t,X), \; \mathbb{P}\text{–a.s.}$$

Therefore, (8) and (9) imply that, for all $\mathbb{P} \in \mathcal{P}_\delta$,

$$V(t,X) - V(s,X) \;\le\; \int_s^t \nabla_\omega V^-(r,X) \cdot dX_r, \;\; \mathbb{P}\text{–a.s.} \tag{10}$$

(c) We now consider $\mathbb{P} \in \mathcal{P}$, under which X is a general semimartingale taking value in the interior of the set E. Under $\mathbb{P}$, X can be uniquely decomposed as the sum of a continuous martingale X^c and a purely discontinuous semimartingale X^d. Recall that every purely discontinuous semimartingale can be approximated uniformly, on $[0,T]$, by processes with finite variation (see e.g. [15, Section I.4 and Theorem II.2.34]). Namely, by keeping only the (compensated) small jumps in X^d, one can find a sequence $(Z^n)_{n\ge 1}$ of purely discontinuous semimartingales, together with a sequence of positive real numbers $(\delta_n)_{n\ge 1}$, such that $\delta_n \longrightarrow 0$, and $\mathbb{P}$–a.s.

$$|\Delta Z^n_t| \le \delta_n, \; \forall t \in [0,T]; \;\text{ and }\; \|Z^n\| + [Z^n]_T \longrightarrow 0, \text{ as } n \longrightarrow \infty, \tag{11}$$

and $Y^n := X - Z^n$ has only jumps bigger than δ_n. Notice that Y^n may not take value in E when $E \ne \mathbb{R}^d$. Let us define

$$\tau_n \;:=\; \inf\{r \ge s \;: Y^n_r \notin E\}, \;\; \overline{Y}^n_r := Y^n_r \mathbf{1}_{\{r < \tau_n\}} + Y^n_{\tau_n -} \mathbf{1}_{\{r \ge \tau_n\}}, \; r \in [s,t],$$

so that $\mathbb{P} \circ (\overline{Y}^n)^{-1} \in \mathcal{P}_\delta$. Then, applying (10) to $\overline{Y}^n$ leads to

$$V(t, \overline{Y}^n) \;\le\; V(s, \overline{Y}^n) + \int_s^t \nabla_\omega V^-(r, \overline{Y}^n) \cdot d\overline{Y}^n_r, \; \mathbb{P}\text{–a.s.}, \; n \ge 1. \tag{12}$$

As X takes values in the interior of E, and $\|X - Y^n\| \longrightarrow 0$, $\mathbb{P}$–a.s., then, for $\mathbb{P}$–a.e. ω, there exists $n_0(\omega)$ such that $\tau_n(\omega) = \infty$ for all $n \ge n_0(\omega)$. Moreover, since Z^n is a purely discontinuous semimartingale with jumps no bigger than δ_n, one can localise the process, so that both $\nabla_\omega V^-(\cdot, \overline{Y}^n)$, Z^n and $[Z^n]_T$ are uniformly bounded. Taking the limit $n \longrightarrow \infty$, we deduce from (11), (12) and [15, Theorem I.4.31] that (10) holds true for all $\mathbb{P} \in \mathcal{P}$ under which X takes values in the interior of E.

(d) We finally consider an arbitrary $\mathbb{P} \in \mathcal{P}$ under which X a semimartingale taking values in E. For all $\varepsilon > 0$, let $X^\varepsilon := (1-\varepsilon) X + \varepsilon x_0$, where x_0 is a given point belonging to the interior of E. Then X^ε is a semimartingale taking values in the interior of E. Applying (10) to X^ε and then letting $\varepsilon \longrightarrow 0$, it follows that (10) holds true for all $\mathbb{P} \in \mathcal{P}$. By the arbitrariness of $s \le t$ and $\mathbb{P} \in \mathcal{P}$, this proves (5) under the additional condition that $V \in \mathbb{C}^{0,1}_{\mathrm{r}}(\Theta)$.

(ii) Let us now consider the general case (in particular without the condition $V \in \mathbb{C}^{0,1}_{\mathrm{r}}(\Theta)$). Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a compactly supported smooth density function satisfying (3), we define $\phi^\varepsilon(y) := \varepsilon^{-d}\phi(\varepsilon^{-1}y)$, $y \in \mathbb{R}^d$, $\varepsilon > 0$. Without loss of generality, we assume that ϕ is supported in $B_1(0) := \{y \in \mathbb{R}^d \ : |y| \le 1\}$. Let us define $V^\varepsilon : \Theta \longrightarrow \mathbb{R}$ by

$$V^\varepsilon(r,\omega) \ := \ \int_{\mathbb{R}^d} V\big(r, \omega \boxplus_r y'\big)\phi^\varepsilon(y' - \omega_r)dy'.$$

First, it is clear that V^ε is Dupire-concave as soon as V is. Next, notice that V is right equi-continous, then by a direct adaptation of Proposition 2.6 of Saporito [24], it follows that

$$V^\varepsilon \in \mathbb{C}^{0,1}_{\mathrm{r}}(\Theta), \ \text{ and } \ V^\varepsilon(r,\omega) \longrightarrow V(r,\omega), \ \ \forall (r,\omega) \in \Theta.$$

Moreover, recall that V is locally equi-nonincreasing in time (in the sense of (2) with functions $(\rho_K, \mathrm{r}_K)_{K>0}$), and that, by (4), the maps $y \mapsto V(t+h, (\omega_{t\wedge\cdot}) \boxplus_{t+h} y)$ are Lipschitz-continuous uniformly in h. It follows that, for each $K > 0$, the functional $V^\varepsilon(t,\omega) - \rho_K(\varepsilon)\mathrm{r}_K(t)$ is Dupire-nonincreasing in time for all $\omega \in \Omega$ satisfying $\|\omega\| \le K$. Further, by Stokes formula and a change of variables,

$$\nabla_\omega V^\varepsilon(r,\omega) \ := \ \int_{\mathbb{R}^d} \varepsilon^{-1}\Big[V\big(r, \omega \boxplus_r (\omega_r + \varepsilon y)\big) - V(r,\omega)\Big]\big(-\nabla\phi(y)\big)dy,$$

in which $\nabla\phi$ is the gradient of ϕ. Then, for all $(r,\omega) \in \Theta$,

$$\nabla_\omega V^\varepsilon(r,\omega) \longrightarrow H(r,\omega) := \int_{\mathbb{R}^d} \partial^+ V(r,\omega;y)\big(-\nabla\phi(y)\big)dy, \ \ \text{as } \varepsilon \longrightarrow 0, \tag{13}$$

where

$$\partial^+ V(r,\omega;y) \ := \ \lim_{\varepsilon \searrow 0} \frac{V\big(r, \omega \boxplus_r (\omega_r + \varepsilon y)\big) - V(t,\omega)}{\varepsilon}, \ \text{for all } y \text{ in the support of } \phi,$$

is well-defined since V is Dupire-concave (see also Remark 2.7 below). Using (4), up to a localisation argument, one can assume w.l.o.g. that $(\nabla_\omega V^\varepsilon(\cdot, X^{\cdot-}))_{\varepsilon>0}$ is uniformly bounded. Then, using the decomposition result (10) on $V^\varepsilon - \rho_K(\varepsilon)\mathrm{r}_K$, and then letting $\varepsilon \longrightarrow 0$, we can apply [15, Theorem I.4.31] to conclude that, for all $\mathbb{P} \in \mathcal{P}$,

$$V(t,X) \le V(s,X) + \int_s^t H(r, X^{r-}) \cdot dX_r, \ \mathbb{P}\text{–a.s.}$$

As $\mathbb{P} \in \mathcal{P}$ and $s < t$ are arbitrary and H does not depend on $\mathbb{P}$ and $s < t$, this proves the decomposition result (5).

(iii) Finally, recalling the definition of the super-differential of a concave function, the fact that $H(r,\omega) \in \partial V(r,\omega)$ is an immediate consequence of (13) and the fact that the Dupire-concave functionals $V^\varepsilon \longrightarrow V$ pointwisely. Moreover, it is clear that the process $(H(s, X^{s-})_{s\in[0,T]}$ is $\mathbb{F}$–predictable, and is a locally bounded process by (4). $\square$

Remark 2.7 *One can check that*[2]

$$\partial^+ V(r,\omega;y) = \min\{y \cdot z : z \in \partial V(r,\omega)\}.$$

[2]We would like to thank Pierre Cardaliaguet who pointed out to us this identity and its proof.

Indeed, first, it is clear from the definition of $\partial^+ V(r,\omega;y)$ that $\partial^+ V(r,\omega;y) \le \min\{y\cdot z : z \in \partial V(r,\omega)\}$. Next, let us consider $z^\varepsilon \in \arg\min\{y\cdot z : z \in \partial V\big(r, \omega \boxplus_r (\omega_r + \varepsilon y)\big)\}$, so that $z^\varepsilon \cdot (\varepsilon y) \le V\big(r, \omega \boxplus_r (\omega_r + \varepsilon y)\big) - V(r,\omega)$. By (4), one can then find a sequence $(\varepsilon_n)_{n\ge1}$ converging to 0 such that $z^{\varepsilon_n} \longrightarrow z \in \partial V(r,\omega)$, and $z \cdot y \le \partial^+ V(r,\omega;y)$.

Remark 2.8 *Notice that the condition (3) is only used to regularize V into a function with continuous first order vertical Dupire derivative. It is not necessary if $\partial V(t,\omega)$ admits a unique element for all $(t,\omega) \in \Theta$. In this case, the vertical Dupire derivative inherits the regularity of V automatically, and there is no need for the intermediate smoothing procedure in part (ii) of the proof of Theorem 2.2.*

3 Super-hedging duality

Let us now turn to the main motivation of this paper. From Theorem 2.2, we derive in this section a robust super-hedging problem and provide a duality result. We first state it under general abstract conditions, Theorem 3.6, and then discuss a typical example of applications in Proposition 3.12.

3.1 Abstract framework

Let $\Phi : \Omega \longrightarrow \mathbb{R}$ be a payoff function and $\mathcal{M}_0 = (\mathcal{M}(0,x))_{x\in E}$ be a family of collections of probability measures $\mathbb{Q}$ on Ω such that X is a $\mathbb{Q}$-local martingale with $X_0 = x$, $\mathbb{Q}$–a.s. We assume that, for all $x \in E$ and $\mathbb{Q} \in \mathcal{M}(0,x)$,

$$\mathbb{E}^{\mathbb{Q}}\big[\big|\Phi(X)\big|\big] < \infty,\ \big(\mathbb{E}^{\mathbb{Q}}[\Phi(X)^-|\mathcal{F}_t]\big)_{t\le T} \text{ is a } \mathbb{Q}\text{-martingale, and } \sup_{\mathbb{Q}\in\mathcal{M}(0,x)} \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big] < \infty. \quad (14)$$

The super-hedging price of a derivative option with payoff $\Phi(X)$ is defined by

$$\mathrm{v}(0,x) \ := \ \inf\big\{v \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ s.t. } Y_T^{v,H} \ge \Phi(X),\ \mathcal{M}(0,x) - \text{q.s.}\big\},$$

in which

$$Y^{v,H} := v + \int_0^{\cdot} H_r \cdot dX_r$$

and $\mathcal{H}$ is the collection of all locally bounded $\mathbb{F}$-predictable processes such that $Y^{v,H}$ is $\mathbb{Q}$-a.s. bounded from below by a $\mathbb{Q}$-martingale, for all $\mathbb{Q} \in \mathcal{M}(0,x)$.

The aim of this section is to prove the following super-hedging duality:

$$\mathrm{v}(0,x) \ = \ V(0,x) \ := \sup_{\mathbb{Q}\in\mathcal{M}(0,x)} \mathbb{E}^{\mathbb{Q}}[\Phi(X)]. \quad (15)$$

As usual, one can easily obtain the weak duality

$$\mathrm{v}(0,x) \ \ge \ V(0,x) \ := \sup_{\mathbb{Q}\in\mathcal{M}(0,x)} \mathbb{E}^{\mathbb{Q}}[\Phi(X)]. \quad (16)$$

Indeed, for all $(v,H) \in \mathbb{R}\times\mathcal{H}$ such that $Y_T^{v,H} \ge \Phi(X)$, $\mathcal{M}(0,x)$-q.s., one has $v \ge \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big]$ for all $\mathbb{Q} \in \mathcal{M}(0,x)$, since $Y^{v,H}$ is a $\mathbb{Q}$-local-martingale bounded from below by a $\mathbb{Q}$-martingale, and therefore a $\mathbb{Q}$-supermartingale, for any $\mathbb{Q} \in \mathcal{M}(0,x)$, whenever $H \in \mathcal{H}$.

To prove the converse inequality, we will rely on Theorem 2.2. Assuming that V defined above satisfies all the conditions of Theorem 2.2, then there exists a $\mathbb{F}$-predictable process H such that

$Y^{V(0,x),H} \geq V(\cdot, X)$ on $[0,T]$, and in particular that $Y_T^{V(0,x),H} \geq \Phi(X)$, $\mathcal{M}(0,x)$-q.s. Since, for all $\mathbb{Q} \in \mathcal{M}(0,x)$, $V(t,X) \geq \mathbb{E}^{\mathbb{Q}}[\Phi(X)|\mathcal{F}_t] \geq \mathbb{E}^{\mathbb{Q}}[\Phi(X)^-|\mathcal{F}_t]$ $\mathbb{Q}$-a.s., and $\big(\mathbb{E}^{\mathbb{Q}}[\Phi(X)^-|\mathcal{F}_t]\big)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale, it follows that $H \in \mathcal{H}$, and therefore that $V(0,x) \geq \mathrm{v}(0,x)$. Together with the weak duality (16), this implies the duality result (15).

To ensure that the conditions of Theorem 2.2 hold, we need to assume more structure conditions on $\mathcal{M}(0,x)$ and Φ. Recall that, given a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ and a $\mathbb{F}$–stopping time τ taking value in $[0,T]$, a r.c.p.d. (regular conditional probability distribution) of $\mathbb{P}$ conditional to $\mathcal{F}_\tau$ is a family $(\mathbb{P}_\omega)_{\omega\in\Omega}$ of probability measure on $(\Omega,\mathcal{F}_T)$, such that $\omega \mapsto \mathbb{P}_\omega$ is $\mathcal{F}_\tau$–measurable, $\mathbb{P}_\omega[X_s = \omega_s,\ s \leq \tau(\omega)] = 1$ for all $\omega \in \Omega$, and $\mathbb{E}^{\mathbb{P}}[\mathbf{1}_A|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}_\omega}[\mathbf{1}_A]$ for $\mathbb{P}$–a.e. $\omega \in \Omega$ for all $A \in \mathcal{F}_T$.

Assumption 3.1 *There exists a family $(\mathcal{M}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ of collections of probability measures on Ω such that, for all $(t,\omega) \in \Theta$:*

(i) $\mathcal{M}(0,\omega) = \mathcal{M}(0,\omega_0)$ *and* $\mathcal{M}(t,\omega) = \mathcal{M}(t,\omega_{t\wedge\cdot})$.

(ii) *For all $\mathbb{Q} \in \mathcal{M}(t,\omega)$, X is a $\mathbb{Q}$–local martingale on $[t,T]$ and $\mathbb{Q}[X_s = \omega_s,\ s \leq t] = 1$.*

(iii) *Given $\mathbb{Q} \in \mathcal{M}(t,\omega)$ and a $\mathbb{F}$–stopping time τ taking values in $[t,T]$:*

  (a) There exists a family $(\mathbb{Q}_\omega)_{\omega\in\Omega}$ of r.c.p.d. of $\mathbb{Q}$ conditional to $\mathcal{F}_\tau$ such that

$$\mathbb{Q}_\omega \in \mathcal{M}(\tau(\omega),\omega),\ \textit{for } \mathbb{Q}\text{–a.e. } \omega \in \Omega.$$

  (b) For all $\varepsilon > 0$, there exists $\mathbb{Q}^\varepsilon \in \mathcal{M}(t,\omega)$ such that $\mathbb{Q}|_{\mathcal{F}_\tau} = \mathbb{Q}^\varepsilon|_{\mathcal{F}_\tau}$ and a family $(\mathbb{Q}^\varepsilon_\omega)_{\omega\in\Omega}$ of r.c.p.d. of $\mathbb{Q}^\varepsilon$ conditional to $\mathcal{F}_\tau$ such that

$$\mathbb{E}^{\mathbb{Q}^\varepsilon}\big[\Phi(X)\big] \geq V(t,\omega) - \varepsilon \textit{ and } \mathbb{Q}^\varepsilon_\omega \in \mathcal{M}(\tau(\omega),\omega),\ \textit{for } \mathbb{Q}\text{–a.e. } \omega \in \Omega,$$

  where

$$V(t,\omega) := \sup_{\mathbb{Q}\in\mathcal{M}(t,\omega)} \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big]. \tag{17}$$

Remark 3.2 *Assumption 3.1.*(iii) *is a standard condition to ensure that the dynamic programming principle holds true for the optimization problem in* (16). *Namly, it ensures that the families $\mathcal{M}(t,\omega)$ of probability measures is stable under conditioning and concatenation. It could be compared to the conditions used in Biagini, Bouchard, Kardaras and Nutz [1], or Nutz [22].*

We also need the following additional conditions. Let $\mathrm{conv}(A)$ denote the convex envelope of a set $A \subset \mathbb{R}^d$, δ_ω denote the Dirac measure at $\omega \in \Omega$, and, for all $x \in \mathbb{R}^d$, $(t,\omega) \in \Theta$ and $\eta > 0$, set

$$B_\eta(x) := \{y \in \mathbb{R}^d\ : |y-x| \leq \eta\}, \quad B_\eta(t,\omega) := \big\{(t',\omega') \in \Theta\ : t' \geq t,\ |t'-t| + \|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\| \leq \eta\big\}.$$

Let $(\rho_K, \mathrm{r}_K)_{K>0}$ denote a family of maps such that, for each $K > 0$, $\mathrm{r}_K : [0,T] \mapsto \mathbb{R}$ is non-decreasing and $\rho_K : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a modulus of continuity.

Assumption 3.3 *With some constant $C > 0$, the following holds for all $(t,\omega) \in [0,T) \times \Omega$, $y \in B_1(\omega_t) \cap E$ and $x^1, x^2 \in E$ such that $\omega_t \in \mathrm{conv}\{x^1, x^2\}$:*

(i) *for all $\varepsilon > 0$,*

(a) *for all* $\mathbb{Q} \in \mathcal{M}(t, \omega \boxplus_t y)$*, we can find* $\eta > 0$ *such that for all* $(t', \omega') \in B_\eta(t,\omega)$ *there exists* $\mathbb{Q}' \in \mathcal{M}(t', \omega' \boxplus_{t'} y)$ *satisfying* $\mathbb{E}^{\mathbb{Q}}[\Phi(X)] \leq \mathbb{E}^{\mathbb{Q}'}[\Phi(X)] + \varepsilon$.

(b) *there is* $\eta > 0$ *such that for all* $(t', \omega') \in B_\eta(t,\omega)$ *and* $\mathbb{Q}' \in \mathcal{M}(t', \omega' \boxplus_{t'} y)$ *we can find* $\mathbb{Q} \in \mathcal{M}(t, \omega \boxplus_t y)$ *satisfying* $\mathbb{E}^{\mathbb{Q}'}[\Phi(X)] \leq \mathbb{E}^{\mathbb{Q}}[\Phi(X)] + \varepsilon$.

(ii) *there exists a family* $(\mathbb{Q}_h, A_h^1, A_h^2)_{h \in (0,h_0)}$ *for some* $h_0 \leq T - t$*, such that* $\mathbb{Q}_h \in \mathcal{M}(t,\omega)$*,* $\mathbb{Q}_h[X \in A_h^1 \cup A_h^2] = 1$*, and*

$$A_h^i \subset \{\omega' \in \Omega \ : \omega'_s = \omega_s \text{ on } [0,t],\ |\omega'_s| \leq |x^1| + |x^2| \text{ on } (t, t+h),\ \omega'_{t+h} = x^i\},\ i = 1,2.$$

Moreover, for each $\varepsilon > 0$*,* $i = 1,2$*,* $\mathbb{Q} \in \mathcal{M}(t, \omega \boxplus_t x^i)$ *and* $h_1 > 0$*, there exists* $h < h_1$ *such that for all* $\omega' \in A_h^i$ *one can find* $\mathbb{Q}' \in \mathcal{M}(t+h, \omega')$ *satisfying* $\mathbb{E}^{\mathbb{Q}}[\Phi(X)] \leq \mathbb{E}^{\mathbb{Q}'}[\Phi(X)] + \varepsilon$.

(iii) *for all* $h \in (0, T-t)$ *and* $\mathbb{Q} \in \mathcal{M}(t+h, (\omega_{t\wedge\cdot}) \boxplus_{t+h} y)$*, there exists* $\mathbb{Q}' \in \mathcal{M}(t, \omega \boxplus_t y)$ *such that*

$$\mathbb{E}^{\mathbb{Q}'}[\Phi(X)] \ \geq \ \mathbb{E}^{\mathbb{Q}}[\Phi(X)] \ - \ \rho_K(|y - \omega_t|)\big(\mathrm{r}_K(t+h) - \mathrm{r}_K(t)\big),$$

whenever $\|\omega\| \leq K \in (0,\infty)$.

Remark 3.4 *In Assumption 3.3, Condition* (i) *is used to ensure that* V *is right equi-continuous. Condition* (ii) *is used to prove that* V *is Dupire-concave. Condition* (iii) *is used to ensure that* V *is locally equi-nonincreasing in time. In particular, taking* $y = \omega_t$*, this implies that for all* $\mathbb{Q} \in \mathcal{M}(t+h, \omega_{t\wedge\cdot})$ *there exists* $\mathbb{Q}' \in \mathcal{M}(t,\omega)$ *such that* $\mathbb{E}^{\mathbb{Q}'}[\Phi(X)] \ \geq \ \mathbb{E}^{\mathbb{Q}}[\Phi(X)]$*. To check this, a convenient sufficient condition is to assume that* $\delta_{\omega_{t\wedge\cdot}} \in \mathcal{M}(t,\omega)$*, so that one can concatenate* $\delta_{\omega_{t\wedge\cdot}}$ *and* $\mathbb{Q}$ *at* $t+h$ *to obtain* $\mathbb{Q}' \in \mathcal{M}(t,\omega)$.

In this case, the condition $\delta_{\omega_{t\wedge\cdot}} \in \mathcal{M}(t,\omega)$ *is almost a necessary condition to ensure that* V *is Dupire-nonincreasing in time. For a simple counter-example, let us consider a one-dimensional Markovian case where* $\Phi(\omega) = \phi(\omega_T)$*, for some concave function* $\phi : \mathbb{R} \to \mathbb{R}$*, and* $\mathcal{M}(t,\omega)$ *denotes the collection of all probability measures* $\mathbb{Q} \in \mathcal{P}(\Omega)$ *such that* $\mathbb{Q}[X_s = \omega_s, s \in [0,t]] = 1$*, where* X *is* $\mathbb{Q}$*–diffusion martingale process on* $[t,T]$ *with volatility greater than* 1*. As* ϕ *is concave, it is easy to see that*

$$V(t,\omega) := \sup_{\mathbb{Q} \in \mathcal{M}(t,\omega)} \mathbb{E}^{\mathbb{Q}}[\Phi(X)] = \mathbb{E}^{\mathbb{P}}[\phi(\omega_t + W_{T-t})], \text{ for all } (t,\omega) \in \Theta,$$

where W *is a standard Brownian motion under* $\mathbb{P}$*. When* ϕ *is strictly concave, by Jensen's inequality, one has*

$$V(t,\omega) = \mathbb{E}^{\mathbb{P}}[\phi(\omega_t + W_{T-t})] < \mathbb{E}^{\mathbb{P}}[\phi(\omega_t + W_{T-t-h})] = V(t+h, \omega_{t\wedge\cdot}),$$

i.e. V *is not non-increasing in time.*

Remark 3.5 *When* $\Phi : \Omega \to \mathbb{R}$ *is a concave function, Condition* (ii) *of Assumption 3.3 is not always necessary to ensure that* V *is Dupire concave. For typical financial derivatives with payoff function* $\Phi(X) = \phi(X_T, A_T, M_T, m_T)$*, where* A_T *(resp.* M_T*,* m_T*) represents the running average (resp. maximum, minimum) of the underlying process* X*, the concavity of* ϕ *may not propagate to* V *as soon as it depends on* M_T *or* m_T*. But when* ϕ *is just a concave function of* X_T *and* A_T*, the optimality in* (17) *is achieved by the constant martingale measure* $\delta_{\omega_{t\wedge\cdot}}$*, because* $\mathbb{E}^{\mathbb{Q}}[(X_T, A_T)|X_0] = (X_0, X_0 T)$ *for any martingale measure* $\mathbb{Q}$*, so that the problem becomes trivial.*

Under Assumptions 3.1 and 3.3 , we can now state our main result which is an immediate consequence of the discussion above, combined with Theorem 2.2, Lemma 3.8 and Lemma 3.9 below. Two examples of applications will be studied in Sections 3.2 and 3.3 below.

Theorem 3.6 *Let Assumptions 3.1 and 3.3 hold true. Assume in addition that V satisfies (4), and that $y \in E \mapsto \Phi(\omega \boxplus_T y)$ is concave for all $\omega \in \Omega$. Then the duality (15) holds true, and there exists a $\mathbb{F}$-predictable process $H \in \mathcal{H}$ such that $H_s \in \partial V(s, X^{s-})$ for all $s \in [0,T]$, $\mathcal{M}(0,x)$–q.s., and $Y_T^{V(0,x),H} \geq \Phi(X)$, $\mathcal{M}(0,x)$–q.s., for all $x \in E$.*

Remark 3.7 *The condition that $y \mapsto \Phi(\omega \boxplus_T y)$ is concave for all $\omega \in \Omega$ is not important as soon as the collection $\mathcal{M}(0,x)$ is rich enough. For a general payoff function $\Phi : \Omega \longrightarrow \mathbb{R}$, let us denote by $\widehat{\Phi} : \Omega \longrightarrow \mathbb{R}$ the smallest function dominating Φ and such that $y \in E \mapsto \widehat{\Phi}(\omega \boxplus_T y)$ is concave. In many situations, such as in the examples of Sections 3.2 and 3.3, one can show that*

$$V(t,\omega) = \widehat{V}(t,\omega) := \sup_{\mathbb{Q} \in \mathcal{M}(t,\omega)} \mathbb{E}^{\mathbb{Q}}\big[\widehat{\Phi}(X)\big], \text{ for all } (t,\omega) \in [0,T) \times \Omega.$$

Then, one only needs to work on $\widehat{\Phi}$ and $\widehat{V}$ to obtain the duality result (15) for $\widehat{\Phi}$, and then to use the weak duality (16) and the above identity to deduce that (15) holds for Φ as well. See the proof of Propositions 3.12 and 3.16 below for more details.

The rest of this section is dedicated to the proof of the two lemmas mentioned above.

Lemma 3.8 *The value function V is non-anticipative and right equi-continuous, i.e. $V \in \mathbb{C}^{\mathrm{e}}_{\mathrm{r}}(\Theta)$. Further, for all $(t,\omega) \in \Theta$ and all $\mathbb{F}$-stopping times τ taking values in $[t,T]$,*

$$V(t,\omega) = \sup_{\mathbb{Q} \in \mathcal{M}(t,\omega)} \mathbb{E}^{\mathbb{Q}}\big[V(\tau, X)\big]. \tag{18}$$

Finally, V is locally equi-nonincreasing in t.

Proof. (i). First, it is clear that V is non-anticipative by the condition $\mathcal{M}(t,\omega) = \mathcal{M}(t, \omega_{t\wedge\cdot})$ in Assumption 3.1.(i).

(ii). We next prove that V is right equi-continuous. Let $(t,\omega) \in \Theta$, $y \in B_1(\omega_t) \cap E$, $(t^n, \omega^n)_{n \geq 1} \subset \Theta$ be a sequence such that $t_n \searrow t$ and $\|\omega^n_{t^n\wedge\cdot} - \omega_{t\wedge\cdot}\| \longrightarrow 0$. By Assumption 3.3(i), for any $\varepsilon > 0$ and $\mathbb{Q} \in \mathcal{M}(t, \omega \boxplus_t y)$ such that $\mathbb{E}^{\mathbb{Q}}[\Phi(X)] \geq V(t, \omega \boxplus_t y) - \varepsilon$, there exists a sequence of $(\mathbb{Q}_n)_{n\geq 1}$ such that $\mathbb{Q}_n \in \mathcal{M}(t^n, \omega^n \boxplus_{t^n} y)$ and $\mathbb{E}^{\mathbb{Q}_n}[\Phi(X)] \geq \mathbb{E}^{\mathbb{Q}}[\Phi(X)] - \varepsilon$ for n large enough. This implies that

$$\liminf_{n\longrightarrow\infty} V(t^n, \omega^n \boxplus_{t^n} y) \;\geq\; \liminf_{n\longrightarrow\infty} \mathbb{E}^{\mathbb{Q}_n}[\Phi(X)] \;\geq\; \mathbb{E}^{\mathbb{Q}}[\Phi(X)] - \varepsilon \;\geq\; V(t, \omega \boxplus_t y) - 2\varepsilon.$$

Next, let $(\mathbb{Q}'_n)_{n\geq 1}$ be a sequence such that $\mathbb{Q}'_n \in \mathcal{M}(t^n, \omega^n \boxplus_{t^n} y)$ and $\lim_{n\longrightarrow\infty} \mathbb{E}^{\mathbb{Q}'_n}[\Phi(X)] = \limsup_{n\longrightarrow\infty} V(t^n, \omega^n \boxplus_{t^n} y)$. By Assumption 3.3(i) again, for all $\varepsilon > 0$, there exists a sequence $(\mathbb{Q}_n)_{n\geq 1} \subset \mathcal{M}(t, \omega \boxplus_t y)$ such that $\mathbb{E}^{\mathbb{Q}_n}[\Phi(X)] \geq \mathbb{E}^{\mathbb{Q}'_n}[\Phi(X)] - \varepsilon$ for $n \geq 1$ large enough. Hence,

$$\limsup_{n\longrightarrow\infty} V(t^n, \omega^n \boxplus_{t^n} y) \;=\; \lim_{n\longrightarrow\infty} \mathbb{E}^{\mathbb{Q}'_n}[\Phi(X)] \;\leq\; \limsup_{n\longrightarrow\infty} \mathbb{E}^{\mathbb{Q}_n}[\Phi(X)] + \varepsilon \;\leq\; V(t, \omega \boxplus_t y) + \varepsilon.$$

By arbitrariness of ε, one concludes that $\lim_{n\longrightarrow\infty} V(t^n, \omega^n \boxplus_{t^n} y) = V(t, \omega \boxplus_t y)$. Taking $y = \omega_t$, this implies that $V \in \mathbb{C}_{\mathrm{r}}(\Theta)$. Further, by (4), the maps $y \mapsto V(t^n, \omega^n \boxplus_{t^n} y)$ is Lipschtiz, uniformly in n, it follows that $V(t^n, \omega^n \boxplus_{t^n} y) \longrightarrow V(t, \omega \boxplus_t y)$ uniformly in $y \in B_1(\omega_t) \cap E$, i.e. V is right equi-continuous.

(iii). Finally, the dynamic programming principle (18) is a direct consequence of Assumption 3.1.(iii). Moreover, using Assumption 3.3.(iii), it is direct to obtain that

$$V(t+h, \omega_{t\wedge\cdot} \boxplus_{t+h} y) \;\leq\; V(t, \omega \boxplus_t y) + \rho_K(|y - \omega_t|)\big(\mathrm{r}_K(t+h) - \mathrm{r}_K(t)\big),$$

that is, V is locally equi-nonincreasing in t. $\square$

Lemma 3.9 *The value function V is Dupire-concave.*

Proof. We first notice that $V(T,\omega) = \Phi(\omega)$ by definition, so that $y \in E \mapsto V(T, \omega \boxplus_T y)$ is concave, for all $\omega \in \Omega$. Let us now set $(t,\omega) \in [0,T) \times \Omega$ and ω^1, ω^2 such that $\omega^1_s = \omega^2_s = \omega_s$ for all $s \in [0,t)$ and $\omega_t = \theta\omega^1_t + (1-\theta)\omega^2_t$ for some $\theta \in (0,1)$. Set $x^1 := \omega^1_t$, $x^2 := \omega^2_t$. By Assumption 3.3.(ii), there exists a family $(\mathbb{Q}_h, A^1_h, A^2_h)_{h\in(0,T-t)}$ such that, for all $h \in (0, T-t)$, one has $\mathbb{Q}_h \in \mathcal{M}(t,\omega)$, $\mathbb{Q}_h[X_{t+h} = \omega^1_t] = \theta$ and $\mathbb{Q}_h[X_{t+h} = \omega^2_t] = 1-\theta$. By (18),

$$\begin{aligned} V(t,\omega) &\geq \mathbb{E}^{\mathbb{Q}_h}\big[V(t+h,X)\big] \\ &= \theta\mathbb{E}^{\mathbb{Q}_h}\big[V(t+h,X)\big|X_{t+h} = \omega^1_t\big] + (1-\theta)\mathbb{E}^{\mathbb{Q}_h}\big[V(t+h,X)\big|X_{t+h} = \omega^2_t\big] \\ &\geq \theta \inf_{\omega' \in A^1_h} V(t+h,\omega') + (1-\theta) \inf_{\omega'\in A^2_h} V(t+h,\omega'). \end{aligned}$$

Fix $i \in \{1,2\}$ and $\varepsilon > 0$. Let $(\omega^{h,i})_{h>0}$ be such that $\omega^{h,i} \in A^i_h$ and

$$\inf_{\omega'\in A^i_h} V(t+h,\omega') \;\geq\; V(t+h,\omega^{h,i}) - \varepsilon,$$

for each $h > 0$. Let $(\mathbb{Q}^i_n)_{n\geq1} \subset \mathcal{M}(t,\omega^i)$ be such that $V(t,\omega^i) = \lim_{n\longrightarrow\infty} \mathbb{E}^{\mathbb{Q}^i_n}[\Phi(X)]$. Then, by Assumption 3.3(ii), we can find $h_n \longrightarrow 0$ and a sequence $(\mathbb{Q}'_{h_n})_{n\geq1}$ such that $\mathbb{Q}'_{h_n} \in \mathcal{M}(t + h_n, \omega^{h_n,i})$ and $\mathbb{E}^{\mathbb{Q}^i_n}[\Phi(X)] \leq \mathbb{E}^{\mathbb{Q}'_{h_n}}[\Phi(X)] + \varepsilon$ for all $n \geq 1$. It follows that

$$V(t,\omega^i) \;\leq\; \limsup_{n\longrightarrow\infty} \mathbb{E}^{\mathbb{Q}'_{h_n}}[\Phi(X)] + \varepsilon \;\leq\; \limsup_{n\longrightarrow\infty} V(t+h_n,\omega^{h_n,i}) + \varepsilon.$$

Combining the above implies that

$$V(t,\omega) \;\geq\; \theta V(t,\omega^1) + (1-\theta)V(t,\omega^2),$$

i.e. V is Dupire-concave. $\square$

3.2 Example 1: robust hedging with positive martingales

We now provide a first typical example of application, where we consider the one-dimensional ($d = 1$ for simplicity) non-negative martingales. Let $E = \mathbb{R}_+ = [0,\infty)$, so that $\Omega = D([0,T],\mathbb{R}_+)$. Let

$$\mathcal{M}_+(t,\omega) := \big\{\mathbb{Q} \;:\mathbb{Q}[X_{t\wedge\cdot} = \omega_{t\wedge\cdot}] = 1,\; X \text{ is } \mathbb{Q}\text{–martingale on } [t,T]\big\}.$$

Given

$$M_t(\omega) := \sup_{0\leq s\leq t} \omega_s, \quad m_t(\omega) := \inf_{0\leq s\leq t} \omega_s, \quad A_t(\omega) := \int_0^t \omega_s\mu(ds),\; t \leq T,$$

in which μ is a finite signed measure on $[0,T]$ without atom, and a uniformly continuous function $\phi : \mathbb{R}^4 \longrightarrow \mathbb{R}$, we define

$$\Phi(\omega) \;:=\; \phi\big(M_T(\omega), m_T(\omega), A_T(\omega), \omega_T\big).$$

We assume that ϕ is uniformly Lipschitz in a, and that there exists some constant $K > 0$ such that, for all $0 \leq M_0 \leq M_1$, $0 \leq m_1 \leq w_1 \wedge \varepsilon$ and $a_0, a_1 \in \mathbb{R}$,

$$\Big|\phi(M_1, m_1, a_1, w_1) - \phi(M_0, 0, a_0, 0)\Big| \leq K\big(|a_1 - a_0| + w_1\big). \tag{19}$$

Let us then introduce the value function

$$V_+(t,\omega) := \sup_{\mathbb{Q}\in\mathcal{M}_+(t,\omega)} \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big], \text{ for all } (t,\omega) \in [0,T) \times \Omega,$$

and $V_+(0,x) := V_+(0, x\mathbf{1}_{[0,T]})$ as well as $\mathcal{M}_+(0,x) := \mathcal{M}_+(0, x\mathbf{1}_{[0,T]})$, for each $x \in \mathbb{R}_+$.

Remark 3.10 *The technical condition* (19) *will be used to ensure Condition* (4). *First,* (19) *implies that*

$$\big|\Phi(\omega)\big| \;\le\; K\Big(1+\omega_T+\int_0^T \omega_t|\mu|(dt)\Big), \quad \text{for all } \omega\in\Omega, \tag{20}$$

for some constant $K>0$, and hence

$$|V_+(t,\omega)| \;\le\; K\Big(1+\omega_t+\int_0^T \omega_{t\wedge\cdot}|\mu|(dt)\Big), \tag{21}$$

as X is a non-negative martingale on $[t,T]$ under each $\mathbb{Q}\in\mathcal{M}_+(t,\omega)$. When V_+ is Dupire-concave, one has

$$\max_{z\in\partial V_+(t,\omega)}|z| \le \max\big\{|V_+(t,\omega\boxplus_t(\omega_t+1))-V_+(t,\omega)|\vee|z_0| \;:z_0\in\partial V_+(t,\omega\boxplus_t 0)\big\}. \tag{22}$$

Next, we will use Condition (19) *again to obtain a bound on $|z_0|$ for $z_0\in\partial V_+(t,\omega\boxplus_t 0)$, which then ensures that the super-gradient in $\partial V_+(t,\omega)$ are also locally bounded.*

Example 3.11 *Let $\phi(M,m,a,w)=f(a)+(w-g(M))_+$ for some Lipschitz function $f:\mathbb{R}\longrightarrow\mathbb{R}$ and a uniformly continuous non-negative function $g:\mathbb{R}_+\longrightarrow\mathbb{R}_+$, then* (20) *and* (19) *hold true.*

Proposition 3.12 *Let the conditions of this subsection hold. Then, for all $x\in\mathbb{R}_+$,*

$$V_+(0,x) \;=\; \inf\big\{v\in\mathbb{R}\;:\exists H\in\mathcal{H}\ s.t.\ Y_T^{v,H}\ge\Phi(X),\ \mathcal{M}_+(0,x)-\text{q.s.}\big\}. \tag{23}$$

Moreover, there exists a $\mathbb{F}$-predictable process $H\in\mathcal{H}$ such that $H_s\in\partial V_+(s,X^{s-})$[3] *for all $s\in[0,T]$, $\mathcal{M}_+(0,x)$–q.s., and*

$$Y_T^{V_+(0,x),H}\ge\Phi(X),\ \mathcal{M}_+(0,x)\text{–}q.s.$$

Remark 3.13 *A duality result similar to* (23) *has been proved in Guo, Tan and Touzi [14, Theorem 5.3], using the discretization technique of Dolinsky and Soner [8] together with the S-topology technique of Jakubowski [16]. In [14, Theorem 5.3], the payoff function Φ is essentially assumed to be upper semi-continuous w.r.t. the S-topology and uniformly continuous w.r.t. the Skorokhod topology. They define the super-hedging price in terms of dynamic trading strategies H that are restricted to be piecewisely constant, so that the integration $\int_0^T H_t\cdot dX_t$ can be defined ω by ω, and the super-hedging property $Y_T^{v,H}\ge\Phi(X)$ also holds ω by ω. Our super-hedging property $Y_T^{v,H}\ge\Phi(X)$ holds in a quasi-sure sense, but we do not require the (semi-)continuity property w.r.t. the S-topology (note that a uniformly continuous function of $(M_T(\omega),m_T(\omega))$ is generally not upper semi-continuous in ω under the S-topology). Meanwhile, we are able to prove the existence of an optimal super-hedging strategy, which can not hold in general in the setting of [14]. Such an optimal strategy is even given by an explicit expression, recall Remark 2.3.*

Remark 3.14 *The duality result in Proposition 3.12 covers a class of derivative options depending on the running underlying, running average/maximum/minimum, under additional continuity conditions. In this aspect, it is much less general than classical results such as in Nutz [22], where no regularity condition on the payoff function is required. The continuity condition is mainly due to our approach, which relies on the functional Itô analysis technique. This approach nevertheless does not require the "domination" condition in [22] and provides an explicit expression of the optimal superhedging strategy in terms of V_+, which is trackable from a computational point of view.*

[3] See also (13) and Remark 2.7 for an explicit expression of H in terms of V_+.

Proof. (of Proposition 3.12). (i) Let $\widehat{\Phi} : \Omega \longrightarrow \mathbb{R}$ be the smallest function dominating Φ and such that $y \mapsto \widehat{\Phi}(\omega \boxplus_T y)$ is concave on $\mathbb{R}_+$. We claim that

$$V_+(t,\omega) = \widehat{V}_+(t,\omega) := \sup_{\mathbb{P}\in\mathcal{M}_+(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[\widehat{\Phi}(X)\big], \text{ for all } (t,\omega) \in [0,T) \times \Omega. \tag{24}$$

Indeed, by the definition of $\widehat{\Phi}$, there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and a measurable map $\xi : \Omega \times \Omega^* \longrightarrow \mathbb{R}_+$ such that, for all $\omega \in \Omega$,

$$\mathbb{E}^{\mathbb{P}^*}[\xi(\omega,\cdot)] = \omega_T, \text{ and } \widehat{\Phi}(\omega) = \mathbb{E}^{\mathbb{P}^*}[\Phi(\omega^{T-} \boxplus_T \xi(\omega,\cdot))].$$

Let $(t,\omega) \in [0,T) \times \Omega$ and $\mathbb{P} \in \mathcal{M}_+(t,\omega)$, we consider the product space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega^*, \mathcal{F}_T \otimes \mathcal{F}^*, \mathbb{P} \times \mathbb{P}^*)$, and define the process

$$\overline{X}_t := X_t \mathbf{1}_{\{t\in[0,T)\}} + \big(X_T + \xi\big)\mathbf{1}_{\{t=T\}}, \ t \le T.$$

Then,

$$\mathbb{Q} := \overline{\mathbb{P}} \circ \overline{X}^{-1} \in \mathcal{M}_+(t,\omega) \text{ and } \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi(\overline{X})\big] = \mathbb{E}^{\mathbb{P}}\big[\widehat{\Phi}(X)\big].$$

This implies (26). We next set $\widehat{V}_+(T,\omega) := \widehat{\Phi}(\omega)$ and claim that the conditions of Theorem 3.6 are satisfied for $\widehat{\Phi}$ and $\widehat{V}_+$, so that

$$\begin{aligned}\sup_{\mathbb{Q}\in\mathcal{M}_+(0,x)} \mathbb{E}^{\mathbb{Q}}\big[\widehat{\Phi}(X)\big] &= \inf\big\{v \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ s.t. } Y_T^{v,H} \ge \widehat{\Phi}(X), \ \mathcal{M}_+(0,x) - \text{q.s.}\big\} \\ &\ge \inf\big\{v \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ s.t. } Y_T^{v,H} \ge \Phi(X), \ \mathcal{M}_+(0,x) - \text{q.s.}\big\},\end{aligned}$$

and we conclude by appealing to (16) applied to $(\widehat{V}_+, \widehat{\Phi})$, and the existence result of Theorem 3.6.

(ii) It remains to check that the conditions of Theorem 3.6 are satisfied for $\widehat{\Phi}$ and $\widehat{V}_+$. By construction $\widehat{\Phi}$ is Dupire-concave and it is straightforward to check that it inherits the uniform continuity of Φ, as a function of $(M_T(\omega), m_T(\omega), A_T(\omega), \omega_T)$, as well as the bound (20) on Ω. First, it is easy to see that (14) holds true. In the following, we check the remaining conditions in Theorem 3.6.

(a) As for Assumption 3.1, it is obvious that Items (i) and (ii) hold for $\mathcal{M}_+(t,\omega)$. To check Item (iii).(a), we notice that a martingale is still a martingale under the r.c.p.d. For Item (iii).(b), one can apply measurable selection arguments, as in e.g. El Karoui and Tan [11], in which the essential argument is to check that the graph set $[[\mathcal{M}_+]] := \{(t,\omega,\mathbb{Q}) \in [0,T] \times \Omega \times \mathcal{P}(\Omega) \ : \mathbb{Q} \in \mathcal{M}_+(t,\omega)\}$ is a Borel (or only analytic) subset of $[0,T] \times \Omega \times \mathcal{P}(\Omega)$, where $\mathcal{P}(\Omega)$ denotes the space of all Borel probability measures on Ω. Indeed, $[[\mathcal{M}_+]]$ is a Borel set as it can rewritten as

$$\begin{aligned}[[\mathcal{M}_+]] = \big\{(t,\omega,\mathbb{Q}) \in [0,T] \times \Omega \times \mathcal{P}(\Omega) \ : \mathbb{Q}[X_{s\wedge t} = \omega_{s\wedge t}] = 1, \ \mathbb{E}^{\mathbb{Q}}[|X_r| + |X_s|] < \infty, \\ \mathbb{E}^{\mathbb{Q}}\big[(X_{t\vee s} - X_{t\vee r})\xi\big] = 0, \text{ for all } (r,s,\xi) \in L\big\},\end{aligned}$$

where $L := L_1 \cup L_2$, L_1 is a countable dense subset of

$$\big\{(r,s,\xi) \ : 0 \le r < s \le T, \ \xi \text{ being bounded continuous and } \mathcal{F}_r\text{–measurable}\big\},$$

and L_2 is a countable dense subset of

$$\big\{(r,T,\xi) \ : 0 \le r < T, \ \ \xi \text{ being bounded continuous and } \mathcal{F}_r\text{–measurable}\big\}.$$

(b) For Assumption 3.3.(i), we shall use constructions that preserve the max, the min, the T-value and the integral w.r.t. μ of a given path up at a uniform distance, possibly up to an event with vanishing probability. The uniform continuity of $\omega \mapsto \widehat{\Phi}(\omega)$ as a function of $(M_T(\omega), m_T(\omega), A_T(\omega), \omega_T)$ will then allow us to conclude.

Let us first consider $(t,\omega) \in [0,T) \times \Omega$, $\eta > 0$, $(t',\omega') \in B_{2\eta}(t,\omega)$, i.e. $t' \geq t$, $|t'-t| + \|\omega_{t\wedge\cdot} - \omega'_{t'\wedge\cdot}\| \leq 2\eta$ and $y \in B_1(\omega_t) \cap E$. For simplicity, we can assume that $t = 0$, $t' = \eta$ and $\|\omega'_{\eta\wedge\cdot} - \omega_{0\wedge\cdot}\| \leq \eta$. Let us fix $\mathbb{Q} \in \mathcal{M}_+(0, \omega \boxplus_0 y)$, we construct a process $\overline{X}^\eta$ as follows:

$$\overline{X}^\eta_s := \begin{cases} \omega'_s, & \text{when } s \in [0,\eta), \\ X_{2(s-\eta)}, & \text{when } s \in [\eta, 2\eta), \\ X_s, & \text{when } s \in [2\eta, T]. \end{cases}$$

Then $\mathbb{Q}'_\eta := \mathbb{Q} \circ (\overline{X}^\eta)^{-1} \in \mathcal{M}_+(t', \omega' \boxplus_{t'} y)$ and $\lim_{\eta\longrightarrow 0} \mathbb{E}^{\mathbb{Q}}\big[\widehat{\Phi}(\overline{X}^\eta)\big] = \mathbb{E}^{\mathbb{Q}}\big[\widehat{\Phi}(X)\big]$ (recall that $\widehat{\Phi}$ is uniformly continuous). Thus, for all $\varepsilon > 0$, $\mathbb{E}^{\mathbb{Q}}[\widehat{\Phi}(X)] \leq \mathbb{E}^{\mathbb{Q}'_\eta}[\widehat{\Phi}(X)] + \varepsilon$ for $\eta > 0$ small enough. This proves Item (i).(a) of Assumption 3.3.

Next, let $(t,\omega) \in [0,T) \times \Omega$, $\eta > 0$, $(t',\omega') \in B_{2\eta}(t,\omega)$ and $y \in B_1(\omega_t) \cap E$. W.l.o.g., let us assume that $t = 0$ and $t' = \eta$. Then, for each $\mathbb{Q}'_\eta \in \mathcal{M}_+(\eta, \omega' \boxplus_\eta y)$, we construct a process $\overline{X}^\eta$ by

$$\overline{X}^\eta_s := \begin{cases} y, & \text{when } s \in [0,\eta), \\ X_s, & \text{when } s \in [\eta, T]. \end{cases}$$

Then it is easy to check that $\mathbb{Q}_\eta := \mathbb{Q}'_\eta \circ (\overline{X}^\eta)^{-1} \in \mathcal{M}_+(0, \omega \boxplus_0 y)$, and $\lim_{\eta\longrightarrow 0}(\mathbb{E}^{\mathbb{Q}'_\eta}[\widehat{\Phi}(X)] - \mathbb{E}^{\mathbb{Q}_\eta}[\widehat{\Phi}(X)]) = 0$. This shows that, for all $\varepsilon > 0$, $\mathbb{E}^{\mathbb{Q}'_\eta}[\widehat{\Phi}(X)] \leq \mathbb{E}^{\mathbb{Q}_\eta}[\widehat{\Phi}(X)] + \varepsilon$ when $\eta > 0$ is small enough. We hence proved Item (ii).(b) of Assumption 3.3.

(c) Let us then check Item (ii) of Assumption 3.3. We use a similar type of construction as in step (b) above. Let $(t,\omega) \in [0,T) \times \Omega$ and $x^1, x^2 \in \mathbb{R}_+$ be such that $x^1 < \omega_t < x^2$. For each $h > 0$, $i = 1,2$, we define A^i_h by

$$A^i_h := \big\{\omega' \in \Omega\ : \omega' = \omega_{t\wedge\cdot} \text{ on } [0, t+h),\ \omega'_{t+h} = x^i\big\},$$

and let $\mathbb{Q}_h \in \mathcal{M}_+(t,\omega)$ be such that $\mathbb{Q}_h[X_s = \omega_t,\ s \in [t,t+h)] = 1$ and $\mathbb{Q}_h[X_{t+h} = x^1] + \mathbb{Q}_h[X_{t+h} = x^2] = 1$. Then, for each $h > 0$ and $i = 1,2$, we define $\overline{X}^{h,i}$ by

$$\overline{X}^{h,i}_s := \omega_{t\wedge s}\mathbf{1}_{\{s\in[0,t+h)\}} + \big(x^i + X_{t+2(s-t-h)} - X_t\big)\mathbf{1}_{\{s\in[t+h,t+2h)\}} + \big(x^i + X_s - X_t\big)\mathbf{1}_{\{s\in[t+2h,T]\}}.$$

For every $\mathbb{Q} \in \mathcal{M}_+(t, \omega \boxplus_t x^i)$, we notice that $\mathbb{Q}'_h := \mathbb{Q} \circ (\overline{X}^{h,i})^{-1} \in \mathcal{M}_+(t+h, \omega_{t\wedge\cdot} \boxplus_{t+h} x^i)$ and that $\lim_{h\longrightarrow 0} \mathbb{E}^{\mathbb{Q}'_h}[\widehat{\Phi}(X)] = \lim_{h\longrightarrow 0} \mathbb{E}^{\mathbb{Q}}[\widehat{\Phi}(\overline{X}^{h,i})] = \mathbb{E}^{\mathbb{Q}}[\widehat{\Phi}(X)]$, which is enough to conclude that Item (ii) of Assumption 3.3 holds true.

(d) To prove Assumption 3.3.(iii), we consider $t \in [0,T)$, $h \in (0, T-t]$, $\omega \in \Omega$, and $y \in B_1(\omega_t) \cap E$. Let $\mathbb{Q} \in \mathcal{M}_+(t+h, \omega_{t\wedge\cdot} \boxplus_{t+h} y)$, we define $\overline{X}^h$ by

$$\overline{X}^h_t := \begin{cases} \omega_s, & \text{when } s \in [0,t), \\ y, & \text{when } s \in [t, t+h), \\ X_s, & \text{when } s \in [t+h, T]. \end{cases}$$

One observes that $\overline{X}^h_T = X_T$, $m_T(\overline{X}^h) = m_T(X)$, $M_T(\overline{X}^h) = M_T(X)$, and $A_T(\overline{X}^h) = A_T(X) + (y - \omega_t)h$. Moreover, $\mathbb{Q}_h := \mathbb{Q} \circ (\overline{X}^h)^{-1} \in \mathcal{M}_+(t, \omega \boxplus_t y)$. As ϕ is uniformly Lipschitz in a, we then conclude that, for some constant L,

$$\mathbb{E}^{\mathbb{Q}_h}[\Phi(X)] \;\ge\; \mathbb{E}^{\mathbb{Q}}[\Phi(X)] \;-\; L|y-\omega_t|h.$$

This proves Assumption 3.3.(iii).

(e) Finally, we prove that V_+ satisfies (4). As discussed in Remark 3.10, the growth condition (20) implies the locally boundedness of the function V_+, c.f. (21). Further, in view of (22), it is enough to prove that $y \mapsto V_+(t, \omega \boxplus_t y)$ is Lipschitz on $[0,\varepsilon]$ for some $\varepsilon > 0$. This is true since, for $y \in [0,\varepsilon]$,

$$\big|V_+(t,\omega\boxplus_t y) - V_+(t,\omega\boxplus_t 0)\big| \le \sup_{\mathbb{Q}\in\mathcal{M}_+(t,\omega\boxplus_t y)} K\mathbb{E}^{\mathbb{Q}}\Big[|A_T - A_t| + X_T\Big] \;\le\; 2K\big(1\vee|\mu|([t,T])\big)y.$$

□

3.3 Example 2: robust hedging with continuous positive martingales

We can adapt the results of Section 3.2 to the case of continuous martingales, for Asian type options. Let $E = \mathbb{R}_+ = [0,\infty)$, so that $\Omega = D([0,T],\mathbb{R}_+)$. Let

$$\mathcal{M}^c_+(t,\omega) \;:=\; \big\{\mathbb{Q}\in\mathcal{P}(\Omega)\;:\mathbb{Q}[X_{t\wedge\cdot}=\omega_{t\wedge\cdot}]=1,\; X \text{ is a } \mathbb{Q}\text{–continuous martingale on } [t,T]\big\}.$$

We consider the payoff function

$$\Phi(\omega) := \phi\big(A_T(\omega),\omega_T\big), \quad \text{with } A_t(\omega) := \int_0^t \omega_s\mu(ds), \quad t\le T,$$

where μ is a finite signed measure on $[0,T]$ without atom, and $\phi:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is a Lipschtiz function. Similarly, we define the value function

$$V^c_+(t,\omega) := \sup_{\mathbb{Q}\in\mathcal{M}^c_+(t,\omega)} \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big], \text{ for all } (t,\omega)\in[0,T)\times\Omega,$$

and $V^c_+(0,x) := V^c_+(0,x\mathbf{1}_{[0,T]})$ as well as $\mathcal{M}^c_+(0,x) := \mathcal{M}^c_+(0,x\mathbf{1}_{[0,T]})$, for each $x\in\mathbb{R}_+$.

Remark 3.15 (i) *As ϕ is Lipschitz, it follows that the growth conditions* (20) *and* (21) *hold true in our context.*

(ii) *We only consider Asian type payoff functions in this context of continuous martingales, which excludes in practice the lookback type options. The main reason is that the Dupire concavity condition involves path jumps in its definition. In our proof, one needs to approximate paths with jumps by continuous martingales, which changes both the running maximum and the running minimum at the same time, at a non neglectable order. The approximation of paths would not imply the approximation of the value function whenever it depends on M_T and m_T.*

Proposition 3.16 *Let the conditions of this subsection hold. Then, for all $x\in\mathbb{R}_+$,*

$$V^c_+(0,x) \;=\; \inf\big\{v\in\mathbb{R}\;:\exists H\in\mathcal{H} \text{ s.t. } Y^{v,H}_T \ge \Phi(X),\; \mathcal{M}^c_+(0,x)-\text{q.s.}\big\}. \tag{25}$$

Moreover, there exists a $\mathbb{F}$-predictable process $H\in\mathcal{H}$ such that $H_s\in\partial V^c_+(s,X^{s-})$ for all $s\in[0,T]$, $\mathcal{M}^c_+(0,x)$–q.s., and

$$Y^{V^c_+(0,x),H}_T \ge \Phi(X),\; \mathcal{M}^c_+(0,x)\text{–}q.s.$$

Remark 3.17 *The duality result in* (25) *has already been established in e.g. [1, 20], under much more general conditions. The only new point is somehow the expected fact that* $H_s \in \partial V_+^c(s, X^{s-})$*, for all* $s \in [0,T]$.

Proof. (of Proposition 3.16). We follow the same main steps as in the proof of Proposition 3.12.

(i) Let $\widehat{\Phi} : \Omega \longrightarrow \mathbb{R}$ be the smallest function dominating Φ and such that $y \mapsto \widehat{\Phi}(\omega \boxplus_T y)$ is concave on $\mathbb{R}_+$. We first show that

$$V_+^c(t,\omega) = \widehat{V}_+^c(t,\omega) := \sup_{\mathbb{P} \in \mathcal{M}_+^c(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[\widehat{\Phi}(X)\big], \text{ for all } (t,\omega) \in [0,T) \times \Omega. \tag{26}$$

By the definition of $\widehat{\Phi}$, there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and a measurable map $\xi : \Omega \times \Omega^* \longrightarrow \mathbb{R}_+$ such that, for all $\omega \in \Omega$,

$$\mathbb{E}^{\mathbb{P}^*}[\xi(\omega,\cdot)] = \omega_T, \quad \text{and} \quad \widehat{\Phi}(\omega) = \mathbb{E}^{\mathbb{P}^*}[\Phi(\omega^{T-} \boxplus_T \xi(\omega,\cdot))].$$

Assuming that $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ is the canonical space of continuous functions endowed with the Wiener measure, and using the martingale representation theorem, one can construct a diffusion martingale process $\big(Z_t^\varepsilon(\omega,\omega^*)\big)_{t \in [T, T+\varepsilon]}$ such that

$$Z_T^\varepsilon(\omega,\omega^*) = \omega_T, \quad \text{and } Z_{T+\varepsilon}^\varepsilon(\omega,\omega^*) = \xi(\omega,\omega^*).$$

We now extend the canonical process X with Z^ε on $[0, T+\varepsilon]$ and then rescale it to $[0,T]$. More precisely, let $(t,\omega) \in [0,T) \times \Omega$ and $\mathbb{P} \in \mathcal{M}_+^c(t,\omega)$, we consider the product space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) := (\Omega \times \Omega^*, \mathcal{F}_T \otimes \mathcal{F}^*, \mathbb{P} \times \mathbb{P}^*)$, and define the process

$$\widehat{X}_t^\varepsilon \;:=\; X_t \mathbf{1}_{\{t \in [0,T]\}} + Z_t \mathbf{1}_{\{t \in (T, T+\varepsilon]\}}, \; t \in [0, T+\varepsilon], \quad \text{and then} \;\; \overline{X}_t^\varepsilon := \widehat{X}^\varepsilon_{t(T+\varepsilon)/T}, \; t \in [0,T].$$

One can easily check that

$$\mathbb{Q}^\varepsilon := \overline{\mathbb{P}} \circ (\overline{X}^\varepsilon)^{-1} \in \mathcal{M}_+^c(t,\omega) \;\; \text{and} \;\; \lim_{\varepsilon \searrow 0} \mathbb{E}^{\mathbb{Q}^\varepsilon}\big[\Phi(X)\big] = \lim_{\varepsilon \searrow 0} \mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi(\overline{X}^\varepsilon)\big] = \mathbb{E}^{\mathbb{P}}\big[\widehat{\Phi}(X)\big].$$

This implies (26). We next set $\widehat{V}_+^c(T,\omega) := \widehat{\Phi}(\omega)$ and claim that all the conditions of Theorem 3.6 are satisfied for $\widehat{\Phi}$ and $\widehat{V}_+^c$, so that

$$\begin{aligned}
\sup_{\mathbb{Q} \in \mathcal{M}_+^c(0,x)} \mathbb{E}^{\mathbb{Q}}\big[\widehat{\Phi}(X)\big] &= \inf\big\{v \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ s.t. } Y_T^{v,H} \geq \widehat{\Phi}(X), \; \mathcal{M}_+^c(0,x) - \text{q.s.}\big\} \\
&\geq \inf\big\{v \in \mathbb{R} : \exists\, H \in \mathcal{H} \text{ s.t. } Y_T^{v,H} \geq \Phi(X), \; \mathcal{M}_+^c(0,x) - \text{q.s.}\big\}.
\end{aligned}$$

We can then conclude by appealing to (16) applied to $(\widehat{V}_+^c, \widehat{\Phi})$, and the existence result of Theorem 3.6.

(ii) As in Proposition 3.12, it remains to check that the conditions of Theorem 3.6 are satisfied for $\widehat{\Phi}$ and $\widehat{V}_+^c$. In fact, most of the arguments are the same as in the proof of Proposition 3.12, except that we need to adapt the arguments in Step (ii).(a) to check Assumption 3.1, and those in Step (ii).(c) to check Assumption 3.3.(ii).

For Assumption 3.1, we notice that $\Omega_c := C([0,T], \mathbb{R}_+)$ is a closed subset of $\Omega = D([0,T], \mathbb{R}_+)$, and that

$$\begin{aligned}
[[\mathcal{M}_+^c]] \;&:=\; \big\{(t,\omega,\mathbb{Q}) \in [0,T] \times \Omega \times \mathcal{P}(\Omega) \;: \mathbb{Q} \in \mathcal{M}_+^c(t,\omega)\big\} \\
&=\; \big\{(t,\omega,\mathbb{Q}) \in [[\mathcal{M}_+]] \;:\; \mathbb{Q}\big[\omega_t \mathbf{1}_{[0,t]} + X \mathbf{1}_{(t,T]} \in \Omega_c\big] = 1\big\},
\end{aligned}$$

Then $[[\mathcal{M}_+^c]]$ is Borel measurable as $[[\mathcal{M}_+]]$ is, and, by following the arguments in Proposition 3.12, one can check that Assumption 3.1 holds.

(c) For Assumption 3.3.(ii). Let $(t,\omega) \in [0,T) \times \Omega$ and $x^1, x^2 \in \mathbb{R}_+$ be such that $x^1 < \omega_t < x^2$. For each $h > 0$, $i = 1, 2$, we define A_h^i by

$$A_h^i := \{\omega' \in \Omega \ : \omega'_s = \omega_s \text{ for } s \in [0,t],\ \omega'_s \in [x^1, x^2] \text{ for } s \in [t, t+h), \text{ and } \omega'_{t+h} = x^i\}.$$

As in Step (i), for every $h \in [t, T-t]$, one can construct a process M^h equal to ω on $[0,t]$, which is a martingale on $[t,T]$ taking values in $[x^1, x^2]$ on $[t, t+h]$ and satisfying $M_{t+h}^h \in \{x^1, x^2\}$ a.s. In other words, $\mathbb{Q}_h := \mathbb{Q} \circ (M^h)^{-1} \in \mathcal{M}_+^c(t,\omega)$ satisfies that $\mathbb{Q}_h[A_h^1] + \mathbb{Q}_h[A_h^2] = 1$. Next, let $\mathbb{Q} \in \mathcal{M}_+^c(t, \omega \boxplus_t x^i)$. For each $h > 0$, $i = 1, 2$ and $\omega' \in A_h^i$, we define $\overline{X}^{h,i,\omega'}$ by

$$\overline{X}_s^{h,i,\omega'} := \omega'_s \mathbf{1}_{\{s \in [0,t+h)\}} + \big(x^i + X_{t+2(s-t-h)} - X_t\big)\mathbf{1}_{\{s \in [t+h, t+2h)\}} + \big(x^i + X_s - X_t\big)\mathbf{1}_{\{s \in [t+2h, T]\}}.$$

Then it is easy to check that $\mathbb{Q}'_h := \mathbb{Q} \circ (\overline{X}^{h,i,\omega'})^{-1} \in \mathcal{M}_+^c(t+h, \omega')$ and that $\lim_{h \longrightarrow 0} \mathbb{E}^{\mathbb{Q}'_h}[\widehat{\Phi}(X)] = \lim_{h \longrightarrow 0} \mathbb{E}^{\mathbb{Q}}[\widehat{\Phi}(\overline{X}^{h,i,\omega'})] = \mathbb{E}^{\mathbb{Q}}[\widehat{\Phi}(X)]$. This proves the conditions in Assumption 3.3.(ii). $\square$

References

[1] Sara Biagini, Bruno Bouchard, Constantinos Kardaras and Marcel Nutz. Robust fundamental theorem for continuous processes. *Mathematical Finance*, 27(4):963-987, 2017.

[2] Bruno Bouchard and Marcel Nutz. Arbitrage and duality in non-dominated discrete-time models. *The Annals of Applied Probability*, 25(2):823-859, 2015.

[3] Bruno Bouchard and Xiaolu Tan. Understanding the dual formulation for the hedging of path-dependent options with price impact. arXiv:1912.03946, 2019.

[4] Eric Carlen and Philip Protter. On semimartingale decompositions of convex functions of semimartingales. *Illinois journal of mathematics*, 36(3), 1992.

[5] Rama Cont and David-Antoine Fournié. Change of variable formulas for non-anticipative functionals on path space. *Journal of Functional Analysis*, 259(4):1043–1072, 2010.

[6] Rama Cont and David-Antoine Fournié. Functional Itô calculus and stochastic integral representation of martingales. *The Annals of Probability*, 41(1):109–133, 2013.

[7] Freddy Delbaen and Walter Schachermayer. A compactness principle for bounded sequences of martingales with applications. *Seminar on stochastic analysis, random fields and applications.* Birkhäuser, Basel, 1999.

[8] Yan Dolinsky and H. Mete Soner. Martingale optimal transport in the Skorokhod space. *Stochastic Processes and their Applications*, 125(10):3893-3931, 2015.

[9] Bruno Dupire. Functional Itô calculus. *Portfolio Research Paper*, 04, 2009.

[10] Nicole El Karoui and Marie-Claire Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM journal on Control and Optimization*, 33(1):29–66, 1995.

[11] Nicole El Karoui and Xiaolu Tan. Capacities, measurable selection and dynamic programming Part I: abstract framework. arXiv:1310.3363, 2013.

[12] Hans Föllmer and Yuri Kabanov. Optional decomposition and Lagrange multipliers. *Finance and Stochastics*, 2(1):69-81, 1997.

[13] Hans Föllmer and Dmitry Kramkov. Optional decompositions under constraints. *Probability Theory and Related Fields*, 109(1):1-25, 1997.

[14] Gaoyue Guo, Xiaolu Tan and Nizar Touzi. Tightness and duality of martingale transport on the Skorokhod space. *Stochastic Processes and their Applications*, 127(3):927-956, 2017.

[15] Jean Jacod and Albert Shiryaev. Limit theorems for stochastic processes. *Springer Science & Business Media*, 288, 2013.

[16] Adam Jakubowski. A non-Skorohod topology on the Skorohod space. *Electronic journal of probability*, 2, 1997.

[17] Benedikt Köpfer and Ludger Rüschendorf. Comparison of path-dependent functionals of semimartingales. *arXiv preprint* arXiv:1908.10076, 2019.

[18] Dmitry Kramkov. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields*, 105(4):459–479, 1996.

[19] Paul-André Meyer. Un cours sur les intégrales stochastiques (exposés 1 à 6). *Séminaire de probabilités de Strasbourg*, 10:245–400, 1976.

[20] Ariel Neufeld and Marcel Nutz. Superreplication under volatility uncertainty for measurable claims. *Electronic journal of probability*, 18, 2013.

[21] Marcel Nutz. Pathwise construction of stochastic integrals. *Electronic Communications in Probability*, 17(24):1-7, 2012.

[22] Marcel Nutz. Robust superhedging with jumps and diffusion. *Stochastic Processes and their Applications*, 125(12), 4543-4555, 2015.

[23] Dylan Possamaï, Guillaume Royer, and Nizar Touzi. On the robust superhedging of measurable claims. *Electronic Communications in Probability*,18, 2013.

[24] Yuri F. Saporito. The functional Meyer-Tanaka formula. *Stochastics and Dynamics*, 18(04): 1850030, 2018.