Introduction to stochastic control of mixed diffusion processes, viscosity solutions and applications in finance and insurance

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Introduction and notations

These lecture notes have been written as a support for the lecture on stochastic control of the master program Masef of Paris Dauphine.

Our aim is to explain how to relate the value function associated to a stochastic control problem to a well suited PDE. This allows, at least, to approximate it numerically, and, in good cases, to retrieve the optimal control through the explicit resolution of the PDE and a verification argument.

The main tool for deriving the PDE is the dynamic programming principle. It essentially relates the value function at time t to its expected value a time $t+h$ for $h>0$. The link between the PDE and the control problem is then obtained through an application of Itô's Lemma on a small time interval.

We shall first discuss this approach in the case where the stochastic process is not controlled, i.e. we just compute an expectation. This corresponds to the well-known Feynman-Kac formula. This first chapter is crucial in the sense that all the fundamental tools will be presented here.

We first derive the PDE under suitable smoothness assumptions and then explain how to construct a verification argument. We also discuss the question of the uniqueness of the solution.

All this is done under the assumption that the value function (or the solution of the PDE) is at least $C^{1,2}$. This is not the case in general and one has in general to rely on a weak notion of solution. Here, we discuss the notion of viscosity solutions which has become very popular in finance. We show how to prove that the value function is a viscosity solution of the associated PDE and explain how to derive a uniqueness result for viscosity solutions through a comparison theorem.

We then repeat these arguments for various control problems in standard form with finite or infinite time horizon. We also discuss some specific control problems leading to

PDEs with free boundary (optimal stopping, optimal switching and optimal dividend payment) and present a direct approach for solving a general class of stochastic target problems for which the dynamic programming principle takes a very particular form. We finally discuss the stochastic maximum principle which is an extension of the Pontryagin principle in deterministic control.

These different analysis are carried out in different settings in order to introduce various techniques and are illustrated by examples of application in finance and insurance.

Notations:
Any element $x \in \mathbb{R}^d$ will be identified to a column vector with i-th component x^i and Euclidean norm $\|x\|$. The scalar product is denoted by $\langle \cdot, \cdot \rangle$. The set of $d \times d$ (resp. symmetric) matrices is denoted by $\mathbb{M}^d$ (resp. $\mathbb{S}^d$), the superscript * stands for transposition, A^j is the j-th column of A. The norm $\|\cdot\|$ on $\mathbb{M}^d$ is the Euclidean norm obtained when $\mathbb{M}^d$ is identified to $\mathbb{R}^{d\times d}$. For a smooth map $\psi : (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto \psi(t, y)$, we denote by $D\psi$ (resp. $D^2\psi$) its partial gradient (resp. Hessian) with respect to y. When it depends on more variables, we use the more explicit notations D_y, D_{yy}, etc... We denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel tribe associated to $\mathbb{R}^d$. For a set $B \subset \mathbb{R}^d$, $\bar{B}$ and ∂B stands for its closure and boundary. The open ball of center x and radius $\eta > 0$ is denoted by $B(x, \eta)$.

We always work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, which satisfies the usual assumptions. On this space, we shall consider:
- a $\mathbb{R}^d$-*market point process* μ with predictable intensity kernel $\tilde{\mu}$ of the form $\tilde{\mu}_t(dz) = \lambda_t \Phi(dz)$ where Φ is a probability distribution on $\mathbb{R}^d$ and λ a bounded Lipschitz-continuous map from $\mathbb{R}_+$ into $\mathbb{R}_+$. We note $\bar{\mu} = \mu - \tilde{\mu}$ the compensated jump measure.
- a d-dimensional Brownian motion W, independent of μ.

Basic properties of Itô's integral and random measures are presented in the Appendix.

Contents

Chapter 1

Conditional expectations and the Feynman-Kac Formula

In this Chapter, we consider a family of processes $(X_{t,x})_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ defined by the SDEs :

$$X(v) = x + \int_t^v b\left(X(s)\right) ds + \int_t^v \sigma\left(X(s)\right) dW(s) + \int_t^v \int_{\mathbb{R}^d} \beta\left(X(s-), z\right) \mu(dz, ds) \; . \tag{1.0.1}$$

Note that the coefficients may actually depend on time by choosing it in such a way that $X^1(t) = t$. We could also choose them so that the effective dimension of X is different from $\mathbb{R}^d$.

Our aim is to characterize *value functions* of the form

$$v(t,x) := \mathbb{E}\left[\mathcal{E}_{t,x}(T) g(X_{t,x}(T)) + \int_t^T \mathcal{E}_{t,x}(s) f(X_{t,x}(s)) ds\right] \quad (t,x) \in [0,T] \times \mathbb{R}^d \; , \tag{1.0.2}$$

where $\mathcal{E}_{t,x}(s) := e^{-\int_t^s \rho(X_{t,x}(v))dv}$, as solutions of PDEs of the form

$$\mathcal{L}v + f = \rho v \tag{1.0.3}$$

on $[0,T) \times \mathbb{R}^d$ with the boundary condition $v(T,\cdot) = g$. Here, $\mathcal{L}$ is the *Dynkin operator* associated to X:

$$\begin{aligned} \mathcal{L}\varphi(t,x) &:= \frac{\partial}{\partial t}\varphi(t,x) + \langle b(x), D\varphi(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(x) D^2\varphi(t,x)\right] \\ &+ \int_{\mathbb{R}^d} \left(\varphi(t, x+\beta(x,z)) - \varphi(t,x)\right) \lambda_t \Phi(dz) \; . \end{aligned}$$

This allows at least to compute v numerically by using standard PDE solvers. In particular cases, we shall see that (1.0.3) can be solved explicitly, thus providing a closed form formula for v.

1 Markov property and formal derivation

Before turning to the technical proof, let us briefly justify this relation. Under the Lipschitz continuity assumptions that will be imposed in the next section, it is well known that the process $X_{t,x}$ is a *strong Markov process*. In particular, for all stopping times $\tau \geq t$ $\mathbb{P}-$a.s. and $h \geq 0$

$$\mathbb{P}\left[X_{t,x}(\tau+h) \in A \mid \mathcal{F}_\tau\right] = \mathbb{P}\left[X_{t,x}(\tau+h) \in A \mid X_{t,x}(\tau)\right] \ \ \mathbb{P}-\text{a.s.} \tag{1.1.1}$$

Moreover, for all continuous and bounded function g

$$\mathbb{E}\left[g(X_{t,x}(\tau+h)) \mid X_{t,x}(\tau)\right] \ = \ \varphi(\tau, X_{t,x}(\tau); g, h) \ \ \mathbb{P}-\text{a.s.} \tag{1.1.2}$$

where

$$\varphi(s,y;g,h) = \mathbb{E}\left[g(X_{s,y}(s+h))\right] \ .$$

Since $X_{t,x}$ is right-continuous, this also implies that for all $T \geq 0$

$$\mathbb{E}\left[g(X_{t,x}(T)) \mid X_{t,x}(\tau)\right] \mathbf{1}_{T\geq\tau} \ = \ \tilde{\varphi}(\tau, X_{t,x}(\tau); g, T)\mathbf{1}_{T\geq\tau} \ \ \mathbb{P}-\text{a.s.} \tag{1.1.3}$$

where for $s \leq T$

$$\tilde{\varphi}(s,y;g,T) = \mathbb{E}\left[g(X_{s,y}(T))\right] \ .$$

See [21] for more details. Under the same Lipschitz continuity assumptions, (1.0.1) admits a unique solution, even if we replace (t,x) by (θ,ξ) where θ is a stopping time and ξ a $\mathcal{F}_\theta$-measurable $\mathbb{R}^d$-valued random variable. It follows that the family X satisfies the *flow property*

$$X_{t,x} = X_{\theta, X_{t,x}(\theta)} \quad \text{on } [\theta,\infty) \ \mathbb{P}-\text{a.s.} \tag{1.1.4}$$

for all stopping times $\theta \geq t$ $\mathbb{P}-$a.s.

For simplicity, let us for the moment assume that ρ, g and f are bounded. Then, the process Z defined by

$$Z(t) := \mathbb{E}\left[\mathcal{E}_{0,x}(T)g(X_{0,x}(T)) + \int_0^T \mathcal{E}_{0,x}(s)f(X_{0,x}(s))ds \mid \mathcal{F}_t\right] \quad t \in [0,T]$$

is well defined and is a martingale. But, it follows from the flow property (1.1.4) and (1.1.1)-(1.1.2)-(1.1.3) that

$$Z(t) = \mathcal{E}_{0,x}(t)v(t,X_{0,x}(t)) + \int_0^t \mathcal{E}_{0,x}(s)f(X_{0,x}(s))ds \ \ \mathbb{P}-\text{a.s.} \tag{1.1.5}$$

Thus, the right-hand side term is a martingale. Since by Itô's Lemma, see the Appendix, its dynamics has the form

$$dZ(s) = \mathcal{E}_{0,x}(s)\left((\mathcal{L}v + f)(s, X_{0,x}(s)) - (\rho v)(s, X_{0,x}(s))\right) ds + (\cdots)\, dW(s) + (\cdots)\, \bar{\mu}(dz, ds)$$

we see that $(\mathcal{L}v + f)(s, X_{0,x}(s)) - (\rho v)(s, X_{0,x}(s))$ must be equal to 0. This formally leads to (1.0.3).

The rest of this chapter is dedicated to the technical justification of this argument. After having derived some standard controls on $X_{t,x}$ and a *dynamic programming principle* of the form (1.1.5), we first show that v solves (1.0.3) whenever it is $C^{1,2}$. It is completed by a *Comparison Theorem* which implies that there is at most one solution to (1.0.3) satisfying the boundary condition $\varphi(T, \cdot) = g$, in a suitable class of functions. We then provide a *verification argument*, i.e. we show that if φ is a sufficiently regular solution of (1.0.3) satisfying the boundary condition $\varphi(T, \cdot) = g$ then $\varphi = v$. This result is here essentially a consequence of the *Comparison Theorem* but this approach will be useful when we will study control problems. Finally, we study the case where v is not smooth (possibly even not continuous). In this case, we still characterize v as the (unique) solution of (1.0.3) but in the sense of *viscosity solutions*.

2 Assumptions on the coefficients and a-priori estimates on the SDE

From now on, we shall always assume that the coefficients satisfy, for some $L > 0$,

$$\begin{aligned}
&\|\beta(x,z)\| \leq L(1 + \|x\|) \\
&\|\beta(x,z) - \beta(y,z)\| \leq L\|y - x\| \\
&\|b(x)\| + \|\sigma(x)\| \leq L(1 + \|x\|) \\
&\|b(x) - b(y)\| + \|\sigma(x) - \sigma(y)\| \leq L\|y - x\|
\end{aligned} \tag{1.2.1}$$

for all $(x, y, z) \in (\mathbb{R}^d)^2 \times \mathbb{R}^d$, so that existence and uniqueness of a solution to (1.0.1) is guaranteed. Existence follows from a standard fixed point procedure, and uniqueness from estimates similar to the one presented in the next proposition.

Proposition 1.2.1 *Fix $T > 0$ and $p \geq 2$. Then, there is $C > 0$ such that, for all*

$(t,x,y) \in [0,T) \times (\mathbb{R}^d)^2$ *and* $h \in [0, T-t]$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le s\le T}\|X_{t,x}(s)\|^p\right] &\le C\,(1+\|x\|)^p \\
\mathbb{E}\left[\sup_{t\le s\le t+h}\|X_{t,x}(s)-x\|^p\right] &\le C\,h^{\frac{p}{2}}(1+\|x\|^p) \\
\mathbb{E}\left[\sup_{t+h\le s\le T}\|X_{t,x}(s)-X_{t+h,x}(s)\|^p\right] &\le C\,h^{\frac{p}{2}}(1+\|x\|^p) \\
\mathbb{E}\left[\sup_{t\le s\le T}\|X_{t,x}(s)-X_{t,y}(s)\|^p\right] &\le C\,\|x-y\|^p\,.
\end{aligned}
$$

Proof. We denote by $C>0$ a generic constant whose value may change from line to line but which does not depend on x, y or t. Using Jensen's inequality, Proposition A.1.1 and Proposition A.2.3 in the Appendix, we first observe that, for $t \le s \le T$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le u\le s}\|X_{t,x}(u)\|^p\right] &\le C\,\mathbb{E}\left[\|x\|^p+\int_t^s\|b\left(X_{t,x}(v)\right)\|^p\,dv+\left(\int_t^s\|\sigma\left(X_{t,x}(v)\right)\|^2\,dv\right)^{\frac{p}{2}}\right] \\
&+ C\,\mathbb{E}\left[\int_t^s\int_{\mathbb{R}^d}\|\beta\left(X_{t,x}(v),z\right)\|^p\,\lambda_v\Phi(dz)dv\right] \\
&\le C\,\mathbb{E}\left[\|x\|^p+\int_t^s\left(\|b\left(X_{t,x}(v)\right)\|^p+\|\sigma\left(X_{t,x}(v)\right)\|^p\right)dv\right] \\
&+ C\,\mathbb{E}\left[\int_t^s\int_{\mathbb{R}^d}\|\beta\left(X_{t,x}(v),z\right)\|^p\,\lambda_v\Phi(dz)dv\right]\,.
\end{aligned}
$$

In view of (1.2.1), this shows that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le u\le s}\|X_{t,x}(u)\|^p\right] &\le C\,\mathbb{E}\left[1+\|x\|^p+\int_t^s\sup_{t\le u\le v}\|X_{t,x}(u)\|^p\,dv\right] \\
&+ C\,\mathbb{E}\left[\int_t^s\int_{\mathbb{R}^d}\left(1+\sup_{t\le u\le v}\|X_{t,x}(u)\|^p\right)\lambda_v\Phi(dz)dv\right]\,.
\end{aligned}
$$

Since λ is bounded, the first assertion is a consequence of Gronwall's Lemma, see Lemma 1.2.1 below.

By the same arguments, we obtain, for $t \le s \le t+h$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le u\le s}\|X_{t,x}(u)-x\|^p\right] &\le C\,\mathbb{E}\left[h^{\frac{p}{2}}+\int_t^{t+h}\sup_{t\le u\le v}\|X_{t,x}(u)\|^p\,dv\right] \\
&+ C\,\mathbb{E}\left[\int_t^{t+h}\int_{\mathbb{R}^d}\left(1+\sup_{t\le u\le v}\|X_{t,x}(u)\|^p\right)\lambda_v\Phi(dz)dv\right]
\end{aligned}
$$

and deduce the second assertion by using the first one. The two last estimates are obtained by similar arguments. $\square$

We now state the lemma which was used in the above proof.

Lemma 1.2.1 *(Gronwall's Lemma) Fix $T > 0$. Let g be a non-negative measurable real map such that*

$$g(t) \leq \alpha(t) + \kappa \int_0^t g(s) ds \quad \forall\, t \in [0, T],$$

where $\kappa \geq 0$ and $\alpha : [0, T] \mapsto \mathbb{R}$ is integrable. Then,

$$g(t) \leq \alpha(t) + \kappa \int_0^t \alpha(s) e^{\kappa(t-s)} ds \quad \forall\, t \in [0, T].$$

In the following, we shall also assume that
(i) ρ is bounded from below,
(ii) g and f have at most polynomial growth,
(iii) ρ, g and f are continuous.

In view of Proposition 1.2.1, this ensures that the value function v is well defined.

3 Feynman-Kac formula in the regular case

3.1 Derivation of the PDE in the regular case

Our first result relates the value function v at time t in terms of its value at some future time. In terms of processes, it corresponds to (1.1.5). In stochastic control, this kind of equation is called *dynamic programming equation* and is here a simple consequence of (1.1.1)-(1.1.2)-(1.1.3)-(1.1.4).

Proposition 1.3.2 *Let θ be a stopping time such that $\theta \in [t, T]$ $\mathbb{P} - a.s.$ and $X_{t,x}$ is bounded on $[t, \theta]$. Then,*

$$v(t, x) = \mathbb{E}\left[\mathcal{E}_{t,x}(\theta) v(\theta, X_{t,x}(\theta)) + \int_t^\theta \mathcal{E}_{t,x}(s) f(X_{t,x}(s)) ds\right] . \tag{1.3.1}$$

Proof. For ease of notations, we only consider the case where $f = 0$. By the flow property of X, the usual tower property and (1.1.1), we have

$$\begin{aligned} v(t,x) &= \mathbb{E}\left[\mathcal{E}_{t,x}(\theta)\mathbb{E}\left[\mathcal{E}_{\theta,X_{t,x}(\theta)}(T) g(X_{\theta,X_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta\right]\right] \\ &= \mathbb{E}\left[\mathcal{E}_{t,x}(\theta)\mathbb{E}\left[\mathcal{E}_{\theta,X_{t,x}(\theta)}(T) g(X_{\theta,X_{t,x}(\theta)}(T)) \mid X_{t,x}(\theta)\right]\right] . \end{aligned}$$

It then follows from (1.1.3) that

$$\mathbb{E}\left[\mathcal{E}_{\theta,X_{t,x}(\theta)}(T)g(X_{\theta,X_{t,x}(\theta)}(T)) \mid X_{t,x}(\theta)\right] = v(\theta, X_{t,x}(\theta)) \ \ \mathbb{P} - \text{a.s.}$$

□

Using the above proposition, we can now relate v with a suitable PDE in the case where it is smooth enough.

Theorem 1.3.1 *(Feynman-Kac) Assume that v is continuous on $[0,T] \times \mathbb{R}^d$ and $v \in C^{1,2}([0,T)\times\mathbb{R}^d)$. Then, v is a solution on $[0,T)\times\mathbb{R}^d$ of* (1.0.3) *and satisfies the boundary condition* $\lim_{t\nearrow T} v(t,x) = g(x)$ *on $\mathbb{R}^d$.*

Proof. The boundary condition is a consequence of the continuity assumption on v.
It remains to show that v solves (1.0.3).
We now fix $(t,x) \in [0,T) \times \mathbb{R}^d$. Let θ be the first time when $(s, X_{t,x}(s))_{s\geq t}$ exits a given bounded open neighborhood of (t,x). Set $\theta^h = \theta \wedge (t+h)$ for $h > 0$ small. Using (1.3.1), Itô's Lemma, Corollary A.2.1, (A.1.1), we deduce that

$$0 = \mathbb{E}\left[\frac{1}{h}\int_t^{\theta^h} \mathcal{E}_{t,x}(s)\left((\mathcal{L}v+f)(s,X_{t,x}(s)) - (\rho v)(s,X_{t,x}(s))\right)ds\right] .$$

Now, we observe that Proposition 1.2.1 implies that, after possibly passing to a subsequence, $\sup_{t\leq s\leq t+h}\|X_{t,x}(s)-x\| \to 0$ $\mathbb{P}-$a.s. as $h \to 0$. Moreover, $\theta > 0$ $\mathbb{P}-$a.s. so that $(\theta^h - t)/h \to 1$. Using the mean value theorem and the continuity of $\mathcal{L}v + f - \rho v$, we then deduce that, after possibly passing to a subsequence,

$$\begin{aligned}&\frac{1}{h}\int_t^{\theta^h} \mathcal{E}_{t,x}(s)\left((\mathcal{L}v+f)(s,X_{t,x}(s)) - (\rho v)(s,X_{t,x}(s))\right)ds\\ &\to (\mathcal{L}v+f)(t,x) - (\rho v)(t,x) \ \ \mathbb{P}-\text{a.s.}\end{aligned}$$

as $h \to 0$. The required result is then obtained by applying the dominated convergence theorem. □

In order to complete the proof, it remains to show that v is the unique solution of (1.0.3) satisfying the boundary condition $\lim_{t\nearrow T} v(t,x) = g(x)$ on $\mathbb{R}^d$. To this purpose, we first state a *comparison principle* also called *maximum principle*.

Theorem 1.3.2 *(Comparison principle) Assume that U and V are continuous on $[0,T]\times\mathbb{R}^d$ and $C^{1,2}$ on $[0,T)\times\mathbb{R}^d$. Assume further that, on $[0,T)\times\mathbb{R}^d$,*

$$\mathcal{L}U + f \leq \rho U \quad \textit{and} \quad \mathcal{L}V + f \geq \rho V \tag{1.3.2}$$

and that $U(T,x) \geq V(T,x)$ on $\mathbb{R}^d$. Finally assume that U and V have polynomial growth. Then, $U \geq V$ on $[0,T] \times \mathbb{R}^d$.

Remark 1.3.1 When U and V satisfy (1.3.2), we say that U is a super-solution and that V is a sub-solution of (1.0.3).

Remark 1.3.2 The above theorem can be restated as follows. If U and V are $C^0([0,T]\times \mathbb{R}^d) \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ super- and sub-solutions of (1.3.2) on $[0,T) \times \mathbb{R}^d$, then the max of $V - U$ can only be attained on the boundary $\{T\} \times \mathbb{R}^d$. This explains why the above comparison theorem is also called the *Maximum Principle*.

Remark 1.3.3 If we set $\tilde{U}(t,x) = e^{\kappa t}U(t,x)$ and $\tilde{V}(t,x) = e^{\kappa t}V(t,x)$ then

$$\mathcal{L}\tilde{U} + \tilde{f} \leq (\rho + \kappa)\tilde{U} \quad \text{and} \quad \mathcal{L}\tilde{V} + \tilde{f} \geq (\rho + \kappa)\tilde{V}$$

where $\tilde{f}(t,x) = e^{\kappa t}f(x)$. After possibly replacing (U,V) by $(\tilde{U},\tilde{V})$ and taking $\kappa > \rho^-$, we can always assume that $\rho > 0$.

Proof. In view of Remark 1.3.3, we may assume that $\rho > 0$. Assume now that for some $(t_0,x_0) \in [0,T] \times \mathbb{R}^d$ we have $U(t_0,x_0) < V(t_0,x_0)$. We shall show that this leads to a contradiction. Fix $\varepsilon > 0$, $\kappa > 0$ and p an integer greater that 1 such that $\limsup_{\|x\|\to\infty} \sup_{t\leq T}(|U(t,x)| + |V(t,x)|)/(1+\|x\|^p) = 0$. Then, there is $(\hat{t},\hat{x}) \in [0,T] \times \mathbb{R}^d$ such that, for ε small enough,

$$0 \;<\; V(\hat{t},\hat{x}) - U(\hat{t},\hat{x}) - \phi(\hat{t},\hat{x}) = \max_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(V(t,x) - U(t,x) - \phi(t,x)\right) ,$$

where

$$\phi(t,x) := \varepsilon e^{-\kappa t}(1 + \|x\|^{2p}) .$$

Since $U \geq V$ on $\{T\} \times \mathbb{R}^d$, we must have $\hat{t} < T$. Moreover, the one and second order conditions of optimality imply

$$(\partial_t V, DV)(\hat{t},\hat{x}) = (\partial_t U + \partial_t \phi, DU + D\phi)(\hat{t},\hat{x})$$

and

$$D^2V(\hat{t},\hat{x}) \;\leq\; (D^2U + D^2\phi)(\hat{t},\hat{x})$$

in the sense of matrices. Combined with (1.3.2) and the fact that $(V-U-\phi)(\hat{t},\hat{x}+\beta(\hat{t},\hat{x},\cdot))\le(V-U-\phi)(\hat{t},\hat{x})$, this leads to

$$\begin{aligned}
\rho(V-U)(\hat{t},\hat{x}) &\le \mathcal{L}(V-U)(\hat{t},\hat{x}) \\
&\le \partial_t\phi(\hat{t},\hat{x})+\langle b(\hat{x}),D\phi(\hat{t},\hat{x})\rangle+\mathrm{Tr}\left[\sigma\sigma^*(x)D^2\phi(\hat{t},\hat{x})\right] \\
&+ \int_{\mathbb{R}^d}\big((V-U)(\hat{t},\hat{x}+\beta(\hat{t},\hat{x},z))-(V-U)(\hat{t},\hat{x})\big)\,\lambda_{\hat{t}}\Phi(dz) \\
&\le \mathcal{L}\phi(\hat{t},\hat{x})\ .
\end{aligned}$$

Since b,σ and β have polynomial growth, we can choose $\kappa>0$ sufficiently large so that the term in the last bracket is (strictly) negative. This contradicts $(V-U)(\hat{t},\hat{x})>0$ since $\rho>0$. □

Remark 1.3.4 In the above proof, the penalization term ϕ is introduced only to ensure that the max is attained. If we were working on a bounded domain, instead of $\mathbb{R}^d$, this term would not be necessary.

Corollary 1.3.1 *Assume that v is $C^{1,2}([0,T)\times\mathbb{R}^d)\cap C^0([0,T]\times\mathbb{R}^d)$, then it is the unique $C^{1,2}([0,T)\times\mathbb{R}^d)\cap C^0([0,T]\times\mathbb{R}^d)$ solution of* (1.0.3) *satisfying $v(T,\cdot)=g$ in the class of solutions with polynomial growth.*

Proof. Since f and g have polynomial growth and ρ is bounded from below, we deduce from the estimates of Proposition 1.2.1 that v polynomial growth too. The result then follows from Theorems 1.3.1 and 1.3.2. □

3.2 Verification theorem

In practice, the regularity assumptions of the above theorem are very difficult to check and we have to rely on a weaker definition of solutions, like viscosity solutions (see e.g. [7] and [6]), or to use a *verification theorem* which essentially consists in showing that, if a smooth solution of (1.0.3) exists, then it coincides with v.

Theorem 1.3.3 *(Verification) Assume that there exists a $C^{1,2}([0,T)\times\mathbb{R}^d)$ solution φ to* (1.0.3) *with polynomial growth such that*

$$\lim_{t\nearrow T,x'\to x}\varphi(t,x')=g(x)\quad \text{on } \mathbb{R}^d\ . \tag{1.3.3}$$

Then, $v=\varphi$.

Proof. Given $n \geq 1$, set

$$\theta_n := \inf\{s \geq t : \ \|X_{t,x}(s)\| \geq n\} .$$

Note that (1.2.1) implies that $X_{t,x}$ is bounded on $[t, \theta \wedge T]$. By Itô's Lemma, Corollary A.2.1, (A.1.1) and the fact that φ solves (1.0.3), we obtain

$$\varphi(t,x) = \mathbb{E}\left[\mathcal{E}_{t,x}(\theta_n \wedge T)\varphi(\theta_n \wedge T, X_{t,x}(\theta_n \wedge T)) + \int_t^{\theta_n \wedge T} \mathcal{E}_{t,x}(s) f(X_{t,x}(s))ds\right] \quad (1.3.4)$$

for each n. Now, observe that $\theta_n \to \infty$ as $n \to \infty$. In view of (1.3.3), this implies that

$$\begin{aligned}
&\mathcal{E}_{t,x}(\theta_n \wedge T)\varphi(\theta_n \wedge T, X_{t,x}(\theta_n \wedge T)) + \int_t^{\theta_n \wedge T} \mathcal{E}_{t,x}(s) f(X_{t,x}(s))ds \\
&\longrightarrow \mathcal{E}_{t,x}(T) g(X_{t,x}(T)) + \int_t^{T} \mathcal{E}_{t,x}(s) f(X_{t,x}(s))ds \ \ \mathbb{P}-\text{a.s.}
\end{aligned}$$

Using Proposition 1.2.1, we then deduce that $\varphi = v$ by sending $n \to \infty$ in (1.3.4) and using the dominated convergence theorem. $\Box$

Remark 1.3.5 In the case where $\sigma = 0$, then we only need v to be C^1 since no second order term appears in the PDE and in Itô's Lemma. If X takes values in $(0,\infty)^d$ then the PDE has to hold on $[0,T) \times (0,\infty)^d$ and the boundary condition has to be written on $(0,\infty)^d$ as well. We shall discuss in Section 5.1 a case where the PDE is satisfied on a cylindrical set $[0,T) \times \mathcal{O}$ for some open set $\mathcal{O} \subset \mathbb{R}^d$. In this case, one has to specify a boundary condition on $[0,T) \times \partial\mathcal{O}$.

4 Viscosity solutions: definitions and derivation of the PDE in the non-regular case

As explained above, it is in general very difficult to derive some a-priori regularity on v or on the solutions of (1.0.3). When σ is *uniformly elliptic*, i.e. there is $c > 0$ such that for all $\xi \in \mathbb{R}^d$

$$\xi^* \sigma \sigma^* \xi \geq c\|\xi\|^2 , \quad (1.4.1)$$

and the coefficients are smooth enough, general results for PDEs can be used, see e.g. [13] or [14], but in general a solution of (1.0.3) needs not to be regular.
In this case, we can still characterize v as a solution of (1.0.3) in a weak sense. In these notes, we present the notion of viscosity solution which has become very popular in finance. We refer to [2] or [6] for more details.

4.1 Definition

Let F be an operator from $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d\times\mathbb{R}$ into $\mathbb{R}$ where $\mathbb{S}^d$ denotes the set of d-dimensional symmetric matrices. In this chapter, we will be mostly interested by the case

$$F(t,x,u,q,p,A,I) = \rho(t,x)u - q - \langle b(t,x),p\rangle - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(x)A\right] - I - f(x)\ , \qquad (1.4.2)$$

so that v solves (1.0.3) means

$$F[v](t,x) := F(t,x,v(t,x),\partial_t v(t,x),Dv(t,x),D^2v(t,x),I[t,x;v(t,\cdot);v(t,x)]) = 0 \qquad (1.4.3)$$

on $[0,T)\times\mathbb{R}^d$ with

$$I[t,x;\varphi;u] = \int_{\mathbb{R}^d}(\varphi(x+\beta(x,z))-u)\,\lambda_t\Phi(dz)\ .$$

We say that F is *elliptique* if it is non increasing with respect to $A\in\mathbb{S}^d$ and its last variable I. This is clearly the case for F defined as in (1.4.2). In the following, F will always be assumed to be elliptic and non-increasing in its q-variable.

Let us assume for a moment that v is smooth. Let φ be $C^{1,2}$ and $(\hat{t},\hat{x})\in[0,T)\times\mathbb{R}^d$ be a (global) minimum point of $v-\varphi$ on $[0,T]\times\mathbb{R}^d$. After possibly adding a constant to φ, one can always assume that $(v-\varphi)(\hat{t},\hat{x})=0$. The first and second order optimality conditions imply

$$\partial_t v(\hat{t},\hat{x}) \ge \partial_t\varphi(\hat{t},\hat{x}),\ Dv(\hat{t},\hat{x}) = D\varphi(\hat{t},\hat{x}) \text{ and } D^2v(\hat{t},\hat{x}) \ge D^2\varphi(\hat{t},\hat{x})\ .$$

Since F is elliptic and non-increasing in its q-variable, we deduce that

$$F(\hat{t},\hat{x},\varphi(\hat{t},\hat{x}),\partial_t\varphi(\hat{t},\hat{x}),D\varphi(\hat{t},\hat{x}),D^2\varphi(\hat{t},\hat{x}),I[\hat{t},\hat{x};\varphi(\hat{t},\cdot);\varphi(\hat{t},\hat{x})]) \ge 0$$

whenever

$$F(t,x,v(t,x),\partial_t v(t,x),Dv(t,x),D^2v(t,x),I[t,x;v(t,\cdot);v(t,x)]) = 0\ .$$

Conversely, if $(\hat{t},\hat{x})$ is a (global) maximum point of $v-\varphi$ then

$$F(\hat{t},\hat{x},\varphi(\hat{t},\hat{x}),\partial_t\varphi(\hat{t},\hat{x}),D\varphi(\hat{t},\hat{x}),D^2\varphi(\hat{t},\hat{x}),I[\hat{t},\hat{x};\varphi(\hat{t},\cdot);\varphi(\hat{t},\hat{x})]) \le 0\ .$$

This leads to the following notion of viscosity solution.

Definition 1.4.1 *Let F be an elliptic operator as defined as above. We say that a l.s.c. (resp. u.s.c) function U is a super-solution (resp. sub-solution) of* (1.4.3) *on* $[0,T)\times\mathbb{R}^d$ *if for all* $\varphi\in C^{1,2}$ *and* $(\hat t,\hat x)\in[0,T)\times\mathbb{R}^d$ *such that* $0=\min_{[0,T]\times\mathbb{R}^d}(U-\varphi)=(U-\varphi)(\hat t,\hat x)$ *(resp.* $0=\max_{[0,T]\times\mathbb{R}^d}(U-\varphi)=(U-\varphi)(\hat t,\hat x)$*), we have:*

$$F(\hat t,\hat x,\varphi(\hat t,\hat x),\partial_t\varphi(\hat t,\hat x),D\varphi(\hat t,\hat x),D^2\varphi(\hat t,\hat x),I[\hat t,\hat x;\varphi(\hat t,\cdot);\varphi(\hat t,\hat x)])\geq 0 \quad (\ \textit{resp.}\ \leq 0)\ . \tag{1.4.4}$$

Note that a smooth solution is also a viscosity solution. We shall say that a locally bounded function is a *discontinuous viscosity solution* of $F=0$ if U_* and U^* are respectively super- and sub-solution, where, for $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$U_*(t,x)=\liminf_{(s,y)\in[0,T)\times\mathbb{R}^d\to(t,x)}U(s,y)\ \text{ and }\ U^*(t,x)=\limsup_{(s,y)\in[0,T)\times\mathbb{R}^d\to(t,x)}U(s,y)\ . \tag{1.4.5}$$

If U is continuous, we simply say that it is a *viscosity solution*.

Remark 1.4.6 If $(\hat t,\hat x)\in[0,T)\times\mathbb{R}^d$ realize a minimum of $U-\varphi$ then it realize a strict minimum of $U-\varphi^\varepsilon$ where $\varphi^\varepsilon(t,x)=\varphi(t,x)-\varepsilon\|x-\hat x\|^4-|t-\hat t|^2$ and $\varepsilon>0$. Moreover, if φ^ε satisfies (1.4.4) at $(\hat t,\hat x)$ then φ satisfies the same equation whenever F is continuous in its last variable (take the limit $\varepsilon\to 0$). In this case, it is clear that the notion of minimum can be replaced by that of strict minimum. Similarly, we can replace the notion of maximum by the one of strict maximum in the definition of sub-solutions.

Remark 1.4.7 Assume that F is continuous in its last variable. Let B_η be the ball of center $(\hat t,\hat x)$ and radius $\eta>0$. It is clear from the super-solution definition that we can always assume that $\varphi=U_*$ on B_η^c if we replace the property $\varphi\in C^{1,2}$ by $\varphi\in C^{1,2}(B_{\eta/4})$. Indeed, let U_n be a sequence of smooth functions such that $U_n\leq U_*$ for each n and $U_n\to U_*$. Let now χ be a smooth non-increasing function such $\chi(z)=1$ if $z>\eta/4$ and $\chi(z)=0$ if $z<-\eta/4$. Let $d_{\eta/2}$ be the algebraic distance to $\partial B_{\eta/2}$ ($d_{\eta/2}>0$ on $B_{\eta/2}$ and $d_{\eta/2}\leq 0$ on $B^c_{\eta/2}$). This function is smooth. We then set $\varphi^n_\eta:=\varphi\ \chi\circ d_{\eta/2}+U_n(1-\chi\circ d_{\eta/2})$. Then, $(\hat t,\hat x)$ is still a minimum point for $U_*-\varphi^n_\eta$ and $(U_*-\varphi^n_\eta)(\hat t,\hat x)=0$. We can then apply the definition to $(\hat t,\hat x,\varphi^n_\eta)$. By sending $n\to\infty$, this shows that (1.4.4) holds for $\varphi_\eta:=\varphi\ \chi\circ d_{\eta/2}+U_*(1-\chi\circ d_{\eta/2})$ which is smooth in $B_{\eta/4}$ and satisfies $\varphi=U_*$ on B^c_η.

Remark 1.4.8 Observe that, in the *discontinuous* viscosity solutions approach, there is no need to prove the a-priori continuity of the value function since we work directly with the l.s.c. and u.s.c. envelope of the value function v. The continuity will actually be a consequence of the maximum principle (see Theorem 1.4.2 below) which, under suitable conditions, implies that $v_*\geq v^*$ and thus $v_*=v^*=v$ is continuous (at least inside the domain, with continuous extension at the boundary).

4.2 PDE derivation

We can now characterize v as a discontinuous viscosity solution of (1.0.3).

Theorem 1.4.1 *The value function v is a discontinuous viscosity solution on $[0,T)\times\mathbb{R}^d$ of (1.0.3). Moreover, $v_*(T,\cdot)\ge g$ and $v^*(T,\cdot)\le g$ on $\mathbb{R}^d$.*

Proof. We only prove the super-solution property of v_* and the fact that $v_*(T,\cdot)\ge g$. The proof of the other assertions is symmetric. Let $(t_n,x_n)_{n\ge1}$ be a sequence of $[0,T)\times\mathbb{R}^d$ such that $(t_n,x_n,v(t_n,x_n))\to(\hat t,\hat x,v_*(\hat t,\hat x))$. We first assume that $\hat t=T$. In this case, we deduce from Proposition 1.2.1 and a dominated convergence argument that

$$
\begin{aligned}
v_*(\hat t,\hat x) &= \lim_n \mathbb{E}\left[\mathcal{E}_{t_n,x_n}(T)g(X_{t_n,x_n}(T))+\int_{t_n}^T \mathcal{E}_{t_n,x_n}(s)f(X_{t_n,x_n}(s))ds\right]\\
&= \mathbb{E}\left[\lim_n\left(\mathcal{E}_{t_n,x_n}(T)g(X_{t_n,x_n}(T))+\int_{t_n}^T \mathcal{E}_{t_n,x_n}(s)f(X_{t_n,x_n}(s))ds\right)\right]\\
&= g(T,\hat x)\ .
\end{aligned}
$$

We now assume that $\hat t<T$. Let $\varphi\in C^{1,2}$ be such that $0=\min_{[0,T]\times\mathbb{R}^d}(v_*-\varphi)=(v_*-\varphi)(\hat t,\hat x)$. We proceed by contradiction, i.e. we assume that for some $\eta>0$

$$\rho\varphi(\hat t,\hat x)-\mathcal{L}\varphi(\hat t,\hat x)-f(\hat x)<-2\eta$$

and show that this contradicts (1.3.1). Indeed, if the above inequality holds at $(\hat t,\hat x)$, then

$$\rho\varphi(t,x)-\mathcal{L}\varphi(t,x)-f(x)\le 0$$

on a neighborhood of $(\hat t,\hat x)$ of the form $B:=B(\hat t,r)\times B(\hat x,r)$, $r\in(\hat t,T-\hat t)$, containing, without loss of generality, the sequence $(t_n,x_n)_n$. By Remark 1.4.6, we can then assume, after possibly changing the value of η, that

$$v\ge v_*\ge\varphi+\eta \quad \text{on } \partial_p B$$

where $\partial_p B$ is the *parabolic boundary* of B, i.e. $(B(\hat t,r)\times\partial B(\hat x,r))\cup(\{\hat t+r\}\times\bar B(\hat x,r))$. We can also assume that it holds on

$$\tilde B:=\{(t,x+\beta(x,z))\ :\ (t,x,z)\in B\times\mathbb{R}^d \text{ and } x+\beta(x,z))\notin B(\hat x,r))\}$$

which is also bounded thanks to the linear growth assumption on β.

Let θ_n be the first exit time of $(t, X_{t_n,x_n}(t))_{t\geq t_n}$ from B. By Itô's Lemma applied to φ and the above inequalities, we then obtain

$$
\begin{aligned}
\varphi(t_n, x_n) &= \mathbb{E}\left[\mathcal{E}_{t_n,x_n}(\theta_n)\varphi(\theta_n, X_{t_n,x_n}(\theta))\right] \\
&- \mathbb{E}\left[\int_{t_n}^{\theta_n} \mathcal{E}_{t_n,x_n}(s)\left(\mathcal{L}\varphi(s, X_{t_n,x_n}(s)) - \rho\varphi(s, X_{t_n,x_n}(s))\right) ds\right] \\
&\leq \mathbb{E}\left[\mathcal{E}_{t_n,x_n}(\theta_n)\left(v(\theta_n, X_{t_n,x_n}(\theta)) - \eta\right) + \int_{t_n}^{\theta_n} \mathcal{E}_{t_n,x_n}(s) f(X_{t_n,x_n}(s))ds\right] .
\end{aligned}
$$

Since X_{t_n,x_n} is uniformly bounded on $[t_n, \theta_n]$, uniformly in n, and ρ is continuous, we can then find $\varepsilon > 0$, independent of n, such that

$$
\varphi(t_n, x_n) \leq -\varepsilon\eta + \mathbb{E}\left[\mathcal{E}_{t_n,x_n}(\theta_n) v(\theta_n, X_{t_n,x_n}(\theta)) + \int_{t_n}^{\theta_n} \mathcal{E}_{t_n,x_n}(s) f(X_{t_n,x_n}(s)ds\right] .
$$

Since $\varphi(t_n, x_n) \to \varphi(\hat{t}, \hat{x}) = v_*(\hat{t}, \hat{x})$ and $v(t_n, x_n) \to v_*(\hat{t}, \hat{x})$, we deduce that for n large enough

$$
v(t_n, x_n) < \mathbb{E}\left[\mathcal{E}_{t_n,x_n}(\theta_n) v(\theta_n, X_{t_n,x_n}(\theta)) + \int_{t_n}^{\theta_n} \mathcal{E}_{t_n,x_n}(s) f(X_{t_n,x_n}(s)ds\right] ,
$$

which contradicts (1.3.1). □

Remark 1.4.9 The introduction of the sequence $(t_n, x_n)_{n\geq 1}$ is only used in order to approximate v_* by v on which the dynamic programming principle is stated. Obviously, this approximation argument is not required if v is already l.s.c.

Remark 1.4.10 If g is not continuous, similar arguments as above show that $v_*(T, \cdot) \geq g_*$ and $v^*(T, \cdot) \leq g^*$

4.3 Comparison principle

An equivalent definition of viscosity solutions

In order to complete the characterization of v, it remains to show that it is the unique solution of (1.0.3) satisfying the boundary condition $v(T, \cdot) = g$. For this purpose, we need an alternative definition of viscosity solutions in terms of super- et subjets.

Note first that if U is l.s.c., $\varphi \in C^{1,2}$ and $(\hat{t}, \hat{x}) \in [0, T) \times \mathbb{R}^d$ is such that $0 = \min_{[0,T]\times\mathbb{R}^d}(U - \varphi) = (U - \varphi)(\hat{t}, \hat{x})$ then a second order Taylor expansion implies

$$
\begin{aligned}
U(t, x) &\geq U(\hat{t}, \hat{x}) + \varphi(t, x) - \varphi(\hat{t}, \hat{x}) \\
&= U(\hat{t}, \hat{x}) + \partial_t\varphi(\hat{t}, \hat{x})(t - \hat{t}) \\
&+ \langle D\varphi(\hat{t}, \hat{x}), x - \hat{x}\rangle + \frac{1}{2}\langle D^2\varphi(\hat{t}, \hat{x})(x - \hat{x}), x - \hat{x}\rangle + o(|t - \hat{t}| + \|x - \hat{x}\|^2) .
\end{aligned}
$$

This naturally leads to the notion of *subjet* defined as the set $\mathcal{P}^- U(\hat{t},\hat{x})$ of points $(q,p,A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ satisfying

$$U(t,x) \geq U(\hat{t},\hat{x}) + q(t-\hat{t}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle A(x-\hat{x}), x-\hat{x}\rangle + o(|t-\hat{t}| + \|x-\hat{x}\|^2) \; .$$

We define similarly the *superjet* $\mathcal{P}^+ U(\hat{t},\hat{x})$ as the collection of points $(q,p,A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ such that

$$U(t,x) \leq U(\hat{t},\hat{x}) + q(t-\hat{t}) + \langle p, x-\hat{x}\rangle + \frac{1}{2}\langle A(x-\hat{x}), x-\hat{x}\rangle + o(|t-\hat{t}| + \|x-\hat{x}\|^2) \; .$$

For technical reasons related to Ishii's Lemma, see below, we will also need to consider the "limit" super- and subjets. More precisely, we define $\bar{\mathcal{P}}^+ U(\hat{t},\hat{x})$ as the set of points $(q,p,A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ for which there exists a sequence $(t_n, x_n, q_n, p_n, A_n)_n$ of $[0,T) \times \mathbb{R}^d \times \mathcal{P}^+ U(t_n,x_n)$ satisfying $(t_n, x_n, U(t_n,x_n), q_n, p_n, A_n) \to (\hat{t},\hat{x},U(\hat{t},\hat{x}),q,p,A)$. The set $\bar{\mathcal{P}}^- U(\hat{t},\hat{x})$ is defined similarly.

We can now state the alternative definition of viscosity solutions.

Lemma 1.4.2 *Assume that F is continuous. A l.s.c. (resp. u.s.c.) function U is a super-solution (resp. sub-solution) of* (1.4.3) *on* $[0,T) \times \mathbb{R}^d$ *if and only if for all* $(\hat{t},\hat{x}) \in [0,T) \times \mathbb{R}^d$ *and all* $(\hat{q},\hat{p},\hat{A}) \in \bar{\mathcal{P}}^- U(\hat{t},\hat{x})$ *(resp.* $\bar{\mathcal{P}}^+ U(\hat{t},\hat{x})$ *)*

$$F(\hat{t},\hat{x},U(\hat{t},\hat{x}),\hat{q},\hat{p},\hat{A},I[\hat{t},\hat{x};U(\hat{t},\cdot),U(\hat{t},\hat{x})]) \geq 0 \quad (\textit{ resp. } \leq 0) \; . \tag{1.4.6}$$

Proof. We only consider the super-solution property. It is clear that the definition of the Lemma implies the Definition 1.4.1. Indeed, if $(\hat{t},\hat{x}) \in [0,T) \times \mathbb{R}^d$ is a minimum of $U - \varphi$ then $(\partial_t\varphi, D\varphi, D^2\varphi)(\hat{t},\hat{x}) \in \bar{\mathcal{P}}^- U(\hat{t},\hat{x})$. It follows that

$$F(\hat{t},\hat{x},U(\hat{t},\hat{x}),\hat{q},\hat{p},\hat{A},I[\hat{t},\hat{x};U(\hat{t},\cdot);U(\hat{t},\hat{x})]) \geq 0$$

with $(\hat{q},\hat{p},\hat{A}) = (\partial_t\varphi, D\varphi, D^2\varphi)(\hat{t},\hat{x})$. Since $U \geq \varphi$, I is non-decreasing in its third argument and F is elliptic, this implies the required result.
We now prove the converse implication. Fix $(\hat{t},\hat{x}) \in [0,T) \times \mathbb{R}^d$ and $(\hat{q},\hat{p},\hat{A}) \in \bar{\mathcal{P}}^- U(\hat{t},\hat{x})$. It is clear that, if $(\hat{q},\hat{p},\hat{A}) \in \mathcal{P}^- U(\hat{t},\hat{x})$, then we can find φ locally $C^{1,2}$ such that $(\hat{q},\hat{p},\hat{A}) = (\partial_t\varphi, D\varphi, D^2\varphi)(\hat{t},\hat{x})$, $\varphi = U$ at $(\hat{t},\hat{x})$ and $U \geq \varphi$ (see e.g. [7] page 225 for an example of construction). We then have

$$F(\hat{t},\hat{x},U(\hat{t},\hat{x}),\hat{q},\hat{p},\hat{A},I[\hat{t},\hat{x};\varphi(\hat{t},\cdot);U(\hat{t},\hat{x})]) \geq 0 \; .$$

By Remark 1.4.7, we can also assume that $\varphi = U$ outside a neighborhood of radius $\eta > 0$. By taking the limit when $\eta \to 0$, we then get

$$F(\hat{t},\hat{x},U(\hat{t},\hat{x}),\hat{q},\hat{p},\hat{A},I[\hat{t},\hat{x};U(\hat{t},\cdot);U(\hat{t},\hat{x})]) \geq 0 \; .$$

The extension to $(\hat{q},\hat{p},\hat{A}) \in \bar{\mathcal{P}}^- U(\hat{t},\hat{x})$ is immediate since F is continuous. □

Ishii's Lemma and Comparison Theorem

The last ingredient to prove a comparison theorem is the so-called *Ishii's Lemma*.

Lemma 1.4.3 (Ishii's Lemma) *Let U (resp. V) be a l.s.c. super-solution (resp. u.s.c. subsolution) of* (1.4.3) *on* $[0,T) \times \mathbb{R}^d$*. Assume that F is continuous and satisfies*

$$F(t,x,u,q,p,A,I) = F(t,x,u,0,p,A,I) - q$$

for all $(t,x,u,q,p,A,I) \in [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times \mathbb{R}$*. Let* $\phi \in C^{1,2,2}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ *and* $(\hat{t},\hat{x},\hat{y}) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$ *be such that*

$$W(t,x,y) := V(t,x) - U(t,y) - \phi(t,x,y) \quad \leq \quad W(\hat{t},\hat{x},\hat{y}) \quad \forall\, (t,x,y) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^d\,.$$

Then, for all $\eta > 0$, there is $(q_1,p_1,A_1) \in \bar{\mathcal{P}}^+ V(\hat{t},\hat{x})$ *and* $(q_2,p_2,A_2) \in \bar{\mathcal{P}}^- U(\hat{t},\hat{y})$ *such that*

$$q_1 - q_2 = \partial_t \phi(\hat{t},\hat{x},\hat{y}) \quad , \quad (p_1,p_2) = (D_x\phi, -D_y\phi)(\hat{t},\hat{x},\hat{y})$$

and

$$\begin{pmatrix} A_1 & 0 \\ 0 & -A_2 \end{pmatrix} \leq D_{(x,y)}\phi(\hat{t},\hat{x},\hat{y}) + \eta \left(D_{(x,y)}\phi(\hat{t},\hat{x},\hat{y})\right)^2 \,.$$

Proof. The proof is technical and long, we refer to [6] for details. □

Remark 1.4.11 *Let F be as in Lemma 1.4.3. Let U be a viscosity supersolution of* $F[\varphi] = 0$ *on* $[0,T) \times \mathbb{R}^d$*. Set* $\tilde{U}(t,x) := U(t,x) + \frac{\varepsilon}{t}$ *for some* $\varepsilon > 0$*. If* $(\hat{t},\hat{x})$ *reaches the infimum of* $\tilde{U} - \tilde{\varphi}$ *over* $(0,T) \times \mathbb{R}^d$ *for some smooth function* $\tilde{\varphi}$*, such that* $(\tilde{U} - \tilde{\varphi})(\hat{t},\hat{x}) = 0$*, then it also reaches a minimum of* $U - \varphi$ *with* $\varphi(t,x) := \tilde{\varphi}(t,x) - \frac{\varepsilon}{t}$*. Thus,* $F[\varphi](\hat{t},\hat{x}) \geq 0$*. If F is also also non-decreasing in its u-argument, then*

$$\begin{aligned} 0 &\leq F(\hat{t},\hat{x},\tilde{\varphi}(\hat{t},\hat{x}) - \frac{\varepsilon}{\hat{t}}, 0, D\tilde{\varphi}(\hat{t},\hat{x}), D^2\tilde{\varphi}(\hat{t},\hat{x}), I[\hat{t},\hat{x};\tilde{\varphi}(\hat{t},\cdot);\tilde{\varphi}(\hat{t},\hat{x})]) - \partial_t\tilde{\varphi}(\hat{t},\hat{x}) - \frac{\varepsilon}{\hat{t}^2} \\ &\leq F(\hat{t},\hat{x},\tilde{\varphi}(\hat{t},\hat{x}), D\tilde{\varphi}(\hat{t},\hat{x}), D^2\tilde{\varphi}(\hat{t},\hat{x}), I[\hat{t},\hat{x};\tilde{\varphi}(\hat{t},\cdot);\tilde{\varphi}(\hat{t},\hat{x})]) - \partial_t\tilde{\varphi}(\hat{t},\hat{x}). \end{aligned}$$

Hence, $\tilde{U}$ *is still a supersolution, but we know that* $\hat{t} > 0$ *since* $\lim_{t\downarrow 0} \tilde{U}(t,\cdot) = +\infty$*.*
If V is a subsolution of $F[\varphi] = 0$ *on* $[0,T) \times \mathbb{R}^d$*, then saying that* $\sup_{(0,T)\times\mathbb{R}^d}(V - U) > 0$ *is the same as saying that* $\sup_{(0,T)\times\mathbb{R}^d}(V - \tilde{U}) > 0$*, upon choosing* $\varepsilon > 0$ *small enough. The latter is also the same as saying that* $\sup_{[0,T)\times\mathbb{R}^d}(V - U) > 0$ *upon considering the same PDE on* $[-1,T]$*.*

We now prove the expected *comparison theorem* also called *maximum principle*.

Theorem 1.4.2 *(Comparison) Let U (resp. V) be a l.s.c. super-solution (resp. u.s.c. subsolution) with polynomial growth of* (1.0.3) *on* $[0,T)\times\mathbb{R}^d$. *If $U \geq V$ on $\{T\}\times\mathbb{R}^d$, then $U\geq V$ on $[0,T)\times\mathbb{R}^d$.*

Proof. By the same arguments as in Remark 1.3.3, we can assume that $\rho > 0$. Assume now that there is some point $(t_0, x_0) \in [0,T]\times\mathbb{R}^d$ such that $U(t_0,x_0) < V(t_0,x_0)$. We shall prove that it leads to a contradiction. Let $\varepsilon > 0$, $\kappa > 0$ and p an integer greater than 1 be such that $\limsup_{\|x\|\to\infty}\sup_{t\leq T}(|U(t,x)|+|V(t,x)|)/(1+\|x\|^p) = 0$. Then there exists $(\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}^d$ such that

$$0 \;<\; V(\hat{t},\hat{x}) - U(\hat{t},\hat{x}) - \phi(\hat{t},\hat{x},\hat{x}) = \max_{(t,x)\in[0,T]\times\mathbb{R}^d}\left(V(t,x)-U(t,x)-\phi(t,x,x)\right)\;,$$

where

$$\phi(t,x,y) := \varepsilon e^{-\kappa t}(1+\|x\|^{2p}+\|y\|^{2p})$$

and ε is chosen small enough. Since $U\geq V$ on $\{T\}\times\mathbb{R}^d$, it is clear that $\hat{t} < T$. By the arguments of Remark 1.4.11, we can restrict to the case where $\hat{t} > 0$.
For all $n\geq 1$, we can also find $(t_n,x_n,y_n)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ such that

$$0 \;<\; \Gamma_n(t_n,x_n,y_n) = \max_{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d}\Gamma_n(t,x,y) \tag{1.4.7}$$

where

$$\begin{aligned}\Gamma_n(t,x,y) \;&:=\; V(t,x)-U(t,y)-\phi(t,x,y)-n\|x-y\|^2\\ &-\;\; (|t-\hat{t}|^2+\|x-\hat{x}\|^4)\;.\end{aligned}$$

It is easily checked that, after possibly passing to a subsequence,

$$(t_n,x_n,y_n,\Gamma_n(t_n,x_n,y_n)) \to (\hat{t},\hat{x},\hat{x},\Gamma_0(\hat{t},\hat{x},\hat{x})) \;\text{ and }\; n\|x_n-y_n\|^2\to 0\;. \tag{1.4.8}$$

Moreover, Ishii's Lemma implies that for all $\eta>0$, we can find $(q_1^n,p_1^n,A_1^n)\in\bar{\mathcal{P}}^+V(t_n,x_n)$ and $(q_2^n,p_2^n,A_2^n)\in\bar{\mathcal{P}}^-U(t_n,y_n)$ such that

$$q_1^n-q_2^n = \partial_t\varphi_n(t_n,x_n,y_n) \quad,\quad (p_1,p_2) = (D_x\varphi_n, -D_y\varphi_n)(t_n,x_n,y_n)$$

and

$$\left(\begin{array}{cc} A_1^n & 0\\ 0 & -A_2^n\end{array}\right) \leq D^2_{(x,y)}\varphi_n(t_n,x_n,y_n) + \eta\left(D^2_{(x,y)}\varphi_n(t_n,x_n,y_n)\right)^2\;.$$

where

$$\varphi_n(t,x,y) := \phi(t,x,y) + n\|x-y\|^2 + |t-\hat{t}|^2 + \|x-\hat{x}\|^4 \ .$$

In order to obtain the required contradiction, it now suffices to appeal to Lemma 1.4.2 and to argue as in the proof of Theorem 1.3.2. By (1.4.8), we obtain that for all $\eta > 0$

$$\rho(V-U)(\hat{t},\hat{x}) \quad \leq \quad \varepsilon_n + \eta C_n + \mathcal{L}\phi(\hat{t},\hat{x},\hat{x})$$

where $\varepsilon_n \to 0$ is independent of η and C_n does neither depend of η. By sending $\eta \to 0$, we deduce that

$$\rho(V-U)(\hat{t},\hat{x}) \quad \leq \quad \varepsilon_n + \mathcal{L}\phi(\hat{t},\hat{x},\hat{x}) \ .$$

For $\kappa > 0$ big enough so that the second term in the right-hand side is strictly negative and n large enough, we get $\rho(V-U)(\hat{t},\hat{x}) < 0$. This contradicts the fact that $(V-U)(\hat{t},\hat{x}) > 0$ since ρ is assumed to be (strictly) positive. □

Corollary 1.4.2 *The value function v is continuous and is the unique viscosity solution on $[0,T)\times\mathbb{R}^d$ of (1.0.3) satisfying $\lim_{s\uparrow T,\, y\to x} v(s,y) = g(x)$ in the class of discontinuous viscosity solutions with polynomial growth.*

Proof. Recall that since f and g have polynomial growth and ρ is bounded from below, the estimates of Proposition 1.2.1 imply that v has polynomial growth too. The above assertion is then a consequence of Theorem 1.4.1 and Theorem 1.4.2. □

5 Examples of application

5.1 Plain Vanilla and Barrier options in a local volatility model

Let us consider the following financial market composed by one non-risky asset with instantaneous constant interest rate $\rho > 0$ and d risky asset whose dynamics under the risk neutral probability measure (here $\mathbb{P}$) is given by

$$dX_{t,x}(s) = X_{t,x}(s)\rho ds + \text{diag}[X_{t,x}(s)]\sigma(X_{t,x}(s))dW(s) \ \ , \ s \geq t \geq 0 \ .$$

Here, $x \in (0,\infty)^d$, diag$[x]$ denotes the element of $\mathbb{M}^d$ whose i-th diagonal argument is given by x^i, and we assume that the map $x \mapsto \text{diag}[x]\sigma(x)$ is Lipschitz.

Plain Vanilla options

Given a payoff function g, the price at time t of the European random claim $g(X_{t,x}(T))$ is given by:

$$v(t,x) := \mathbb{E}\left[e^{-\rho(T-t)}g(X_{t,x}(T))\right] .$$

If g is continuous with polynomial growth, then it follows from the same arguments as those used to prove Theorem 1.4.1 that it is a discontinuous viscosity solution of

$$\rho\varphi - \frac{\partial}{\partial t}\varphi(t,x) - \rho\langle x, D\varphi(t,x)\rangle - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[x]\sigma\sigma^*(x)\mathrm{diag}[x]D^2\varphi(t,x)\right] = 0 \quad (1.5.1)$$

on $[0,T) \times (0,\infty)^d$ with the terminal condition $\lim_{s\uparrow T,\, y\to x} v(s,y) = g(x)$ on $(0,\infty)^d$. However, we can not apply directly the argument of the proof of Theorem 1.4.2 because the domain in the space variable is not closed. In particular, there is no reason why the max in (1.4.7) should be attained on $[0,T] \times (0,\infty)^{2d}$ and not on $[0,T] \times [0,\infty)^{2d}$.
In order to avoid this problem, we could specify a boundary condition. This is possible when $d = 1$. In this case, standard estimates implies that $\lim_{(s,y)\to(t,0)} v(s,y) = e^{-\rho(T-t)}g(0)$ on $[0,T]$. This provides uniqueness in the class of solutions satisfying this additional boundary condition: since $\hat{x}$ can not be equal to 0, it follows that, after possibly passing to a subsequence $x_n, y_n > 0$, recall (1.4.8), and the remaining arguments can be applied.
For $d \geq 2$, this is much more difficult. An other way to adapt the proof of Theorem 1.4.2 is to start with a point $(\hat{t}_c, \hat{x}_c)$ satisfying

$$\begin{aligned} 0 &< V(\hat{t}_c,\hat{x}_c) - U(\hat{t}_c,\hat{x}_c) - \left(\varepsilon e^{-\kappa\hat{t}_c}(1+2\|\hat{x}_c\|^{2p}) + c\sum_{i=1}^d(\hat{x}_c^i)^{-2}\right) \\ &= \max_{(t,x)\in[0,T]\times(0,\infty)^d}\left(V(t,x) - U(t,x) - \varepsilon e^{-\kappa t}(1+2\|x\|^{2p}) - 2c\sum_{i=1}^d(x^i)^{-2}\right) , \end{aligned}$$

for some $c > 0$. In this case, $\hat{x}_c \in (0,\infty)^d$. We can then repeat the argument of the proof of Theorem 1.4.2. The only difference is that we have to take the limit $c \to 0$ to conclude and control the additional term (which is easy once we have observed that $\lim_{c\to 0} c\sum_{i=1}^d(\hat{x}_c^i)^{-2} = 0$).

Barrier options

In the case of Barrier options, the payoff is typically of the form $g(X_{t,x}(T))\mathbf{1}_{T\leq\tau_{t,x}}$ where $\tau_{t,x} := \inf\{s \geq t \ :\ X_{t,x}(s) \notin \mathcal{O}\}$ for some given open subset $\mathcal{O} \subset (0,\infty)^d$.

Here again, one can easily adapt the proof of Theorem 1.4.1 to show that the price

$$v(t,x) \quad := \quad \mathbb{E}\left[e^{-\rho(T-t)} g(X_{t,x}(T))\mathbf{1}_{T\le\tau_{t,x}}\right]$$

is a discontinuous viscosity solution of (1.5.1) on $[0,T)\times\mathcal{O}$ with the terminal condition $v(T-,\cdot) = g$ on $(0,\infty)^d$. It remains to specify a boundary condition on $[0,T)\times\partial\mathcal{O}$. From now on, we define v^* and v_* as in (1.4.5) except that we take the limit over $y\in\mathcal{O}$. For a smooth test function φ and $(\hat t,\hat x)\in[0,T)\times\partial\mathcal{O}$ such that $(v_*-\varphi)(\hat t,\hat x) = \min_{[0,T]\times\bar{\mathcal{O}}}(v_*-\varphi)(t,x)$, we can then show that, at $(\hat t,\hat x)\in[0,T)\times\partial\mathcal{O}$,

$$\max\left\{\varphi\ ,\ \rho\varphi - \frac{\partial}{\partial t}\varphi - \rho\langle\hat x, D\varphi\rangle - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[\hat x]\sigma\sigma^*\mathrm{diag}[\hat x]D^2\varphi\right]\right\} \ge 0$$

by adapting the arguments used to prove Theorem 1.4.1. Similarly, for a smooth test function φ and $(\hat t,\hat x)\in[0,T)\times\partial\mathcal{O}$ such that $(v^*-\varphi)(\hat t,\hat x) = \max_{[0,T]\times\bar{\mathcal{O}}}(v^*-\varphi)(t,x)$, we must have, at $(\hat t,\hat x)$,

$$\min\left\{\varphi\ ,\ \rho\varphi - \frac{\partial}{\partial t}\varphi - \rho\langle\hat x, D\varphi\rangle - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[\hat x]\sigma\sigma^*\mathrm{diag}[\hat x]D^2\varphi\right]\right\} \le 0\ .$$

Let us now assume that the algebraic distance d to $\partial\mathcal{O}$ is C^2 ($d>0$ on $\mathcal{O}$, $d<0$ on $\mathcal{O}^c$ and $d=0$ on $\partial\mathcal{O}$). Then, if φ is a test function at $(\hat t,\hat x)$ for v_*, then so is $\varphi-(d-d^2/\varepsilon)$ for $\varepsilon>0$ in the sense that $(\hat t,\hat x)$ is still a local minimum point. Applying the above characterization, we obtain that, if $\varphi(\hat t,\hat x)<0$, then, at $(\hat t,\hat x)$

$$\begin{aligned}
0 \quad &\le \quad \rho\varphi - \frac{\partial}{\partial t}\varphi - \rho\langle\hat x, D\varphi\rangle - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[\hat x]\sigma\sigma^*\mathrm{diag}[\hat x]D^2\varphi\right] \\
&+ \quad \rho\langle x, Dd - 2dDd/\varepsilon\rangle + \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[\hat x]\sigma\sigma^*\mathrm{diag}[\hat x](D^2d(1-2d/\varepsilon) - 2Dd\otimes Dd/\varepsilon)\right] \\
&= \quad \rho\varphi - \frac{\partial}{\partial t}\varphi - \rho\langle\hat x, D\varphi\rangle - \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[\hat x]\sigma\sigma^*\mathrm{diag}[\hat x]D^2\varphi\right] \\
&+ \quad \rho\langle\hat x, Dd\rangle + \frac{1}{2}\mathrm{Tr}\left[\mathrm{diag}[\hat x]\sigma\sigma^*\mathrm{diag}[\hat x](D^2d - 2Dd\otimes Dd/\varepsilon)\right]\ .
\end{aligned}$$

If moreover, the *non-characteristic boundary condition*

$$\exists\, c>0 \ \text{ s.t. } \|Dd\mathrm{diag}[x]\sigma\| \ge c \quad \text{on } \partial\mathcal{O}\ , \tag{1.5.2}$$

holds, then we obtain a contradiction by sending $\varepsilon\to 0$ in the above inequality. This shows that $v_*(\hat t,\hat x) = \varphi(\hat t,\hat x)\ge 0$. We can similarly show that $v^*(\hat t,\hat x)\le 0$ which provides the required boundary condition whenever (1.5.2) holds.

5.2 The Lundberg model in insurance

In the Lundberg model the evolution of the reserve X of the insurance company is given by

$$X(t) = x + pt - \int_0^t z\mu(dz, ds) ,$$

where p is the premium rate and μ models the arrival of sinisters.

A reinsurance rule can be described as a map

$$R \ : \ \mathbb{R}^d \times U \mapsto \mathbb{R}^d$$

which to a size of claim z and a level of reinsurance $u \in U \subset \mathbb{R}^d$ associate a retention level, i.e. the part which is not reinsured.

Two typical examples are:

- Proportional reinsurance: $U = [0,1]$, $R(z,u) = (1-u)z$.

- Excess of Loss reinsurance: $U = \mathbb{R}_+$, $R(z,u) = \min\{z \ , \ u\}\mathbf{1}_{z-u\leq L} + L\mathbf{1}_{z-u>L}$ where $L > 0$ stands for the maximum amount of claims insured by the reinsurance company.

To a reinsurance rule, we can associate a reinsurance premium function q which depends on the level of reinsurance.

The evolution of the reserve in the Lundberg model with reinsurance is:

$$X(t) = x + \int_0^t (p - q(\nu_s))\, ds - \int_0^t R(z, \nu_s)\mu(dz, ds) ,$$

where ν is a U-valued predictable process which models the evolution in time of the level of retention.

Evaluation of reinsurance premiums

We take $d = 1$ and assume that the level of reinsurance ν is constant, equal to $u \in U$. Then, the part of the claim paid by the reinsurance company up to time T is:

$$\int_0^T (z - R(z,u))\mu(dz, ds) .$$

If the reinsurance premium is paid once for all at time 0 for all the period $[0,T]$, then the fair premium is:

$$v(0,0) := \mathbb{E}\left[\int_0^T (z - R(z,u))\mu(dz, ds)\right] .$$

The associated PDE is

$$\frac{\partial}{\partial t}v(t,x) + \int_{\mathbb{R}^d} \left(v(t, x + (z - R(z,u))) - v(t,x)\right) \lambda_t \Phi(dz) = 0 \quad \text{on } [0,T) \times \mathbb{R}_+$$

with the boundary condition

$$v(T,x) = x \; .$$

If we look for a solution in the form $v(t,x) = x + f(t)$, for some smooth function f, we obtain

$$\frac{\partial}{\partial t}f(t) + \lambda_t \mathbb{E}\left[Z_1 - R(Z_1,u)\right] = 0$$

which implies

$$v(t,x) = x + \mathbb{E}\left[Z_1 - R(Z_1,u)\right] \int_t^T \lambda_s ds \; ,$$

where Z_1 has the distribution Φ.

Risk evaluation

We now compute the level of remaining risk for the insurance company. It is defined as

$$v(0,0) := \mathbb{E}\left[V\left(\int_0^T R(z,u)\mu(dz,ds)\right)\right]$$

for some increasing convex function V.
The associated PDE is

$$\frac{\partial}{\partial t}v(t,x) + \int_{\mathbb{R}^d} \left(v(t, x + R(z,u)) - v(t,x)\right) \lambda_t \Phi(dz) = 0 \quad \text{on } [0,T) \times \mathbb{R}_+$$

with the boundary condition

$$v(T,x) = V(x) \; .$$

In the special case where $V(x) = e^{\eta x}$, we can look for a solution in the form $v(t,x) = e^{\eta x} f(t)$, for some smooth function f. We obtain

$$\frac{\partial}{\partial t}f(t) + \lambda_t \mathbb{E}\left[e^{\eta R(Z_1,u)} - 1\right] = 0$$

which implies

$$v(t,x) = e^{\eta x} \mathbb{E}\left[e^{\eta R(Z_1,u)} - 1\right] \int_t^T \lambda_s ds \; .$$

Chapter 2

Hamilton-Jacobi-Bellman equations and control problems in standard form

1 Controlled diffusions: definition and a-priori estimates

We now consider the case where the process X is controlled. The set of controls $\mathcal{U}$ is defined as the set of all locally bounded square integrable predictable processes $\nu = \{\nu_t, t \geq 0\}$ valued in a given subset U of $\mathbb{R}^d$.

Given a control process $\nu \in \mathcal{U}$, $t \geq 0$ and $x \in \mathbb{R}^d$, we define the controlled process $X^\nu_{t,x}$ as the solution on $[t, \infty)$ of the stochastic differential system :

$$\begin{aligned} X(v) &= x + \int_t^v b\left(X(s), \nu_s\right) ds + \int_t^v \sigma\left(X(s), \nu_s\right) dW(s) \\ &\quad + \int_t^v \int_{\mathbb{R}^d} \beta\left(X(s-), \nu_s, z\right) \mu(dz, ds) \ . \end{aligned} \tag{2.1.1}$$

In the rest of this chapter, we shall always assume that there is some $L > 0$ such that

$$\begin{aligned} &\|\beta\left(x, u, z\right)\| \leq L\left(1 + \|x\| + \|u\|\right) \\ &\|\beta\left(x, u, z\right) - \beta\left(y, v, z\right)\| \leq L\left(\|y - x\| + \|u - v\|\right) \\ &\|b\left(x, u\right)\| + \|\sigma\left(x, u\right)\| \leq L\left(1 + \|x\| + \|u\|\right) \\ &\|b\left(x, u\right) - b\left(y, v\right)\| + \|\sigma\left(x, u\right) - \sigma\left(y, v\right)\| \leq L\left(\|y - x\| + \|u - v\|\right) \end{aligned} \tag{2.1.2}$$

for all $(x, y, u, v, z) \in (\mathbb{R}^d)^2 \times U^2 \times \mathbb{R}^d$. These conditions ensure the existence of a unique solution to (2.1.1) which, in particular, has the *flow property* (1.1.4).

By the same arguments as those used to prove Proposition 1.2.1, we can also obtain the following a-priori controls on X^ν.

Proposition 2.1.3 *Fix $\nu, \tilde{\nu} \in \mathcal{U}$ and $T > 0$. Then, for all $p \geq 1$, there is $C > 0$ such that, for all $(t,x,y) \in [0,T) \times (\mathbb{R}^d)^2$ and $h \in [0, T-t]$,*

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\leq s\leq T} \|X^\nu_{t,x}(s)\|^p\right] &\leq C\,(1 + \|x\|^p + \mathbb{E}\left[\int_t^T \|\nu_s\|^p ds\right]) \\
\mathbb{E}\left[\sup_{t\leq s\leq t+h} \|X^\nu_{t,x}(s) - x\|^p\right] &\leq C\,\left(h^{\frac{p}{2}}(1 + \|x\|^p) + \mathbb{E}\left[\int_t^{t+h} \|\nu_s\|^p ds\right]\right) \\
\mathbb{E}\left[\sup_{t\leq s\leq T} \|X^\nu_{t,x}(s) - X^\nu_{t,y}(s)\|^p\right] &\leq C\,\|x-y\|^p \\
\mathbb{E}\left[\sup_{t\leq s\leq T} \|X^\nu_{t,x}(s) - X^{\tilde{\nu}}_{t,x}(s)\|^p\right] &\leq C\,\mathbb{E}\left[\int_t^T \|\nu_s - \tilde{\nu}_s\|^p ds\right] .
\end{aligned}
$$

2 Optimal control with finite time horizon

In this section, we focus on stochastic control problem in standard form with a finite (and fixed) time horizon $T > 0$. We first study a general control problem and then discuss in more details some applications in insurance.

The aim of the controller is to maximize the quantity

$$
J(t,x;\nu) := \mathbb{E}\left[\mathcal{E}^\nu_{t,x}(T) g(X^\nu_{t,x}(T)) + \int_t^T \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s), \nu_s) ds\right]
$$

over the set of controls $\mathcal{U}$. Here, g, f and ρ are locally Lipschtiz, ρ is bounded from below, g and f have at most polynomial growth, uniformly in ν.

The function J is called the *gain function*. In minimization problem, we call it the *cost function*.

The associated *value function* is:

$$
v(t,x) := \sup_{\nu \in \mathcal{U}_t} J(t,x;\nu)
$$

where $\mathcal{U}_t$ denotes the set of controls in $\mathcal{U}$ which are independent of $\mathcal{F}_t$.

The aim of this section is to characterize v in terms of the *Hamilton-Jacobi-Bellman equation*

$$
\mathcal{H} v = \rho v \tag{2.2.1}
$$

where

$$\mathcal{H}\varphi(t,x) \quad := \quad \sup_{u\in U} \left(\mathcal{L}^u\varphi(t,x) + f(x,u)\right)$$

and, for a smooth function φ and $u \in U$,

$$\begin{aligned}\mathcal{L}^u\varphi(t,x) \quad &:= \quad \frac{\partial}{\partial t}\varphi(t,x) + \langle b(x,u), D\varphi(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(x,u)D^2\varphi(t,x)\right] \\ &+ \quad \int_{\mathbb{R}^d} \left(\varphi(t,x+\beta(x,u,z)) - \varphi(t,x)\right)\lambda_t\Phi(dz) \ .\end{aligned}$$

This relation will be obtained by using essentially the same arguments as in Chapter 1. We shall first prove a dynamic programming principle, see (2.2.3) below. It will imply that, for all $\nu \in \mathcal{U}$, the process Z^ν defined by

$$Z^\nu(t) := \mathcal{E}^\nu_{0,x}(t)v(t, X^\nu_{0,x}(t)) + \int_0^t \mathcal{E}^\nu_{0,x}(s)f(X^\nu_{0,x}(s),\nu_s)ds \ \ , \ t \geq 0 \ , \qquad (2.2.2)$$

is a super-martingale. Since its dynamics is given by

$$\begin{aligned}dZ(s) \quad &= \quad \mathcal{E}^\nu_{0,x}(s)\left((\mathcal{L}^{\nu_s}v + f)(s, X^\nu_{0,x}(s)) - (\rho v)(s, X^\nu_{0,x}(s),\nu_s)\right)ds \\ &+ \quad (\cdots)\,dW(s) + (\cdots)\,\bar{\mu}(dz,ds) \ ,\end{aligned}$$

we must have $(\mathcal{L}^{\nu_s}v + f)(s, X^\nu_{0,x}(s)) - (\rho v)(s, X^\nu_{0,x}(s),\nu_s) \leq 0$ which formally leads to

$$\sup_{u\in U}\left(\mathcal{L}^u\varphi(t,x) + f(x,u)\right) \leq \rho v \ .$$

Moreover, if for some $\hat{\nu}$, $Z^{\hat{\nu}}$ is a martingale, then $\hat{\nu}$ should be the optimal control in (2.2.3) and should satisfy

$$(\mathcal{L}^{\hat{\nu}_s}v + f)(s, X^{\hat{\nu}}_{0,x}(s),\hat{\nu}_s) = (\rho v)(s, X^{\hat{\nu}}_{0,x}(s)) \ ,$$

thus leading to (2.2.1).

From the technical point of view, we first prove the *dynamic programming principle* which justifies the above super-martingale property for Z. Then, we show that v solves (2.2.1) in the classical sense if it is $C^{1,2}$, in the viscosity sense otherwise. The proof of a uniqueness result for (2.2.1) is left to the reader. Then, we prove a *verification theorem*. When the solution of (2.2.1) with the boundary condition $\varphi(T,\cdot) = g$ can be computed explicitly (and is sufficiently nice), this allows to retrieve the optimal control associated to v.

2.1 Dynamic programming

We first prove the so-called *dynamic programming principle*:

$$v(t,x) = \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta) v(\theta, X^{\nu}_{t,x}(\theta)) + \int_t^{\theta} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] \tag{2.2.3}$$

which, as in the non-controlled case, is the key ingredient to relate control problems to PDEs, compare with Proposition 1.3.2.

Interpretation

Its interpretation is the following. First note that the strong Markov and flow properties of X^{ν} imply that

$$v(t,x) = \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta) J(\theta, X^{\nu}_{t,x}(\theta); \nu) + \int_t^{\theta} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] .$$

Let $\mathcal{U}(\theta, \nu)$ denotes the set of controls of $\mathcal{U}$ which coincides with ν on $[0, \theta)$, then (2.2.3) can be interpreted as

$$v(t,x) = \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta) \left(\operatorname{ess} \sup_{\tilde{\nu} \in \mathcal{U}(\theta,\nu)} J(\theta, X^{\nu}_{t,x}(\theta); \tilde{\nu})\right) + \int_t^{\theta} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] .$$

Here, $\operatorname{ess} \sup_{\tilde{\nu} \in \mathcal{U}(\theta,\nu)} J(\theta, X^{\nu}_{t,x}(\theta); \tilde{\nu})$ denotes the smallest random variables which dominates the family $\left\{J(\theta, X^{\nu}_{t,x}(\theta); \tilde{\nu}),\ \tilde{\nu} \in \mathcal{U}(\theta, \nu)\right\}$ and should be interpreted as a sup[1].
Thus, (2.2.3) means that the optimization problem can be split in two parts, i.e. there is no difference between:
1. Looking directly for an optimal control on the whole time interval $[t, T]$.
2. First, searching for an optimal control starting from time θ, given the value of $X^{\nu}_{t,x}$ at time θ, i.e. compute $\tilde{\nu}(\theta, X^{\nu}_{t,x}(\theta))$ that maximizes $J(\theta, X^{\nu}_{t,x}(\theta); \cdot)$. Second, maximizing over ν the quantity

$$\mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta) J(\theta, X^{\nu}_{t,x}(\theta); \tilde{\nu}(\theta, X^{\nu}_{t,x}(\theta)) + \int_t^{\theta} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] .$$

Morally speaking, if $\hat{\nu}$ maximizes the last quantity, then the control $\hat{\nu}\mathbf{1}_{[t,\theta)} + \tilde{\nu}(\theta, X^{\hat{\nu}}_{t,x}(\theta))\mathbf{1}_{[\theta,T]}$ should be the optimal control on $[t, T]$ associated to $v(t, x)$.

[1] The reason why we have to consider this notion of esssup is that the sup may not be well defined as a random variable, see [16]

Rigorous proof

In the following, we denote by $\mathcal{U}_{t,b}$ the subset of elements of $\mathcal{U}_t$ which are bounded in L^∞ and by $\mathcal{U}_{t,b}^K$ the subset of the elements bounded by K, $K \geq 1$. We omit the subscript t if the controls are allowed to depend on $\mathcal{F}_t$. We first show that we can restrict to elements of $\mathcal{U}_{t,b}$ to compute the value function. This will allow us to work with the family $\{J(t,x;\nu),\ \nu \in \mathcal{U}_{t,b}^K\}$ instead of $\{J(t,x;\nu),\ \nu \in \mathcal{U}_t\}$. The main advantage is that the elements of the first family are continuous in (t,x) uniformly in the control parameter. This point will be essential in the proof of the dynamic programming principle.

Proposition 2.2.4 *For all $(t,x) \in [0,T) \times \mathbb{R}^d$,*

$$v(t,x) = \sup_{\nu \in \mathcal{U}_{t,b}} J(t,x;\nu) = \sup_K v_K(t,x)$$

where

$$v_K(t,x) := \sup_{\nu \in \mathcal{U}_{t,b}^K} J(t,x;\nu) \ .$$

Proof. Clearly, $v(t,x) \geq \sup_{\nu \in \mathcal{U}_{t,b}} J(t,x;\nu)$. Given some $\nu \in \mathcal{U}_t$, set $\nu^K = \nu \wedge (-K) \vee K$. By dominated convergence,

$$\lim_{K \to \infty} \mathbb{E}\left[\int_t^T \|\nu_s - \nu_s^K\|^2 ds\right] = 0 \ .$$

By Proposition 1.2.1, this implies that

$$\lim_{K \to \infty} \sup_{t \leq s \leq T} \mathbb{E}\left[\|X_{t,x}^{\nu}(s) - X_{t,x}^{\nu^K}(s)\|^2\right] \quad = \quad 0 \ .$$

Since f^- and g^- have at most polynomial growth, Fatou's Lemma and the dominated convergence theorem show that

$$\liminf_{K \to \infty} \mathbb{E}\left[\mathcal{E}_{t,x}^{\nu^K}(T) g(X_{t,x}^{\nu^K}(T)) + \int_t^T \mathcal{E}_{t,x}^{\nu^K}(s) f(X_{t,x}^{\nu^K}(s), \nu_s^K) ds\right] \geq J(t,x;\nu)$$

which proves that $v(t,x) \leq \sup_K \sup_{\nu \in \mathcal{U}_{t,b}^K} J(t,x;\nu)$. $\square$

Proposition 2.2.5 *For all $K \geq 1$ and compact set $\Theta \subset [0,T] \times \mathbb{R}^d$, there is a real map $\varepsilon_{K,\Theta}$ such that $\varepsilon_{K,\Theta}(r) \to 0$ as $r \to 0$ for which:*

$$\sup_{\nu \in \mathcal{U}_b^K} |J(t,x;\nu) - J(s,y;\nu)| \leq \varepsilon_{K,\Theta}(|s-t| + \|x-y\|) \ \ \textit{for all } (s,y,t,x) \in \Theta^2 \ .$$

The value function v_K is locally uniformly continuous, v_K^- has at most polynomial growth and v is lower-semicontinuous.

Proof. The first assertion is a consequence of the estimates of Proposition 2.1.3 and the assumptions on ρ, f and g. Indeed, since they are locally Lipschitz with polynomial growth, Proposition 2.1.3 shows that, for each N, there is $C_N > 0$, $C > 0$ and $p \geq 1$, independent on N, such that (if $s \geq t$)

$$\begin{aligned}
& \sup_{\nu \in \mathcal{U}_b^K} |J(t,x;\nu) - J(s,y;\nu)| \\
\leq \; & C_N \left(|s-t|^{\frac{1}{2}}(1+\|x\|) + \|x-y\| + |s-t|K\right) \\
+ \; & C(1+\|x\|+\|y\|+K)^p \, \mathbb{P}\left[\sup_{t \leq u \leq T} \min\{\|X^\nu_{t,x}(u)\| \; ; \; \|X^\nu_{s,y}(u \vee s)\|\} \geq N\right] \\
\leq \; & C_N \left(|s-t|^{\frac{1}{2}}(1+\|x\|) + \|x-y\| + |s-t|K\right) \\
+ \; & C(1+\|x\|+\|y\|+K)^{p+1}/N \, .
\end{aligned}$$

Similar estimates shows that $J(\cdot;\nu)^-$ has polynomial growth for each $\nu \in \mathcal{U}_b$. Since $v_K(t,x) = \sup_{\nu \in \mathcal{U}^K_{t,b}} J(t,x;\nu)$ and $v(t,x) = \sup_K v_K(t,x)$, by Proposition 2.2.4, the lower semi-continuity of v and the polynomial growth of v_K^- follow. $\square$

We now turn to the proof of (2.2.3).

Theorem 2.2.1 *For all $(t,x) \in [0,T) \times \mathbb{R}^d$ and all family of stopping times $\{\theta^\nu, \nu \in \mathcal{U}\}$ with values in $[t,T]$, we have*

$$v(t,x) = \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[\mathcal{E}^\nu_{t,x}(\theta^\nu) v(\theta^\nu, X^\nu_{t,x}(\theta^\nu)) + \int_t^{\theta^\nu} \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s),\nu_s) ds\right] \, .$$

Proof. For ease of notations, we omit the dependence of θ with respect to ν.
1. Since a control ν has to be a measurable function of the Brownian motion W, one easily checks that for $\mathbb{P}$-almost each $\omega \in \Omega$, on can find $\tilde{\nu}_\omega \in \mathcal{U}_{\theta(\omega)}$ such that

$$\begin{gathered}
\mathbb{E}\left[\mathcal{E}^\nu_{t,x}(T) g(X^\nu_{t,x}(T)) + \textstyle\int_t^T \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s),\nu_s) ds \mid \mathcal{F}_\theta\right](\omega) \\
= \\
\mathcal{E}^\nu_{t,x}(\theta)(\omega) J(\theta(\omega), X^\nu_{t,x}(\theta)(\omega); \tilde{\nu}_\omega) + \textstyle\int_t^{\theta(\omega)} \mathcal{E}^\nu_{t,x}(s)(\omega) f(X^\nu_{t,x}(s)(\omega), \nu_s(\omega)) ds \, .
\end{gathered}$$

Since $J \leq v$, this shows that

$$v(t,x) \leq \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[\mathcal{E}^\nu_{t,x}(\theta) v(\theta, X^\nu_{t,x}(\theta) + \int_t^\theta \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s),\nu_s) ds\right] \, .$$

2. We now prove the converse inequality. In view of Proposition 2.2.4 and the arguments used in its proof it suffices to prove the result for v_K, $K \geq 1$, and then to pass to the

limit. Fix $\nu \in \mathcal{U}_b^K$, $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$, a ball B_0 of radius r centered on (t_0, x_0) and fix a compact set $\Theta \supset B_0$ such that

$$\{(t, x + \beta(x,u,z)) \ : \ (t,x,u,z) \in B_0 \times A \times \mathbb{R}^d\} \subset \Theta \subset [0,T] \times \mathbb{R}^d \ , \tag{2.2.4}$$

recall (2.1.2). Let $(B_n)_{n\geq 1}$ be a partition of Θ and $(t_n, x_n)_{n\geq 1}$ be a sequence such that $(t_n, x_n) \in B_n$ for each $n \geq 1$. By definition, for each $n \geq 1$, we can find $\nu^n \in \mathcal{U}_{t_n,b}^K$ such that

$$J(t_n, x_n; \nu^n) \geq v_K(t_n, x_n) - \varepsilon \ , \tag{2.2.5}$$

where $\varepsilon > 0$ is a fix parameter. Moreover, by uniform continuity of v_K and $J(\cdot;\nu)$ for $\nu \in \mathcal{U}_b^K$ on Θ, see Proposition 2.2.5, we can choose $(B_n, t_n, x_n)_{n\geq 1}$ in such a way that $B_n = [t_n - r, t_n] \times B(x_n, r)$, for some $r > 0$, and

$$|v_K(\cdot) - v_K(t_n, x_n)| + |J(\cdot;\nu^n) - J(t_n, x_n; \nu^n)| \leq \varepsilon \quad \text{on } B_n \ , \tag{2.2.6}$$

Let us now define

$$\vartheta \ := \ \inf\left\{s \in [t_0, T] \ : \ (s, X_{t_0,x_0}^\nu(s)) \notin B_0\right\} \wedge \theta$$

where θ is a given stopping time with values in $[t_0, T]$. For $\nu \in \mathcal{U}_{t_0,b}^K$, we define $\bar\nu \in \mathcal{U}_b^K$ by

$$\bar\nu_t \ := \ \nu_t \mathbf{1}_{t<\vartheta} + \mathbf{1}_{t\geq\vartheta}\left(\sum_{n\geq 1} \nu_t^n \ \mathbf{1}_{\{(\vartheta, X_{t_0,x_0}^\nu(\vartheta)) \in B_n\}}\right) \ .$$

It follows from (2.2.4), (2.2.5), (2.2.6) and the fact that ν^n is independent of $\mathcal{F}_{t_n}$ that, for all $\nu \in \mathcal{U}_{t_0,b}^K$,

$$\begin{aligned}
& J(t_0, x_0; \bar\nu) \\
\geq \ & \mathbb{E}\left[\mathcal{E}_{t_0,x_0}^{\bar\nu}(\vartheta) J(\vartheta, X_{t_0,x_0}^{\bar\nu}(\vartheta); \bar\nu) + \int_t^\vartheta \mathcal{E}_{t_0,x_0}^{\bar\nu}(s) f(X_{t_0,x_0}^{\bar\nu}(s), \bar\nu_s) ds\right] \\
\geq \ & \mathbb{E}\left[\sum_n \left\{\mathcal{E}_{t_0,x_0}^{\nu}(\vartheta) J(t_n, x_n; \nu^n) - \varrho\varepsilon + \int_t^\vartheta \mathcal{E}_{t_0,x_0}^{\nu}(s) f(X_{t_0,x_0}^{\nu}(s), \nu_s) ds\right\} \mathbf{1}_{(\vartheta, X_{t_0,x_0}^\nu(\vartheta)) \in B_n}\right] \\
\geq \ & \mathbb{E}\left[\sum_n \left\{\mathcal{E}_{t_0,x_0}^{\nu}(\vartheta) v_K(t_n, x_n) - 2\varrho\varepsilon + \int_t^\vartheta \mathcal{E}_{t_0,x_0}^{\nu}(s) f(X_{t_0,x_0}^{\nu}(s), \nu_s) ds\right\} \mathbf{1}_{(\vartheta, X_{t_0,x_0}^\nu(\vartheta)) \in B_n}\right] \\
\geq \ & \mathbb{E}\left[\mathcal{E}_{t_0,x_0}^{\nu}(\vartheta) v_K(\vartheta, X_{t_0,x_0}^\nu(\vartheta)) + \int_t^\vartheta \mathcal{E}_{t_0,x_0}^{\nu}(s) f(X_{t_0,x_0}^{\nu}(s), \nu_s) ds\right] \ - 3\varrho\varepsilon
\end{aligned}$$

where ϱ is a bounded parameter depending on the bound on ρ^- and T. By arbitrariness of $\varepsilon > 0$, this shows that

$$v_K(t_0, x_0) \geq \mathbb{E}\left[\mathcal{E}^{\nu}_{t_0,x_0}(\vartheta) v_K(\vartheta, X^{\nu}_{t_0,x_0}(\vartheta)) + \int_t^{\vartheta} \mathcal{E}^{\nu}_{t_0,x_0}(s) f(X^{\nu}_{t_0,x_0}(s), \nu_s) ds\right] . \quad (2.2.7)$$

Letting r go to infinity in the definition of B_0 and using Proposition 2.1.3 again, we deduce from the above inequality, the lower semi-continuity of v_K and the polynomial growth of v_K^-, see Proposition 2.2.5, that

$$v_K(t_0, x_0) \geq \mathbb{E}\left[\mathcal{E}^{\nu}_{t_0,x_0}(\theta) v_K(\theta, X^{\nu}_{t_0,x_0}(\theta)) + \int_t^{\theta} \mathcal{E}^{\nu}_{t_0,x_0}(s) f(X^{\nu}_{t_0,x_0}(s), \nu_s) ds\right] .$$

□

Remark 2.2.12 We refer to [4] for an easy proof of a weak version of the dynamic programming principle which pertains to consider much more general settings.

2.2 Direct derivation of the Hamilton-Jacobi-Bellman equations

We can now show that, if v is smooth enough, then it solves the Hamilton-Jacobi-Bellman equation (2.2.1).

Theorem 2.2.2 *Assume that v is continuous on $[0, T] \times \mathbb{R}^d$ and $v \in C^{1,2}([0, T) \times \mathbb{R}^d)$. Then, v is a solution on $[0, T) \times \mathbb{R}^d$ of* (2.2.1) *and satisfies the boundary condition $\lim_{t \nearrow T} v(t, x) = g(x)$ on $\mathbb{R}^d$.*

Proof. Fix $(t, x) \in [0, T) \times \mathbb{R}^d$ and assume that $\mathcal{H}v(t, x) \neq \rho(t, x)v(t, x)$.
1. We assume that $\mathcal{H}v(t, x) > \rho(t, x)v(t, x)$ and work toward a contradiction. Fix $u \in U$ such that $\mathcal{L}^u v(t, x) + f(x, u) > \rho(t, x)v(t, x)$. By continuity of the involved functions, we can assume that

$$\mathcal{L}^u v + f(\cdot, u) > \rho v \quad (2.2.8)$$

on a compact neighborhood $V \subset [0, T) \times \mathbb{R}^d$ of (t, x). Set $\nu = u$, a constant control of $\mathcal{U}$, and let θ be the first exit time of $(s, X^{\nu}_{t,x}(s))$ from V. Observe that $(\theta, X^{\nu}_{t,x}(\theta-)) \in V$. Using Itô's Lemma, Corollary A.2.1, (A.1.1) and the fact that $\mathcal{E}^{\nu}_{t,x}(s)$, $D\varphi(s, X^{\nu}_{t,x}(s-))$, $\sigma(s, X^{\nu}_{t,x}(s-))$ and $\varphi(s, X^{\nu}_{t,x}(s-) + \beta(s, X^{\nu}_{t,x}(s-), \nu_s, \cdot)) - \varphi(s, X^{\nu}_{t,x}(s-))$ are bounded on $[t, \theta]$, recall (1.2.1), we observe that

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta) v(\theta, X^{\nu}_{t,x}(\theta)) + \int_t^{\theta} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), u) ds\right] \\
&= \ v(t, x) + \mathbb{E}\left[\int_t^{\theta} \mathcal{E}^{\nu}_{t,x}(s) \left(\mathcal{L}^u v(s, X^{\nu}_{t,x}(s)) - (\rho v)(s, X^{\nu}_{t,x}(s)) + f(X^{\nu}_{t,x}(s), u)\right) ds\right] .
\end{aligned}$$

In view of (2.2.8), this contradicts Theorem 2.2.1.
2. We now assume that $\mathcal{H}v(t,x) < \rho(t,x)v(t,x)$. This implies that

$$\mathcal{H}v < \rho v$$

on a neighborhood V of (t,x) of radius $r > 0$. Moreover, for r small enough, we must have

$$\mathcal{H}w \leq \rho w \quad \text{on } V ,$$

with $w(s,y) = v(s,y) + (s-t) + \|x-y\|^2$. Given $\nu \in \mathcal{U}_t$, let θ be the first exit time of $(s, X^\nu_{t,x}(s))$ from V. Using Itô's Lemma and the above inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{E}^\nu_{t,x}(\theta)w(\theta, X^\nu_{t,x}(\theta)) + \int_t^\theta \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s), \nu_s)ds\right] \\
&= \; w(t,x) + \mathbb{E}\left[\int_t^\theta \mathcal{E}^\nu_{t,x}(s)\left(\mathcal{L}^{\nu_s}w(s, X^\nu_{t,x}(s)) - (\rho w)(s, X^\nu_{t,x}(s)) + f(X^\nu_{t,x}(s), \nu_s)\right)\right) ds\right] \\
&\leq v(t,x)
\end{aligned}$$

for some $C > 0$. By definition of w and θ, it follows that, for some $C', C'' > 0$,

$$\begin{aligned}
v(t,x) \;\; &\geq \;\; C'\mathbb{E}\left[(\theta - t) + \|X^\nu_{t,x}(\theta) - x\|^2\right] \\
&+ \;\; \mathbb{E}\left[\mathcal{E}^\nu_{t,x}(\theta)v(\theta, X^\nu_{t,x}(\theta)) + \int_t^\theta \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s), \nu_s)ds\right] \\
&\geq \;\; C'' + \mathbb{E}\left[\mathcal{E}^\nu_{t,x}(\theta)v(\theta, X^\nu_{t,x}(\theta)) + \int_t^\theta \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s), \nu_s)ds\right] .
\end{aligned}$$

By arbitrariness of ν, this contradicts Theorem 2.2.1. □

In the case where v may not be smooth, it can still be characterize as a solution of (2.2.1) in the viscosity sense.

Theorem 2.2.3 *The value function v is a discontinuous viscosity solution of* (2.2.1) *on* $[0,T) \times \mathbb{R}^d$. *Moreover,* $v_*(T,x) \geq g(x)$ *and* $v^*(T,x) \leq g(x)$ *on* $\mathbb{R}^d$.

Proof. The viscosity property inside the domain can be easily obtained by combining the arguments used in the proofs of Theorem 1.4.1 and Theorem 2.2.2. Since g is continuous, and, ρ, g and f have polynomial growth the property $v_*(T,x) \geq g(x)$ follows from Theorem 2.2.1, the estimates of Proposition 2.1.3 and the dominated convergence

Theorem. It remains to prove that $v^*(T,x) \le g(x)$ on $\mathbb{R}^d$. We argue by contradiction and assume that there is $x_0 \in \mathbb{R}^d$ such that

$$(v^* - g)(T, x_0) \quad =: \quad 2\varepsilon > 0 \; .$$

Let $(t_k, x_k)_{n\ge1}$ be a sequence in $[0,T] \times \mathbb{R}^d$ satisfying

$$(t_k, x_k) \;\longrightarrow\; (T, x_0) \;\text{ and }\; v(t_k, x_k) \;\longrightarrow\; v^*(T, x_0) \quad \text{as } k \longrightarrow \infty \; . \tag{2.2.9}$$

We can then find a sequence of smooth functions $(\varphi^n)_{n\ge0}$ on $[0,T] \times \mathbb{R}^d$ such that $\varphi^n(T, x_0) \to v^*(T, x_0)$, and

$$\varphi^n - g \ge \varepsilon \tag{2.2.10}$$

on some neighborhood B_n of (T, x_0). After possibly passing to a subsequence of $(t_k, x_k)_{k\ge1}$, we can then assume that it holds on $B_n^k := [t_k, T] \times B(x_k, \delta_n^k)$ for some sufficiently small $\delta_n^k \in (0,1]$ such that $B_n^k \subset B_n$. Since v^* is locally bounded, there is some $\zeta > 0$ such that $|v^*| \le \zeta$ on B_n. We can then assume that $\varphi^n \ge -2\zeta$ on B_n. Let us define $\tilde\varphi_k^n$ by

$$\tilde\varphi_k^n(t,x) := \varphi^n(t,x) + 4\zeta|x - x_k|^2/(\delta_n^k)^2 + \sqrt{T - t} \; ,$$

and observe that

$$(v^* - \tilde\varphi_k^n)(t,x) \le -\zeta < 0 \quad \text{for } (t,x) \in [t_k, T] \times \partial B(x_k, \delta_n^k) \; . \tag{2.2.11}$$

Since $(\partial/\partial t)(\sqrt{T-t}) \to -\infty$ as $t \to T$, we can choose t_k large enough in front of δ_n^k and the derivatives of φ^n to ensure that

$$(\mathcal{H} - \rho)\tilde\varphi_k^n \le 0 \quad \text{on } B_n^k \; . \tag{2.2.12}$$

It then suffices to argue as in the proof of Theorem 2.2.2 to obtain for some $C > 0$

$$\begin{aligned}
\tilde\varphi_k^n(t_k, x_k) \;&\ge\; \mathbb{E}\left[\mathcal{E}^\nu_{t_k,x_k}(\theta)\tilde\varphi_k^n(\theta, X^\nu_{t_k,x_k}(\theta)) + \int_{t_k}^\theta \mathcal{E}^\nu_{t_k,x_k}(s) f(X^\nu_{t_k,x_k}(s), \nu_s)ds\right] \\
&\ge\; C(\varepsilon \wedge \zeta) + \mathbb{E}\left[\mathcal{E}^\nu_{t_k,x_k}(\theta) v(\theta, X^\nu_{t_k,x_k}(\theta)) + \int_{t_k}^\theta \mathcal{E}^\nu_{t_k,x_k}(s) f(X^\nu_{t_k,x_k}(s), \nu_s)ds\right]
\end{aligned}$$

which contradicts Theorem 2.2.1 for k and n large enough so that $|\tilde\varphi_k^n(t_k, x_k) - v(t_k, x_k)| \le C(\varepsilon \wedge \zeta)/2$. □

Remark 2.2.13 (Comparison theorem) A comparison theorem can be easily obtained for (2.2.1) by arguing as in the proof of Theorem 1.4.2 in the case where U is compact. We leave the proof to the reader as an exercise.

2.3 Verification theorem

As for the Feynman-Kac representation, we can also state a verification result. In good cases, it allows to exhibit an optimal control strategy of the form $(\hat{\nu}_s)_{s\geq 0}$ as defined below. Such a control is called a *Markovian control* because its value at time s depends only on $(s, X(s))$.

Theorem 2.2.4 *(Verification) Assume that there exists a* $C^{1,2}([0,T)\times\mathbb{R}^d)$ *solution* φ *to* (2.2.1) *such that*

$$\liminf_{t\nearrow T, x'\to x} \varphi(t,x') \geq g(x) \quad \text{on } \mathbb{R}^d \tag{2.2.13}$$

and φ^- *has polynomial growth. Assume further that*
1. There is a measurable map $\hat{u}\ :\ [0,T)\times\mathbb{R}^d \mapsto U$ *such that* $\mathcal{H}\varphi = \mathcal{L}^{\hat{u}}\varphi + f(\cdot,\hat{u})$ *on* $[0,T)\times\mathbb{R}^d$.
2. For all initial conditions $(t,x)\in[0,T)\times\mathbb{R}^d$, *there is a solution to* (2.1.1) *with* $\hat{\nu}$ *defined by* $\hat{\nu}_s := \hat{u}(s, X^{\hat{\nu}}_{t,x}(s))$.
3. Given $(t,x)\in[0,T)\times\mathbb{R}^d$, *there is an increasing sequence of stopping times* $(\theta_n)_n$ *such that* $X^{\hat{\nu}}_{t,x}$ *is bounded on* $[t,\theta_n]$, $\theta_n\to T$ $\mathbb{P}-a.s.$ *and*

$$\begin{aligned}&\mathbb{E}\left[\mathcal{E}^{\hat{\nu}}_{t,x}(\theta_n)\varphi(\theta_n, X^{\hat{\nu}}_{t,x}(\theta_n)) + \int_t^{\theta_n}\mathcal{E}^{\hat{\nu}}_{t,x}(s) f(X^{\hat{\nu}}_{t,x}(s),\hat{\nu}_s)ds\right]\\ &\to\ \mathbb{E}\left[\mathcal{E}^{\hat{\nu}}_{t,x}(T)g(X^{\hat{\nu}}_{t,x}(T)) + \int_t^{T}\mathcal{E}^{\hat{\nu}}_{t,x}(s) f(X^{\hat{\nu}}_{t,x}(s),\hat{\nu}_s)ds\right] \quad \text{as } n \text{ goes to } \infty.\end{aligned}$$

Then, $v=\varphi$.

Proof. By Itô's Lemma, Corollary A.2.1, (A.1.1), the fact that φ solves (2.2.1) and the assumptions 1, 2 and 3 of the Theorem, we obtain

$$\varphi(t,x) = \mathbb{E}\left[\mathcal{E}^{\hat{\nu}}_{t,x}(\theta_n)\varphi(\theta_n, X^{\hat{\nu}}_{t,x}(\theta_n)) + \int_t^{\theta_n}\mathcal{E}^{\hat{\nu}}_{t,x}(s) f(X^{\hat{\nu}}_{t,x}(s),\hat{\nu}_s)ds\right]$$

for each n. Using assumption 3, we then deduce that

$$\varphi(t,x) = \mathbb{E}\left[\mathcal{E}^{\hat{\nu}}_{t,x}(T)g(X^{\hat{\nu}}_{t,x}(T)) + \int_t^{T}\mathcal{E}^{\hat{\nu}}_{t,x}(s) f(X^{\hat{\nu}}_{t,x}(s),\hat{\nu}_s)ds\right]$$

by sending n to ∞. This shows that $\varphi(t,x)\leq v(t,x)$. We now prove the converse inequality. Fix $\nu\in\mathcal{U}_b$, let τ_n be the first time where $\|X^{\nu}_{t,x}(s)\|\geq n$ and set $\theta_n := T\wedge\tau_n$.

By the same arguments as in the proof of Theorem 2.2.2, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta_n)\varphi(\theta_n, X^{\nu}_{t,x}(\theta_n)) + \int_t^{\theta_n} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] \\
&= \varphi(t,x) + \mathbb{E}\left[\int_t^{\theta_n} \mathcal{E}^{\nu}_{t,x}(s)\left(\mathcal{L}^{\nu_s}\varphi(s, X^{\nu}_{t,x}(s)) - (\rho\varphi)(s, X^{\nu}_{t,x}(s)) + f(X^{\nu}_{t,x}(s), \nu_s)\right) ds\right] \\
&\leq \varphi(t,x) \ .
\end{aligned}
$$

Since ν is bounded, φ^- has polynomial growth and $\tau_n \to \infty$, we deduce from the first estimate of Proposition 1.2.1, the boundedness of ρ^- and the boundary condition (2.2.13) that

$$
\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta_n)\varphi(\theta_n, X^{\nu}_{t,x}(\theta_n)) + \int_t^{\theta_n} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] \\
&\geq \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(T) g(X^{\nu}_{t,x}(T)) + \int_t^{T} \mathcal{E}^{\nu}_{t,x}(s) f(X^{\nu}_{t,x}(s), \nu_s) ds\right] \ .
\end{aligned}
$$

In view of Proposition 2.2.4, this shows that $\varphi(t,x) \geq v(t,x)$. □

Remark 2.2.14 If $\hat{u}$ is Lipschitz continuous, then the condition 2. is satisfied. Moreover, a direct extension of the arguments used in the proof of Proposition 1.2.1 shows that

$$
\sup_{t\leq s\leq T} \mathbb{E}\left[\|X^{\hat{\nu}}_{t,x}(s)\|^2\right]^{\frac{1}{2}} \quad \leq \quad C\,(1 + \|x\|)
$$

for some $C > 0$ independent of x. Let τ_n be the first time where $\|X^{\hat{\nu}}_{t,x}(s)\| \geq n$ and set $\theta_n := T \wedge \tau_n$. If φ as polynomial growth, the dominated convergence theorem then implies that

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta_n)\varphi(\theta_n, X^{\hat{\nu}}_{t,x}(\theta_n)) + \int_t^{\theta_n} \mathcal{E}^{\hat{\nu}}_{t,x}(s) f(X^{\hat{\nu}}_{t,x}(s), \hat{\nu}_s) ds\right] \\
&\to \mathbb{E}\left[g(X^{\hat{\nu}}_{t,x}(T)) + \int_t^{T} \mathcal{E}^{\hat{\nu}}_{t,x}(s) f(X^{\hat{\nu}}_{t,x}(s), \hat{\nu}_s) ds\right] \quad \text{as } n \text{ goes to } \infty
\end{aligned}
$$

if $\lim_{t\nearrow T, x'\to x} \varphi(t,x') = g(x)$ on $\mathbb{R}^d$.

Remark 2.2.15 Observe that the process $Z^{\hat{\nu}}$ defined as in (2.2.2) for the optimal control exhibited in the previous Theorem is a martingale, while Z^{ν} is only a supermartingale for any other admissible control. This corroborates the interpretation given in introduction of this chapter. We refer to [11] for a proof of this phenomenon is general (non Markovian) control problems.

3 Examples of applications in optimal control with finite time horizon (exercices)

3.1 Super-replication under portfolio constraints

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one dimensional Brownian motion W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \le T}$ the natural filtration generated by W, satisfying the usual assumptions, where $T > 0$ is a fixed time horizon.
We consider a financial market model where the risk free interest rate is zero and the dynamics of the stock price, under the risk neutral measure $\mathbb{P}$, is given by the process $X_{t,x}$ solution of

$$X(s) = x + \int_t^s X(r)\sigma(X(r))dW_r \ \ , \ t \le s \le T$$

where $x \in (0, \infty)$ and $\sigma : (0, \infty) \mapsto (0, \infty)$ is continuous, bounded, with bounded inverse $\sigma^{-1} := 1/\sigma$. We also assume that $x \in (0, \infty) \mapsto x\sigma(x)$ is uniformly Lipschitz.
Fix $A \subset \mathbb{R}$. We denote by $\mathcal{A}$ the set of predictable processes ϕ with values in A such that

$$\int_t^T |\phi_s|^2 ds < \infty \ \mathbb{P} - \text{a.s. } \ \forall \, t \le T \ .$$

Given $(t, x, v) \in [0, T] \times (0, \infty) \times \mathbb{R}_+$ and $\phi \in \mathcal{A}$, we define $V^{\phi}_{t,x,v}$ as

$$V^{\phi}_{t,x,v}(s) \ \ = \ \ v + \int_t^s \phi_r V^{\phi}_{t,x,v}(r)\sigma(X_{t,x}(r))dW_r \ \ , \ t \le s \le T \ .$$

1. Give a financial interpretation of $V^{\phi}_{t,x,v}$ and $\phi \in \mathcal{A}$.
2. Set
$$A^* := \left\{ \xi \in \mathbb{R} \ : \ \delta(\xi) := \sup_{a \in A} \langle \xi, a \rangle < \infty \right\}$$
We denote by $\mathcal{A}^*$ the set of progressively measurable processes ν with values in A^* for which there exists $C > 0$, which may depend on ν, such that $\sup_{s \le T} |\nu_s| \le C$ $\mathbb{P} -$ a.s. Given $\nu \in \mathcal{A}^*$ and $(t, x, y) \in [0, T) \times (0, \infty) \times (0, \infty)$, we define $Y^{\nu}_{t,x,y}$ by
$$Y^{\nu}_{t,x,y}(s) := y e^{-\int_t^s \delta(\nu_r)dr - \frac{1}{2}\int_t^s |\sigma(X_{t,x}(r))^{-1}\nu_r|^2 dr + \int_t^s \sigma(X_{t,x}(r))^{-1}\nu_r dW_r} \ \ , \ s \in [t, T] \ .$$
   (a) What can we say on $\delta(\xi) - \langle \xi, a \rangle$ when $a \in A$ and $\xi \in A^*$?

   (b) Sow that $\mathbb{E}\left[\sup_{s \in [t,T]} \left(|Y^{\nu}_{t,x,y}(s)|^q + |X_{t,x}(s)|^q\right)\right] < \infty$ for all $(t, x, y) \in [0, T) \times (0, \infty) \times (0, \infty)$, $\nu \in \mathcal{A}^*$ and $q \in \mathbb{N}$.

(c) Show that, for all $(t,x,v,y) \in [0,T)\times(0,\infty)\times\mathbb{R}_+\times(0,\infty)$ and $(\nu,\phi)\in\mathcal{A}^*\times\mathcal{A}$, the process $V^{\phi}_{t,x,v}Y^{\nu}_{t,x,y}$ is a non-negative local super-martingale.

(d) Deduce that, for all $(t,x,v,y) \in [0,T)\times(0,\infty)\times\mathbb{R}_+\times(0,\infty)$ and $(\nu,\phi)\in\mathcal{A}^*\times\mathcal{A}$, the process $V^{\phi}_{t,x,v}Y^{\nu}_{t,x,y}$ is a super-martingale.

3. Let g be a map $\mathbb{R}\mapsto\mathbb{R}_+$, which is lower semi-continuous with linear growth.

(a) Show that if $\phi\in\mathcal{A}$ is is such that $V^{\phi}_{t,x,v}(T)\geq g(X_{t,x}(T))$ $\mathbb{P}-$ a.s., then $v\geq p(t,x,1)$ where the function p is defined by

$$p(t,x,y) := \sup_{\nu\in\mathcal{A}^*} J(t,x,y;\nu) \text{ where } J(t,x,y;\nu) := \mathbb{E}\left[Y^{\nu}_{t,x,y}(T)g(X_{t,x}(T))\right] \ ,$$

for all $(t,x,y)\in[0,T]\times(0,\infty)\times(0,\infty)$.

(b) Deduce that

$$\hat{p}(t,x) := \inf\left\{v\in\mathbb{R}_+ \ : \ \exists\,\phi\in\mathcal{A} \text{ s.t. } V^{\phi}_{t,x,v}(T)\geq g(X_{t,x}(T))\ \mathbb{P}-\text{a.s.}\right\}\geq\bar{p}(t,x)\geq 0$$

where $\bar{p}(t,x) := p(t,x,1)$.

4. In the following, we admit that

$$p(t,x,y)\geq\sup_{\nu\in\mathcal{A}^*}\mathbb{E}\left[p(\theta,X_{t,x}(\theta),Y^{\nu}_{t,x,y}(\theta))\right] \ ,$$

for all $(t,x,y)\in[0,T]\times(0,\infty)\times(0,\infty)$ and all stopping time θ with values in $[t,T]$.

(a) Show that p is a viscosity supersolution on $[0,T)\times(0,\infty)\times(0,\infty)$ of

$$\begin{aligned} 0 &= -\mathcal{L}_X\varphi(t,x,y) \\ &+ \inf_{u\in A^*}\left(y\delta(u)\frac{\partial}{\partial y}\varphi(t,x,y)-\frac{1}{2}|y\sigma^{-1}(x)\nu|^2\frac{\partial^2}{\partial y^2}\varphi(t,x,y)-uxy\frac{\partial^2}{\partial x\partial y}\varphi(t,x,y)\right)\end{aligned}$$

where $\mathcal{L}_X$ is the Dynkin operator associated to X.

(b) Show that if $\bar{\varphi}\in C^{1,2}$ and $(t_0,x_0)\in[0,T]\times(0,\infty)$ satisfy $\min_{[0,T]\times(0,\infty)}(\bar{p}-\bar{\varphi}) = (\bar{p}-\bar{\varphi})(t_0,x_0) = 0$ then φ defined as $\varphi(t,x,y) = y\bar{\varphi}(t,x)$ satisfies $\min_{[0,T]\times(0,\infty)\times(0,\infty)}(p-\varphi) = (p-\varphi)(t_0,x_0,y_0) = 0$ for all $y_0>0$.

(c) Deduce that $\bar{p}$ is a viscosity supersolution on $[0,T)\times(0,\infty)$ of

$$\inf_{u\in A^*}\left(-y\mathcal{L}_X\bar{\varphi}(t,x)+y\delta(u)\bar{\varphi}(t,x)-uxy\frac{\partial}{\partial x}\bar{\varphi}(t,x)\right)=0$$

for all $y>0$.

(d) Deduce that $\bar{p}$ is a viscosity supersolution on $[0, T) \times (0, \infty)$ of

$$\min \left\{ -\mathcal{L}_X \bar{\varphi}(t, x) \ , \ \inf_{u \in A_1^*} \left(\delta(u) \bar{\varphi}(t, x) - ux \frac{\partial}{\partial x} \bar{\varphi}(t, x) \right) \right\} = 0$$

where $A_1^* := \{\xi \in A^* \ : \ |\xi| = 1\}$.

5. We now study the boundary condition at $t = T$.

(a) Let $x_0 \in (0, \infty)$ and $(t_n, x_n)_{n \geq 1} \subset [0, T) \times (0, \infty)$ be such that $(t_n, x_n) \to (T, x_0)$ and

$$\bar{p}(t_n, x_n) \to \liminf_{\substack{(t', x') \to (T, x_0) \\ (t', x') \in [0, T) \times (0, \infty)}} \bar{p}(t', x') =: \bar{p}(T-, x_0) \ .$$

By considering the sequence of controls

$$\nu^n := \frac{1}{T - t_n} u \mathbf{1}_{[t_n, T]} \ , \ n \geq 1 \ ,$$

for $u \in A^*$, and by using an appropriate Girsanov transformation, show that

$$\bar{p}(T-, x_0) \geq \hat{g}(x_0) := \sup_{u \in A^*} e^{-\delta(u)} g(x e^u) \geq 0 \ .$$

(b) From now on, we assume that $\hat{g}$ is differentiable on $(0, \infty)$.

i. By using the fact that A^* is a convex cone, show that $\hat{g}(x) \geq e^{-\lambda \delta(\varepsilon)} \hat{g}(x e^{\lambda \varepsilon})$ for all $\varepsilon \in A^*$, $\lambda > 0$ and $x > 0$.

ii. Deduce that $\delta(u) \hat{g}(x) - ux \frac{\partial}{\partial x} \hat{g}(x) \geq 0$ for all $x > 0$ and $u \in A^*$.

6. From now on, we assume that there exists a function with linear growth $w \in C^{1,2}([0, T) \times (0, \infty)) \cap C^{0,0}([0, T] \times (0, \infty))$ solution of

$$-\mathcal{L}_X w = 0 \text{ sur } [0, T) \times (0, \infty) \ \text{ and } w(T, \cdot) = \hat{g} \text{ sur } (0, \infty) \ .$$

(a) Under the assumption that $\bar{p} \in C^{1,2}([0, T) \times (0, \infty))$, show that $\bar{p} \geq w$ on $[0, T] \times (0, \infty)$.

(b) Explain briefly why $\bar{p} \geq w$ even if $\bar{p} \notin C^{1,2}([0, T) \times (0, \infty))$. We shall assume from now on that the above inequality hold.

7. We now assume that σ does not depend on x and we simply write $\sigma = \sigma(x)$. We also assume that A is of the form $[-m, M]$ where $M, m \geq 0$.

(a) Compute δ in this case.

(b) Show by a verification argument that $w(t,x) = \mathbb{E}\left[\hat{g}(X_{t,x}(T))\right]$ for all $(t,x) \in [0,T] \times (0,\infty)$.

(c) Under the assumption that $\frac{\partial}{\partial x}\hat{g}$ is uniformly bounded, show that

$$\frac{\partial}{\partial x} w(t,x) = \mathbb{E}\left[\frac{\partial}{\partial x}\hat{g}(X_{t,x}(T)) X_{t,1}(T)\right] ,$$

for all $(t,x) \in [0,T) \times (0,\infty)$.

(d) Deduce that $\delta(u)w(t,x) - ux\frac{\partial}{\partial x}w(t,x) \geq 0$ for all $u \in A^*$ and $(t,x) \in [0,T) \times (0,\infty)$.

(e) By also assuming that $w > 0$ sur $[0,T)\times(0,\infty)$, deduce that $(x\frac{\partial}{\partial x}w(t,x))/w(t,x) \in [-m, M]$ for all $(t,x) \in [0,T) \times (0,\infty)$.

(f) Show that, if

$$\int_t^T |X_{t,x}(s)\frac{\partial}{\partial x}w(s,X_{t,x}(s)))|^2/w(s,X_{t,x}(s))^2 ds < \infty \;\; \mathbb{P}-\text{a.s.}$$

for all $(t,x) \in [0,T) \times (0,\infty)$, then $w \geq \hat{p}$.

(g) Conclude that, in this case, $w = \hat{p} = \bar{p}$.

3.2 Super-hedging with unbounded stochastic volatility (Exercise in French)

Le but de cet exercice est de caractériser, déterminer un prix de sur-réplication d'un actif financier risqué dans un modèle à volatlité stochastique non-bornée. Pour se faire, nous allons introduire un problème de contrôle stochastique et l'étudier. Nous allons notamment dériver une équation aux dérivées partielles *via* un principe de programmation dynamique *ad hoc*. Commençons par introduire le problème et les notations.
On se place sur une espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ muni de la filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfaisant les conditions *habituelles*. On se donne un horizon de temps $T > 0$. On considère un marché financier constitué d'un actif sans risque, de taux d'intérêt nul $r = 0$, et d'un actif risqué X dont la dynamique est donnée par le modèle à volatilité stochatique :

$$dX(t) = \sigma(Y(t))dW_t \;\; , \; X(0) = x_0 \in \mathbb{R} \; , \; dY(t) = \eta dt + \gamma d\bar{W}_t \;\; , \; Y(0) = y_0 \in \mathbb{R} \quad (2.3.1)$$

où $\eta \in \mathbb{R}$, $\gamma > 0$ et $\sigma : \mathbb{R} \mapsto \mathbb{R}_+$ est une fonction continue uniformément Lipschitz, W et $\bar{W}$ sont deux mouvements browniens indépendants. Soit g une fonction continue, bornée. On s'intéresse au problème de sur-réplication de l'option de payoff $g(X(T))$

lorsque l'on ne peut acheter et vendre que l'actif financier X. Le prix de sur-réplication est donné par

$$p(0, x_0, y_0) = \sup_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}} [g(X(T))] \tag{2.3.2}$$

où $\mathcal{M}(S)$ est l'ensemble des mesures de probabilité sous lesquelles X est une martingale. Afin d'étudier ce problème par les techniques de contrôle stochastique, on commence par étendre la définition des dynamiques de X et Y à des conditions initiales $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ quelconques. Par la suite, on note $(X_{t,x,y}, Y_{t,y})$ la solution de (2.3.1) sur $[t, T]$ vérifiant la condition initiale $(X_{t,x,y}(t), Y_{t,y}(t)) = (x, y)$.
On note $\mathcal{U}$ (reps. $\mathcal{U}_t$) l'ensemble des processus ν progressivement mesurables (resp. indépendants de $\mathcal{F}_t$) à valeurs réelles tels qu'il existe une constante $C > 0$ pour laquelle $|\nu_t| \leq C$ pour tout $t \geq 0$ $\mathbb{P}-$a.s. Etant donnés $\nu \in \mathcal{U}$ et $(t, h) \in [0, T] \times (0, \infty)$, on définit $H^\nu_{t,h}$ par

$$H^\nu_{t,h}(r) := h \exp\left(-\frac{1}{2} \int_t^r |\nu_s|^2 ds + \int_t^r \nu_s d\bar{W}_s \right) \ , \ r \in [t, T] \ .$$

On considère finalement la **fonction coût** J suivante

$$J(t, x, y, h; \nu) := \mathbb{E}\left[H^\nu_{t,h}(T) g(X_{t,x,y}(T)) \right]$$

et on lui associe alors la **fonction valeur** v suivante

$$v(t, x, y, h) := \sup_{\nu \in \mathcal{U}_t} J(t, x, y, h; \nu) \, .$$

Partie I : fonction valeur et programmation dynamique

1. Etant donnés $(t, x, y, h) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times (0, \infty)$ et $\nu \in \mathcal{U}$, montrer que $X_{t,x,y} H^\nu_{t,h}$ est une martingale sur $[t, T]$ sous $\mathbb{P}$.

2. Déduire de la question précédente que

$$p(0, x_0, y_0) \geq v(0, x_0, y_0, 1) \tag{2.3.3}$$

3. Montrer que, pour tout $\nu \in \mathcal{U}$ et tout ensemble borné $B \subset [0, T] \times \mathbb{R} \times \mathbb{R} \times (0, \infty)$, il existe une constante $K > 0$ telle que, pour tout $(t_i, x_i, y_i, h_i)_{i=1,2} \subset B$ vérifiant

$t_2 \geq t_1$, on a

$$\begin{aligned}
\mathbb{E}\left[|H^{\nu}_{t_2,h_2}(T)/H^{\nu}_{t_1,h_1}(T)-1|^2\right] &\leq K((h_2/h_1-1)^2+|t_1-t_2|)\ ,\\
\mathbb{E}\left[\sup_{s\in[t_1,t_2]}|Y_{t_1,y_1}(s)-y_1|^2\right] &\leq K(|t_1-t_2|)\ ,\\
\mathbb{E}\left[\sup_{s\in[t_2,T]}|Y_{t_1,y_1}(s)-Y_{t_2,y_2}(s)|^2\right] &\leq K(|y_1-y_2|^2+|t_1-t_2|)\ ,\\
\mathbb{E}\left[|X_{t_1,x_1,y_1}(T)-X_{t_2,x_2,y_2}(T)|^2\right] &\leq K(|t_1-t_2|+|x_1-x_2|^2+|y_1-y_2|^2)\ .
\end{aligned}$$

4. En déduire que $J(\cdot;\nu)$ est semi-continue inférieurement sur $[0,T]\times\mathbb{R}\times\mathbb{R}\times(0,\infty)$ à $\nu\in\mathcal{U}$ fixé (on rappelle que g est bornée). Que peut-on en déduire sur la régularité de v ?

5. Justifier (sans rentrer dans les détails) que pour tout $(t_0,x_0,y_0,h_0)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times(0,\infty)$ et pour toute fonction $\varphi\in C^2$ telle que $(v-\varphi)(t_0,x_0,y_0,h_0)=0$ et $v\geq\varphi$ sur $[t_0,T]\times\mathbb{R}\times\mathbb{R}\times(0,\infty)$, on a

$$v(t_0,x_0,y_0,h_0)\geq\mathbb{E}\left[\varphi\left(\tau,X_{t_0,x_0,y_0}(\tau),Y_{t_0,y_0}(\tau),H^{\nu}_{t_0,y_0,h_0}(\tau)\right)\right] \tag{2.3.4}$$

pour tout $\nu\in\mathcal{U}$ et tout temps d'arrêt τ à valeurs dans $[t_0,T]$.

Dans la suite du problème, on note

$$w_*(t,x,y):=\liminf_{(t',x',y')\to(t,x,y),\ t'<T} w(t',x',y')\ \ ,\ (t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}\ .$$

6. On note $w:=v(\cdot,1)$. Montrer que $v(\cdot,h)=hw$ pour tout $h>0$, que w est semi-continue inférieurement sur $[0,T)\times\mathbb{R}\times\mathbb{R}$ et que

$$w_*(T,x,y)\geq g(x)\ . \tag{2.3.5}$$

Partie II: dérivation d'une équation aux dérivées partielles

Dans cette partie, il s'agit de montrer que la fonction w_* ne dépend pas de y. Pour cela, on va montrer que $\frac{\partial w_*}{\partial y}=0$ au sens des solutions de viscosité.

7. Déduire de l'inégalité de programmation dynamique (2.3.4) sur v et d'une transformation de Girsanov astucieuse que w_* est sur-solution de viscosité de l'équation

$$\begin{aligned}
\inf_{u\in\mathbb{R}}\left(-\frac{\partial}{\partial t}-\frac{1}{2}\sigma(y)^2\frac{\partial^2}{\partial x^2}-u\frac{\partial}{\partial y}-\frac{1}{2}\gamma^2\frac{\partial^2}{\partial y^2}\right)\varphi(t,x,y) &\geq 0\ \text{ sur } [0,T)\times\mathbb{R}\times\mathbb{R}\ ,\\
\varphi(T,\cdot) &\geq g\ \text{ sur } \mathbb{R}\times\mathbb{R}\ .
\end{aligned}$$

8. En déduire que w_* est sur-solution de viscosité de $-\frac{\partial}{\partial y}\varphi(t,x,y) \geq 0$ et $\frac{\partial}{\partial y}\varphi(t,x,y) \geq 0$ sur $[0,T) \times \mathbb{R} \times \mathbb{R}$.

Le but est maintenant de montrer que si l'on fixe t_0 et x_0, alors la fonction $y \mapsto w_*(t_0,x_0,y)$ est une solution de viscosité de $-\frac{\partial}{\partial y}\varphi(y) \geq 0$ et $\frac{\partial}{\partial y}\varphi(y) \geq 0$ sur $\mathbb{R}$.

9. Soit $(t_0,x_0) \in [0,T) \times \mathbb{R}$, $\varphi \in C^2$, une fonction bornée de $\mathbb{R}$ dans $\mathbb{R}$, et $y_0 \in \mathbb{R}$ tels que $w_*(t_0,x_0,y_0) - \varphi(y_0) = 0$ et $w_*(t_0,x_0,y) - \varphi(y) \geq 0$ pour tout $y \in \mathbb{R}$. A $n \geq 1$ fixé on associe la fonction φ_n définie par $\varphi_n(t,x,y) := \varphi(y) - n(|t-t_0|^2 + |x-x_0|^4) - |y-y_0|^4$.
   (a) Montrer que w_* est bornée et qu'il existe $(t_n,x_n,y_n) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ atteignant le minimum de $w_* - \varphi_n$.
   (b) En utilisant l'inégalité
   $$(w_*(t_n,x_n,\cdot)-\varphi)(y_n)+n(|t_n-t_0|^2+|x_n-x_0|^4)+|y_n-y_0|^4 \leq (w_*(t_0,x_0,\cdot)-\varphi)(y_0)$$
   montrer que $(t_n,x_n) \to (t_0,x_0)$, et que $(y_n, n(|t_n-t_0|^2+|x_n-x_0|^4)) \to (y_\infty, m) \in \mathbb{R} \times \mathbb{R}_+$ quand $n \to \infty$, quite à passer à une sous-suite.
   (c) Que peut-on dire sur $\liminf_{n\to\infty}(w_*(t_n,x_n,\cdot)-\varphi)(y_n)$?
   (d) En déduire que $(w_*(t_0,x_0,\cdot)-\varphi)(y_0) \leq (w_*(t_0,x_0,\cdot)-\varphi)(y_\infty) + m + |y_\infty - y_0|^4 \leq (w_*(t_0,x_0,\cdot)-\varphi)(y_0)$.
   (e) En déduire que $y_\infty = y_0$ et $m = 0$.
10. Déduire des deux questions précédentes que, à $(t_0,x_0) \in [0,T) \times \mathbb{R}$, l'application $y \mapsto w_*(t_0,x_0,y)$ est sur-solution de viscosité de $-\frac{\partial}{\partial y}\varphi(y) \geq 0$ et $\frac{\partial}{\partial y}\varphi(y) \geq 0$ sur $\mathbb{R}$.
11. Supposons que w_* est dérivable. Montrer que w_* est alors indépendante de la variable y.

Partie III : identification du prix de sur-réplication

12. On écrit maintenant $w_*(t,x)$ en omettant l'argument y (voir partie II). Montrer que w_* est sur-solution de viscosité de l'équation
$$\left(-\frac{\partial}{\partial t} - \frac{1}{2}\sigma(y)^2\frac{\partial^2}{\partial x^2}\right)\varphi(t,x) \quad \geq \quad 0 \;\text{ sur } [0,T) \times \mathbb{R} \;\;,\; \varphi(T,\cdot) \geq g \;\text{ sur } \mathbb{R} \times \mathbb{R}\;.$$

13. On suppose à partir de maintenant que $\sup_{y\in\mathbb{R}} \sigma^2(y) = \infty$ et $\inf_{y\in\mathbb{R}} \sigma^2(y) = 0$. Montrer que si w_* est régulière alors elle est concave en x et décroissante en temps sur $[0, T) \times \mathbb{R}$.

14. A partir de maintenant, on admet que w_* est concave en x et décroissante en temps sur $[0, T) \times \mathbb{R}$. Montrer que $w_* \geq \hat{g}$ sur $[0, T] \times \mathbb{R}$ où $\hat{g}$ et la plus petite fonction concave qui majore g.

15. En utilisant (2.3.2), montrer que $p(0, x_0, y_0) \leq \hat{g}(x_0)$ et en déduire que $p(0, x_0, y_0) = \hat{g}(x_0)$.

3.3 Calibration of local volatility surface

Let $0 < \underline{\sigma} < \bar{\sigma}$ be two real constants. We denote by $\mathcal{A}$ (resp. $\mathcal{A}_t$) the set of predictable processes σ taking values in $[\underline{\sigma}, \bar{\sigma}]$ (resp. independants de $\mathcal{F}_t$). Given $(t, x) \in [0, T] \times (0, \infty)$ and $\sigma \in \mathcal{A}$, we denote by $X^\sigma_{t,x}$ the solution of

$$X(s) = x + \int_t^s X(r)\sigma_r dW_r \ , \ t \leq s \leq T \ . \tag{2.3.6}$$

We observe on the market the prices p_i of plain vanilla options of payoff $g_i(S(T_i))$, $i \in I := \{1, \ldots, i_{\max}\}$. The functions g_i are assumed to be Lipschitz and bounded, $T_0 := 0 < T_1 < T_2 < \ldots < T_{i_{\max}} = T$ are the maturities, S is the underlying stock price process with initial value at 0 given by $x_0 > 0$.
We want to model S as a diffusion of type X^σ satisfying (2.3.6) so as to calibrate the prices of the options on the market data p_i, $i \in I$.
Following Avellaneda, Friedman, Holmes and Samperi (1997)[2], we compute σ as the solution of the problem :

$$\sup_{\lambda\in\mathbb{R}^{i_{\max}}} \inf_{\sigma\in\mathcal{A}_t} \left[\left(\sum_{i=1}^{i_{\max}} \lambda_i \mathbb{E}\left[g_i(X^\sigma_{t,x}(T_i)) - p_i\right] \right) + \mathbb{E}\left[\int_0^T \eta(\sigma_s^2) ds\right] \right]$$

where $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is a given convex function.
Given $\lambda \in \mathbb{R}^{i_{\max}}$, we define on $[0, T] \times (0, \infty)$

$$v(t, x) := \inf_{\sigma\in\mathcal{A}_t} \mathbb{E}\left[\int_t^T \eta(\sigma_s^2) ds + \sum_{i=1}^{i_{\max}} \mathbf{1}_{t<T_i} \lambda_i g_i(X^\sigma_{t,x}(T_i))\right] \tag{2.3.7}$$

the associated value function.

[2]M. Avellaneda, C. Friedman, R. Holmes and D. Samperi (1997). Calibrating volatility surfaces via relative-entropy minimization. *Appl. Math. Finance*, 4(1), 37-64. We refer to this paper for a description of the full methodology.

1. **A-priori estimates :**

   (a) Justify the existence of a solution to (2.3.6) for all $\sigma \in \mathcal{A}$.

   (b) Show that there exists $C > 0$ such that, for all $(t,x) \in [0,T] \times (0,\infty)$ and $\sigma \in \mathcal{A}$,

   $$\mathbb{E}\left[\sup_{s\in[t,T]} |X^\sigma(s)|^2\right] \le C\left(1+x^2\right) .$$

   (c) Show that there exists $C > 0$ such that, for all $(t,x) \in [0,T] \times (0,\infty)$, $\sigma \in \mathcal{A}$, and $h > 0$ satisfying $t+h \le T$, we have

   $$\mathbb{E}\left[\sup_{s\in[t,t+h]} |X^\sigma(s) - x|^2\right] \le C\left(1+x^2\right) h .$$

   (d) Let B be a compact subset of $(0,\infty)$. Show that there exists $C(B) > 0$ such that, for all $(t_1,x_1),(t_2,x_2) \in [0,T] \times B$ satisfying $t_1 \le t_2$ and $\sigma \in \mathcal{A}$,

   $$\mathbb{E}\left[\sup_{s\in[t_2,T]} |X^\sigma_{t_1,x_1}(s) - X^\sigma_{t_2,x_2}(s)|^2\right] \le C(B)\left(|x_1-x_2|^2 + |t_1-t_2|\right) .$$

   (e) Let B be a compact subset of $(0,\infty)$. Show that there exists $C(B) > 0$ such that, for all $(t_1,x_1),(t_2,x_2) \in [0,T] \times B$ satisfying $T_i \le t_1 \le t_2 < T_{i+1}$ for some $i \in I$, we have

   $$|v(t_1,x_1) - v(t_2,x_2)| \le C(B)\left(|x_1-x_2|^2 + |t_1-t_2|\right)^{\frac{1}{2}} .$$

   (Hint : use an argument of the type $\inf_a f(b,a) - \inf_a f(c,a) \le \sup_a (f(b,a) - f(c,a))$.)

2. **Dynamic programming :**

   (a) Show that for $(t,x) \in [0,T) \times (0,\infty)$ and $0 < h \le T-t$ we have

   $$\begin{aligned} v(t,x) &= \inf_{\sigma\in\mathcal{A}_t} \mathbb{E}\Bigg[\int_t^{t+h} \eta(\sigma_s^2)ds + \sum_{i=1}^{i_{\max}} \mathbf{1}_{t<T_i\le t+h}\lambda_i g_i(X^\sigma_{t,x}(T_i)) \\ &\qquad + \mathbb{E}[\int_{t+h}^T \eta(\sigma_s^2)ds + \sum_{i=1}^{i_{\max}} \mathbf{1}_{t+h<T_i}\lambda_i g_i(X^\sigma_{t+h,X^\sigma(t+h)}(T_i)) \mid \mathcal{F}_{t+h}]\Bigg] . \end{aligned}$$

   (b) Deduce by a formal argument that

   $$v(t,x) = \inf_{\sigma\in\mathcal{A}_t} \mathbb{E}\left[v(t+h, X^\sigma(t+h)) + \int_t^{t+h} \eta(\sigma_s^2)ds + \sum_{i=1}^{i_{\max}} \mathbf{1}_{t<T_i\le t+h}\lambda_i g_i(X^\sigma_{t,x}(T_i))\right] . \tag{2.3.8}$$

3. **H.-J.-B. equation :** We now assume that $v \in C_b^{1,2}(D)$ where $D := \bigcup_{i=0}^{i_{\max}-1}[T_i, T_{i+1}) \times (0,\infty)$.

   (a) By considering the case where σ is constant, deduce from (2.3.8) that
   $$\sup_{\sigma \in [\underline{\sigma},\bar{\sigma}]} \left(-\mathcal{L}^\sigma v(t,x) - \eta(\sigma^2)\right) \leq 0$$
   for all $(t,x) \in D$. Here, for a real σ and $(t,x) \in D$, we use the notation
   $$\mathcal{L}^\sigma v(t,x) := \frac{\partial}{\partial t} v(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} v(t,x) \; .$$

   (b) We assume that the inf in (2.3.8) is achieved by a Markovian control $\sigma_s = \tilde{\sigma}(s, X_{t,x}^\sigma(s))$ where $\tilde{\sigma}$ is C_b^1. Deduce from (2.3.8) that
   $$\sup_{\sigma \in [\underline{\sigma},\bar{\sigma}]} \left(-\mathcal{L}^\sigma v(t,x) - \eta(\sigma^2)\right) \geq 0 \; .$$

   (c) Show that $\lim_{t\uparrow T_i} v(t,x) \leq v(T_i,x) + \lambda_i g_i(x)$, for all $i \leq i_{\max}$ and $x \in (0,\infty)$.

   (d) What should the terminal condition as $t \uparrow T_i$ look like ?

   (e) We now assume that η is strictly convex, C^1 on $(\underline{\sigma},\bar{\sigma})$ and such that $\eta'(\underline{\sigma}+) = -\infty$ and $\eta'(\bar{\sigma}-) = +\infty$. We also admit that v solves on D
   $$\sup_{\sigma \in [\underline{\sigma},\bar{\sigma}]} \left(-\mathcal{L}^\sigma v(t,x) - \eta(\sigma^2)\right) = 0 \; . \tag{2.3.9}$$
   Show that v solves on D
   $$-\frac{\partial}{\partial t} v(t,x) - \Phi\left(\frac{1}{2} x^2 \frac{\partial^2}{\partial x^2} v(t,x)\right) = 0 \; .$$
   where
   $$\Phi(y) := \inf_{z \in (\underline{\sigma}^2,\bar{\sigma}^2)} \left(zy + \eta(z)\right) \; .$$
   Admitting that Φ' is the inverse of $-\eta'$, show that the sup in (2.3.9) is achieved by
   $$\hat{\sigma}(t,x) := \Phi'\left(x^2 \frac{\partial^2}{\partial x^2} v(t,x)\right) \; .$$

4. **Verification argument :** We now assume that η is strictly convex, C^1 on $(\underline{\sigma},\bar{\sigma})$ and such that $\eta'(\underline{\sigma}+) = -\infty$ and $\eta'(\bar{\sigma}-) = +\infty$. We also assume that there exists $\varphi \in C_b^{1,2}(D)$ which solves (2.3.9) on D and satisfies $\lim_{t\uparrow T_i, x'\to x} \varphi(t,x') = \varphi(T_i,x) + \lambda_i g_i(x)$, for all $i \leq i_{\max}$ and $x \in (0,\infty)$.

(a) Show that (2.3.6) admits a solution, $\hat{X}_{t,x}$, if we replace σ_r by $\hat{\sigma}(r, X(r))$ in (2.3.6).

(b) Deduce that $\varphi \geq v$.

(c) Show that $\varphi \leq v$.

(d) What can we say on the solution of the control problem (2.3.7) ?

5. **Numerical resolution:** Assume that you have a software which can solve equations of the form (2.3.9). How can you use it to estimate v and $\hat{\sigma}$ (explain how to treat the boundary conditions at T_i, $i \in I$) ?

3.4 Optimal Importance sampling

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one-dimensional Brownian motion W. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ the completed natural filtration induced by W, where $T > 0$. Let $X_{t,x}$ be the solution

$$X(s) = x + \int_t^s \sigma(X(r))dW_r \ , \ t \leq s \leq T$$

where $x \in \mathbb{R}$, $t \in [0, T]$ and σ is Lipschitz and satisfies $\inf_{x \in \mathbb{R}} \sigma(x) =: \bar{\sigma} > 0$.
We consider the optimal importance sampling problem for the Monte-Carlo computation of

$$m(t, x) := \mathbb{E}\left[g(X_{t,x}(T))\right],$$

where g is continuous and bounded.
The idea consists in introduction a family of density processes $\{H^\nu_{t,x}, \nu \in \mathcal{U}\}$ defined by

$$H^\nu_{t,x}(s) := 1 + \int_t^s \frac{\nu_r}{\sigma(X_{t,x}(r))} H^\nu_{t,x}(r)dW_r \ , t \leq s \leq T \ ,$$

where $\mathcal{U}$ is the set of progressively measurable processes ν with values in $[-M, M]$, where $M > 0$ is a fixed constant. We then observe that

$$m(t, x) = \mathbb{E}^\nu\left[g(X_{t,x}(T))/H^\nu_{t,x}(T)\right] \tag{2.3.10}$$

where $\mathbb{E}^\nu$ is the expectation under $\mathbb{Q}^\nu$ defined as

$$d\mathbb{Q}^\nu/d\mathbb{P} = H^\nu_{t,x}(T). \tag{2.3.11}$$

We then look for $\hat{\nu} \in \mathcal{U}$ such that

$$\mathrm{Var}^{\hat{\nu}}\left[g(X_{t,x}(T))/H^{\hat{\nu}}_{t,x}(T)\right] = \inf_{\nu \in \mathcal{U}} \mathrm{Var}^\nu\left[g(X_{t,x}(T))/H^\nu_{t,x}(T)\right] =: w(t, x) \ , \tag{2.3.12}$$

where Var^ν denotes the variance under $\mathbb{Q}^\nu$. We finally replace the standard Monte-Carlo estimator of $\mathbb{E}[g(X_{t,x}(T))]$ by the one associated to $\mathbb{E}^{\hat{\nu}}\left[g(X_{t,x}(T))/H^{\hat{\nu}}_{t,x}(T)\right]$, obtained by sampling $g(X_{t,x}(T))/H^{\hat{\nu}}_{t,x}(T)$ under $\mathbb{Q}^{\hat{\nu}}$.
The aim of the above question is to treat problem (2.3.12) by stochastic control technics.

1. Show that $\mathbb{Q}^\nu$ is actually a probability measure equivalent to $\mathbb{P}$ and that (2.3.10) holds true, for $\nu \in \mathcal{U}$.

2. Show that the problem defined in the right-hand side of (2.3.12) is equivalent to
$$v(t,x) := \inf_{\nu\in\mathcal{U}} \mathbb{E}^\nu\left[\left(g(X_{t,x}(T))/H^\nu_{t,x}(T)\right)^2\right] .$$
and that
$$v(t,x) = w(t,x) + m(t,x)^2 . \tag{2.3.13}$$

In the following, we set
$$J(t,x;\nu) := \mathbb{E}^\nu\left[\left(g(X_{t,x}(T))/H^\nu_{t,x}(T)\right)^2\right] \ , \ (t,x) \in [0,T]\times\mathbb{R} ,$$
and we admit that v is continuous on $[0,T)\times\mathbb{R}$.

3. Show that
$$J(t,x;\nu) = \mathbb{E}\left[g(X_{t,x}(T))^2/H^\nu_{t,x}(T)\right] .$$

4. Let $\nu_1,\nu_2 \in \mathcal{U}$, $(t,x) \in [0,T]\times\mathbb{R}$ et $\tau \in \mathcal{T}_{[t,T]}$, where $\mathcal{T}_{[t,T]}$ denotes the set of stopping times with values in $[t,T]$.

   (a) Show that $\nu := \nu_1\mathbf{1}_{[0,\tau)} + \nu_2\mathbf{1}_{[\tau,T]} \in \mathcal{U}$.

   (b) Show that
   $$J(t,x;\nu) = \mathbb{E}\left[\mathbb{E}\left[g(X_{\tau,X_{t,x}(\tau)}(T))^2/H^{\nu_2}_{\tau,X_{t,x}(\tau)}(T) \mid \mathcal{F}_\tau\right]/H^{\nu_1}_{t,x}(\tau)\right] .$$

   (c) Deduce by a formal argument that
   $$v(t,x) \geq \inf_{\nu\in\mathcal{U}} \mathbb{E}\left[v(\tau,X_{t,x}(\tau))/H^\nu_{t,x}(\tau)\right] .$$

   (d) We assume that there exists a measurable map $\phi : [0,T]\times\mathbb{R} \mapsto \mathcal{U}$ such that
   $$v(\theta,\xi) = \mathbb{E}\left[g(X_{\theta,\xi}(T))^2/H^{\phi(\theta,\xi)}_{\theta,\xi}(T) \mid \mathcal{F}_\theta\right] \quad \mathbb{P}-a.s.$$
   for all $\theta \in \mathcal{T}_{[0,T]}$ and all real valued $\mathcal{F}_\theta$-mesurable random variable ξ. Deduce that
   $$v(t,x) \leq \inf_{\nu\in\mathcal{U}} \mathbb{E}\left[v(\tau,X_{t,x}(\tau))/H^\nu_{t,x}(\tau)\right] .$$

From now on, we admit that v is continuous and satisfies

$$v(t,x) = \inf_{\nu \in \mathcal{U}} \mathbb{E}\left[v(\tau, X_{t,x}(\tau))/H^{\nu}_{t,x}(\tau)\right] \tag{2.3.14}$$

for $(t,x) \in [0,T] \times \mathbb{R}$ and $\tau \in \mathcal{T}_{[t,T]}$. We denote, for $\varphi \in C^{1,2}([0,T]\times\mathbb{R})$, $u \in \mathbb{R}$ and $(t,x) \in [0,T] \times \mathbb{R}$,

$$\mathcal{L}^u\varphi(t,x) := \frac{\partial}{\partial t}\varphi(t,x) - u\frac{\partial}{\partial x}\varphi(t,x) + \frac{1}{2}\sigma(x)^2\frac{\partial^2}{\partial x^2}\varphi(t,x) + (u/\sigma(x))^2\varphi(t,x)\ .$$

5. By considering controls of the form $\nu \equiv u$ with $u \in [-M,M]$, deduce from (2.3.14) that v is a viscosity subsolution on $[0,T) \times \mathbb{R}$ of

$$-\inf_{u\in[-M,M]} \mathcal{L}^u\varphi = 0\ . \tag{2.3.15}$$

6. Let $(t_0,x_0) \in [0,T) \times \mathbb{R}$ and $\varphi \in C^{1,2}([0,T]\times\mathbb{R})$ be such that (t_0,x_0) achieves a strict local minimum of $v - \varphi$ and $(v-\varphi)(t_0,x_0) = 0$.
   (a) Show that the assertion $-\inf_{u\in[-M,M]}\mathcal{L}^u\varphi(t_0,x_0) < 0$ leads to a contradiction to (2.3.14).
   (b) Deduce that v is a viscosity supersolution on $[0,T) \times \mathbb{R}$ of (2.3.15).
7. We now study the terminal condition at T.
   (a) Deduce from (2.3.13) that $\liminf_{t'\uparrow T, x'\to x} v(t',x') \geq g(x)^2$ for all $x \in \mathbb{R}$.
   (b) Show that $v(t,x) \leq \mathbb{E}\left[g(X_{t,x}(T))^2\right]$ for all $(t,x) \in [0,T] \times \mathbb{R}$.
   (c) Deduce that $\limsup_{t'\uparrow T, x'\to x} v(t',x') = \liminf_{t'\uparrow T, x'\to x} v(t',x') = v(T,x) = g(x)^2$ for all $x \in \mathbb{R}$.
8. State a verification theorem for the above problem.

3.5 Optimal insurance

Let us consider the problem of a financial agent whose wealth is submitted to some exogenous risks, seen as accidents. The times arrival of the accidents are described by a Poisson process N of intensity $\lambda > 0$. If X is his wealth just before the sinister, its size is δX where $\delta \in (0,1)$ is a fixed constant. To protect himself against the exogenous risk, he can buy an insurance at a level $u \in U := [0,1]$. If a sinister arrives at time t and his level of protection is u then his wealth is diminished by $(1-u)\delta X(t-)$. To obtain a protection u at time t, he has to pay a premium puX (paid continuously), with $p > 0$ a fixed parameter.

To sum up, we assume that the wealth process $X^{\nu}_{t,x}$ evolves on $[t, T]$ according to

$$X^{\nu}_{t,x}(s) = x - \int_t^s p\nu_v X_{t,x}(v)dv - \int_t^s \delta(1-\nu_v)X_{t,x}(v-)dN_v \ .$$

Observe that, by Itô's Lemma,

$$X^{\nu}_{t,x}(s) = xe^{-\int_t^s p\nu_v dv} \prod_{k=1}^{N_t} (1 - \delta(1 - \nu_{T_k})) \tag{2.3.16}$$

where T_k denote the time of the k-th jump of N.

The aim of the agent is to maximize the utility of his terminal wealth

$$v(t, x) := \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left[V(X^{\nu}_{t,x}(T)\right]$$

where V is a concave increasing function defined on $(0, \infty)$.

In this case, the Hamilton-Jacobi-Bellman equation reads:

$$\frac{\partial}{\partial t}v(t, x) + \max_{u\in[0,1]} \{-upxDv(t, x) + \lambda (v(t, x(1 - (1 - u)\delta)) - v(t, x))\} = 0 \tag{2.3.17}$$

a. Power utility: We first consider the case where $V(x) = x^\gamma$ with $\gamma \in (0, 1)$.
We try to find a solution to (2.3.17) satisfying the terminal condition $\lim_{t\nearrow T, x'\to x} v(t, x') = x^\gamma$. In view of (2.3.16), we have $v(t, x) = x^\gamma v(t, 1)$. Thus, it is natural to look for a solution of the form $x^\gamma\varphi(t)$ with φ a smooth function. In this case, (2.3.17) reduces to

$$x^\gamma \frac{\partial}{\partial t}\varphi(t) + x^\gamma\varphi(t) \max_{u\in[0,1]} \{-\gamma up + \lambda ((1 - (1 - u)\delta)^\gamma - 1)\} = 0$$

so that φ must satisfies

$$\frac{\partial}{\partial t}\varphi(t) + \varphi(t)m(\hat{u}) = 0$$

where $\hat{u}$ maximizes over $u \in [0, 1]$ the quantity

$$m(u) := -\gamma up + \lambda ((1 - (1 - u)\delta)^\gamma - 1) \ .$$

Since we must have $\varphi(T) = 1$, this implies that

$$\varphi(t) = e^{(T-t)m(\hat{u})} \ .$$

The arguments used in the proof of Theorem 2.2.4 shows that the optimal control is constant and equal to $\hat{u}$ and that the value function v is actually given by $x^\gamma e^{(T-t)m(\hat{u})}$.

b. Log utility: We now take $V(x) = \ln(x)$ and look for a solution of the form $\ln(x) + \varphi(t)$ with φ a smooth function. In this case, (2.3.17) reduces to

$$\frac{\partial}{\partial t}\varphi(t) + \max_{u\in[0,1]} \{-\gamma up + \lambda \ln(1 - (1-u)\delta)\} = 0$$

so that

$$\varphi(t) = (T-t)m(\hat{u})$$

where $\hat{u}$ maximizes over $u \in [0,1]$ the quantity

$$m(u) := -\gamma up + \lambda \ln(1 - (1-u)\delta) \ .$$

3.6 Optimal reinsurance with exponential utility

We now consider the problem of an insurance company. It receives premiums at a rate p. The times arrival of the sinisters are described by a Poisson process N of intensity $\lambda > 0$ and each sinister as a constant size δ. The company can reinsure a proportion $u \in [0,1]$ of these risks against the payment of a premium uq with $q \geq p$. The dynamic of the wealth is

$$X^\nu_{t,x}(s) = x + \int_t^s (p - \nu_v q)\, dv - \int_t^s \delta(1-\nu_v) dN_v \ .$$

We consider the exponential utility maximization problem:

$$v(t,x) := \sup_{\nu\in\mathcal{U}_t} \mathbb{E}\left[-e^{-\eta X^\nu_{t,x}(T)}\right] \ , \ \eta > 0 \ .$$

The associated Hamilton-Jacobi-Bellman equation is

$$\frac{\partial}{\partial t}v(t,x) + \max_{u\in[0,1]} \{(p-uq)Dv(t,x) + \lambda\left(v(t, x-(1-u)\delta) - v(t,x)\right)\} = 0$$

and it is natural to look for a solution of the form $-e^{-\eta x}\varphi(t)$, for a smooth function φ satisfying $\varphi(T) = 1$. This implies that

$$\frac{\partial}{\partial t}\varphi(t) + \varphi(t)m(\hat{u}) = 0$$

where $\hat{u}$ maximizes the quantity

$$m(u) := \eta(p - uq) + \lambda\left(1 - e^{\eta(1-u)\delta}\right) \ .$$

Thus, we must have

$$\varphi(t) = e^{(T-t)m(\hat{u})} \ .$$

3.7 Optimal reinsurance with possible investments in a financial asset

We consider the previous model except that we now assume that the company can invest on a financial market which consists in two assets: a risk free asset which provides a constant instantaneous rate of return $r > 0$, and a risky asset S which evolves as in the *Black and Scholes model*:

$$S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t} .$$

The amount of money invested in the risky asset is described by a predictable process ϕ satisfying

$$\mathbb{E}\left[\int_0^T |\phi_s|^2 ds\right] < \infty .$$

The remaining part of the wealth is invested in the non-risky asset.

The dynamics of the wealth is given by

$$X_{t,x}^{\phi,\nu}(s) \quad = \quad x + \int_t^s \phi_v \frac{dS_v}{S_v} + \int_t^s \left\{(X_{t,x}^{\phi,\nu}(s) - \phi_v)r + p - \nu_v q\right\} dv - \int_t^s \delta(1-\nu_v)dN_v .$$

The aim of the company is to maximize

$$J(t,x;\phi,\nu) := \mathbb{E}\left[-e^{-\eta X_{t,x}^{\phi,\nu}(T)}\right] .$$

The associated Hamilton-Jacobi-Bellman equation is

$$\begin{aligned} 0 \quad &= \quad \frac{\partial}{\partial t}v(t,x) + \sup_{\phi\in\mathbb{R}}\left\{(xr + \phi(\mu - r))Dv(t,x) + \frac{1}{2}\phi^2\sigma^2 D^2 v(t,x)\right\} \\ &+ \quad \max_{u\in[0,1]}\left\{(p-uq)Dv(t,x) + \lambda\left(v(t,x-(1-u)\delta) - v(t,x)\right)\right\} . \end{aligned}$$

Here again, we look for a solution of exponential type in x. To take into account the return of the non-risky asset, we look for v in the form $-f(t)e^{-\eta x e^{r(T-t)}}$. In this case, we must have:

$$\begin{aligned} 0 \quad &= \quad -\frac{\partial}{\partial t}f(t)/f(t) + \sup_{\phi\in\mathbb{R}}\left\{\eta\phi(\mu-r)e^{r(T-t)} - \eta^2\frac{1}{2}\phi^2\sigma^2 e^{2r(T-t)}\right\} \\ &+ \quad \max_{u\in[0,1]}\left\{\eta(p-uq)e^{r(T-t)} + \lambda\left(1 - e^{\eta(1-u)e^{r(T-t)}}\right)\right\} . \end{aligned}$$

The sup over ϕ is attained by

$$\hat{\phi} := \frac{\mu - r}{\eta\sigma^2}e^{-r(T-t)}$$

so that f must satisfy

$$0 \quad = \quad -\frac{\partial}{\partial t}f(t)/f(t) + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} + m_t(\hat{u}(t))$$

where $\hat{u}(t)$ maximizes

$$m_t(u) := \left\{\eta(p-uq)e^{r(T-t)} + \lambda\left(1 - e^{\eta(1-u)e^{r(T-t)}}\right)\right\} .$$

4 Optimal control with infinite time horizon

We come back to the framework of Section 1 except that we assume that λ does not depend on the time variable t.

Our aim is to maximize the functional

$$J(t,x;\nu) := \mathbb{E}\left[\int_t^\infty \mathcal{E}^\nu_{t,x}(s) f(X^\nu_{t,x}(s),\nu_s)ds\right] \quad x \in \mathbb{R}^d$$

We now assume that f and ρ are non-negative continuous functions. This ensures that the expectation is well defined, taking possibly the value $+\infty$. We can therefore define the value function

$$v(t,x) := \sup_{\nu\in\mathcal{U}_t} J(t,x;\nu) \ .$$

Clearly, v does not depend on t and it is equivalent to consider the problem

$$v(x) := \sup_{\nu\in\mathcal{U}} J(x;\nu) \ ,$$

where

$$J(x;\nu) := \mathbb{E}\left[\int_0^\infty \mathcal{E}^\nu_{x}(s) f(X^\nu_{x}(s),\nu_s)ds\right] \quad x \in \mathbb{R}^d$$

with $(X^\nu_x, \mathcal{E}^\nu_x) := (X^\nu_{0,x}, \mathcal{E}^\nu_{0,x})$.

We shall follow the steps of Section 2. The only difference is that we shall derive an *elliptic* equation (time independent) without terminal condition (since there is no time horizon). In practice, the terminal condition has to be replaced by an analysis of the behavior of the value function v as $t \to \infty$.

4.1 Dynamic programming and Hamilton-Jacobi-Bellman equation

We start with the dynamic programming principle.

Proposition 2.4.6 *Assume that for each $\nu \in \mathcal{U}$, $J(\cdot;\nu)$ is finite and continuous, and that v is locally bounded. Then, for all uniformly bounded family of stopping times $\{\theta^\nu, \nu \in \mathcal{U}\}$,*

$$v(x) = \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\mathcal{E}^\nu_x(\theta^\nu) v(X^\nu_x(\theta^\nu)) + \int_0^{\theta^\nu} \mathcal{E}^\nu_x(s) f(X^\nu_x(s),\nu_s)ds\right] \ .$$

Proof. This follows from similar arguments as in the proof of Theorem 2.2.1. □

As in the finite horizon case, this leads to the derivation of an Hamilton-Jacobi-Bellman equation. As in Section 2.2, we use the notations

$$\begin{aligned}\mathcal{L}^u\varphi(x) &:= \langle b(x,u), D\varphi(x)\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(x,u)D^2\varphi(x)\right] \\ &+ \int_{\mathbb{R}^d} \left(\varphi(x+\beta(x,u,z)) - \varphi(x)\right)\lambda\Phi(dz)\end{aligned}$$

and

$$\mathcal{H}\varphi(x) := \sup_{u\in U}\left(\mathcal{L}^u\varphi(x) + f(x,u)\right) .$$

Proposition 2.4.7 *Let the conditions of Proposition 2.4.6 hold. Assume further that v is C^2. Then, it satisfies*

$$\mathcal{H}v = \rho v . \tag{2.4.1}$$

Proof. Given a constant control $\nu = u$, $u \in \mathcal{U}$, we deduce from Proposition 2.4.6 that

$$v(x) \geq \mathbb{E}\left[\mathcal{E}^\nu_x(\theta)v(X^\nu_x(\theta)) + \int_0^\theta \mathcal{E}^\nu_x(s)f(X^\nu_x(s),\nu_s)ds\right]$$

for all stopping time θ. Then, the arguments used in the proof of Theorem 1.3.1 implies that

$$0 \geq \mathcal{L}^u v(x) + f(x,u) - \rho(x)v(x) .$$

It remains to show that

$$\mathcal{H}v(x) \geq \rho(x)v(x) .$$

Assume to the contrary that

$$\mathcal{H}v(x) < \rho(x)v(x) .$$

Then, the same arguments as in the second part of the proof of Theorem 2.2.2 leads to a contradiction to Proposition 2.4.6. □

In the case where v may not be smooth, it can still be characterize as a solution in the viscosity sense, we leave the proof to the reader, see the previous section.

We now turn to the verification argument.

Proposition 2.4.8 *(Verification) Assume that there exists a C^2 locally bounded function φ satisfying* (2.4.1)*. Assume further that*
1. There is a measurable map $\hat{u} : \mathbb{R}^d \mapsto U$ such that $\mathcal{H}\varphi = \mathcal{L}^{\hat{u}}\varphi + f(\cdot, \hat{u})$ on $\mathbb{R}^d$.
2. For all initial conditions $x \in \mathbb{R}^d$, there is a solution to (2.1.1) *starting at $t = 0$ with $\hat{\nu}$ defined by $\hat{\nu}_s := \hat{u}(X_x^{\hat{\nu}}(s))$.*
3. For all $(x, \nu) \in \mathbb{R}^d \times \mathcal{U}$, there is an increasing sequence of stopping times $(\theta_n)_n$ such that X_x^ν is bounded on $[0, \theta_n]$, $\theta_n \to \infty$ $\mathbb{P}-$ a.s. and

$$\mathbb{E}\left[\mathcal{E}_x^\nu(\theta_n)\varphi(X_x^\nu(\theta_n))\right] \to 0 \quad \text{as } n \text{ goes to } \infty.$$

Then, $v = \varphi$.

Proof. By Itô's Lemma, Corollary A.2.1, (A.1.1), the fact that φ solves (2.4.1) and the assumptions 1., 2. and 3., we obtain

$$\varphi(x) = \mathbb{E}\left[\mathcal{E}_x^{\hat{\nu}}(\theta_n)\varphi(X_x^{\hat{\nu}}(\theta_n)) + \int_0^{\theta_n} \mathcal{E}_x^{\hat{\nu}}(s) f(X_x^{\hat{\nu}}(s), \hat{\nu}_s) ds\right]$$

for each n, where $(\theta_n)_n$ is associated to $\hat{\nu}$. Using assumption 3, we then deduce that

$$\varphi(x) = \mathbb{E}\left[\int_0^\infty \mathcal{E}_x^{\hat{\nu}}(s) f(X_x^{\hat{\nu}}(s), \hat{\nu}_s) ds\right]$$

by sending n to ∞ and using the monotone convergence theorem, recall that $f \geq 0$. This shows that $\varphi(x) \leq v(x)$. We now prove the converse inequality. Fix $\nu \in \mathcal{U}$, let $(\theta_n)_n$ be the sequence associated to ν. By the same arguments as in the proof of Theorem 2.2.2, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{E}_x^\nu(\theta_n)\varphi(X_x^\nu(\theta_n)) + \int_0^{\theta_n} \mathcal{E}_x^\nu(s) f(X_{t,x}^\nu(s), \nu_s) ds\right] \\
&= \varphi(x) + \mathbb{E}\left[\int_0^{\theta_n} \mathcal{E}_x^\nu(s)\left(\mathcal{L}^{\nu_s}\varphi(X_x^\nu(s)) - (\rho\varphi)(X_x^\nu(s)) + f(X_x^\nu(s), \nu_s)\right) ds\right] \\
&\leq \varphi(x)\ .
\end{aligned}$$

Using the assumption 3. and sending $n \to \infty$ leads to

$$\mathbb{E}\left[\int_0^\infty \mathcal{E}_x^\nu(s) f(X_{t,x}^\nu(s), \nu_s) ds\right] \leq \varphi(x)\ .$$

Since ν is arbitrary, this implies that $v(x) \leq \varphi(x)$. □

Remark 2.4.16 As in the finite time horizon case, the value function can be characterize as a discontinuous viscosity solution of $\mathcal{H}v = \rho v$ and a comparison result can be established, under suitable growth assumptions. We leave this to the reader.

4.2 Application to the optimal investment with random time horizon

Let us consider the problem of Section 3.7 where the dynamics of the wealth is given by

$$X_x^{\phi,\nu}(t) = x + \int_0^t \phi_s \frac{dS_s}{S_s} + \int_0^t \left\{(X_x^{\phi,\nu}(s) - \phi_s)r + p - \nu_s q\right\} ds - \int_0^t \delta(1-\nu_s)dN_s$$

with

$$S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t} .$$

Instead of fixing a time horizon, we now consider the problem of an investor who wants to maximize the expected value of his wealth at his death time τ. We assume that τ is independent of (W, N) and admits an exponential distribution of parameter κ. We take the utility of exponential type. In this case, one can show that

$$J(x;\phi,\nu) := \mathbb{E}\left[-e^{-\eta X_x^{\phi,\nu}(\tau)}\right] = \mathbb{E}\left[\int_0^\infty \kappa e^{-\kappa t}\left\{-e^{-\eta X_x^{\phi,\nu}(t)}\right\} dt\right] .$$

i.e. the original problem is equivalent to an optimal control problem in infinite horizon. The value function is given by

$$v(x) := \sup_{(\phi,\nu)\in\mathcal{U}} \mathbb{E}\left[\int_0^\infty \kappa e^{-\kappa t}\left\{-e^{-\eta X_x^{\phi,\nu}(t)}\right\} dt\right] ,$$

where $\mathcal{U}$ is the set of predictable processes (ν,ϕ) with values in $[0,1]\times\mathbb{R}$ satisfying

$$\int_0^t |\phi_s|^2 ds < \infty \ , \ t \geq 0$$

and

$$\mathbb{E}\left[\int_0^\infty \kappa e^{-\kappa t} e^{-\eta X_x^{\phi,\nu}(t)} dt\right] < \infty \ , \ \lim_{t\to\infty} \mathbb{E}\left[e^{-\kappa t} e^{-\eta X_x^{\phi,\nu}(t)} dt\right] = 0 \ , \qquad (2.4.2)$$

The associated Hamilton-Jacobi-Bellman equation is

$$\begin{aligned} \kappa v &= -e^{-\eta x} + \sup_{\phi\in\mathbb{R}}\left\{(xr + \phi(\mu - r))Dv(t,x) + \frac{1}{2}\phi^2\sigma^2 D^2 v(x)\right\} \\ &+ \max_{u\in[0,1]} \left\{(p-uq)Dv(x) + \lambda\left(v(x-(1-u)\delta) - v(x)\right)\right\} \qquad (2.4.3) \end{aligned}$$

In the case where $r = 0$, it is natural to search a solution of the form $-be^{-\eta x}$, $b > 0$, which leads to

$$-\kappa b = -1 + \sup_{\phi\in\mathbb{R}}\left\{\eta b\phi\mu - \frac{1}{2}\phi^2 b\eta^2\sigma^2\right\} + \max_{u\in[0,1]}\left\{\eta b(p-uq) + b\lambda\left(1 - e^{\eta(1-u)\delta}\right)\right\} .$$

The sup over ϕ is attained by

$$\hat{\phi} := \frac{\mu}{\eta\sigma^2}$$

so that b must satisfy

$$b = \hat{b} := \left(\kappa + \frac{1}{2}\frac{\mu^2}{\sigma^2} + m(\hat{u})\right)^{-1} ,$$

where $\hat{u}$ maximizes over u

$$m(u) := \left\{\eta(p - uq) + \lambda\left(1 - e^{\eta(1-u)\delta}\right)\right\} .$$

We now provide the verification argument in the case where $\hat{b} > 0$. This condition is satisfied in particular if $p = q$. We then set $\psi(x) := -\hat{b}e^{-\eta x}$.
Using Itô's Lemma, a localization argument and the fact that ψ satisfies (2.4.3), we first observe that for any $(\phi, \pi) \in \mathcal{U}$

$$\mathbb{E}\left[e^{-\kappa t}\psi(X_x^{\phi,\nu}(t)) + \int_0^t \kappa e^{-\kappa s}\left\{-e^{-\eta X_x^{\phi,\nu}(s)}\right\} ds\right] \leq \psi(x)$$

Using (2.4.2), the particular form of ψ and sending $t \to \infty$, we obtain

$$\mathbb{E}\left[\int_0^\infty \kappa e^{-\kappa s}\left\{-e^{-\eta X_x^{\phi,\nu}(s)}\right\} ds\right] \leq \psi(x) .$$

On the other hand, the same arguments shows that for $(\phi, \nu) := (\hat{\phi}, \hat{u})$

$$\mathbb{E}\left[e^{-\kappa t}\psi(X_x^{\hat{\phi},\hat{u}}(t)) + \int_0^t \kappa e^{-\kappa s}\left\{-e^{-\eta X_x^{\hat{\phi},\hat{u}}(s)}\right\} ds\right] = \psi(x) ,$$

and, if $(\hat{\phi}, \hat{u})$ satisfies (2.4.2),

$$\mathbb{E}\left[\int_0^\infty \kappa e^{-\kappa s}\left\{-e^{-\eta X_x^{\hat{\phi},\hat{u}}(s)}\right\} ds\right] = \psi(x) .$$

Thus, it remains to check the admissibility condition. We have

$$X_x^{\hat{\phi},\hat{u}}(t) = x + \frac{\mu}{\eta\sigma^2}(\mu t + \sigma W_t) + (p - \hat{u}q)t - \delta(1 - \hat{u})N_t$$

so that

$$\mathbb{E}\left[e^{-\eta X_x^{\hat{\phi},\hat{u}}(t)}\right] = e^{-\eta x}e^{-\eta t\left(\frac{\mu^2}{2\sigma^2} + p - \hat{u}q\right)}e^{\lambda t\left(e^{\eta\delta(1-\hat{u})} - 1\right)} = e^{-\eta x}e^{-t\left(\frac{\mu^2}{2\sigma^2} + m(\hat{u})\right)} .$$

Thus, the admissibility of the candidate to be the optimal strategy reads

$$\kappa > -\left(\frac{\mu^2}{2\sigma^2} + m(\hat{u})\right) \Leftrightarrow \hat{b} > 0 .$$

Thus $\psi = v$ and the optimal strategy is given by $(\hat{\phi}, \hat{u})$.

4.3 Optimization of the survival probability

We consider the same model as in Section 3.7 except that:

1. There is no reinsurance.
2. The non risky rate of return r is equal to 0.
3. The size of each claim has a bounded density f on $(0,\infty)$.
4. We want to maximize the survival probability: $J(x;\phi) := \mathbb{P}\left[X_x^\phi(t) \geq 0 \ \ \forall\, t \geq 0\right]$.

The dynamics of the wealth is given by

$$X_x^\phi(t) \quad = \quad x + \int_0^t \phi_s \frac{dS_s}{S_s} + pt - \int_0^t z\mu(dz,dt)$$

where μ admits $\lambda f(z)dz$ as predictable kernel intensity and

$$S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t} \ ,$$

where W a standard Brownian motion. Here, ϕ belongs to the set $\mathcal{U}$ of predictable processes satisfying

$$\int_0^t |\phi_s|^2 ds < \infty \quad , \ t \geq 0 \ .$$

We assume that $\lambda, \sigma > 0$.

The value function is: $v(x) = \sup_{\phi\in\mathcal{U}} J(x;\phi)$.

Observing that, for all stopping time θ,

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{X_x^\phi(t)\geq 0 \ \ \forall\, t\geq 0\}}\right] &= \mathbb{E}\left[\mathbf{1}_{\{X_x^\phi(t)\geq 0 \ \ \forall\, t\in[0,\theta]\}}\mathbf{1}_{\{X_x^\phi(t)\geq 0 \ \ \forall\, t\in[\theta,\infty)\}}\right] \ , \\
&= \mathbb{E}\left[\mathbf{1}_{\{X_x^\phi(t)\geq 0 \ \ \forall\, t\in[0,\theta]\}}\mathbb{E}\left[\mathbf{1}_{\{X^\phi_{X_x^\phi(\theta)}(\theta+t)\geq 0 \ \ \forall\, t\geq 0\}} \mid X_x^\phi(\theta)\right]\right]
\end{aligned}$$

we end up, formally, with the dynamic programming equation:

$$v(x) \quad = \quad \sup_{\phi\in\mathcal{U}} \mathbb{E}\left[\mathbf{1}_{\{X_x^\phi(t)\geq 0 \ \ \forall\, t\in[0,\theta]\}} v(X_x^\phi(\theta))\right] \ .$$

Thus, still formally, v is associated to the PDE on $(0,\infty)$:

$$0 = \sup_{\phi\in\mathbb{R}}\left\{\phi\mu Dv(x) + \frac{1}{2}\phi^2\sigma^2 D^2 v(x)\right\} + \left\{pDv(x) + \lambda\int_0^\infty (v(x-z)-v(x))f(z)dz\right\} \tag{2.4.4}$$

with $v = 0$ on $\mathbb{R}_-$. Direct computation shows that the optimal ϕ in the above equation should be given by

$$\hat{\phi}(x) = -\frac{\mu Dv(x)}{\sigma^2 D^2 v(x)} \ ,$$

which leads to the equation

$$0 \quad = \quad pDv(t,x) - \frac{1}{2}\frac{\mu^2 Dv(t,x)^2}{\sigma^2 D^2 v(t,x)} + \lambda \int_0^\infty (v(x-z) - v(x)) f(z) dz \ . \qquad (2.4.5)$$

Moreover, one could expect to have $v(\infty) = 1$.

In [8], it is actually shown that there exists a solution $\psi \in C^2((0,\infty)) \cap C^1([0,\infty))$ to (2.4.5) which is positive, strictly increasing and strictly concave on $[0,\infty)$ and satisfies $\psi = 0$ on $\mathbb{R}_-$. In the case where $1 - \Phi$ is integrable, then it is bounded. In this case, we can choose it to satisfy $\psi(\infty) = 1$ by a simple normalization.

We now show that, if existence holds for

$$\hat{X}_x(t) \quad = \quad x + \int_0^t \hat{\phi}(\hat{X}_x(t-)) \frac{dS_s}{S_s} + pt - \int_0^t z\mu(dz, dt)$$

with $\hat{\phi}(x) := -\mu Dv(x)/(\sigma^2 D^2 v(x))$, then $\psi = v$ and the optimal strategy is given by $(\hat{\phi}(\hat{X}_x(t-))_{t \geq 0}$.

Let ϕ be a given strategy and let τ be the first time X_x^ϕ goes below 0. By dominated convergence, the fact that ψ satisfies (2.4.4), Itô's Lemma and a localization argument, we obtain

$$\mathbb{E}\left[\lim_{t\to\infty} \psi(X_x^\phi(t\wedge\tau))\right] = \lim_{t\to\infty} \mathbb{E}\left[\psi(X_x^\phi(t\wedge\tau))\right] \leq \psi(x) \ .$$

But, on $\{\tau = \infty\}$ one can show that $\lim_{t\to\infty} X_x^{\hat{\phi}}(t) = \infty$. Since $\psi = 0$ on $\mathbb{R}_-$, this implies that

$$\psi(x) \geq \mathbb{P}\left[\tau = \infty\right] \ .$$

Now, let $\hat{\tau}$ be the first time $\hat{X}_x$ goes below 0. By the same arguments as above, one obtains

$$\mathbb{E}\left[\psi(\lim_{t\to\infty} \hat{X}_x(t\wedge\tau))\right] = \lim_{t\to\infty} \mathbb{E}\left[\psi(\hat{X}_x(t\wedge\tau))\right] = \psi(x)$$

where $\lim_{t\to\infty} \hat{X}_x(t) = \infty$ on $\{\tau = \infty\}$. Thus,

$$\psi(x) = \mathbb{P}\left[\hat{\tau} = \infty\right] \ .$$

Chapter 3

Free boundary problems

1 Optimal stopping

In this section, we study an optimal stopping problem. Namely, given X defined as in Chapter 1, we want to optimize

$$\mathbb{E}\left[\mathcal{E}_{t,x}(\tau)g(X_{t,x}(\tau)) + \int_t^\tau \mathcal{E}_{t,x}(s)f(X_{t,x}(s))ds\right]$$

for τ running in the set $\mathcal{T}_t$ of stopping times with values in $[t,T]$, that are independent of $\mathcal{F}_t$. This problem naturally appears in finance in the computation of American option prices, see e.g. [3].

As in the previous chapters, we denote by v the associated value function:

$$v(t,x) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\mathcal{E}_{t,x}(\tau)g(X_{t,x}(\tau)) + \int_t^\tau \mathcal{E}_{t,x}(s)f(X_{t,x}(s))ds\right] .$$

The aim of this chapter is to relate v to the PDE in variational form

$$\min\left\{\rho\varphi - \mathcal{L}\varphi - f \ , \ \varphi - g\right\} = 0 \ , \tag{3.1.1}$$

in the viscosity sense. Note that, formally, the PDE $\rho\varphi - \mathcal{L}\varphi - f = 0$ is satisfied on the domain $\mathcal{C} := \{(t,x) \in [0,T) \times \mathbb{R}^d \ : \ v(t,x) > g(t,x)\}$ with the boundary condition $v = g$ on the parabolic boundary of $\mathcal{C}$. Since this boundary is not know, we call this problem a *free boundary* problem.

As in control problems in standard form, this formulation has a nice probabilistic interpretation. Let us define

$$Z(t) := \mathcal{E}^\nu_{0,x}(t)v(t, X^\nu_{0,x}(t)) + \int_0^t \mathcal{E}^\nu_{0,x}(s)f(X^\nu_{0,x}(s))ds \ \ , \ t \geq 0 \ .$$

The PDE (3.1.1) means that this process is a super-martingale $[0,T]$ and a martingale on $[0,\hat{\tau}]$ where $\hat{\tau}$ is the first time s for which $v(s,X_{0,x}(s)) = g(s,X_{0,x}(s))$ or $v(s,X_{0,x}(s-)) = g(s,X_{0,x}(s-))$. We refer to [11] for a general analysis of optimal stopping problems.

For sake of simplicity, for the remaining of this Chapter, we assume that g and f are Lipschitz continuous with Lipschitz constant $L>0$, that ρ is constant and $g \geq 0$.

1.1 Continuity of the value function

We first prove that the value function v is continuous. This is due to the Lipschitz continuity of f and g. A similar analysis could be carried out in the previous Chapters.

Lemma 3.1.4 *There is a constant $C>0$ such that for all $(t_1,t_2,x_1,x_2) \in [0,T]^2 \times (\mathbb{R}^d)^2$ we have*

$$|v(t_1,x_1) - v(t_2,x_2)| \quad \leq \quad C\left((1+\|x\|)|t_1-t_2|^{\frac{1}{2}} + \|x_1-x_2\|\right) .$$

Proof. Without loss of generality we can take $t_1 \leq t_2$. Since ρ is constant, we have

$$\begin{aligned} v(t_i,x_i) &= \sup_{\tau \in \mathcal{T}_{t_i}} \mathbb{E}\left[\mathcal{E}_{t_i,x_i}(\tau) g(X_{t_i,x_i}(\tau)) + \int_{t_i}^{\tau} \mathcal{E}_{t_i,x_i}(s) f(X_{t_i,x_i}(s)) ds\right] \\ &= e^{\rho t_i} w(t_i,x_i) \ , \ i=1,2 \ , \end{aligned}$$

with

$$w(t,x) := \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-\rho\tau} g(X_{t,x}(\tau) + \int_t^{\tau} e^{-\rho s} f(X_{t,x}(s)) ds\right] .$$

Thus, it suffices to prove the result for w. Since $g \geq 0$, $\mathcal{T}_{t_2} \subset \mathcal{T}_{t_1}$ and $\tau \vee t_2 \in \mathcal{T}_{t_2}$ if $\tau \in \mathcal{T}_{t_1}$, we have

$$\begin{aligned} 0 &\leq w(t_1,x_1) - w(t_2,x_1) \\ &\leq \sup_{\tau \in \mathcal{T}_{t_1}} \mathbb{E}\left[e^{-\rho\tau} g(X_{t_1,x_1}(\tau)) - e^{-\rho(\tau \vee t_2)} g(X_{t_2,x_1}(\tau \vee t_2)) + \int_{t_1}^{\tau \wedge t_2} e^{-\rho s} |f(X_{t_1,x_1}(s))| ds\right] \\ &+ \sup_{\tau \in \mathcal{T}_{t_1}} \mathbb{E}\left[\int_{t_2}^{\tau \vee t_2} e^{-\rho s} |f(X_{t_1,x_1}(s)) - f(X_{t_2,x_1}(s))| ds\right] . \end{aligned}$$

In view of Proposition 1.2.1, the Lipschitz continuity assumption on g and f implies that

$$|w(t_1,x_1) - w(t_2,x_1)| \quad \leq \quad C(1+\|x_1\|)|t_1-t_2|^{\frac{1}{2}} .$$

Similarly,

$$\begin{aligned}|w(t_1,x_1)-w(t_1,x_2)| &\leq \sup_{\tau\in\mathcal{T}_{t_1}} \mathbb{E}\left[e^{-\rho\tau}|g(X_{t_1,x_1}(\tau))-g(X_{t_1,x_2}(\tau))|\right] \\ &+ \sup_{\tau\in\mathcal{T}_{t_1}} \mathbb{E}\left[\int_{t_1}^{\tau} e^{-\rho s}|f(X_{t_1,x_1}(s))-f(X_{t_1,x_2}(s))|ds\right] \\ &\leq C\|x_1-x_2\| \ .\end{aligned}$$

□

1.2 Dynamic programming and viscosity property

As in the previous chapters, we shall appeal to the dynamic programming principle which takes here the following form.

Lemma 3.1.5 *(Dynamic programming) Fix $(t,x)\in[0,T]\times\mathbb{R}^d$. For all $[t,T]$-valued stopping time θ, we have*

$$v(t,x)=\sup_{\tau\in\mathcal{T}_t}\mathbb{E}\left[e^{-\rho(\theta-t)}v(\theta,X_{t,x}(\theta))\mathbf{1}_{\theta\leq\tau}+e^{-\rho(\tau-t)}g(X_{t,x}(\tau))\mathbf{1}_{\theta>\tau}+\int_t^{\theta\wedge\tau}e^{-\rho(s-t)}f(X_{t,x}(s))ds\right] \ .$$

Proof. It suffices to adapt the arguments used to prove Theorem 2.2.1. □

We can now prove that v is a viscosity solution of (3.1.1).

Theorem 3.1.1 *The function v is a viscosity solution of* (3.1.1) *and satisfies* $\lim_{t\to T} v(t,\cdot) =$ *g on $\mathbb{R}^d$.*

Proof. The super-solution property is clear. First, $v\geq g$ by construction. Second, Lemma 3.1.5 implies that

$$v(t,x)\geq\mathbb{E}\left[e^{-\rho(\theta-t)}v(\theta,X_{t,x}(\theta))+\int_t^{\theta}e^{-\rho(s-t)}f(X_{t,x}(s))ds\right]$$

for all $\theta\in\mathcal{T}_t$ (take $\tau=\theta$). It thus suffices to repeat the arguments of the proof of supersolution property of Theorem 1.4.1 . As for the subsolution property, we have to prove that for $(\hat{t},\hat{x})\in[0,T)\times\mathbb{R}^d$ and a smooth function φ such that $0=\max(v-\varphi)=(v-\varphi)(\hat{t},\hat{x})$, we have

$$\rho\varphi(\hat{t},\hat{x})-\mathcal{L}\varphi(\hat{t},\hat{x})-f(\hat{x})\leq 0 \quad \text{if} \quad v(\hat{t},\hat{x})>g(\hat{x}) \ .$$

We can argue by contradiction and assume that

$$\rho\varphi(\hat{t},\hat{x})-\mathcal{L}\varphi(\hat{t},\hat{x})-f(\hat{x})>0 \quad \text{while} \quad v(\hat{t},\hat{x})>g(\hat{x}) \ .$$

Then, by choosing a suitable stopping time $\theta \in \mathcal{T}_{\hat{t}}$ as done in the proof of Theorem 1.4.1, we can find $\eta > 0$ such that for all $s \in [\hat{t}, \theta]$

$$\rho\varphi(X_{\hat{t},\hat{x}}(s)) - \mathcal{L}\varphi(s, X_{\hat{t},\hat{x}}(s)) - f(X_{\hat{t},\hat{x}}(s)) \geq \eta \text{ and } v(s, X_{\hat{t},\hat{x}}(s)) - g(X_{\hat{t},\hat{x}}(s)) \geq \eta$$

while $(v - \varphi)(\theta, X_{\hat{t},\hat{x}}(\theta)) \leq -\eta$. By Itô's Lemma, this implies that, for all $\tau \in \mathcal{T}_{\hat{t}}$,

$$\begin{aligned}
\varphi(\hat{t}, \hat{x}) &\geq \mathbb{E}\left[e^{-\rho(\theta-\hat{t})}\varphi(\theta, X_{\hat{t},\hat{x}}(\theta))\mathbf{1}_{\theta\leq\tau} + e^{-\rho(\tau-\hat{t})}(g(X_{\hat{t},\hat{x}}(\tau)) + \eta)\mathbf{1}_{\theta>\tau}\right] \\
&+ \mathbb{E}\left[\int_{\hat{t}}^{\theta\wedge\tau} e^{-\rho(s-\hat{t})} f(X_{\hat{t},\hat{x}}(s))ds\right] \\
&\geq \mathbb{E}\left[e^{-\rho(\theta-\hat{t})}(v(\theta, X_{\hat{t},\hat{x}}(\theta)) + \eta)\mathbf{1}_{\theta\leq\tau} + e^{-\rho(\tau-\hat{t})}(g(X_{\hat{t},\hat{x}}(\tau)) + \eta)\mathbf{1}_{\theta>\tau}\right] \\
&+ \mathbb{E}\left[\int_{\hat{t}}^{\theta\wedge\tau} e^{-\rho(s-\hat{t})} f(X_{\hat{t},\hat{x}}(s))ds\right] .
\end{aligned}$$

This contradicts Lemma 3.1.5. □

In order to complete this characterization, we now provide a comparison result.

Theorem 3.1.2 *Let U (resp. V) be a l.s.c. super-solution (resp. u.s.c. sub-solution) with polynomial growth of* (3.1.1) *on $[0, T) \times \mathbb{R}^d$. If $U \geq V$ on $\{T\} \times \mathbb{R}^d$, then $U \geq V$ on $[0, T) \times \mathbb{R}^d$.*

Proof. We only explain how to adapt the proof of Theorem 1.4.2 from which we take the notations. The only difference appears if (t_n, x_n) is such that $V(t_n, x_n) \leq g(x_n)$. But, in this case, since $U(t_n, y_n) \geq g(x_n)$, we obtain that $U(t_n, x_n) \geq V(t_n, x_n)$ which contradicts (1.4.7). □

1.3 Construction of the optimal stopping strategy: formal derivation in the smooth case

For sake of simplicity, we only consider the case where there is no jumps, i.e. $\beta = 0$. Assume that v is $C^{1,2}$ on the set $\mathcal{C} := \{(t, x) \in [0, T] \times \mathbb{R}^d \;:\; v(t, x) > g(x)\}$ and is continuous on $[0, T] \times \mathbb{R}^d$. Assume further that is solves (3.1.1) on $\mathcal{C}$ and satisfies $v = g$ on $\mathcal{C}^c := ([0, T] \times \mathbb{R}^d) \setminus \mathcal{C}$. We assume that the distance function d to $\mathcal{C}^c$ is well defined and is continuous. We fix the initial conditions $(0, x_0)$, with $(0, x_0) \in \mathcal{C}$, and write X for X_{0,x_0}. Let $\tau_\varepsilon := \inf\{s \geq 0 \;:\; d(s, X(s)) \leq \varepsilon\}$. Then, it follows from the fact that v solves (3.1.1) on $\mathcal{C}$ that

$$v(0, x_0) = \mathbb{E}\left[v(\tau^\varepsilon, X(\tau^\varepsilon)) + \int_0^{\tau^\varepsilon} f(X(s))ds\right] .$$

Since $\tau^\varepsilon \to \tau^0$ $\mathbb{P}-$a.s. as $\varepsilon \to 0$, by continuity of the path of X, we deduce from the continuity of v and the fact that $v = g$ on $\mathcal{C}^c$ that

$$v(0, x_0) = \mathbb{E}\left[g(X(\tau^0)) + \int_0^{\tau^0} f(X(s))ds\right] .$$

It follows that $\tau^0 = \inf\{s \geq 0 \ : \ (s, X_s) \in \mathcal{C}^c\}$ is an optimal stopping strategy. The domain $\mathcal{C}$ is therefore called the *continuity region*, it is the region where it is never optimal to stop.

2 Optimal switching problems

In this section, we consider a diffusion whose coefficients may take different values, called regimes, depending on the value taken by an underlying control. More precisely, in the i-th regime, the drift is given by $b(\cdot, i)$ and the volatility by $\sigma(\cdot, i)$. The aim of the controller is to manage the different regimes of the process in order to maximize a mean gain.

2.1 Definitions

We first define a set of possible *regimes* $E = \{0, \ldots, \kappa\}$. Given the initial regime $e_0 \in E$, we say that an adapted process ξ is a regime control with initial condition $e_0 \in E$ if it takes the form

$$\xi_t \quad = \quad e_0 \mathbf{1}_{[0,\tau_1)} + \sum_{i \geq 1} \mathcal{E}_i \mathbf{1}_{\tau_i \leq t < \tau_{i+1}} \quad , \ t \leq T \ ,$$

where $(\tau_i)_{i\geq 1}$ is a sequence of (strictly) increasing stopping times such that $\tau_i \to \infty$ $\mathbb{P}-$a.s., and $(\mathcal{E}_i)_{i\geq 1}$ is a sequence of random variables with values in E such that $\mathcal{E}_i$ is $\mathcal{F}_{\tau_i}$-measurable for all $i \geq 1$. We denote by $(\tau_i^\xi)_{i\geq 1}$ the associates sequence of stopping times.

The controlled process $X_{t,x}^\xi$ is defined as the solution of:

$$\begin{aligned} X_{t,x}(s) \quad &= \quad x + \int_t^s b(X_{t,x}(u), \xi(u))du + \int_0^t \sigma(X_{t,x}(u), \xi(u))dW_s \qquad (3.2.1) \\ &+ \quad \sum_{\tau_i^\xi \leq t} \beta(X_{t,x}(\tau_i^\xi -), \xi(\tau_i^\xi -), \xi(\tau_i^\xi)) \ . \end{aligned}$$

Here, we do not introduce exogenous jumps in the dynamics of X. However, X may jump when we pass from a regime to another one.

As in the preceding chapters, we assume that b, σ (resp. β) are uniformly Lipschitz on $\mathbb{R}^d \times [0,\kappa]$ (resp. $\mathbb{R}^d \times [0,\kappa]^2$). We also assume that there is Ψ defined on E^2 such that

$$\sup_{x\in\mathbb{R}^d} |\beta(x,i,j)| \vee 1 \le \Psi(i,j) \quad \text{for all} \quad i,j \in E\ . \tag{3.2.2}$$

We say that a control ξ is admissible if

$$\mathbb{E}\left[|\sum_{\tau_i^\xi \le T} \Psi(\xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi})|^{2\bar{p}} \right] \;<\; \infty\ , \tag{3.2.3}$$

for some fixed $\bar{p} \ge 1$, and we denote by $\mathcal{S}_0(e_0)$ the set of admissible controls with initial condition $\xi_0 = e_0$.
The aim of the controller is to maximize the mean of

$$\Pi_{t,x}(\xi) := \quad g(X^\xi_{t,x}(T), \xi(T)) + \int_0^T f(X^\xi_{t,x}(s), \xi_s) ds - \sum_{\tau_i^\xi \le T} c\left(X^\xi_{t,x}(\tau_i^\xi -), \xi_{\tau_i^\xi -}, \xi_{\tau_i^\xi} \right) \tag{3.2.4}$$

where g, f and c are locally Lipschitz functions such that

$$\sup_{(x,i,j)\in\mathbb{R}^d\times E^2} \frac{|g(x,i)| + |f(x,i)| + |c(x,i,j)|}{1+|x|^{\bar{p}}} \;<\; \infty \tag{3.2.5}$$

and

$$\sup_{x\in\mathbb{R}^d} c(x,i,j)^+ \le \Psi(i,j) \quad \text{for all} \quad i,j\in E\ . \tag{3.2.6}$$

The associated value function is defined as

$$v(t,x,e) := \sup_{\xi\in\mathcal{S}_0(e_0)} \mathbb{E}\left[\, \Pi(\xi)\, \right]\ . \tag{3.2.7}$$

We shall always assume that

$$\beta(\cdot,e,e) = 0 \quad \text{and} \quad c(\cdot,e,e) = 1 \quad \text{for all } e \in E\ . \tag{3.2.8}$$

This allows to avoid strategies ξ for which $\mathbb{P}[\exists\ i \ge 1 \text{ s.t. } \tau_i^\xi \le T \text{ and } \xi_{\tau_i^\xi -} = \xi_{\tau_i^\xi}] > 0$ which makes no sense.

The aim of this Section is to prove that the value function solves on $[0,T) \times \mathbb{R}^d \times E$

$$\min\{-\mathcal{L}\varphi\ ,\ \mathcal{G}\varphi\} = 0 \tag{3.2.9}$$

where

$$\mathcal{G}^e\varphi(t,x,e) \;\; := \;\; \min_{j\in E\setminus\{e\}} \left(\varphi(t,x,e) - \varphi(t,x+\beta(x,e,j),j) + c(x,e,j)\right) \;,$$

and

$$\mathcal{L}^e\varphi \;\; := \;\; \frac{\partial}{\partial t}\varphi + \langle b(\cdot,e), D\varphi\rangle + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^*(\cdot,e)D^2\varphi\right] + f(\cdot,e) \;,$$

with the boundary condition on $\{T\}\times\mathbb{R}^d\times E$

$$\min\left\{\varphi(T,x,e) - g(x,e)\;,\;\mathcal{G}^e\varphi(T,x,e)\right\} = 0 \quad . \tag{3.2.10}$$

Note that, formally, this shows that $v(\cdot,e)$ solves the *free boundary* problem

$$\min\left\{-\mathcal{L}^e v(t,x,e)\;,\;v(t,x,e) - \Psi^e(t,x)\right\} = 0$$

where

$$\Psi^e(t,x) := \max_{j\in E\setminus\{e\}} \left(v(t,x+\beta(x,e,j),j) - c(x,e,j)\right) \;. \tag{3.2.11}$$

This problem is therefore very close from the optimal control problem of the previous section. The main difficulty here comes from the fact that the boundary is related to each $v(\cdot,j)$, $j\neq e$, which also solve free boundary problems which boundaries depend on $v(\cdot,e)$.

2.2 Dynamic programming

Some useful qualitative properties

We start with a Remark on the controlled process.

Remark 3.2.17 It follows from the admissibility condition (3.2.3) and similar arguments as the one used to prove Proposition 1.2.1 that

$$\mathbb{E}\left[\sup_{t\le s\le T}\|X^\xi_{t,x}(s)\|^{2\bar{p}}\right] < \infty \;.$$

Moreover, if $\mathrm{card}\{i\geq 1\;:\;t<\tau^\xi_i\leq T\}\leq K$ for some $K>0$, then

$$\mathbb{E}\left[\sup_{t\le s\le T}\|X^\xi_{t,x}(s)\|^{p}\right] \;\;\leq\;\; C^p_K(1+\|x\|^p) \;,$$

where the constant $C^p_K>0$ depends only on $b,\sigma,\beta,T,$ K and $p\geq 1$.

We first derive some useful properties for the functional

$$J(t,x,\xi) := \mathbb{E}\left[g(X_T^{(t,x),\xi},\xi_T) + \int_t^T f(X_s^{(t,x),\xi},\xi_s)ds - \sum_{t<\tau_i^\xi\le T} c\left(X^{(t,x),\xi}_{\tau_i^\xi -},\xi_{\tau_i^\xi -},\xi_{\tau_i^\xi}\right)\right]$$

defined for $(t,x)\in[0,T]\times\mathbb{R}^d$ and $\xi\in\mathcal{S}_0$. For $t_1\le t_2\le T$, we set

$$I^\xi_{t_1,t_2} \;:=\; \mathrm{card}\{i\ge 1 \;:\; t_1<\tau_i^\xi\le t_2\}\;,$$

and we denote by $\mathcal{S}_0^b$ the set of elements $\xi\in\mathcal{S}_0$ such that $I^\xi_{0,T}$ is essentially bounded. We set $\mathcal{S}_t^b(e) := \mathcal{S}_t(e)\cap\mathcal{S}_0^b$.

Lemma 3.2.6 *Fix $(t,x,e)\in[0,T)\times\mathbb{R}^d\times E$. Then,*
(i) For all $\xi\in\mathcal{S}_0^b$, $J(\cdot,\xi)$ is jointly continuous in x and right-continuous in t. If ξ is such that $\mathbb{P}\left[\tau_i^\xi = t\right]=0$ for all $i\ge1$, then $J(\cdot,\xi)$ is continuous at (t,x).
(ii) $\sup_{\xi\in\mathcal{S}_t^b(e)} J(t,x,\xi) = v(t,x,e)$.
(iii) $v(\cdot,e)$ is lower semicontinuous.

Proof. (i) We start with the first assertion. Fix $\xi\in\mathcal{S}_0^b$, $t_1\le t_2$, $x_1,x_2\in\mathbb{R}^d$ and write (X^1,X^2) for $(X^{(t_1,x_1),\xi},X^{(t_2,x_2),\xi})$. We define the sequence

$$\vartheta_{i+1} \;:=\; \inf\{s>\vartheta_i \;:\; \xi_s\neq\xi_{s-}\} \quad\text{for}\;\; i\ge 0\;, \quad \text{with } \vartheta_0 = t_2\;.$$

Standard computations based on Burkholder-Davis-Gundy's inequality, Gronwall's Lemma and the Lipschitz continuity of b,a,β shows that

$$\mathbb{E}\left[\sup_{t_2\le s\le\vartheta_{i+1}\wedge T}|X_s^1-X_s^2|^{2\bar p}\right] \;\le\; C\,\mathbb{E}\left[\sup_{t_2\le s\le\vartheta_i\wedge T}|X_s^1-X_s^2|^{2\bar p}\right] \quad i\ge 0\;,$$

where $C>0$ denotes a generic constant which may change from line to line. Since $I^\xi_{0,T}$ is essentially bounded and $\vartheta_0=t_2$, we deduce that

$$\mathbb{E}\left[\sup_{t_2\le s\le T}|X_s^1-X_s^2|^{2\bar p}\right] \;\le\; C\,\mathbb{E}\left[|X_{t_2}^1-x_2|^{2\bar p}\right]\;, \tag{3.2.12}$$

where, by Remark 3.2.17 and (3.2.2),

$$\mathbb{E}\left[\sup_{t_1\le s\le t_2}|X_s^1-x_1|^{2\bar p}\right] \;\le\; C\;\left(|t_2-t_1|^{2\bar p}+\mathbb{E}\left[|I^\xi_{t_1,t_2}|^{2\bar p}\right]\right)\;. \tag{3.2.13}$$

We now fix $(t,x) \in [0,T] \times \mathbb{R}^d$ and a sequence $(t_n, x_n)_{n\geq 1}$ such that $t_n \downarrow t$ and $x_n \to x$, we write X and X^n for $X^{(t,x),\xi}$ and $X^{(t_n,x_n),\xi}$. In view of (3.2.12)-(3.2.13), we can find a subsequence such that $\sup_{t\vee t_n\leq s\leq T} |X^n_s - X_s| \to 0$ $\mathbb{P}-$a.s. Moreover, if follows from Remark 3.2.17 that $\mathbb{E}\left[\sup_{t_n\leq s\leq T} |X^n_s|^{2\bar{p}}\right]$ is bounded, uniformly in $n \geq 1$. Recalling the growth condition (3.2.5) and the fact that $I^{\xi}_{0,T}$ is bounded, we deduce that

$$\liminf_{n\to\infty} J(t_n, x_n, \xi) \quad \geq \quad J(t,x,\xi) - \limsup_{n\to\infty} \mathbb{E}\left[\Big| \sum_{t\wedge t_n < \tau^{\xi}_i \leq t\vee t_n} \Psi(\xi_{\tau^{\xi}_i -}, \xi_{\tau^{\xi}_i}) \Big| \right] = J(t,x,\xi) \; .$$

We obtain similarly that $\limsup_{n\to\infty} J(t_n,x_n,\xi) \leq J(t,x,\xi)$. In the case where the control ξ satisfies $\mathbb{P}\left[\tau^{\xi}_i = t\right] = 0$ for all $i \geq 1$, the term $\mathbb{E}\left[\big|\sum_{t\wedge t_n<\tau^{\xi}_i\leq t\vee t_n} \Psi(\xi_{\tau^{\xi}_i -}, \xi_{\tau^{\xi}_i})\big|\right]$ goes to 0 even if t_n approximate t from the left. The above argument can then be repeated without modification for any sequence $(t_n,x_n)_n$ such that $t_n \to t$ and $x_n \to x$.
(ii) Fix $\xi \in \mathcal{S}_0$ and let $\xi^k \in \mathcal{S}^b_0$ be defined by $\xi^k_t = \xi_{t\wedge\tau^{\xi}_k}$, $k \geq 1$. Arguing as in Remark 3.2.17, we obtain that

$$\sup_{k\geq 1} \mathbb{E}\left[\sup_{t\leq s\leq T} |X^{(t,x),\xi^k}_s|^{2\bar{p}}\right] < \infty \; . \tag{3.2.14}$$

Moreover, it follows from a similar induction argument as above that

$$\mathbb{E}\left[\sup_{t\leq s\leq t\vee\tau^{\xi}_i} |X^{(t,x),\xi^k}_s - X^{(t,x),\xi}_s|^2\right] \longrightarrow 0 \quad \text{for all } i \geq 1 \; .$$

After possibly passing to a subsequence, we can then assume that

$$\sup_{t\leq s\leq t\vee\tau^{\xi}_i} |X^{(t,x),\xi^k}_s - X^{(t,x),\xi}_s| \longrightarrow 0 \;\; \mathbb{P}-\text{a.s.} \;\; \forall\, i \geq 1 \; .$$

In view of (3.2.14), we deduce from (3.2.5), (3.2.3), (3.2.6) and the continuity of g, f and c that

$$\liminf_{k\to\infty} J(t,x,\xi^k) \quad \geq \quad J(t,x,\xi) \; .$$

This proves (ii).
(iii) By using a continuity argument as in (ii) above, we can restrict to ξ such that $\mathbb{P}\left[\tau^{\xi}_i = t\right] = 0$ for all $i \geq 1$ in the definition of $v(t,x)$. The last assertion is then an immediate consequence of (i) and (ii). □

We conclude this section with an easy result which will also be useful in the following.

Proposition 3.2.9 *v^- has polynomial growth.*

Proof. This is an easy consequence of (3.2.5) since a constant control is admissible. $\Box$

Dynamic programming principle

We now turn to the proof of the dynamic programming principle.

Remark 3.2.18 The inequality

$$v(t,x,e) \leq \sup_{\xi \in \mathcal{S}_t(e)} \mathbb{E}\left[v(\theta^\xi, X^\xi_{t,x}(\theta^\xi), \xi_{\theta^\xi}) + \int_t^{\theta^\xi} f(X^\xi_{t,x}(s), \xi_s) ds - \sum_{t < \tau^\xi_i \leq \theta^\xi} c\left(X^\xi_{t,x}(\tau^\xi_i -), \xi_{\tau^\xi_i -}, \xi_{\tau^\xi_i}\right)\right]$$

for all family of $[t,T]$-valued stopping times $\{\theta^\xi, \xi \in \mathcal{S}_0\}$, follows from the Markov feature of our model.

Thus it suffices to prove the following Lemma.

Lemma 3.2.7 *Fix $(t,x,e) \in [0,T] \times \mathbb{R}^d \times E$. For all $[t,T]$-valued stopping time θ and $\xi \in \mathcal{S}_t(e)$, we have*

$$v(t,x,e) \geq \mathbb{E}\left[v(\theta, X^\xi_{t,x}(\theta), \xi_\theta) + \int_t^{\theta} f(X^\xi_{t,x}(s), \xi_s) ds - \sum_{t < \tau^\xi_i \leq \theta} c\left(X^\xi_{t,x}(\tau^\xi_i -), \xi_{\tau^\xi_i -}, \xi_{\tau^\xi_i}\right)\right] .$$

Proof. In view of the previous results, it suffices to adapt the proof of Theorem 2.2.1. $\Box$

2.3 PDE characterization

Definition of viscosity solutions

In this section, we adapt the notion of viscosity solutions introduced in Chapter 1 to our context.

Definition 3.2.2 *We say that a lower-semicontinuous (resp. upper-semicontinuous) function U on $[0,T) \times \mathbb{R}^d \times E$ is a viscosity super-solution (resp. subsolution) of* (3.2.9) *if, for all $e \in E$, $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and all $(t,x) \in [0,T) \times \mathbb{R}^d$ which realizes a local minimum (resp. maximum) of $U(\cdot, e) - \varphi$, we have*

$$\min\left\{-\mathcal{L}^e\varphi(t,x)\ ,\ \mathcal{G}^e U(t,x,e)\right\} \geq 0 \quad (\textit{resp} \leq 0)\ .$$

We say that a locally bounded function w is a discontinuous viscosity solution of (3.2.9) *if w_* (resp. w^*) is a super-solution (resp. subsolution) of* (3.2.9) *where*

$$
\begin{aligned}
w^*(t,x,e) &:= \limsup_{(t',x')\to(t,x),\ t'<T} w(t',x',e) \\
w_*(t,x,e) &:= \liminf_{(t',x')\to(t,x),\ t'<T} w(t',x',e)\ ,\quad (t,x,e)\in[0,T]\times\mathbb{R}^d\times E\ .
\end{aligned}
$$

To complete this characterization, we need to provide a suitable boundary condition. In general, we can not expect to have $v(T-,\cdot)=g$, and we need to consider the relaxed boundary condition given by the equation (3.2.10).

Definition 3.2.3 *We say that a locally bounded map w satisfies the boundary condition* (3.2.10) *if $w_*(T,\cdot)$ (resp. $w^*(T,\cdot)$) is a super-solution (resp. subsolution) of* (3.2.10). *Here the terms super-solution and subsolution are taken in the classical sense.*
If w is a discontinuous viscosity solution of (3.2.9) *and satisfies the boundary condition* (3.2.10), *we just say that w is a (discontinuous) viscosity solution of* (3.2.9)-(3.2.10). *We define similarly the notion of super and subsolution of* (3.2.9)-(3.2.10).

Viscosity properties

We now provide the characterization of v as a viscosity solution of (3.2.9)-(3.2.10).

Proposition 3.2.10 *The function v_* is a viscosity super-solution of* (3.2.9)-(3.2.10).

Proof. Fix $(t_0,x_0,e_0)\in[0,T]\times\mathbb{R}^d\times E$ and let $(t_k,x_k)_{k\geq1}$ be a sequence in $[0,T)\times\mathbb{R}^d$ such that

$$
(t_k,x_k)\ \longrightarrow\ (t_0,x_0)\ \text{ and }\ v(t_k,x_k,e_0)\ \longrightarrow\ v_*(t_0,x_0,e_0)\quad\text{as } k\longrightarrow\infty\ .
$$

Given $\xi^k\in\mathcal{S}_{t_k}(e_0)$ to be chosen later, we write X^k and τ_i^k for $X^{\xi^k}_{t_k,x_k}$ and $\tau_i^{\xi^k}$.
1. We first assume that $t_0=T$. By taking $\xi^k=e_0\in\mathcal{S}_{t_k}(e_0)$, we deduce from the definition of v that

$$
v(t_k,x_k,e_0)\ \geq\ \mathbb{E}\left[g(X_T^k,e_0)+\int_{t_k}^T f(X_s^k,e_0)ds\right]\ .
$$

Using standard estimates on X^k, we deduce from our continuity and growth assumptions on f,g that

$$
v_*(T,x_0,e_0)\ \geq\ \liminf_{k\to\infty}\mathbb{E}\left[\left(g(X_T^k,e_0)+\int_{t_k}^T f(X_s^k,e_0)ds\right)\right]=g(x_0,e_0)\ .
$$

We now fix $j \in E$, set $\tau_k := (T + t_k)/2$ and $\xi^k := (e_0 \mathbf{1}_{t<\tau_k} + j\mathbf{1}_{t\geq\tau_k})_{t\leq T} \in \mathcal{S}_{t_k}(e_0)$. By Lemma 3.2.7

$$v(t_k, x_k, e_0) \geq \mathbb{E}\left[v_*\left(\tau_k, X^k_{\tau_k-} + \beta(X^k_{\tau_k-}, e_0, j), j\right) + \int_{t_k}^{\tau_k} f(X^k_s, e_0)ds - c(X^k_{\tau_k-}, e_0, j)\right] .$$

Sending $k \to \infty$, using Proposition 3.2.9 and standard estimates shows that

$$v_*(T, x_0, e_0) \geq v_*\left(T, x_0 + \beta(x_0, e_0, j), j\right) - c(x_0, e_0, j) .$$

2. We now fix $t_0 < T$. By considering the sequence of controls $\xi^k = (e_0\mathbf{1}_{t<\tau_k} + j\mathbf{1}_{\tau_k\leq t})_{t\leq T} \in \mathcal{S}_{t_k}(e_0)$ where $\tau_k := t_k + k^{-1}$, $j \in E$, using Lemma 3.2.7 and arguing as above, we obtain

$$v_*(t_0, x_0, e_0) \;=\; \lim_{k\to\infty} v(t_k, x_k, e_0) \geq v_*\left(t_0, x_0 + \beta(x_0, e_0, j), j\right) - c(x_0, e_0, j) .$$

The fact that v_* is a super-solution of $-\mathcal{L}\varphi = 0$ is obtained by considering constant control processes, Lemma 3.2.7 and similar arguments as in the proof of Theorem 1.4.1.
□

Proposition 3.2.11 *The function v^* is a viscosity subsolution of* (3.2.9)-(3.2.10).

Proof. 1. We first consider the viscosity property. We argue by contradiction. Fix $(t_0, x_0, e_0) \in [0, T) \times \mathbb{R}^d \times E$ and $\varphi \in C^2_b([0, T] \times \mathbb{R}^d)$ such that

$$0 \;=\; (v^*(\cdot, e_0) - \varphi)(t_0, x_0) = \max_{[0,T]\times\mathbb{R}^d} (v^*(\cdot, e_0) - \varphi)$$

and assume that

$$\min\left\{-\mathcal{L}^{e_0}\varphi(t_0, x_0)\ ,\ \mathcal{G}^{e_0}v^*(t_0, x_0, e_0)\right\} =: 2\varepsilon > 0 .$$

Since $\varphi(t_0, x_0) = v^*(t_0, x_0, e_0)$, it follows from the upper-semicontinuity of v^* that we can find $\delta \in (0, T - t_0)$ for which

$$\min\left\{-\mathcal{L}^{e_0}\varphi\ ,\ \min_{j\in E\setminus\{e_0\}} \left(\varphi - v^*(\cdot, \cdot + \beta(\cdot, e_0, j), j) + c(\cdot, e_0, j)\right)\right\} \geq \varepsilon > 0 \qquad (3.2.15)$$

on $B := B(t_0, \delta) \times B(x_0, \delta)$. Observe that we can assume, without loss of generality, that (t_0, x_0) achieves a strict local maximum so that

$$\sup_{\partial_p B((t_0,x_0),\delta)} (v^*(\cdot, e_0) - \varphi) =: -\zeta < 0 , \qquad (3.2.16)$$

where $\partial_p B = [t_0, t_0+\delta] \times \partial B(x_0,\delta) \cup \{t_0+\delta\} \times B(x_0,\delta)$. Let $(t_k, x_k)_{k\geq 1}$ be a sequence in $[0,T) \times \mathbb{R}^d$ satisfying

$$(t_k, x_k) \longrightarrow (t_0, x_0) \text{ and } v(t_k, x_k, e_0) \longrightarrow v^*(t_0, x_0, e_0) \quad \text{as } k \longrightarrow \infty$$

so that

$$v(t_k, x_k, e_0) - \varphi(t_k, x_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty\,. \tag{3.2.17}$$

Let ξ^k be a k^{-1}-optimal control for $v(t_k, x_k, e_0)$, i.e. such that $v(t_k, x_k, e_0)$ is bounded from above by

$$\mathbb{E}\left[g(X^k_T, \xi(T)) + \int_{t_k}^T f(X^k_s, \xi^k_s)ds - \sum_{t_k < \tau^k_i \leq T} c\left(X^k_{\tau^k_i -}, \xi^k_{\tau^k_i -}, \xi^k_{\tau^k_i}\right)\right] + k^{-1}$$

where X^k denotes $X^{\xi^k}_{t_k, x_k}$ and τ^k_i stands for $\tau^{\xi^k}_i$. Set $\vartheta^k := \inf\{s > t_k \;:\; \xi_k \neq e_0\}$, $\theta^k := \inf\{s \geq t_k \;:\; (s, X^k_s) \notin B\} \wedge \vartheta^k$ and $\mathcal{E}_k := \xi^k_{\theta^k}$. By taking the conditional expectation with respect to $\mathcal{F}_{\theta^k}$ in the above expression and using the Markov property of (X^k, ξ^k), we get

$$\begin{aligned} v(t_k, x_k, e_0) \;&\leq\; \mathbb{E}\left[v\left(\theta^k, X^k_{\theta^k -} + \beta(X^k_{\theta^k -}, e_0, \mathcal{E}_k), \mathcal{E}_k\right)\right] \\ &+\; \mathbb{E}\left[\int_{t_k}^{\theta^k} f(X^k_s, e_0)ds - c\left(X^k_{\theta^k_i -}, e_0, \mathcal{E}_k\right)\mathbf{1}_{\theta^k = \vartheta^k}\right] + k^{-1}\,, \end{aligned} \tag{3.2.18}$$

recall that $\beta(\cdot, e, e) = 0$ for $e \in E$. On the other hand, applying Itô's Lemma to φ and using (3.2.15) and (3.2.16) leads to

$$\begin{aligned} \varphi(t_k, x_k) \;&\geq\; \mathbb{E}\left[\varphi\left(\theta^k, X^k_{\theta^k -}\right) + \int_{t_k}^{\theta^k} f(X^k_s, e_0)ds\right] \\ &\geq\; \mathbb{E}\left[v^*\left(\theta^k, X^k_{\theta^k -} + \beta(X^k_{\theta^k -}, e_0, \mathcal{E}_k), \mathcal{E}_k\right)\right] \\ &+\; \mathbb{E}\left[\int_{t_k}^{\theta^k} f(X^k_s, e_0)ds - c\left(X^k_{\theta^k -}, e_0, \mathcal{E}_k\right)\mathbf{1}_{\theta^k = \vartheta^k}\right] + \varepsilon \wedge \zeta\,. \end{aligned}$$

In view of (3.2.17) and (3.2.18), this leads to a contradiction for k large enough.
2. It remains to show that

$$\min\left\{(v^* - g)(T, x_0, e_0)\,,\; \mathcal{G}^{e_0} v^*(T, x_0, e_0)\right\} \;\leq\; 0\,. \tag{3.2.19}$$

We argue by contradiction and assume that

$$\min\left\{(v^* - g)(T, x_0, e_0)\,,\; \mathcal{G}^{e_0} v^*(T, x_0, e_0)\right\} \;=:\; 2\varepsilon > 0\,.$$

Let $(t_k, x_k)_{k\geq 1}$ be a sequence in $[0,T]\times\mathbb{R}^d$ satisfying

$$(t_k, x_k) \longrightarrow (t_0, x_0) \;\text{ and }\; v(t_k, x_k, e_0) \longrightarrow v^*(t_0, x_0, e_0) \quad \text{as } k \longrightarrow \infty\,. \tag{3.2.20}$$

Under the above assumption, we can find a sequence of smooth functions $(\varphi^n)_{n\geq 0}$ on $[0,T]\times\mathbb{R}^d$ such that $\varphi^n \to v^*(\cdot, e_0)$ uniformly on compact sets and

$$\min\left\{\varphi^n - g(\cdot, e_0)\;,\; \min_{j\in E\setminus\{e_0\}}(\varphi^n - v^*(\cdot, \cdot + \beta(\cdot, e_0, j), j) + c(\cdot, e_0, j))\right\} \geq \varepsilon \tag{3.2.21}$$

on some neighborhood B_n of (T, x_0). After possibly passing to a subsequence of $(t_k, x_k)_{k\geq 1}$, we can then assume that it holds on $B_n^k := [t_k, T]\times B(x_k, \delta_n^k)$ for some sufficiently small $\delta_n^k \in (0,1]$ such that $B_n^k \subset B_n$. Since v^* is locally bounded, there is some $\zeta > 0$ such that $|v^*| \leq \zeta$ on B_n. We can then assume that $\varphi^n \geq -2\zeta$ on B_n. Let us define $\tilde{\varphi}_k^n$ by

$$\tilde{\varphi}_k^n(t,x) := \varphi^n(t,x) + 4\zeta|x - x_k|^2/(\delta_n^k)^2 + \sqrt{T-t}\;,$$

and observe that

$$(v^*(\cdot, e_0) - \tilde{\varphi}_k^n)(t,x) \leq -\zeta < 0 \quad \text{for } (t,x) \in [t_k, T]\times\partial B(x_k, \delta_n^k)\,. \tag{3.2.22}$$

Since $(\partial/\partial t)(\sqrt{T-t}) \to -\infty$ as $t \to T$, we can choose t_k large enough in front of δ_n^k and the derivatives of φ^n to ensure that

$$-\mathcal{L}^{e_0}\tilde{\varphi}_k^n \geq 0 \quad \text{on } B_n^k\,. \tag{3.2.23}$$

Let X^k, ξ^k and ϑ^k be defined as in Step 1 and set $\theta_n^k := \inf\{s \geq t_k \;:\; (s, X_s^k) \notin B_n^k\}\wedge\vartheta^k$. Using Itô's Lemma on $\tilde{\varphi}_k^n$ together with (3.2.21), (3.2.22) and (3.2.23), we obtain that

$$\begin{aligned}
\tilde{\varphi}_k^n(t_k, x_k) \;\geq\;& \mathbb{E}\left[\left(v^*\left(\theta^k, X^k_{\theta_n^k-} + \beta(X^k_{\theta_n^k-}, e_0, \mathcal{E}_k), \mathcal{E}_k\right) - c\left(X^k_{\theta_n^k-}, e_0, \mathcal{E}_k\right)\right)\mathbf{1}_{\vartheta^k\leq\theta_n^k}\right]\\
+\;& \mathbb{E}\left[\left(v^*\left(\theta^k, X^k_{\theta_n^k}, e_0\right)\mathbf{1}_{\theta_n^k<T} + g\left(X_T^k, e_0\right)\mathbf{1}_{\theta_n^k=T}\right)\mathbf{1}_{\theta_n^k<\vartheta^k}\right]\\
+\;& \mathbb{E}\left[\int_{t_k}^{\theta_n^k} f(X_s^k, e_0)ds\right] + \varepsilon\wedge\zeta\,.
\end{aligned}$$

Since $v(T,\cdot) = g$, (3.2.18) implies that

$$\varphi^n(t_k, x_k) + \sqrt{T-t_k} = \tilde{\varphi}_k^n(t_k, x_k) \;\;\geq\;\; v(t_k, x_k, e_0) + \varepsilon\wedge\zeta - k^{-1}\,.$$

We then obtain a contradiction by sending $k\to\infty$ and taking n large enough, recall (3.2.20). $\square$

2.4 A comparison result

In this section, we prove a comparison principle for (3.2.9)-(3.2.10) under the additional assumptions

H1 : For some integer $\gamma \geq 1$, v^+ satisfies the growth condition

$$\sup_{(t,x,e)\in[0,T]\times\mathbb{R}^d\times E} |w(t,x,e)|/(1+|x|^\gamma) \; < \; \infty \,. \tag{3.2.24}$$

H2 : There is a function Λ on $\mathbb{R}^d \times E$ satisfying
(i) $\Lambda(\cdot,e) \in C^2(\mathbb{R}^d)$ for all $e \in E$,
(ii) $b'D\Lambda + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^* D^2\Lambda\right] \leq \varrho\Lambda$ on $\mathbb{R}^d \times E$, for some $\varrho > 0$,
(iii) $\mathcal{G}^e\Lambda(x,e) \geq q(x)$ on $\mathbb{R}^d \times E$ for some continuous function $q > 0$ on $\mathbb{R}^d$,
(iv) $\Lambda \geq g^+$,
(v) $\Lambda(x,e)/|x|^\gamma \to \infty$ as $|x| \to \infty$ for all $e \in E$.

We shall provide below some conditions on the coefficients under which this assumptions hold.

Proposition 3.2.12 *Assume that* **H2** *holds. Let* U *(resp.* V*) be a lower-semicontinuous (resp. upper-semicontinuous) viscosity super-solution (resp. subsolution) of* (3.2.9)-(3.2.10) *such that* V^+ *and* U^- *satisfies the growth condition* (3.2.24)*. Then,* $U \geq V$ *on* $[0,T] \times \mathbb{R}^d \times E$*.*

Proof. 1. As usual, we shall argue by contradiction. We assume that $\sup_{[0,T]\times\mathbb{R}^d\times E}(V - U) > 0$. Recalling the definition of Λ and ϱ in **H2**, it follows from the growth condition on $V-U$ that for $\lambda \in (0,1)$ small enough there is some $(t_0,x_0,e_0) \in [0,T]\times\mathbb{R}^d\times E$ such that

$$\max_{[0,T]\times\mathbb{R}^d\times E}(\tilde{V}-\tilde{W}) \;\; = \;\; (\tilde{V}-\tilde{W})(t_0,x_0,e_0) =: \eta > 0 \tag{3.2.25}$$

where, for a map w on $[0,T]\times\mathbb{R}^d\times E$, we write $\tilde{w}(t,x,e)$ for $e^{\varrho t}w(t,x,e)$, and $\tilde{W} := (1-\lambda)\tilde{U} + \lambda\tilde{\Lambda}$. Let us define $\tilde{\mathcal{G}}^e$ and $\tilde{\mathcal{G}}$ as $\mathcal{G}^e$ and $\mathcal{G}$ with $\tilde{c}$ in place of c and observe that $\tilde{U}$ and $\tilde{V}$ are super and sub-solutions on $[0,T]\times\mathbb{R}^d\times E$ of

$$\min\left\{\varrho\varphi - \mathcal{L}\varphi \;,\; \tilde{\mathcal{G}}\varphi\right\} \;\; = \;\; 0 \tag{3.2.26}$$

and satisfy the boundary condition

$$\min\left\{\varphi - \tilde{g} \;,\; \tilde{\mathcal{G}}\varphi\right\} \;\; = \;\; 0 \,. \tag{3.2.27}$$

2. For $(t,x,y,e) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times E$ and $n \geq 1$, we set

$$\begin{aligned}\Gamma(t,x,y,e) &:= \tilde{V}(t,x,e) - \tilde{W}(t,y,e)\\ \Theta_n(t,x,y,e) &:= \Gamma(t,x,y,e) - \left(n|x-y|^{2\gamma} + |x-x_0|^{2\gamma+2} + |t-t_0|^2 + |e-e_0|\right) .\end{aligned}$$

By the growth assumption on V and U again, there is $(t_n,x_n,y_n,e_n) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times E$ such that

$$\max_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times E} \Theta_n = \Theta_n(t_n,x_n,y_n,e_n) .$$

Since

$$\Gamma(t_n,x_n,y_n,e_n) \geq \Theta_n(t_n,x_n,y_n,e_n) \geq (\tilde{V}-\tilde{W})(t_0,x_0,e_0) ,$$

it follows from the growth assumption on V and U, (v) of **H2**, (3.2.25) and the upper-semicontinuity of Γ that, up to a subsequence,

$$(t_n,x_n,y_n,e_n) \to (t_0,x_0,x_0,e_0) \tag{3.2.28}$$

$$n|x_n-y_n|^{2\gamma} + |t_n-t_0|^2 + |e_n-e_0| \to 0 \tag{3.2.29}$$

$$\Gamma(t_n,x_n,y_n,e_n) \to \Gamma(t_0,x_0,x_0,e_0) . \tag{3.2.30}$$

3. We first assume that, up to a subsequence,

$$\tilde{\mathcal{G}}^{e_n}\tilde{V}(t_n,x_n,e_n) \leq 0 \quad \text{for all } \; n \geq 1 .$$

Then, it follows from the super-solution property of $\tilde{U}$ and (iii) of **H2** that, for some $j_n \in E \notin \{e_n\}$,

$$\begin{aligned}\Gamma(t_n,x_n,y_n,e_n) &\leq \Gamma\left(t_n, x_n+\beta(x_n,e_n,j_n), y_n+\beta(y_n,e_n,j_n), j_n\right)\\ &+ \tilde{c}(y_n,e_n,j_n) - \tilde{c}(x_n,e_n,j_n) - \lambda\tilde{q}(y_n) .\end{aligned}$$

Observe that $j_n \to j_0 \in E \setminus \{e_0\}$, up to a subsequence. Using (3.2.28) and (3.2.30), we then deduce from the upper-semicontinuity of Γ, (iii) of **H2** and the continuity of $\tilde{c}$ that

$$\begin{aligned}\Gamma(t_0,x_0,x_0,e_0) &< \Gamma(t_0,x_0,x_0,e_0) + \lambda\tilde{q}(x_0)\\ &\leq \Gamma\left(t_0, x_0+\beta(x_0,e_0,j_0), x_0+\beta(x_0,e_0,j_0), j_0\right) ,\end{aligned}$$

which contradicts the definition of (t_0,x_0,e_0) in (3.2.25).

4. We now show that there is a subsequence such that $t_n < T$ for all $n \geq 1$. If not, we can assume that $t_n = T$ and it follows from the boundary condition (3.2.27) and the above argument that $\tilde{V}(t_n, x_n, e_n) \le \tilde{g}(x_n, e_n)$ for all $n \ge 1$, up to a subsequence. Since, by step 1 and (iv) of **H2**, $\tilde{W}(t_n, y_n, e_n) \ge \tilde{g}(y_n, e_n)$, it follows that $\Gamma(t_n, x_n, y_n, e_n) \le \tilde{g}(x_n, e_n) - \tilde{g}(y_n, e_n)$. Using (3.2.28), (3.2.30) and the continuity of g, we then obtain a contradiction to (3.2.25).

5. In view of the previous arguments, we may assume that

$$t_n < T \quad \text{ and } \quad \tilde{\mathcal{G}}^{e_n}\tilde{V}(t_n, x_n, e_n) > 0 \quad \text{for all } \ n \ge 1 \ .$$

Using Ishii's Lemma, Remark 1.4.11, and following standard arguments, we deduce from the viscosity property of $\tilde{U}$, $\tilde{V}$, (ii) of **H2** and the Lipschitz continuity assumptions on b, a and f that

$$\varrho\Gamma(t_n, x_n, y_n, e_n) \quad \le \quad O\left(n|x_n - y_n|^{2\gamma} + |x_n - y_n| + |x_n - x_0|\right) \ .$$

In view of (3.2.28), (3.2.29), (3.2.30), this implies that $\varrho\Gamma(t_0, x_0, x_0, e_0) \le 0$ which contradicts (3.2.25). □

Some sufficient conditions for H1 and H2

Our general assumptions **H1** and **H2** hold under various different conditions on the coefficients. In this section, we provide some of them.
In the following, $C > 0$ is a generic constant which depends only on T, κ and b, σ, β, f, g and c.

a. The growth condition H1

Observe that, when $c \ge 0$ and $f^+ + g^+$ is bounded, then v is trivially bounded from above so that **H1** is satisfied. We now consider a case where g is upper-bounded by an affine map.

Proposition 3.2.13 *Assume that there exists real constants $C_1, C_2 > 0$ and some $\eta \in \mathbb{R}^d$ such that*

$$g(x,e) \le C_1 + \langle \eta, x\rangle \quad \textit{for all } \ (x,e) \in \mathbb{R}^d \times E \quad , \quad [\langle \eta, b\rangle + f]^+ \le C_2 \quad \textit{and} \quad \langle \eta, \beta\rangle - c \le 0 \ .$$

Then, v^+ has polynomial growth.

A similar result can be obtained under weaker conditions on c and g whenever b, σ, f are bounded and (3.2.10) admits a C^2_b solution. This follows from the more general result stated in the following proposition.

Proposition 3.2.14 *Assume that there exists a super-solution w to* (3.2.10) *satisfying* (3.2.24) *such that $w(\cdot, e) \in C^2(\mathbb{R}^d)$ for each e and $(\mathcal{L}w)^+ + |Dw^*\sigma|$ is uniformly bounded. Then, v^+ satisfies* (3.2.24).

Proof. Fix $(t, x, e) \in [0, T] \times \mathbb{R}^d \times E$ and $\xi \in \mathcal{S}_t(e)$. We write X for $X^{\xi}_{(t,x)}$ and τ_i for τ^{ξ}_i. Using Itô's Lemma and the super-solution property of w, we obtain that

$$
\begin{aligned}
& w(X(T), \xi(T)) + \int_t^T f(X(s), \xi_s)ds - \sum_{t<\tau_i\leq T} c\left(X(\tau_i-), \xi(\tau_i-), \xi_{\tau_i}\right) \\
&= w(x, e) + \int_t^T \mathcal{L}w(X(s), \xi_s)ds + \int_t^T Dw(X(s), \xi_s)^*\sigma(X(s), \xi_s)dW_s \\
&+ \sum_{t<\tau_i\leq T} \left[w(X(\tau_i-) + \beta(X(\tau_i-), \xi(\tau_i-), \xi_{\tau_i}), \xi_{\tau_i}) - w(X(\tau_i-), \xi(\tau_i-))\right] \\
&- \sum_{t<\tau_i\leq T} c\left(X(\tau_i-), \xi(\tau_i-), \xi(\tau_i)\right) \\
&\leq w(x, e) + \int_t^T \mathcal{L}w(X(s), \xi_s)ds + \int_t^T Dw(X(s), \xi_s)^*\sigma(X(s), \xi_s)dW_s \, .
\end{aligned}
$$

Since $\mathcal{L}w^+$ and $|Dw^*\sigma|$ are uniformly bounded by some $C > 0$ and $g \leq w$ we deduce that the expectation in the definition of v is bounded by $w(x, e) + TC$, uniformly in $\xi \in \mathcal{S}_t(e)$. □

We conclude this section with a last condition which pertains for unbounded coefficients but imposes a restriction on the support of β and the sign of c.

Proposition 3.2.15 *Assume that*

$$
c \geq 0 \quad \text{and} \quad \beta = 0 \quad \text{on} \quad \left(\{x \in \mathbb{R}^d \; : \; |x| \geq K\} \times E^2\right) , \tag{3.2.31}
$$

for some $K > 0$. Then v^+ satisfies the growth condition (3.2.24) *with $\gamma := \bar{p}$.*

Proof. For ease of notations, we write X for $X^{\xi}_{t,x}$ and τ_i for τ^{ξ}_i. It follows from the assumption $c \geq 0$ and (3.2.5) that

$$
\begin{aligned}
& \mathbb{E}\left[g(X(T), \xi(T)) + \int_t^T f(X(s), \xi(s))ds - \sum_{t<\tau_i\leq T} c\left(X(\tau_i-), \xi(\tau_i-), \xi(\tau_i)\right)\right] \\
& \leq C(1 + \sup_{t\leq s\leq T} \mathbb{E}\left[|X(s)|^{\bar{p}}\right]) \, .
\end{aligned}
$$

Thus, it suffices to show that

$$
\sup_{t\leq s\leq T} \mathbb{E}\left[|X(s)|^{\bar{p}}\right] \quad \leq \quad C(1 + |x|^{\bar{p}}) \, .
$$

Since β is uniformly Lipschitz, it follows from its support condition that is it bounded by some constant $K' > 0$. Fix $K" := K + K' + |x|$. Let us introduce the sequence of stopping times $(\vartheta_i)_{i\geq 1}$ by $\vartheta_1 = \inf\{s \geq t \;:\; |X(s)| \geq 2K"\}$ and for $i \geq 1$

$$\vartheta_{2i} := \inf\{s \geq \vartheta_{2i-1} \;:\; |X(s)| \leq K"\} \quad , \quad \vartheta_{2i+1} := \inf\{s \geq \vartheta_{2i} \;:\; |X(s)| \geq 2K"\} \, .$$

It follows from (3.2.31) that $|X_{\vartheta_{2i+1}}| = 2K"$. Fix $s \in [t,T]$ and set $A_s^i := \{\vartheta_{2i-1} \leq s < \vartheta_{2i}\}$, $A_s := \cup_{i\geq 1} A_s^i$ and $B_u^{s,i} := \{\vartheta_{2i-1} \leq u \leq s < \vartheta_{2i}\}$. Then

$$X(s)\mathbf{1}_{A_s} \;=\; \sum_{i\geq 1}\left(X_{\vartheta_{2i-1}}\mathbf{1}_{A_s^i} + \int_t^s \mathbf{1}_{B_u^{s,i}} b(X_u,\xi_u)du + \int_t^s \mathbf{1}_{B_u^{s,i}}\sigma(X_u,\xi_u)dW_u\right) \, ,$$

and it follows from the Lipschitz continuity of b and a that

$$\mathbb{E}\left[|X(s)\mathbf{1}_{A_s}|^{2\bar p}\right] \;\leq\; C\left(1 + (K")^{2\bar p} + \int_t^s \mathbb{E}\left[\sum_{i\geq 1}\mathbf{1}_{B_u^{s,i}}|X_u|^{2\bar p}\right] du\right) \, .$$

Since $B_u^{s,i} \subset \tilde B_u := \{|X_u| \geq K"\}$ and $\tilde B_s \subset A_s \cup \{|X(s)| \leq 2K"\}$, we get

$$\mathbb{E}\left[|X(s)\mathbf{1}_{\tilde B_s}|^{2\bar p}\right] \;\leq\; C\left(1 + (K")^{2\bar p} + \int_t^s \mathbb{E}\left[\mathbf{1}_{\tilde B_u}|X_u|^{2\bar p}\right] du\right) \, .$$

It then follows from Gronwall's Lemma that

$$\mathbb{E}\left[|X(s)|^{2\bar p}\right] \;\leq\; (K")^{2\bar p} + \mathbb{E}\left[|X(s)\mathbf{1}_{\tilde B_s}|^{2\bar p}\right] \leq C(1 + (K")^{2\bar p}) \, .$$

Since $K" = K + K' + |x|$, this leads to the required result. □

b. The strict super-solution condition H2

We now provide a general condition under which **H2** holds.

Proposition 3.2.16 *Fix some integer $\gamma \geq \bar p$. Assume that there is a sequence of real numbers $(d_i)_{i\in E}$ and some $\alpha > 0$ such that*

$$\begin{aligned} -\alpha \;&<\; |x + \beta(x,i,j)|^{2\gamma} - |x|^{2\gamma} \quad \textit{for all } (x,i,j) \in \mathbb{R}^d \times E^2 \\ \eta \;&:=\; \min_{i,j\in E}\inf_{x\in\mathbb{R}^d} \frac{d_i - d_j + c(x,i,j)}{|x+\beta(x,i,j)|^{2\gamma} - |x|^{2\gamma} + \alpha} > 0 \, . \end{aligned}$$

Then, assumption **H2** *holds for γ.*

Proof. We set $\Lambda(t,x,e) := (d + \eta|x|^{2\gamma} + d_e)$ for some $d > 0$ large enough so that $\Lambda \geq g^+$, recall (3.2.5). A straightforward computation shows that (iii) of **H2** is satisfied with $q \equiv \alpha\eta$. Clearly, (i) and (v) hold too. Finally, it follows from the linear growth assumption on b and a that (ii) holds for a sufficiently large parameter ϱ. □

Remark 3.2.19 (i) If $c \geq \varepsilon$ for some $\varepsilon > 0$ and β satisfies the support condition of Proposition 3.2.15, then the conditions of Proposition 3.2.16 trivially hold with $d_i = 0$ for all $i \in E$ and α large enough.
(ii) In the case where $\beta \equiv 0$ and c satisfies a strict triangular condition

$$c(x,i,j) + c(x,j,k) \quad > \quad c(x,i,k) \quad \text{for all } \; x \in \mathbb{R}^d \, , \; i,j,k \in E \, . \tag{3.2.32}$$

When c does not depend on x, they show that the sequence $(d_i)_{i \in E}$ defined by

$$d_i \quad = \quad \min_{j \in E \setminus \{i\}} c(x,i,j) \tag{3.2.33}$$

satisfies $d_i - d_j + c(x,i,j) > 0$. It follows that if c is independent of x, satisfies (3.2.32) and β satisfies the support condition of Proposition 3.2.15, or more generally the first condition of Proposition 3.2.16, then the second condition of this proposition holds too with $(d_i)_{i \in E}$ defined as in (3.2.33) and α large enough.

2.5 Verification argument: formal derivation

We now explain formally how to prove a verification theorem for the PDE (3.2.9) with the boundary condition (3.2.10). Assume that there is a family of smooth functions $\{\varphi(\cdot, e) \; , \; e \in E\}$ which solves (3.2.9) with the boundary condition (3.2.10). We let Ψ^e be defined as in (3.2.11) and call $\hat{\jmath}(t,x,e)$ its argmax. We fix the initial data $(0, x_0)$ and the initial regime e_0. We write $\hat{X}$ for $X^{\hat{\xi}}_{0,x_0}$ where $\hat{\xi}$ is associated to the sequence of stopping times and switches defined by

$$\begin{aligned}
\hat{\tau}_1 \quad &:= \quad \inf\{s > 0 \; : \; \varphi(s, \hat{X}(s), e_0) = \Psi^{e_0}(s, \hat{X}(s))\} \\
\hat{\mathcal{E}}_1 \quad &:= \quad \hat{\jmath}(\hat{\tau}_1, \hat{X}(\hat{\tau}_1-), e_0) \\
\ldots \quad &= \quad \ldots \\
\hat{\tau}_{i+1} \quad &:= \quad \inf\{s > \hat{\tau}_i \; : \; \varphi(s, \hat{X}(s), \hat{\mathcal{E}}_i) = \Psi^{\hat{\mathcal{E}}_i}(s, \hat{X}(s))\} \\
\hat{\mathcal{E}}_{i+1} \quad &:= \quad \hat{\jmath}(\hat{\tau}_{i+1}, \hat{X}(\hat{\tau}_{i+1}-), \hat{\mathcal{E}}_i) \, .
\end{aligned} \tag{3.2.34}$$

Now observe that (3.2.9) and Itô's Lemma imply that

$$\begin{aligned}
\varphi(\hat{\tau}_i, \hat{X}(\tau_i), \hat{\mathcal{E}}_i) \quad &= \quad \mathbb{E}\left[\varphi(\hat{\tau}_{i+1}-, \hat{X}(\hat{\tau}_{i+1}-), \hat{\mathcal{E}}_i) + \int_{\hat{\tau}_i}^{\hat{\tau}_{i+1}} f(\hat{X}(s), \hat{\xi}_s) ds\right] \\
&= \quad \mathbb{E}\left[\varphi(\hat{\tau}_{i+1}, \hat{X}(\hat{\tau}_{i+1}), \hat{\mathcal{E}}_{i+1}) + \int_{\hat{\tau}_i}^{\hat{\tau}_{i+1}} f(\hat{X}(s), \hat{\xi}_s) ds\right] \\
&- \quad \mathbb{E}\left[c\left(\hat{X}(\hat{\tau}_{i+1}-), \hat{\xi}_{\hat{\tau}_{i+1}-}, \hat{\xi}_{\hat{\tau}_{i+1}}\right)\right] \, .
\end{aligned}$$

If we now assume that $\varphi(T,\cdot) = g$ and that $\hat\tau_i \to \infty$ as $i \to \infty$, we deduce by summing up over i in the previous equation that, up to integrability conditions to be specified,

$$\varphi(0, x_0, e_0) \;=\; \mathbb{E}\left[g(\hat X(T), \hat\xi_T) + \int_0^T f(\hat X(s), \hat\xi_s)ds - \sum_{\hat\tau_i \le T} c\left(\hat X(\hat\tau_i-), \hat\xi_{\hat\tau_i-}, \hat\xi_{\hat\tau_i}\right)\right] .$$

This implies that $\varphi \le v$. On the other hand, the same argument but for an arbitrary admissible strategy ξ associated to a sequence $(\tau_i, \mathcal{E}_i)_{i\ge1}$ leads to

$$\begin{aligned}
\varphi(\tau_i, X^\xi(\tau_i), \mathcal{E}_i) \;&\ge\; \mathbb{E}\left[\varphi(\tau_{i+1}-, X^\xi(\tau_{i+1}-), \mathcal{E}_i) + \int_{\tau_i}^{\tau_{i+1}} f(X^\xi(s), \xi_s)ds\right] \\
&\ge\; \mathbb{E}\left[\varphi(\tau_{i+1}, X^\xi(\tau_{i+1}), \mathcal{E}_{i+1}) + \int_{\tau_i}^{\tau_{i+1}} f(X^\xi(s), \xi_s)ds\right] \\
&-\; \mathbb{E}\left[c\left(X^\xi(\tau_{i+1}-), \xi_{\tau_{i+1}-}, \xi_{\tau_{i+1}}\right)\right]
\end{aligned}$$

which implies $\varphi \ge v$ by arbitrariness of ξ. It follows that $\varphi = v$ and that the optimal switching strategy is given by (3.2.34).

3 An example of impulse control problem: Partial hedging with constant transaction costs (exercise)

Let us consider the Black-Scholes financial model where the underlying stock is modeled as the solution $X_{t,x}$ of

$$X(s) = x + \int_t^s X(r)\mu dr + \int_t^s X(r)\sigma dW_r \;\;,\; t \le s \le T$$

where $x > 0$, $\mu \in \mathbb{R}$ et $\sigma > 0$. For sake of simplicity the risk neutral interest rate is set to 0, i.e. $r = 0$. We assume that each transaction is subject to a constant transaction cost $c > 0$ which does not depend on the size of the transaction. A portfolio strategy is defined as a sequence of increasing stopping times $(\tau_n)_{n\ge1}$ and amount of exchanges $(\delta_n)_{n\ge1}$ such that δ_n is $\mathcal{F}_{\tau_n}$-measurable. We denote by $\mathcal{A}^{ad}$ the set of sequences $\nu := ((\tau_n)_{n\ge1}, (\delta_n)_{n\ge1})$ that satisfies the above properties and such that $\mathbb{E}\left[(\sum_{n\ge1}(|\delta_n| + c)\mathbf{1}_{\tau_n\in[0,T]})^2\right] < \infty$. Given an initial amount of money y invested in the risky asset at time t, we denote by $Y^\nu_{t,y}(s)$ the amount of money invested in the risky asset at time $s \ge t$ [1] :

$$Y^\nu_{t,y}(s) = y + \int_t^s \frac{Y^\nu_{t,y}(r)}{X_{t,x}(r)} dX_{t,x}(r) + \sum_{n\ge1} \delta_n \mathbf{1}_{\tau_n\in(t,s]} \;.$$

[1] We should write $Y(r-)$ in the integral by since X has continuous paths and Y only a finite number of jumps, this does not change anything.

If z is the amount invested in the non-risky asset at time t, we denote by $Z^{\nu}_{t,z}(s)$ this amount at time $s \geq t$:

$$Z^{\nu}_{t,z}(s) := z - \sum_{n\geq 1}(\delta_n + c)\mathbf{1}_{\tau_n \in (t,s]} \ .$$

Our aim is to characterize

$$v(t,x,y,z) := \inf_{\nu \in \mathcal{A}^{ad}} J(t,x,y,z;\nu)$$

where

$$J(t,x,y,z;\nu) := \mathbb{E}\left[\left(\left[Z^{\nu}_{t,z}(T) + Y^{\nu}_{t,y}(T) - g(X_{t,x}(T))\right]^{-}\right)^{2}\right] \ .$$

Here, $[\cdot]^{-}$ denotes the negative part and g is a continuous function with linear growth [2].

1. Fix $\nu \in \mathcal{A}^{ad}$. Give a bound in terms of y and ν but independent of T for $\mathbb{E}\left[|Y^{\nu}_{t,y}(T)|^2\right]$.
2. Show that J is finite for all $\nu \in \mathcal{A}^{ad}$.
3. Show that $0 \leq v < \infty$. In the following, we shall admit that v has quadratic growth.
4. Show that for any stopping time θ with values in $[t,T]$
$$v(t,x,y,z) \geq \inf_{\nu \in \mathcal{A}^{ad}} \mathbb{E}\left[v\left(\theta, X_{t,x}(\theta), Y^{\nu}_{t,y}(\theta), Z^{\nu}_{t,z}(\theta)\right)\right] \ .$$
5. Explain why, for any stopping time θ with values in $[t,T]$, we should have
$$v(t,x,y,z) \leq \inf_{\nu \in \mathcal{A}^{ad}} \mathbb{E}\left[v\left(\theta, X_{t,x}(\theta), Y^{\nu}_{t,y}(\theta), Z^{\nu}_{t,z}(\theta)\right)\right] \ .$$
6. By considering a particular form of strategies, show that
$$v(t,x,y,z) \leq \inf_{\delta \in \mathbb{R}} v^{*}(t,x,y+\delta, z - (\delta + c)\mathbf{1}_{\delta \neq 0})$$
where v^{*} is the upper-semicontinuous envelop of v.

[2]Note that $Z^{\nu}_{t,z}(T) + Y^{\nu}_{t,y}(T)$ is the liquidative value of the portfolio at T, up to the payment of the transaction costs due to the final transfer, if required.

7. Show that, if v is C^2, then it satisfies $[0,T) \times (0,\infty) \times \mathbb{R}^2$

$$-\mathcal{L}v \leq 0 \tag{3.3.1}$$

where

$$\begin{aligned}\mathcal{L}v(t,x,y,z) \quad &:= \quad \partial_t v(t,x,y,z) + x\mu D_x v(t,x,y,z) + \frac{1}{2}x^2\sigma^2 D_{xx}v(t,x,y,z) \\ &+ \quad y\mu D_y v(t,x,y,z) + \frac{1}{2}y^2\sigma^2 D_{yy}v(t,x,y,z) + yx\sigma^2 D_{xy}v(t,x,y,z)\ .\end{aligned}$$

8. Explain how to adapt the argument to show that v is only a viscosity subsolution of (3.3.1) in the case where it is only upper-semicontinuous.

9. Fix $(t,x,y,z) \in [0,T) \times (0,\infty) \times \mathbb{R}^2$ et $\nu \in \mathcal{A}^{ad}$ such that $\sum_{n\geq 1} \mathbf{1}_{\tau_n = t} = 0$ $\mathbb{P}-$a.s. Show that $J(\cdot;\nu)$ is upper-semicontinuous at (t,x,y,z). Deduce that v is upper-semicontinuous on $[0,T) \times (0,\infty) \times \mathbb{R}^2$.

10. Deduce that v is a viscosity subsolution on $[0,T) \times (0,\infty) \times \mathbb{R}^2$ of

$$\max\left\{-\mathcal{L}v(t,x,y,z)\ ,\ \sup_{\delta\in\mathbb{R}}\left(v(t,x,y,z) - v(t,x,y+\delta,z-(\delta+c)\mathbf{1}_{\delta\neq 0})\right)\right\} \leq 0\ .$$

11. Show that $\limsup_{t\uparrow T,(x',y',z')\to(x,y,z)} v(t,x',y',z') \leq \left([z+y-g(x)]^-\right)^2$.

12. If v is continuous on $[0,T) \times (0,\infty) \times \mathbb{R}^2$, should it be a viscosity super-solution of

$$\max\left\{-\mathcal{L}v(t,x,y,z)\ ,\ \sup_{\delta\in\mathbb{R}}\left(v(t,x,y,z) - v(t,x,y+\delta,z-(\delta+c)\mathbf{1}_{\delta\neq 0})\right)\right\} \geq 0\ ?$$

Briefly explain why.

13. What can we say if there exists a smooth solution V on $[0,T) \times (0,\infty) \times \mathbb{R}^2$ to the equation

$$\max\left\{-\mathcal{L}V(t,x,y,z)\ ,\ \sup_{\delta\in\mathbb{R}}\left(V(t,x,y,z) - V(t,x,y+\delta,z-(\delta+c)\mathbf{1}_{\delta\neq 0})\right)\right\} = 0$$

such that $\lim_{t\uparrow T,(x',y',z')\to(x,y,z)} V(t,x',y',z') = \left([z+y-g(x)]^-\right)^2$. Explain how to construct an optimal control in this case.

4 A singular control problem: dividend payment optimization

In this section, we consider the problem of a large insurance company whose aim is to maximize the discounted cumulated amount of dividend paid up to bankruptcy and can reinsure part of its portfolio. This leads to a singular control problem where part of the control is defined as a bounded variation process. We shall see that in this case the associated PDE is of variational form as in the two previous chapters.

4.1 Problem formulation

We follow the approach of, e.g., [26] and [9] which consists in approximating the evolution of the reserve process X (before paiment of dividends) by the diffusion

$$X_x^{\pi,0}(t) = x + \int_0^t \gamma\pi_s ds + \int_0^t \pi_s\kappa dW_s^1 + \int_0^t X_x^{\pi,0}(s)\mu ds + \int_0^t X_x^{\pi,0}(s)\sigma dW_s^2$$

where W^1 and W^2 are two independent Brownian motions, $\gamma, \kappa, \mu > 0$, $\sigma \geq 0$.

The term $\gamma\pi_s ds + \pi_s\kappa dW_s^1$ corresponds to the approximation of the instantaneous evolution of the wealth due the received premiums and the paid claims for a level of retention π_s with values in $[\bar{u}, 1]$ in the *Cramer-Lundberg model*. Here, $\bar{u} \in [0, 1]$ denotes the minimal level of retention. In the case where $\bar{u} = 1$, then no reinsurance is possible.

The term $X_s^{\pi,0}\mu ds + X_s^{\pi,0}\sigma dW_s^2$ is due to the investment of the reserve in a risky asset S which evolves as in the Black and Scholes model according to

$$dS_t/S_t = \mu dt + \sigma dW_t^2 \ .$$

The payment of dividends is modeled by a continuous non-decreasing adapted process L, satisfying $L_{0-} = 0$: L_t is the total amount of dividends paid up to time t.

In case where dividends are paid, the reserve evolves according to:

$$X_x^{\pi,L}(t) = x + \int_0^t \gamma\pi_s ds + \int_0^t \pi_s\kappa dW_s^1 + \int_0^t X_x^{\pi,L}(s)\mu ds + \int_0^t X_x^{\pi,L}(s)\sigma dW_s^2 - L_t \ .$$

We denote by $\mathcal{U}$ the set of adapted processes $\nu = (\pi, L)$ satisfying the above properties and such that $X_x^\nu \geq 0$ $\mathbb{P}-$a.s. We note π^ν and L^ν the controls associated to ν, if not clearly given by the context.

The aim of the company is to maximize over $\nu \in \mathcal{U}$

$$J(x;\nu) := \mathbb{E}\left[\int_0^{\tau^\nu} e^{-ct} dL_t^\nu\right]$$

where $c > 0$ is a fixed parameter and τ_x^ν is the first time X_x^ν reaches 0, this is the bankruptcy time.

We define

$$v(x) := \sup_{\nu \in \mathcal{U}} J(x;\nu) \ .$$

Remark 3.4.20 1. If $\bar{u} = 0$, $\mu > c$ and $x > 0$ then $v(x) = \infty$. To see this is suffices to consider the strategy $\nu = (\pi, L)$ defined by

$$\pi = 0 \ \text{ and } \ dL_t = \frac{1}{2}(\mu - c)X_x^\nu dt \ .$$

2. The map $x \geq 0 \mapsto v(x)$ is concave. Indeed, for $\lambda \in [0,1]$, $x^i \geq 0$ and $\nu^i = (\pi^i, L^i) \in \mathcal{U}$, $i = 1,2$, we can set $(x,\nu) = \lambda(x^1,\nu^1) + (1-\lambda)(x^2,\nu^2)$, which implies that $X_x^\nu = \lambda X_{x_1}^{\nu^1} + (1-\lambda)X_{x_2}^{\nu^2}$, $\tau_x = \max\{\tau_{x^1}, \tau_{x^2}\}$ and $J(x;\nu) \geq \lambda J(x_1;\nu^1) + (1-\lambda)J(x_2;\nu^2)$.

4.2 Dynamic programming and Hamilton-Jacobi-Bellman equation with free boundary

In this section, we relate the value function v to a suitable PDE in variational form. We start with the dynamic programming principle.

Proposition 3.4.17 *For all $x \geq 0$ and all uniformly bounded family of stopping times $\{\theta^\nu, \nu \in \mathcal{U}\}$*

$$v(x) = \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\int_0^{\theta^\nu \wedge \tau_x^\nu} e^{-ct} dL_t^\nu + e^{-c\theta^\nu} v(X_x^\nu(\theta^\nu)) \mathbf{1}_{\theta^\nu \leq \tau_x^\nu}\right]$$

Proof. The proof uses similar arguments as the one used in the proof of Theorem 2.2.1. □

We now derive the associated PDE. We restrict to the case where v is smooth enough but a similar analysis can be lead with the notion of viscosity solutions.

Proposition 3.4.18 *If $v \in C^2$, then it satisfies*

$$\min\left\{\min_{u \in [\bar{u},1]}\left(cv - (u\gamma + x\mu)Dv - \frac{1}{2}(u^2\kappa^2 + x^2\sigma^2)D^2v\right) \ , \ Dv - 1\right\} = 0 \ \textit{on} \ (0,\infty) \ . \tag{3.4.1}$$

Proof. 1. Let $\nu = (\pi, L)$ be of the form $\pi = u$ for some constant u and $L_t = \ell(t \wedge \tau_x^\nu)$, $\ell \geq 0$. Fix $h > 0$, let θ be the first time when $X_x^\nu \geq K$ for some $K > 2x$ and set

$\theta^h = \theta \wedge h$. Then, Proposition 3.4.17 and Itô's Lemma imply:

$$
\begin{aligned}
v(x) &\geq v(x) + \mathbb{E}\left[\int_0^{\theta^h \wedge \tau_x^\nu} e^{-ct} \ell dt\right] \\
&+ \mathbb{E}\left[\mathbf{1}_{\theta^h < \tau_x^\nu} \int_0^{\theta^h \wedge \tau_x^\nu} e^{-cs}(u\gamma + X_x^\nu(s)\mu - \ell) Dv(X_x^\nu(s)) ds\right] \\
&+ \mathbb{E}\left[\mathbf{1}_{\theta^h < \tau_x^\nu} \int_0^{\theta^h \wedge \tau_x^\nu} e^{-cs}\left(\frac{1}{2}(u^2\kappa^2 + (X_x^\nu(s))^2\sigma^2) D^2 v(X_x^\nu(s)) - cv(X_x^\nu(s))\right) ds\right] .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
0 &\geq \lim_{h\to 0}\left\{\mathbb{E}\left[h^{-1}\int_0^{\theta^h \wedge \tau_x^\nu} e^{-cs}\ell ds\right]\right. \\
&+ \mathbb{E}\left[\mathbf{1}_{\theta^h < \tau_x^\nu} h^{-1} \int_0^{\theta^h \wedge \tau_x^\nu} e^{-cs}(u\gamma + X_x^\nu(s)\mu - \ell) Dv(X_x^\nu(s)) ds\right] \\
&+ \left.\mathbb{E}\left[\mathbf{1}_{\theta^h < \tau_x^\nu} h^{-1} \int_0^{\theta^h \wedge \tau_x^\nu} e^{-cs}\left(\frac{1}{2}(u^2\kappa^2 + (X_x^\nu(s))^2\sigma^2) D^2 v(X_x^\nu(s)) - cv(X_x^\nu(s))\right) ds\right]\right\} .
\end{aligned}
$$

Since $\tau_x^\nu \wedge \theta > 0$, we have $(\theta^h \wedge \tau_x^\nu)/h \to 1$ $\mathbb{P}-$a.s., by the mean value theorem and the dominated convergence theorem, we then deduce that

$$
\begin{aligned}
0 &\geq \mathbb{E}\left[\lim_{h\to 0} h^{-1}\int_0^h e^{-cs}\ell ds\right] \\
&+ \mathbb{E}\left[\lim_{h\to 0} h^{-1}\int_0^h e^{-cs}(u\gamma + X_x^\nu(s)\mu - \ell) Dv(X_x^\nu(s)) ds\right] \\
&+ \mathbb{E}\left[\lim_{h\to 0} h^{-1}\int_0^h e^{-cs}\left(\frac{1}{2}(u^2\kappa^2 + (X_x^\nu(s))^2\sigma^2) D^2 v(X_x^\nu(s)) - cv(X_x^\nu(s))\right) ds\right] \\
&= \ell + (u\gamma + x\mu - \ell) Dv(x) + \frac{1}{2}(u^2\kappa^2 + x^2\sigma^2) D^2 v(x) - cv(x) .
\end{aligned}
$$

Since $\ell \geq 0$ is arbitrary, we must have $(1 - Dv) \leq 0$ and

$$
\min_{u\in[\bar{u},1]}\left(cv(x) - (u\gamma + x\mu) Dv(x) - \frac{1}{2}(u^2\kappa^2 + x^2\sigma^2) D^2 v(x)\right) \geq 0 .
$$

2. It remains to show that one of the two terms is equal to 0. Assume that for some $x > 0$

$$
\min\left\{\min_{u\in[\bar{u},1]}\left(cv(x) - (u\gamma + x\mu) Dv(x) - \frac{1}{2}(u^2\kappa^2 + x^2\sigma^2) D^2 v(x)\right) ,\ Dv(x) - 1\right\} > 0 .
$$

Then, there is a neighborhood B of x of radius $r \in (0, x/2)$ such that, for some $\varepsilon > 0$,

$$
\min\left\{\min_{u\in[\bar{u},1]}\left(cw(y) - (u\gamma + y\mu) Dw(y) - \frac{1}{2}(u^2\kappa^2 + y^2\sigma^2) D^2 w(y)\right) ,\ Dw(y) - 1\right\} \geq \varepsilon
$$

for all $y \in B$ where $w(y) = v(y) + \|x - y\|^2$. Fix $\nu = (\pi, L) \in \mathcal{U}$ and let θ be the first time when $\|X_x^\nu(t) - x\| \geq r/2$ or $t \geq r/2$. Observe that $\theta \leq \tau_x^\nu$ $\mathbb{P}-$a.s. By Itô's Lemma and the above inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-c\,\theta} w(X_x^\nu(\theta)) + \int_0^\theta e^{-cs} dL_s\right] \\
&= w(x) + \mathbb{E}\left[\int_0^\theta e^{-cs}(1 - Dw(X_x^\nu(s)))dL_s\right] \\
&+ \mathbb{E}\left[\int_0^\theta e^{-cs}(\pi_s\gamma + X_x^\nu(s)\mu)Dw(X_x^\nu(s))ds\right] \\
&+ \mathbb{E}\left[\int_0^\theta e^{-cs}\left(\frac{1}{2}(\pi_s^2\kappa^2 + (X_x^\nu(s))^2\sigma^2)D^2w(X_x^\nu(s)) - cw(X_x^\nu(s))\right)ds\right] \\
&\leq w(x) - \varepsilon C\mathbb{E}\left[\left(\frac{1}{c}(1 - e^{-c\theta}) + e^{-c\theta}L_\theta\right)\right]
\end{aligned}
$$

for some $C > 0$. By definition of θ and w, this implies that

$$
\begin{aligned}
v(x) &\geq \mathbb{E}\left[e^{-c\,\theta} v(X_x^\nu(\theta)) + \int_0^\theta e^{-cs} dL_s\right] \\
&+ \varepsilon\mathbb{E}\left[r\mathbf{1}_{\|X_x^\nu(\theta)-x\|=r/2}\mathbf{1}_{\theta<r/2} + \frac{C}{c}(1 - e^{-c\theta})\mathbf{1}_{\theta=r/2}\right] \\
&\geq \mathbb{E}\left[e^{-c\,\theta} v(X_x^\nu(\theta)) + \int_0^\theta e^{-cs} dL_s\right] + \eta
\end{aligned}
$$

for some $\eta > 0$. Recalling that $\theta \leq \tau_x^\nu$ $\mathbb{P}-$a.s., this contradicts Proposition 3.4.17. □

4.3 Verification theorem

We now explain how to construct a verification argument. Contrary to what was done in the above sections, we shall now allow for a jump of L at the initial time.

Proposition 3.4.19 *Assume that there exists a concave function $w \in C^2((0, \infty))$ satisfying* (3.4.1) *and a constant $b^* > 0$ such that $w(0) = 0$ and*

$$
\begin{aligned}
&\textit{On } 0 < x < b^* \quad \textit{we have} \quad \min_{u\in[\bar{u},1]}\left(cw - (u\gamma + x\mu)Dw - \frac{1}{2}(u^2\kappa^2 + x^2\sigma^2)D^2w\right) = 0\,. \\
&\textit{On } x \geq b^* \quad \textit{we have} \quad Dw = 1\,.
\end{aligned}
\tag{3.4.2}
$$

Assume further that for all $\nu \in \mathcal{U}$ and $x > 0$

$$
\lim_{t\to\infty} e^{-ct}\mathbb{E}\left[X_x^\nu(t)\mathbf{1}_{\tau_x^\nu=\infty}\right] = 0\,. \tag{3.4.3}
$$

Then, $w = v$. Moreover, if

$$\hat{u} = argmin_{[\bar{u},1]} \left(cw - (u\gamma + x\mu)Dw - \frac{1}{2}(u^2\kappa^2 + x^2\sigma^2)D^2 w \right) ,$$

then the optimal strategy associated to the initial condition x is given by $\hat{\nu} = (\hat{\pi}, \hat{L} + (x - b^*)^+)$ where $\hat{\pi} = \hat{u}(X_x^{\hat{\nu}})$ and $(X_x^{\hat{\nu}}, \hat{L})$ is the solution of the Skorokhod problem

$$\begin{cases} X_x^{\hat{\nu}}(t) = x \wedge b^* + \int_0^t \gamma \hat{u}(X_x^{\hat{\nu}}(s))ds + \int_0^t \hat{u}(X_x^{\hat{\nu}}(s))\kappa dW_s^1 + \int_0^t X_x^{\hat{\nu}}(s)\mu ds \\ \qquad\qquad + \int_0^t X_x^{\hat{\nu}}(s)\sigma dW_s^2 - \hat{L}_t \\ X_x^{\hat{\nu}}(t) \in (0, b^*] \\ \hat{L}_t = \int_0^t \mathbf{1}_{\{X_x^{\hat{\nu}}(s) = b^*\}} d\hat{L}_s \ . \end{cases}$$

Proof. 1. We first prove that

$$w(x) = J(x; \hat{\nu})$$

with $\hat{\nu}$ defined as in the Proposition. It follows from the result of [15] that the above Skorokhod problem has a continuous solution $(X_x^{\hat{\nu}}, \hat{L})$. By Itô's formula, (3.4.2) and the condition $w(0) = 0$, we have

$$\mathbb{E}\left[e^{-c(t\wedge\tau_x^{\hat{\nu}})} w(X_x^{\hat{\nu}}(t \wedge \tau_x^{\hat{\nu}}))\right] = w(x) - \mathbb{E}\left[\int_0^{t\wedge\tau_x^{\hat{\nu}}} e^{-cs} Dw(X_x^{\hat{\nu}}(s))d\hat{L}_s\right] - (x - b^*)^+ \ .$$

Since $Dw(b^*) = 1$, it follows from the definition of $\hat{L}$ that

$$\mathbb{E}\left[e^{-c(t\wedge\tau_x^{\hat{\nu}})} w(X_x^{\hat{\nu}}(t \wedge \tau_x^{\hat{\nu}}))\right] = w(x) - \mathbb{E}\left[\int_0^{t\wedge\tau_x^{\hat{\nu}}} e^{-cs} d\hat{L}_s\right] \ .$$

Since $w(X_x^{\hat{\pi},\hat{L}})$ is bounded by continuity of w and the bound on $X_x^{\hat{\pi},\hat{L}}$, sending $t \to \infty$ leads to the required result.

2. We now prove that $w(x) \geq J(x; \nu)$ for all $\nu \in \mathcal{U}$. Fix $\nu = (\pi, L) \in \mathcal{U}$. By the same arguments as in 2. of the proof of Proposition 3.4.18, we observe that

$$\mathbb{E}\left[e^{-c\, t\wedge\tau_x^{\nu}} w(X_x^{\nu}(t \wedge \tau_x^{\nu})) + \int_0^{t\wedge\tau_x^{\nu}} e^{-cs} dL_s\right] \leq w(x) \ .$$

Since w is concave, bounded at 0 and $Dw = 1$ for $x > b^*$, the condition (3.4.3) implies that

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-c\, t\wedge\tau_x^{\nu}} w(X_x^{\nu}(t \wedge \tau_x^{\nu}))\right] = 0 \ ,$$

which leads to the required result. $\square$

Chapter 4

Stochastic target problems

A general stochastic target problem consists in finding the set of initial conditions z such that there exists a control process ν, belonging to a well defined set of admissible controls, for which a given controlled process $Z^{\nu}_{t,z}(T)$ reaches a given target, say for example a Borel subset of $\mathbb{R}^{d+1}$.
In this Chapter, we consider the case where $Z^{\nu}_{t,z} = (X^{\nu}_{t,x},\, Y^{\nu}_{t,x,y})$, where X is $\mathbb{R}^d$-valued and Y is $\mathbb{R}$-valued. We address the problem of finding the minimal initial data y such that $Y^{\nu}_{t,y,x}(T) \geq g(X^{\nu}_{t,x}(T))$ for some admissible control ν, where g is a $\mathbb{R}^d \mapsto \mathbb{R}$ measurable function.
Note that this corresponds to a generalization of the usual super-hedging problem in finance where Y corresponds to the wealth process, X the risky assets and g the payoff of a plain Vanilla option, see [3].

1 The Model

We now assume that the set U in which the admissible controls take values is compact and convex. The controlled process $Z^{\nu}_{t,z} = (X^{\nu}_{t,x}, Y^{\nu}_{t,z})$ is defined as the solution on $[t,T]$ of the stochastic differential system :

$$\begin{aligned} dX(s) &= \rho\left(X(s),\nu(s)\right)ds + \alpha^*\left(X(s),\nu(s)\right)dW(s) \\ &\quad + \int_{\mathbb{R}^d} \beta\left(X(s-),\nu(s),\sigma\right)\mu(d\sigma,ds) \\ dY(s) &= r\left(Z(s),\nu(s)\right)ds + a^*\left(Z(s),\nu(s)\right)dW(s) \\ &\quad + \int_{\mathbb{R}^d} b\left(Z(s-),\nu(s),\sigma\right)\mu(d\sigma,ds) \\ Z(t) &= (x,y) \end{aligned} \tag{4.1.1}$$

where ρ, α, β, r, b and a are continuous with respect to $(t, \nu, \sigma) \in [0, T] \times U \times \mathbb{R}^d$, Lipschitz in t, Lipschitz and polynomialy growing in the variable z, uniformly in the variables (ν, σ), and bounded with respect to σ. This guarantees existence and uniqueness of a strong solution $Z^{\nu}_{t,z}$ to the stochastic differential system (4.1.1) for each control process $\nu \in \mathcal{U}$.

Let $g: \mathbb{R}^d \longrightarrow \mathbb{R}$ be a measurable function. Our stochastic target problem is :

$$v(t,x) := \inf \Gamma(t,x)$$

where

$$\Gamma(t,x) := \left\{ y \in \mathbb{R} \ : \ \exists\, \nu \in \mathcal{U},\ Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right) \ \right\} .$$

Assume that the infimum in the definition of v is attained and let $y = v(t,x)$. Then, we can find some $\nu \in \mathcal{U}$ such that $Y^{\nu}_{t,x,y}(T) \geq g\left(X^{\nu}_{t,x}(T)\right)$. Hence, if we start with y' $> y$, we should be able to find some $\nu' \in \mathcal{U}$ such that $Y^{\nu'}_{t,x,y'}(T) \geq g\left(X^{\nu'}_{t,x}(T)\right)$. If this property does not hold (which can be the case in a jump diffusion model) we are not able to characterize the set $\Gamma(t,x)$ by its lower bound $v(t,x)$.
Hence we assume that, for all $(t,x,y,y') \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$,

$$y' \ \geq \ y \quad \text{and} \quad y \ \in \ \Gamma(t,x) \implies y' \ \in \ \Gamma(t,x) \, .$$

By standard comparison arguments for stochastic differential equations, it will hold in particular if b is independent of y (see e.g. [21]). It will also hold in most financial applications as soon as there is a non-risky asset.
Under the above assumption, for each $[0,T]$-valued stopping time θ and control $\nu \in \mathcal{U}$, $v(\theta, X^{\nu}_{t,x}(\theta))$ corresponds to the minimal condition, when starting at time θ, such that the stochastic target can be reached at time T. This means that, if v is finite, given y $> v(t,x)$, we can find a control ν such that $Y^{\nu}_{t,x,y}(\theta) \geq v(\theta, X^{\nu}_{t,x}(\theta))$ for any $[0,T]$-valued stopping time θ (see Proposition 4.2.20 below). Assume that v is smooth and denote by Dv the gradient of v with respect to x. Applying Itô's Lemma to v shows that the only way to control the Brownian part of $Y^{\nu}_{t,x,y}(\cdot) - v(\cdot, X^{\nu}_{t,x}(\cdot))$ is to define $\nu(\cdot)$ in a Markovian way by $\nu(\cdot) = \psi\big(\cdot,\ X^{\nu}_{t,x}(\cdot),\ Y^{\nu}_{t,x,y}(\cdot),\ Dv(\cdot, X^{\nu}_{t,x}(\cdot))\big)$ where, for all $(t,x,y) \in$ $[0,T] \times \mathbb{R}^d \times \mathbb{R}$,

$$\psi(t,x,y,\cdot) \text{ is the inverse of the mapping } \nu \mapsto \alpha^{-1}(t,x,\nu)a(t,x,y,\nu).$$

Hence, we assume that, either, α is invertible and ψ is well defined, or, $a = \alpha = 0$.

2 Dynamic programming

In order to characterize the value function as a viscosity solution of a suitable PDE, we need a dynamic programming principle. For stochastic target problems, it reads as follows.

Proposition 4.2.20 *Fix $(t,x) \in [0,T] \times \mathbb{R}^d$.*
(DP1) *Let $y \in \mathbb{R}$ and $\nu \in \mathcal{U}$ be such that $Y^\nu_{t,x,y}(T) \geq g\left(X^\nu_{t,x}(T)\right)$. Then, for all stopping time $\theta \geq t$, we have :*

$$Y^\nu_{t,x,y}(\theta) \geq v\left(\theta, X^\nu_{t,x}(\theta)\right) \ .$$

(DP2) *Set $y < v(t,x)$ and let $\theta \geq t$ be an arbitrary stopping time. Then, for all $\nu \in \mathcal{U}$:*

$$\mathbb{P}\left[Y^\nu_{t,x,y}(\theta) > v\left(\theta, X^\nu_{t,x}(\theta)\right)\right] < 1 \ .$$

Proof. The rigorous proof can be found in [25]. We only sketch the main argument. The assertion (DP1) is essentially a consequence of the fact that $Z^\nu_{t,z}(T) = Z^\nu_{\theta, Z^\nu_{t,z}(\theta)}(T)$ and the very definition of v. As for (DP2), we observe that (formally), if $Y^\nu_{t,x,y-\eta}(\theta) > v\left(\theta, X^\nu_{t,x}(\theta)\right)$ $\mathbb{P}-$a.s., then there must be a control $\tilde\nu$ such that $Z^{\tilde\nu}_{\theta, Z^\nu_{t,z}(\theta)}(T)$ reaches the epigraph of g $\mathbb{P}-$a.s.. Thus, setting $\bar\nu = \nu \mathbf{1}_{[0,\theta)} + \tilde\nu \mathbf{1}_{[\theta,T]}$, we get $Z^{\bar\nu}_{t,z}(T)$ reaches the epigraph of g $\mathbb{P}-$a.s. which contradicts the fact that $y < v(t,x)$. □

The interpretation of (DP1) and (DP2) is very natural. If $y > u$, then there should be a control such that the target can be reached. Thus, at any intermediate time θ, we are in position to find a control on $[\theta, T]$ such that we can reach the target at T. This exactly means that $Y^\nu_{t,x,y}(\theta) \geq v\left(\theta, X^\nu_{t,x}(\theta)\right)$. Conversely, if $y < u$ then there is no control such that the target can be reached for sure. But if $Y^\nu_{t,x,y}(\theta) > v\left(\theta, X^\nu_{t,x}(\theta)\right)$ $\mathbb{P}-$a.s., then there is a control such that starting from $Y^\nu_{t,x,y}(\theta)$ the target can be reached for sure. This leads to a contradiction.

3 PDE characterization in the mixed diffusion case

In this section, we assume that ψ is well defined (see the previous section) and that U is convex. We introduce the support function δ_U of the closed convex set U :

$$\delta_U(\zeta) := \sup_{\nu \in U} \langle \zeta, \nu \rangle \ , \ \zeta \in \mathbb{R}^d \ ,$$

and $\tilde U_1$ the restriction to the unit sphere of the effective domain of δ_U :

$$\tilde U_1 := \left\{ \zeta \in \mathbb{R}^d \ , \ \|\zeta\| = 1 \ , \ \delta_U(\zeta) \in \mathbb{R}^d \right\} \ .$$

Clearly, $\tilde{U}_1$ is equal to the unit sphere of $\mathbb{R}^d$. We use this notation since part of our results holds without the compacity assumption on U.
Notice that U and $\mathrm{Int}(U)$ may be characterized in terms of $\tilde{U}_1$:

$$\nu \in U \Longleftrightarrow \chi_U(\nu) \ \geq \ 0 \ \text{ and } \ \nu \in \mathrm{Int}(U) \Longleftrightarrow \chi_U(\nu) \ > \ 0 \ ,$$

where

$$\chi_U(\nu) := \inf_{\zeta \in \tilde{U}_1} \left(\delta_U(\zeta) - \langle \zeta, \nu \rangle\right) \ .$$

Remark 4.3.21 *The mapping $\nu \in U \mapsto \chi_U(\nu)$ is continuous. This follows from the compactness of $\tilde{U}_1$.*

Given a smooth function φ on $[0,T] \times \mathbb{R}^d$, $\nu \in U$ and $\sigma \in \mathbb{R}^d$, we define the operators :

$$\begin{aligned}
\mathcal{L}^\nu \varphi(t,x) &:= r(x, \varphi(t,x), \nu) - \frac{\partial \varphi}{\partial t}(t,x) - \rho(x,\nu)^* D\varphi(t,x) \\
& \quad -\frac{1}{2}\mathrm{Trace}\left(D^2\varphi(t,x)\alpha^*(x,\nu)\alpha(x,\nu)\right) \\
\mathcal{G}^{\nu,\sigma}\varphi(t,x) &:= b(x,\varphi(t,x),\nu,\sigma) - \varphi\left(x + \beta(x,\nu,\sigma)\right) + \varphi\left(t,x\right) \\
\mathcal{T}^\nu \varphi(t,x) &:= \min\left\{ \inf_{\sigma \in \mathbb{R}^d} \mathcal{G}^{\nu,\sigma}\varphi(t,x) \ ; \ \chi_U(\nu) \right\} \\
\mathcal{H}^\nu \varphi(t,x) &:= \min\left\{ \mathcal{L}^\nu \varphi(t,x) \ ; \ \mathcal{T}^\nu \varphi(t,x) \right\}
\end{aligned}$$

where $D\varphi$ and $D^2\varphi$ denote respectively the gradient and the Hessian matrix of φ with respect to x. We also define :

$$\begin{aligned}
\widehat{\mathcal{G}}^\sigma \varphi(t,x) &:= \mathcal{G}^{\nu,\sigma}\varphi(t,x) \quad \text{ for } \nu = \psi(t,x,\varphi(t,x), D\varphi(t,x)) \ , \\
\widehat{\mathcal{T}}\varphi(t,x) &:= \mathcal{T}^\nu \varphi(t,x) \quad \text{ for } \nu = \psi(t,x,\varphi(t,x), D\varphi(t,x)) \ , \\
\widehat{\mathcal{H}}\varphi(t,x) &:= \mathcal{H}^\nu \varphi(t,x) \quad \text{ for } \nu = \psi(t,x,\varphi(t,x), D\varphi(t,x)) \ ,
\end{aligned}$$

and we naturally extend all these operators to functions that are independent of t by replacing t by T in the definition of ν.
In the following, we shall assume that the support of Φ is $\mathbb{R}^d$. We can always reduce to this case, after possibly changing β and b.

3.1 Viscosity property inside the domain

Theorem 4.3.1 *Assume that v^* and v_* are finite. Then, the value function v is a discontinuous viscosity solution on $(0,T) \times \mathbb{R}^d$ of :*

$$\widehat{\mathcal{H}}\varphi(t,x) = 0 \ . \tag{4.3.1}$$

Proof. We shall only prove the sub-solution property. The super-solution property can be proved similarly by appealing to (DP1) instead of (DP2). Let $\varphi \in C^2([0,T] \times \mathbb{R}^d)$ and (t_0, x_0) be a strict global maximizer of $v^* - \varphi$. Without loss of generality, we may assume that $(v^* - \varphi)(t_0, x_0) = 0$.
We argue by contradiction. Set $y_0 := \varphi(t_0, x_0)$, $z_0 := (x_0, y_0)$, and assume that

$$2\varepsilon \; := \; \widehat{\mathcal{H}}\varphi(t_0, x_0) > 0 \; .$$

Then, from our continuity assumptions, there exists some $\eta > 0$ such that for all $(t,x) \in B_0 := B((t_0, x_0), 2\eta)$ and $\delta \in [-\eta, \eta]$:

$$\widehat{\mathcal{H}}(\varphi + \delta)(t,x) > \varepsilon \; . \tag{4.3.2}$$

Let $(t_n, x_n)_{n \geq 0}$ be a sequence such that :

$$(t_n, x_n) \; \longrightarrow \; (t_0, x_0) \;\text{ and }\; v(t_n, x_n) \; \longrightarrow \; v^*(t_0, x_0)$$

as n tends to ∞. Set $y_n := v(t_n, x_n) - n^{-1}$, $z_n := (x_n, y_n)$ and notice that

$$v(t_n, x_n) - n^{-1} - \varphi(t_n, x_n) \;\text{ tends to }\; 0 \;\text{ as }\; n \;\text{ tends to }\; \infty \; . \tag{4.3.3}$$

Since $(t_n, z_n) \longrightarrow (t_0, z_0)$, we may assume without loss of generality that $(t_n, z_n) \in B_1$ $:= B((t_0, z_0), \eta)$. In order to alleviate the notation, we shall denote :

$$Z_n(.) \; = \; (X_n(.), Y_n(.)) := Z^{\hat{\nu}_n}_{t_n, z_n}(.)$$

the state process with initial data (t_n, z_n) and feedback control process $\hat{\nu}_n(\cdot) := \psi(\cdot, X_n(.)$, $Y_n(\cdot)$, $D\varphi(\cdot, X_n(\cdot)))$ (existence of Z_n follows from our Lipschitz and polynomial growth assumptions on the coefficients of the diffusion uniformly in ν, see Section 2). Notice that from (4.3.2) and the characterization of U in terms of its support function (see Section 3)

$$\big[(s, X_n(s)) \in B_0 \text{ and } |Y_n(s) - \varphi(s, X_n(s))| \leq \eta\big] \implies \hat{\nu}_n(s) \in U \; . \tag{4.3.4}$$

Define the stopping times :

$$\begin{aligned}
\theta_n^j \;&:=\; T \wedge \inf\{s > t_n \;:\; \Delta Z_n(s) \neq 0\} \; , \\
\theta_n^d \;&:=\; T \wedge \inf\{s > t_n \;:\; |Y_n(s) - \varphi(s, X_n(s))| \geq \eta\} \; .
\end{aligned}$$

Denote by Z_n^c the continuous part of Z_n. Since θ_n^j is the time of the first jump of Z_n, we have :

$$Z_n(s \wedge \theta_n^j -) = Z_n^c(s \wedge \theta_n^j) \; , \quad s \geq t_n. \tag{4.3.5}$$

Finally, define the sequences :

$$\tau_n \ := \ T \wedge \inf \{s > t_n \ : \ (s, X_n(s)) \notin B_0\} \ , \ \theta_n \ := \ \tau_n \wedge \theta_n^j \wedge \theta_n^d$$

together with the random sets $\mathcal{J}_n := \{\omega \in \Omega \ : \ \tau_n < \theta_n^j \wedge \theta_n^d\}$. Notice that from the definition of θ_n, (4.3.2) and (4.3.4) for all $s \geq t_n$,

$$\begin{aligned}
&\hat{\nu}_n(s \wedge \theta_n{}^-) \in U \\
&\varepsilon \ < \ r\left(Z_n(s \wedge \theta_n{}^-), \nu_n(s \wedge \theta_n{}^-)\right) + \mathcal{L}^{\nu_n(s\wedge\theta_n{}^-)}\varphi\left(s \wedge \theta_n{}^-, X_n(s \wedge \theta_n{}^-)\right) \\
&\qquad -r\left(X_n(s \wedge \theta_n{}^-), \varphi\left(s \wedge \theta_n{}^-, X_n(s \wedge \theta_n{}^-)\right), \nu_n(s \wedge \theta_n{}^-)\right) \qquad (4.3.6) \\
&\varepsilon \ < \ \inf_{\sigma \in \mathbb{R}^d} b\left(Z_n(s \wedge \theta_n{}^-), \nu_n(s \wedge \theta_n{}^-), \sigma\right) \\
&\qquad +\mathcal{G}^{\nu_n(s\wedge\theta_n{}^-),\sigma}\varphi\left(s \wedge \theta_n{}^-, X_n(s \wedge \theta_n{}^-)\right) \\
&\qquad -b\left(X_n(s \wedge \theta_n{}^-), \varphi\left(s \wedge \theta_n{}^-, X_n(s \wedge \theta_n{}^-)\right), \nu_n(s \wedge \theta_n{}^-), \sigma\right) \ .
\end{aligned}$$

By (4.3.5), applying Itô's Lemma to φ on $[t_n, \theta_n)$ leads to

$$\begin{aligned}
\varphi(\theta_n-, X_n(\theta_n-)) \ &= \ \varphi(t_n, x_n) + \int_{t_n}^{\theta_n} r(X_n^c(s), \varphi(s, X_n^c(s)), \hat{\nu}_n(s))ds \\
&\quad - \int_{t_n}^{\theta_n} \mathcal{L}^{\hat{\nu}_n(s)}\varphi(s, X_n^c(s))ds \\
&\quad +D\varphi(s, X_n^c(s))^* \alpha(X_n^c(s), \hat{\nu}_n(s))dW(s) \ ,
\end{aligned}$$

where by definition of Y_n, y_n and $\hat{\nu}_n$

$$\begin{aligned}
Y_n(\theta_n-) \ &= \ y_n + \int_{t_n}^{\theta_n} r(Z_n^c(s), \hat{\nu}_n(s))ds + a^*(Z_n^c(s), \hat{\nu}_n(s))dW(s) \\
&= \ v(t_n, x_n) - n^{-1} \\
&\quad + \ \int_{t_n}^{\theta_n} r(Z_n^c(s), \hat{\nu}_n(s))ds + D\varphi(s, X_n^c(s))^*\alpha(X_n^c(s), \hat{\nu}_n(s))dW(s) \ .
\end{aligned}$$

Then, by a standard comparison result on the dynamics of $\varphi(\cdot, X_n(\cdot))$ and $Y_n(\cdot)$, the definition of $\hat{\nu}_n$, θ_n and (4.3.6), we obtain :

$$Y_n(\theta_n-) - \varphi(\theta_n-, X_n(\theta_n-)) \geq v(t_n, x_n) - \frac{1}{n} - \varphi(t_n, x_n) \ > \ -\eta \ , \qquad (4.3.7)$$

where the last inequality is obtained by taking some sufficiently large n and using (4.3.3).

We now provide a contradiction to (DP2) at stopping time θ_n for some large n. We study separately the case where $\omega \in \mathcal{J}_n$ and the case where $\omega \in \mathcal{J}_n^c$.

Case 1, on $\mathcal{J}_n$: Define

$$-\zeta := \sup_{(t,x)\in\partial_p B_0} (v^* - \varphi)(t,x) \tag{4.3.8}$$

where $\partial_p B_0$ stands for the parabolic boundary of B_0, i.e. $\partial_p B_0 := [t_0 - 2\eta, t_0 + 2\eta] \times \partial B(x_0, 2\eta) \cup \{t_0 + 2\eta\} \times \bar{B}(x_0, 2\eta)$. Since (t_0, x_0) is a strict global maximizer of $v^* - \varphi$, we have $\zeta > 0$.

Recall that on $\mathcal{J}_n$, $\theta_n^j > \theta_n = \tau_n$. Hence, from (4.3.5), $Z_n(\cdot \wedge \theta_n)$ is continuous on $\mathcal{J}_n$. Together with (4.3.7) and the fact that $v \leq v^*$, this leads to

$$\begin{aligned}
[Y_n(\theta_n) - v(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n} &= [Y_n(\tau_n) - v(\tau_n, X_n(\tau_n))]\, 1\!\!1_{\mathcal{J}_n} \\
&\geq [\varphi(\tau_n, X_n(\tau_n)) - v^*(\tau_n, X_n(\tau_n)) \\
&+ v(t_n, x_n) - n^{-1} - \varphi(t_n, x_n)] 1\!\!1_{\mathcal{J}_n} .
\end{aligned}$$

Since by continuity, $(\tau_n, X_n(\tau_n)) \in \partial_p B_0$, on $\mathcal{J}_n$, (4.3.8) implies that

$$[Y_n(\theta_n) - v(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n} \geq \left[\zeta + v(t_n, x_n) - n^{-1} - \varphi(t_n, x_n)\right] 1\!\!1_{\mathcal{J}_n} .$$

Using (4.3.3) and assuming that n is large enough, we get :

$$[Y_n(\theta_n) - v(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n} \geq (\zeta/2) 1\!\!1_{\mathcal{J}_n} \quad \text{for some } \zeta > 0. \tag{4.3.9}$$

Case 2, on $\mathcal{J}_n^c$: Recall that on $\mathcal{J}_n^c$, $\theta_n = (\theta_n^j \wedge \theta_n^d)$. From the definition of θ_n^d, (4.3.5) and (4.3.7), we have

$$\begin{aligned}
[Y_n(\theta_n) - \varphi(\theta_n, X_n(\theta_n))]\, 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d < \theta_n^j} &= \left[Y_n^c(\theta_n^d) - \varphi(\theta_n^d, X_n^c(\theta_n^d))\right] 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d < \theta_n^j} \\
&= \eta 1\!\!1_{\mathcal{J}_n^c} 1\!\!1_{\theta_n^d < \theta_n^j} .
\end{aligned} \tag{4.3.10}$$

On the other hand, on $\mathcal{J}_n^c \cap \{\theta_n^d \geq \theta_n^j\}$

$$\begin{aligned}
& Y_n(\theta_n) - \varphi(\theta_n, X_n(\theta_n)) \\
&= Y_n(\theta_n^j\text{-}) - \varphi(\theta_n^j\text{-}, X_n(\theta_n^j\text{-})) \\
&+ \int_{\mathbb{R}^d} b(Z_n(\theta_n^j\text{-}), \hat{\nu}_n(\theta_n^j\text{-}), \sigma) \mu(\{\theta_n^j\}, d\sigma) \\
&+ \int_{\mathbb{R}^d} \mathcal{G}^{\hat{\nu}_n(\theta_n^j\text{-}),\sigma} \varphi(\theta_n^j\text{-}, X_n(\theta_n^j\text{-})) \mu(\{\theta_n^j\}, d\sigma) \\
&- \int_{\mathbb{R}^d} b(X_n(\theta_n^j\text{-}), \varphi(\theta_n^j\text{-}, X_n(\theta_n^j\text{-})), \hat{\nu}_n(\theta_n^j\text{-}), \sigma) \mu(\{\theta_n^j\}, d\sigma) .
\end{aligned}$$

Using (4.3.6) and (4.3.7), this proves that

$$\begin{aligned}&[Y_n(\theta_n) - \varphi(\theta_n, X_n(\theta_n))]\, \mathbb{1}_{\mathcal{J}_n^c} \mathbb{1}_{\theta_n^d \geq \theta_n^j} \\ &\geq \left[v(t_n, x_n) - \frac{1}{n} - \varphi(t_n, x_n) + \varepsilon\right] \mathbb{1}_{\mathcal{J}_n^c} \mathbb{1}_{\theta_n^d \geq \theta_n^j} \ .\end{aligned}$$

Finally, by (4.3.3), the fact that $v \leq v^* \leq \varphi$ and (4.3.10), this proves that we can find some n such that :

$$[Y_n(\theta_n) - v(\theta_n, X_n(\theta_n))]\, \mathbb{1}_{\mathcal{J}_n^c} \geq (\varepsilon/2 \wedge \eta) \mathbb{1}_{\mathcal{J}_n^c} \tag{4.3.11}$$

for some $\varepsilon > 0$ and $\eta > 0$. □

3.2 Boundary condition

The nonlinear PDE reported in the above theorem does not provide a complete characterization of the value function v. To further characterize it, we need to specify the terminal condition. From the definition of v it is clear that $v(T, x) = g(x)$ but we know that v may be discontinuous in T. Therefore we introduce :

$$\bar{G}(x) \ := \ \limsup_{t \uparrow T, x' \to x} v(t, x') \ \text{ and } \ \underline{G}(x) \ := \ \liminf_{t \uparrow T, x' \to x} v(t, x') \qquad x \in \mathbb{R}^d.$$

Observe that the above functions functions may not be smooth. Since the constraint on the gradient which appears inside the domain through the operator $\widehat{\mathcal{T}}$ should propagates on the boundary, it is natural to consider the equations solved by $\bar{G}$ and $\underline{G}$ in the viscosity sense.

Theorem 4.3.2 *Let the conditions of Theorem 4.3.1 hold. Then, if $\underline{G}$ is finite, it is a viscosity super-solution on $\mathbb{R}^d$ of*

$$\min\left\{\varphi(x) - g_*(x) \ ; \ \widehat{\mathcal{T}}\varphi(x)\right\} = 0 \ , \tag{4.3.12}$$

and, if $\bar{G}$ is finite, it is a viscosity subsolution on $\mathbb{R}^d$ of

$$\min\left\{\varphi(x) - g^*(x) \ ; \ \widehat{\mathcal{T}}\varphi(x)\right\} = 0 \ . \tag{4.3.13}$$

We split the proof in different lemmas.

Terminal condition for $\underline{G}$

Lemma 4.3.8 *For all $x \in \mathbb{R}^d$, we have $\underline{G}(x) \geq g_*(x)$.*

Proof. Fix $x \in \mathbb{R}^d$ and let (t_n, x_n) be a sequence in $(0,T) \times \mathbb{R}^d$ such that $v(t_n, x_n)$ tends to $\underline{G}(x)$ as n tends to ∞. Set $y_n := v(t_n, x_n) + n^{-1}$. By definition of $v(t_n, x_n)$, there exists some control $\nu_n \in \mathcal{U}$ such that :

$$Y^{\nu_n}_{t_n,x_n,y_n}(T) \geq g\left(X^{\nu_n}_{t_n,x_n}(T)\right) .$$

Now, observe that $Z^{\nu_n}_{t_n,x_n,y_n}(T) \longrightarrow (x, \underline{G}(x))$ P-a.s. as $n \to \infty$ after possibly passing to a subsequence. Then, sending $n \to \infty$ in the last inequality provides :

$$\underline{G}(x) \geq \liminf_{x' \to x} g\left(x'\right) \ = \ g_*(x) .$$

$\square$

Lemma 4.3.9 *Let $x_0 \in \mathbb{R}^d$ and $f \in C^2(\mathbb{R}^d)$ be such that :*

$$0 \ = \ (\underline{G} - f)(x_0) = \min_{x \in \mathbb{R}^d} (\underline{G} - f)(x) .$$

Then,

$$\widehat{\mathcal{T}} f\,(x_0) \geq 0 .$$

Proof. Let f and x_0 be as in the above statement. Let $(s_n, \xi_n)_n$ be a sequence in $(0,T) \times \mathbb{R}^d$ satisfying :

$$(s_n, \xi_n) \ \longrightarrow \ (T, x_0) \ , \quad s_n < T \quad \text{and} \quad v_*(s_n, \xi_n) \ \longrightarrow \ \underline{G}(x_0) .$$

The existence of such a sequence is justified by the fact that we may always replace v by v_* in the definition of $\underline{G}$. For all $n \in \mathbb{N}$ and $k > 0$, we define :

$$\varphi^k_n(t,x) := f(x) - \frac{k}{2}\|x - x_0\|^2 + k\,\frac{T-t}{T-s_n} .$$

Since β is continuous in (t,x,ν), bounded in σ and U is compact, we see that :

$$\eta := \sup\Big\{|\beta(x,\sigma,\nu)| \ : \ \sigma \in \mathbb{R}^d, \ \nu \in U, \ \|x - x_0\| \leq C\Big\} \ < \ \infty ,$$

where $C > 0$ is a given constant. Let $\bar{B}_0$ denote the closed ball of radius η centered at x_0. Notice that for $t \in [s_n, T]$, we have $0 \leq (T-t)(T-s_n)^{-1} \leq 1$, and therefore :

$$\lim_{k \to 0} \ \limsup_{n \to \infty} \ \sup_{(t,x) \in [s_n,T] \times \bar{B}_0} |\varphi^k_n(t,x) - f(x)| = 0 . \tag{4.3.14}$$

Next, let (t_n^k, x_n^k) be a sequence of local minimizers of $v_* - \varphi_n^k$ on $[s_n, T] \times \bar{B}_0$ and set $e_n^k := (v_* - \varphi_n^k)(t_n^k, x_n^k)$. We shall prove later that, after possibly passing to a subsequence :

$$\text{for all } \; k > 0\;, \quad (t_n^k, x_n^k) \longrightarrow (T, x_0)\;, \tag{4.3.15}$$

$$\text{for all } \; k > 0\;, \quad t_n^k \; < \; T \;\; \text{for sufficiently large } n\;, \tag{4.3.16}$$

$$v_*(t_n^k, x_n^k) \longrightarrow \underline{G}(x_0) = f(x_0) \;\; \text{as} \;\; n \to \infty \text{ and } k \to 0\;. \tag{4.3.17}$$

First notice from (4.3.15) and a standard diagonalization argument, that we may assume that $x_n^k \in \mathrm{Int}\bar{B}_0$. Therefore, by (4.3.16), for all k, (t_n^k, x_n^k) is a sequence of local minimizers on $[s_n, T) \times \mathrm{Int}\bar{B}_0$.

Also, notice that from (4.3.14), (4.3.15) and (4.3.17)

$$\text{for all } \; k > 0\;, \;\; D\varphi_n^k(t_n^k, x_n^k) \; = \; Df(x_n^k) - k(x_n^k - x_0) \to Df(x_0)\;, \tag{4.3.18}$$

$$\text{and } \;\; \lim_{k \to 0} \lim_{n \to \infty} e_n^k \; = \; 0\;. \tag{4.3.19}$$

Hence, for sufficiently large n, using Theorem 4.3.1, (4.3.16) and the fact that (t_n^k, x_n^k) is a local minimizer for $v_* - \varphi_n^k$, we get,

$$\widehat{\mathcal{T}}(\varphi + e_n^k)(t_n^k, x_n^k) \; \geq \; 0 \;\; \text{for all } \;\; n \in \mathbb{N}, \; k > 0\;.$$

The statement of the lemma is then obtained by taking limits as $n \to \infty$, then as $k \to 0$, and using (4.3.14), (4.3.15), (4.3.17), (4.3.18), (4.3.19) as well as the continuity of the involved functions.

In order to complete the proof, it remains to show that (4.3.15), (4.3.16) and (4.3.17) hold.

Notice, from the convergence assumption on (s_n, ξ_n), that we can find some large integer N (independent of k) such that for all $n \geq N$ and $k > 0$:

$$(v_* - \varphi_n^k)(s_n, \xi_n) \; = \; v_*(s_n, \xi_n) - f(\xi_n) + \frac{k}{2}\|\xi_n - x_0\| - k \leq -\frac{k}{2} \; < \; 0\;.$$

On the other hand, by definition of the test function f,

$$(v_* - \varphi_n^k)(T, x) = \underline{G}(x) - f(x) + \frac{k}{2}\|x - x_0\|^2 \; \geq \; 0 \quad \text{for all } \;\; x \in \mathbb{R}^d\;.$$

Comparing the two inequalities and using the definition of (t_n^k, x_n^k) provides (4.3.16).

For all $k > 0$, let $x^k \in \bar{B}_0$ be the limit of some subsequence of $(x_n^k)_n$. Then by definition of x_0, we have :

$$\begin{aligned} 0 \;\; &\leq \;\; (\underline{G} - f)(x^k) - (\underline{G} - f)(x_0) \\ &\leq \;\; \liminf_{n \to \infty} (v_* - \varphi_n^k)(t_n^k, x_n^k) - (v_* - \varphi_n^k)(s_n, \xi_n) - \frac{k}{2}\|x^k - x_0\|^2 \\ &+ \;\; k\frac{T - t_n^k}{T - s_n} - k\;. \end{aligned}$$

Since $s_n \leq t_n^k < T$, it follows from the definition of (t_n^k, x_n^k) that :

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty}(v_* - \varphi_n^k)(t_n^k, x_n^k) - (v_* - \varphi_n^k)(s_n, \xi_n) - \frac{k}{2}\|x^k - x_0\|^2 \\ &\leq -\frac{k}{2}\|x^k - x_0\|^2 \leq 0 , \end{aligned}$$

This proves that we must have $x^k = x_0$, and therefore (4.3.15) holds since the convergence of the sequence $(t_n^k)_n$ to T is trivial. Notice that the two last terms in the previous inequality tend to 0. This proves that

$$\liminf_{n\to\infty}(v_* - \varphi_n^k)(t_n^k, x_n^k) = 0$$

which, together with (4.3.14) and (4.3.15), implies (4.3.17), after possibly passing to some subsequences. $\square$

Terminal condition for $\bar{G}$

Let f be in $C^2(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ be such that

$$0 = (\bar{G} - f)(x_0) = \max_{\mathbb{R}^d} (\textit{strict})(\bar{G} - f) .$$

We assume that

$$\min\left\{ f(x_0) - g^*(x_0) \ ; \ \widehat{\mathcal{T}} f\,(x_0) \right\} > 0 , \tag{4.3.20}$$

and we work towards a contradiction to (DP2) of Proposition 4.2.20 in the 4th Step of this proof.

1st Step. Fix some arbitrarily small scalar $c > 0$ and define on $[0,T] \times \mathbb{R}^d$:

$$\varphi(t,x) := f(x) + c\|x - x_0\|^2 + (T-t)^{\frac{1}{2}} .$$

Notice that for all $x \in \mathbb{R}^d$:

$$\frac{\partial \varphi(t,x)}{\partial t} \longrightarrow -\infty \ \text{ as } \ t \longrightarrow T . \tag{4.3.21}$$

Recall that r, $D\varphi$, $D^2\varphi$, α and ρ are continuous and therefore locally bounded. Hence, we may assume by (4.3.21) that for all (t,x,y) in a suitable neighborhood of $(T, x_0, f(x_0))$

$$r(z,\nu) - r(x, \varphi(t,x), \nu) + \mathcal{L}^\nu \varphi(t,x) > 0 \quad \text{ for all } \nu \in U. \tag{4.3.22}$$

Now, notice that β is uniformly bounded in σ on any neighborhood of (T, x_0) and that $\varphi(T, x_0) = f(x_0)$. Hence, by (4.3.20) and by taking a sufficiently small c, we may assume that φ satisfies :

$$\varphi(T,x_0) - g^*(x_0) > 0 \ \text{ and } \ \inf_{\sigma\in\mathbb{R}^d} \widehat{\mathcal{G}}^\sigma(\varphi)(T,x_0) > 0 \ .$$

Then, by upper-semicontinuity of g^*, continuity of φ, and continuity of b and β uniformly in σ and ν, there exist some $\varepsilon > 0$ and some $\eta > 0$ such that for all $(t,x) \in \bar{B}_0 := [T-\eta, T] \times \bar{B}(x_0,\eta)$ and $\delta \in [-\eta,\eta]$:

$$\varphi(t,x) + \delta - g^*(x) > \varepsilon \quad \text{and} \quad \inf_{\sigma \in \mathbb{R}^d} \widehat{\mathcal{G}}^\sigma(\varphi+\delta)(t,x) > \varepsilon \ .$$

Finally, by using (4.3.22) and by taking a sufficiently small η, we may assume that for all $(t,x,\delta) \in \bar{B}_0 \times [-\eta,\eta]$:

$$\min\left\{\varphi(t,x) + \delta - g^*(x) \ ; \ \widehat{\mathcal{H}}(\varphi+\delta)(t,x)\right\} > \varepsilon \ . \tag{4.3.23}$$

2nd Step. Let (s_n,ξ_n) be a sequence in $[T-\eta/2, T) \times \bar{B}(x_0,\eta) \subset \bar{B}_0$ satisfying :

$$(s_n,\xi_n) \longrightarrow (T,x_0)\ , \quad s_n < T \quad \text{and} \quad v^*(s_n,\xi_n) \longrightarrow \bar{G}(x_0)\ .$$

Let (t_n,x_n) be a maximizer of $(v^* - \varphi)$ on $[s_n,T] \times \bar{B}(x_0,\eta) \subset \bar{B}_0$. For all n, let $(t_n^k, x_n^k)_k$ be a subsequence in $[s_n,T] \times \bar{B}(x_0,\eta)$ satisfying :

$$(t_n^k, x_n^k) \longrightarrow (t_n,x_n) \quad \text{and} \quad v(t_n^k,x_n^k) \longrightarrow v^*(t_n,x_n)\ .$$

We shall prove later that

$$(t_n,x_n) \longrightarrow (T,x_0) \text{ and } v^*(t_n,x_n) \longrightarrow \bar{G}(x_0) \tag{4.3.24}$$

and that there exists a subsequence of $(t_n^k,x_n^k)_{k,n}$, relabelled (t_n', x_n'), satisfying :

$$(t_n',x_n') \to (T,x_0) \quad \text{and} \quad v(t_n',x_n') \to \bar{G}(x_0) \quad \text{, where for all } n,\ t_n' < T\ . \tag{4.3.25}$$

3rd Step. Consider the sequence (t_n',x_n') of the 2nd Step. Set $y_n' := v(t_n',x_n') - n^{-1}$, $z_n' := (x_n', y_n')$ and notice that

$$v(t_n',x_n') - n^{-1} - \varphi(t_n',x_n') \ \text{ tends to } 0 \text{ as } n \text{ tends to } \infty\ . \tag{4.3.26}$$

Hence $(t_n', z_n') \longrightarrow (T, x_0, f(x_0))$ and we may assume without loss of generality that $(t_n',x_n') \in \mathrm{Int}\bar{B}_0$ and that $|y_n' - \varphi(t_n',x_n')| \le \eta$ for all n. In order to alleviate the notation, we shall denote :

$$Z_n(.) \ = \ (X_n(.), Y_n(.)) := Z_{t_n,z_n}^{\hat{\nu}_n}(.)$$

the state process with initial data (t_n', z_n') and control process $\hat{\nu}_n(\cdot) := \psi(\cdot, X_n(.), Y_n(\cdot), D\varphi(\cdot, X_n(\cdot)))$. Recall that by (4.3.25), $t_n' < T$ for all n and define the stopping times :

$$\begin{aligned}
\theta_n^j &:= T \wedge \inf\left\{s > t_n' \ : \ \Delta Z_n(s) \neq 0\right\}\ , \\
\theta_n^d &:= T \wedge \inf\left\{s > t_n' \ : \ |Y_n(s) - \varphi(s, X_n(s))| \ge \eta\right\} \\
\tau_n &:= T \wedge \inf\left\{s > t_n' \ : \ (s, X_n^c(s)) \notin B_0\right\}\ , \\
\theta_n &:= \tau_n \wedge (\theta_n^j \wedge \theta_n^d)\ ,
\end{aligned}$$

together with the random sets $\mathcal{J}_n := \{\omega \in \Omega \ : \ \tau_n < (\theta_n^j \wedge \theta_n^d)\}$. Finally, define

$$-\zeta := \sup_{x \in \partial \bar{B}(x_0, \eta)} (\bar{G} - f)(x) \ .$$

Since x_0 is a strict maximizer and $(\bar{G} - f)(x_0) = 0$, we have $\zeta > 0$.

<u>4th Step.</u> We can now prove the required contradiction. Arguing like in the proof of Theorem 4.3.1 and using (4.3.23) as well as (4.3.26), it is easily checked that we can find some n such that :

$$\begin{array}{rl} & Y_n(\theta) - v(\theta_n, X_n(\theta_n)) \ \geq \ \zeta/2 \mathbb{1}_{\mathcal{J}_n} + (\varepsilon/2 \wedge \eta) \mathbb{1}_{\mathcal{J}_n^c} \ > \ 0 \\ \text{and} & \hat{\nu}_n(\cdot \wedge \theta_n) \in \mathcal{U} \ \text{ on } \ [t_n, T] \ . \end{array}$$

Since, $y'_n < v(t'_n, x'_n)$ this leads to the required contradiction to (DP2).

<u>5th Step.</u> It remains to prove (4.3.24) and (4.3.25). Clearly, $t_n \to T$. Let $\hat{x} \in [x_0 - \eta, x_0 + \eta]$ be such that $x_n \to \hat{x}$, along some subsequence. Then, by definition of f and x_0 :

$$\begin{aligned} 0 \ &\geq \ (\bar{G} - f)(\hat{x}) - (\bar{G} - f)(x_0) \\ &\geq \ \limsup_{n \to \infty} (v^* - \varphi)(t_n, x_n) + c\|\hat{x} - x_0\|^2 - (v^* - \varphi)(s_n, \xi_n) \\ &\geq \ c\|\hat{x} - x_0\|^2 \geq 0 \ , \end{aligned}$$

where the third inequality is obtained by definition of (t_n, x_n). Hence, $\hat{x} = x_0$ and, by continuity of φ, $v^*(t_n, x_n) \to \bar{G}(x_0)$. This also proves that

$$\lim_n \lim_k (t_n^k, x_n^k) = (T, x_0) \quad \text{and} \quad \lim_n \lim_k v(t_n^k, x_n^k) = \bar{G}(x_0) \ . \tag{4.3.27}$$

Now assume that $\text{card}\{(n,k) \in \mathbb{N} \times \mathbb{N} \ : \ t_n^k = T\} = \infty$. Since $v(T, \cdot) = g(\cdot)$, there exists a subsequence, relabelled (t_n^k, x_n^k), such that :

$$\limsup_n \limsup_k v(t_n^k, x_n^k) \leq g^*(x_0) \ .$$

Since by assumption $g^*(x_0) < f(x_0) = \bar{G}(x_0)$, this leads to a contradiction to (4.3.27). Hence, $\text{card}\{(n,k) \in \mathbb{N} \times \mathbb{N} \ : \ t_n^k = T\} < \infty$, and, using a diagonalization argument, we can construct a subsequence $(t'_n, x'_n)_n$ of $(t_n^k, x_n^k)_{n,k}$ satisfying (4.3.25). $\square$

4 The pure jump model

The proof of Theorem 4.3.1 can be reproduced almost exactly in the case where $\alpha = a = 0$, i.e. in the case of a pure jump model.

Theorem 4.4.1 *Assume that v^* and v_* are finite. Then, the value function v is a discontinuous viscosity solution on $(0,T) \times \mathbb{R}^d$ of :*

$$\sup_{\nu \in U} \min \left\{ \mathcal{L}^\nu \varphi(t,x) \; ; \; \inf_{\sigma \in \mathbb{R}^d} \mathcal{G}^{\nu,\sigma} \varphi(t,x) \right\} = 0 \; . \tag{4.4.1}$$

Notice that since $\alpha = 0$, we have

$$\mathcal{L}^\nu \varphi(t,x) := r(x, \varphi(t,x), \nu) - \frac{\partial \varphi}{\partial t}(t,x) - \rho(t,x,\nu)^* D\varphi(t,x) \; .$$

Theorem 4.4.2 *Let the conditions of Theorem 4.4.1 hold and assume that $\underline{G}$ and $\bar{G}$ are finite. Then :*

$$H_*(\underline{G}(x)) := \min \left\{ \underline{G}(x) - g_*(x) \; ; \; \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} \mathcal{G}^{\nu,\sigma} \underline{G}(x) \right\} \geq 0 \; , \; x \in \mathbb{R}^d \tag{4.4.2}$$

$$H^*(\bar{G}(x)) := \min \left\{ \bar{G}(x) - g^*(x) \; ; \; \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} \mathcal{G}^{\nu,\sigma} \bar{G}(x) \right\} \leq 0 \; , \; x \in \mathbb{R}^d \; . \tag{4.4.3}$$

In contrast to Theorem 4.3.2, the boundary condition is obtained in the classical sense (in opposition to the viscosity sense). This comes from the fact that it does not contain any derivatives term.

In general, we are not able to prove that $\underline{G} = \bar{G}$, and, even if g is continuous, we have no general comparison result for continuous functions satisfying both (4.4.2) and (4.4.3). Nevertheless, the intuition is that if g is continuous and $\bar{G} = \underline{G} =: G$, then G should be interpreted as the smallest solution of (4.4.2)-(4.4.3). In this case, and under mild assumptions, we can construct explicitly a sequence of functions that converge to G.

The existence of a smallest solution for (4.4.2) is easily obtained under (4.4.4) below.

Proposition 4.4.21 *Assume that there exists a strictly increasing function h on $\mathbb{R}$ such that for all $(x,\nu,\sigma) \in \mathbb{R}^d \times U \times \mathbb{R}^d$, the mapping*

$$y \longmapsto y + b(x,y,\nu,\sigma) - h(y) \quad \textit{is non-decreasing.} \tag{4.4.4}$$

Assume further that there exists a finite function f satisfying $H_*(f) \geq 0$ on $\mathbb{R}^d$.

Then, there exists a lower-semicontinuous function ℓ such that $H_*(\ell) \geq 0$ on $\mathbb{R}^d$ and such that $\ell \leq f$ for all function f satisfying $H_*(f) \geq 0$ on $\mathbb{R}^d$, i.e. (4.4.2) *admits a smallest solution which is lower-semicontinuous. Moreover, we have $H_*(\ell(x)) = 0$ for all $x \in \mathbb{R}^d$.*

Remark 4.4.22 (4.4.4) *implies in particular that for all finite function f*

$$y \longmapsto y + \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} b(x, y, \nu, \sigma) - f(x + \beta(x, \nu, \sigma))$$

is strictly increasing. Hence, given $(y_1, y_2) \in \mathbb{R}^2$ and a finite function f such that

$$\begin{aligned} y_1 + \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} b(x, y_1, \nu, \sigma) - f(x + \beta(x, \nu, \sigma)) &\geq 0 \\ y_2 + \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} b(x, y_2, \nu, \sigma) - f(x + \beta(x, \nu, \sigma)) &\leq 0, \end{aligned}$$

(4.4.4) *implies that $y_1 \geq y_2$. Moreover, if such y_1 and y_2 exist, by using the continuity of b in y, uniformly in (ν, σ), we can find some y (which is unique) such that*

$$y + \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} b(x, y, \nu, \sigma) - f(x + \beta(x, \nu, \sigma)) = 0 .$$

We now provide sufficient conditions under which we can explicitly characterize the boundary condition. The assumptions of the following proposition are quite strong but it gives the intuition for the general case.

Proposition 4.4.22 *Let the conditions of Theorem 4.4.2 hold. Assume that g is continuous and that there exists a continuous smallest solution ℓ of* (4.4.2)*. Assume further that there exist a neighborhood V of T and a classical super-solution w of* (4.4.1) *on $V \times \mathbb{R}^d$ such that, for all $x \in \mathbb{R}^d$, $\lim_{t \uparrow T,\, x' \to x} w(t, x') = w(T, x) = \ell(x)$ and for all $(t, x) \in V \times \mathbb{R}^d$*

$$y \mapsto y + \sup_{\nu \in U} \inf_{\sigma \in \mathbb{R}^d} b(x, y, \nu, \sigma) - w(t, x + \beta(x, \nu, \sigma)) \text{ is strictly increasing.} \tag{4.4.5}$$

Then, $\underline{G} = \bar{G} = \ell$.

Remark 4.4.23 *If we combine the conditions of Propositions 4.4.21 and 4.4.22, we obtain that $\underline{G} = \bar{G} = \ell$ where ℓ is the solution of $H_*(\ell) = 0$.*

Proof. Fix $(t, x) \in V \times \mathbb{R}^d$ and set $z := (x, y)$ where $y := w(t, x)$. w satisfies on $V \times \mathbb{R}^d$:

$$\sup_{\nu \in U} \min \left\{ \mathcal{L}^\nu w(t, x) \; ; \; \inf_{\sigma \in \mathbb{R}^d} \mathcal{G}^{\nu, \sigma} w(t, x) \right\} \geq 0 \,, \tag{4.4.6}$$

Define for all $n \in \mathbb{N} \setminus \{0\}$, the sequence of stopping times :

$$\begin{aligned}\theta_1 &:= T \wedge \inf\left\{s > t \ : \ \Delta Z^{\nu}_{t,z}(s) \neq 0\right\} \\ \theta_{n+1} &:= T \wedge \inf\left\{s > \theta_n \ : \ \Delta Z^{\nu}_{t,z}(s) \neq 0\right\} \ ,\end{aligned}$$

where the control process ν is defined in a Markovian way as $\nu(\cdot) := \hat{\nu}(\cdot, X^{\nu}_{t,x}(\cdot))$ and $\hat{\nu}(t,x)$ is the argmax in (4.4.6) for all $(t,x) \in V \times \mathbb{R}^d$. Using (4.4.6), the fact that $y = w(t,x)$ and standard comparison results for stochastic differential equations, we get that $Y^{\nu}_{t,z}(\theta_1{}^-) \geq w(\theta_1{}^-, X^{\nu}_{t,x}(\theta_1{}^-))$. By (4.4.5) and (4.4.6) again we obtain that $Y^{\nu}_{t,z}(\theta_1) \geq w(\theta_1, X^{\nu}_{t,x}(\theta_1))$. Using a recursive argument, we get that, for all $i \geq 1$, $Y^{\nu}_{t,z}(\theta_i) \geq w(\theta_i, X^{\nu}_{t,x}(\theta_i))$. This proves that : $Y^{\nu}_{t,z}(T) \geq w(T, X^{\nu}_{t,x}(T)) \geq \ell(X^{\nu}_{t,x}(T)) \geq g(X^{\nu}_{t,x}(T))$. Hence, by definition of v, for all $(t,x) \in V \times \mathbb{R}^d$, $w(t,x) \geq v(t,x)$ and $\ell(x) = \lim_{t\uparrow T, x' \to x} w(t,x') \geq \limsup_{t\uparrow T, x'\to x} v(t,x) = \bar{G}(x) \geq \underline{G}(x)$, where the last inequality is obtained by definition. The result is finally obtained by noticing that, by definition of ℓ and Theorem 4.4.2, $\underline{G} \geq \ell$. □

Finally, we give some conditions under which we can easily prove the continuity of the smallest solution of (4.4.2).

Proposition 4.4.23 *Under the conditions of Proposition 4.4.21, if g is uniformly continuous and b and β are independent of (x,y), then the smallest solution ℓ of* (4.4.2) *is uniformly continuous.*

5 Applications

In this section, we will always assume that the standing assumptions of this Chapter hold except when the contrary is explicitly specified.

5.1 Optimal insurance and self-protection strategies

We denote by $\mathcal{U}$ the set of all $\mathbb{F}$-predictable processes $\nu = \{\nu(t),\ 0 \leq t \leq T\}$ valued in $U := U_1 \times [0,1]$, where U_1 is defined below. Fix $z := (t,x,y) \in [0,T] \times (0,\infty) \times \mathbb{R}$. We assume that the dynamics of $Y^{\nu,z}$ and $X^{\nu,z}$ are given by

$$\begin{aligned}dY^{\nu}_s &= Y^{\nu}_s r ds - c(\nu^1_s)ds - \pi(\nu^2_s, X^{\nu}_s)ds - \int_{\mathbb{R}^d} (1-\nu^2_s) b(X^{\nu}_s, \sigma)\mu(d\sigma, ds) \\ dX^{\nu}_s &= \nu^1_s ds\end{aligned}$$

together with the initial condition $(X^{\nu,z}_t, Y^{\nu,z}_t) = (x,y)$.

Remark 4.5.24 This dynamics is derived from that of Section 2 by setting $\rho(t,x,\nu) = \nu^1$, $r(x,y,\nu) = ry - c(\nu^1) - \pi(\nu^2,x)$, $\alpha = a = \beta = 0$ and $b(x,y,\nu,\sigma) = -(1-\nu^2)b(x,\sigma)$.

The economic interpretation of the above model is the following. Consider the problem of an agent who wants to protect part of his wealth from a depreciation due to a random event modeled by a point process associated with the mark-space $\mathbb{R}^d$ and the random measure μ.
He has the choice between insurance and self-protection. The level of self-protection is modeled by the controlled process X^ν. The nonnegative insurance premium π is paid continuously and depends on the level of insurance $\nu^2 \in [0,1]$ and self-protection X^ν. We suppose that π, defined on $[0,1] \times [0,\infty)$, is Lipschitz continuous, nondecreasing with respect to its first variable and non-increasing with respect to its second variable. We assume that the level of loss b is decreasing with x, that there exists a level $\hat{x} \in \mathbb{R}^+$ such that, for all $x \geq \hat{x}$, $b(x,\cdot) = 0$ and that $b(x,\sigma) > 0$ for all $x < \hat{x}$ and $\sigma \in \mathbb{R}^d$.
The wealth of the agent Y^ν may be invested in a non-risky asset with instantaneous appreciation rate $r > 0$. Y^ν is used to pay the insurance premium, the non insured losses $(1-\nu^2)b(X^\nu,\cdot)$ and to invest in order to increase the level of self-protection X^ν. The instantaneous level of investment is modeled by the U^1-valued control process ν^1, where $U^1 = [0,\bar{\nu}^1]$ (with $\bar{\nu}^1 > 0$), and the associated instantaneous cost is $c(\nu^1)$ where $c(U^1)$ is bounded, $c(0) = 0$ and $c(\nu^1) > 0$ on $(0,\bar{\nu}^1]$.

The aim of the agent is to compute the minimal initial wealth needed in order to guarantee the non negativity of the terminal wealth Y_T^ν, and therefore, the value function of the associated super-replication problem is :

$$v(t,x) := \inf\left\{y \in \mathbb{R} \ : \ \exists\, \nu \in \mathcal{U},\ Y_T^{\nu,(t,x,y)} \geq 0 \ \right\}.$$

Using Theorem 4.4.1 we can easily prove that :

Theorem 4.5.1 *The value function v is the unique continuous viscosity solution on $(0,T) \times (0,\hat{x})$ of*

$$\varphi(t,x)r - \pi(1,x) - \frac{\partial\varphi(t,x)}{\partial t}(t,x) - \tilde{c}\left(\frac{\partial\varphi(t,x)}{\partial x}\right) = 0\,,$$

where

$$\tilde{c}\left(\frac{\partial\varphi(t,x)}{\partial x}\right) = \inf_{\nu^1 \in U^1}\left(c(\nu^1) + \nu^1\frac{\partial\varphi(t,x)}{\partial x}\right)\,,$$

satisfying $\lim_{t\to T} v(t,x) = 0$, *for all* $x \in (0,\hat{x})$, *and* $\lim_{x\uparrow\hat{x}} v(t,x) = 0$, *for all* $t \in [0,T]$.

Remark 4.5.25 Fix $(t,x) \in [0,T] \times [0,\hat{x})$. Direct computation show that $f(t,x) := \pi(1,x)/r\ (1-\exp(-r(T-t)))$ is the minimal initial capital needed in order to pay full insurance on $[t,T]$ if the level of self-protection remains equal to x, i.e. $\nu^1 = 0$. Therefore, if $f(t,x) > v(t,x)$, it is less expensive to invest in self-protection and the problem is basically a problem of optimal rate of investment. From an economic point of view, $v(t,x)$ may be considered as a upper-bound for the discounted price of full insurance.

Proof. We first prove that for all $(t,x) \in [0,T] \times [0,\hat{x}]$:

$$0 \ \leq\ v(t,x)\ \leq \min\left(f(t,x)\ ,\ \frac{[\hat{x}-x]^+}{\bar{\nu}^1}\left(\pi(1,x)+c(\bar{\nu}^1)\right)\right)\ , \tag{4.5.1}$$

where $f(t,x)$ is defined as in the above Remark. It is clear from the dynamics of Y^ν that $v \geq 0$. To see that $v(t,x) \leq \frac{[\hat{x}-x]^+}{\bar{\nu}^1}\ (\pi(1,x)+\ c(\bar{\nu}^1))$, consider the strategy where $(\nu^1(s),\nu^2(s)) = (\bar{\nu}^1 1\!\!\mathrm{I}_{s-t\leq[\hat{x}-x]^+/\bar{\nu}^1},\ 1\!\!\mathrm{I}_{s-t\leq[\hat{x}-x]^+/\bar{\nu}^1})$ for $s \in [t,T]$ and notice that X^ν is nondecreasing with $X^\nu(t+[\hat{x}-x]^+/\bar{\nu}^1) = \hat{x}$. Then, using the fact that π is non-increasing with respect to x, it is easily checked that starting with $[\hat{x}-x]^+/\bar{\nu}^1\ (\pi(1,x)+\ c(\bar{\nu}^1))$ is more than we need to adopt a full insurance strategy up to $T\wedge\ [\hat{x}-x]^+/\bar{\nu}^1$ and then a full self-protection strategy with no insurance from $T\wedge\ [\hat{x}-x]^+/\bar{\nu}^1$ up to T.
Boundary conditions. This is a direct consequence of (4.5.1).
Super-solution property. From (4.5.1), v_* is finite and, by Theorem 4.4.1, v_* is a viscosity super-solution on $(0,T) \times (0,\hat{x})$ of

$$\sup_{\nu\in U}\min\left\{\varphi(t,x)r - c(\nu^1) - \pi(\nu^2,x) - \frac{\partial\varphi(t,x)}{\partial t} - \nu^1\frac{\partial\varphi(t,x)}{\partial x}\ ;\right.$$
$$\left.\inf_{\sigma\in\mathbb{R}^d} -(1-\nu^2)b(x,\sigma)\right\} = 0\ .$$

Since $\inf_{\sigma\in\mathbb{R}^d} -(1-\nu^2)b(x,\sigma) < 0$ if $x < \hat{x}$ and $\nu^2 < 1$, this proves that v_* is also a viscosity super-solution on $(0,T) \times (0,\hat{x})$ of

$$\sup_{\nu^1\in U^1}\ \varphi(t,x)r - c(\nu^1) - \pi(1,x) - \frac{\partial\varphi(t,x)}{\partial t} - \nu^1\frac{\partial\varphi(t,x)}{\partial x} = 0\ .$$

Subsolution property. From (4.5.1), v^* is finite. Then, the fact that v^* is a viscosity subsolution on $(0,T) \times (0,\hat{x})$ of

$$\sup_{\nu^1\in U^1}\ \varphi(t,x)r - c(\nu^1) - \pi(1,x) - \frac{\partial\varphi(t,x)}{\partial t} - \nu^1\frac{\partial\varphi(t,x)}{\partial x} = 0\ ,$$

is obtained by arguing as as above.

Continuity and uniqueness. Recall that $r > 0$ and $\pi(1, \cdot)$ is Lipschitz continuous. Moreover, from the compacity of U^1 and the boundedness of $c(U^1)$ it is easily checked that $\tilde{c}$ is uniformly Lipschitz. Therefore, the result is a direct consequence of [1, theorem 4.8 p.100]. □

5.2 Option hedging under stochastic volatility and dividend revision process

We consider a financial market with a non-risky asset, normalized to unity, and a risky asset S that pays a dividend $S_{t_1}\delta(X_{t_1}) \in \mathcal{F}(t_1)$ at time $t_1 \in (0, T]$. We assume that the dividend anticipation process X may be modified along the time. The problem consists in finding the minimal initial capital needed in order to hedge the contingent claim $\psi(S_T)$ where ψ is a $\mathbb{R}$-valued function, continuous and bounded from below. We assume that the dynamics of S and X are given on $[0, T]$ by :

$$\begin{aligned} dS_t &= S_{t\text{-}}\left(\alpha(S_t, X_t)dW(t) - \delta(X_t)1\!\!1_{t=t_1}\right) \\ dX_t &:= \int_{\mathbb{R}^d} X_{t\text{-}} b(S_{t\text{-}}, X_{t\text{-}}, \sigma)\mu(d\sigma, dt) \end{aligned}$$

where δ is continuous, valued in $[0, \bar{\delta}]$ with $\bar{\delta} < 1$, and b takes values in $(-1, 1)$. We also assume that for all $(t, s, x) \in [0, T] \times (0, \infty)^2$:

$$\begin{aligned} &(i) \text{ there exists } \sigma_1 \text{ and } \sigma_2 \in \mathbb{R}^d \text{ such that } b(s, x, \sigma_1)b(s, x, \sigma_2) < 0. \\ &(ii) \ \bar{\alpha}(t, s) := \sup_{x \in (0,\infty)} \alpha(s, x) < \infty \\ &\quad \textit{and } \underline{\alpha}(t, s) := \inf_{x \in (0,\infty)} \alpha(s, x) \geq \varepsilon \text{ for some } \varepsilon > 0 \ . \end{aligned}$$

Let ν be a progressively measurable $I\!\!F$-predictable process valued in a convex compact set U with non empty interior corresponding to the proportion of wealth Y^ν invested in the risky asset. Then, under the self-financing condition, the dynamics of Y^ν on $[0, T]$ is given by

$$dY_t^\nu := \nu_{t\text{-}} Y_{t\text{-}}^\nu \left(\frac{dS_t}{S_{t\text{-}}} + \delta(X_t)1\!\!1_{t=t_1} \right) = \nu_t Y_t^\nu \alpha(S_t, X_t)dW(t) \ .$$

Given, $(t, z) = (t, s, x, y) \in [0, T] \times (0, \infty)^2 \times \mathbb{R}$, we denote by $(S^{(t,z)}, X^{(t,z)}, Y^{\nu,(t,z)})$ the previously introduced processes with initial conditions $(S_t^{(t,z)}, X_t^{(t,z)}, Y_t^{\nu,(t,z)}) = (s, x, y)$.

The value function associated with the target problem is defined on $[0, T] \times (0, \infty)^2$ by :

$$v(t, s, x) := \inf \left\{ y \in \mathbb{R} \ : \ Y_T^{\nu,(t,s,x,y)} \geq \psi(S_T^{(t,s,x)}) \ , \text{ for some } \nu \in \mathcal{U} \right\} \ .$$

Remark 4.5.26 *Using standard arguments it is easily checked that for all $\nu \in \mathcal{U}$ and $(t,z) \in [0,T] \times (0,\infty)^2 \times \mathbb{R}$, $Y^{\nu,(t,z)}$ is a super-martingale. It follows, from the definition of v and the fact that ψ is bounded from below, that v is also bounded from below.*

Notice that our continuity assumption of Section 2 does not hold in this model because of the term $\delta(X_t)\mathbb{1}_{t=t_1}$ in the dynamics of S. We show in Lemma 4.5.10 that this difficulty may be avoided.
We first introduce some notation. For all $(t,s,x) \in [0,T] \times (0,\infty)^2$, we set

$$\tilde{v}_*(t,s,x) := \liminf_{(s',x')\to(s,x)} v(t,s',x') \ , \ \tilde{v}^*(t,s,x) := \limsup_{(s',x')\to(s,x)} v(t,s',x') \ ,$$
$$\underline{G}_t(s,x) := \liminf_{t'\uparrow t,\ (s',x')\to(s,x)} v(t',s',x') \ , \ \bar{G}_t(s,x) := \limsup_{t'\uparrow t,\ (s',x')\to(s,x)} v(t',s',x') \ .$$

Then we have the

Lemma 4.5.10 *Assume that v^* is finite on $[0,t_1]$ and $\bar{G}_{t_1}$ is finite, then Theorem 4.3.1 holds for v on $(0,t_1)$. Moreover, Theorem 4.3.2 holds for $\underline{G}_{t_1}$ and $\bar{G}_{t_1}$ with $g_*((s,x)) = \tilde{v}_*(t_1, s(1-\delta(x)), x)$, $g^*((s,x)) = \tilde{v}^*(t_1, s(1-\delta(x)), x)$ and $T = t_1$.*

Proof. The proof is similar to that of Theorem 4.3.1, so we only explain how to adapt it. First notice that (DP1) and (DP2) hold in our framework and that there is no discontinuity on the functions driving the dynamics of S on $(0,t_1)$. Since by Remark 4.5.26, v_* is finite, Theorem 4.3.1 holds for v on $(0,t_1)$. We now consider the boundary conditions.
<u>Super-solution</u>. First notice that, by Remark 4.5.26, $\underline{G}_{t_1}$ is finite. From (DP1), for all $(t,s,x) \in [0,t_1) \times (0,\infty)^2$ and $y > v(t,s,x)$, there exists some $\nu \in \mathcal{U}$ such that :

$$Y_{t_1}^{\nu,(t,s,x,y)} \ \geq \ v(t_1, S_{t_1}^{(t,s,x)}, X_{t_1}^{(t,s,x)}) \ = \ v(t_1, S_{t_1\text{-}}^{(t,s,x)}(1-\delta(X_{t_1}^{(t,s,x)})), X_{t_1}^{(t,s,x)})$$

Hence, the proof of the super-solution property is similar as in the general case. It suffices to replace T by t_1, $g((s,x))$ by $v(t_1, s - s\delta(x), x)$ and consider the continuous part of the state process $(S^{(t,s,x)},\ X^{(t,s,x)})$.
<u>Subsolution</u>. Fix $(t,s,x) \in (0,t_1) \times (0,\infty)^2$ and $y < v(t,s,x)$. From (DP2), for all $\nu \in \mathcal{U}$:

$$P\left(Y_{t_1}^{\nu,(t,s,x,y)} \ > \ v(t_1, S_{t_1\text{-}}^{(t,s,x)}(1-\delta(X_{t_1}^{(t,s,x)})), X_{t_1}^{(t,s,x)})\right) < 1$$

Hence, we may apply the same kind of contradiction argument as in the proof of the subsolution property. Here again, it suffices to replace T by t_1, $g((s,x))$ by $v(t_1, s - s\delta(x), x)$ and consider the continuous part of the state process $(S^{(t,s,x)},\ X^{(t,s,x)})$. □

We can now state the main result of this subsection.

Theorem 4.5.2 *Assume that v^* is finite. Then, the value function v is a discontinuous viscosity solution on $(0,t_1) \times (0,\infty)$ and on $(t_1,T) \times (0,\infty)$ of :*

$$\min\left\{-\frac{\partial\varphi}{\partial t}(t,s) - \frac{1}{2}s^2\hat{\alpha}\frac{\partial^2\varphi}{\partial s^2}(t,s)\ ;\ \chi_U\left(\frac{s}{\varphi(t,s)}\frac{\partial\varphi}{\partial s}(t,s)\right)\right\} = 0 \tag{4.5.2}$$

where

$$\hat{\alpha}(t,s) := \left(\bar{\alpha}^2 1\!\!1_{\frac{\partial^2\varphi}{\partial s^2}\geq 0} + \underline{\alpha}^2 1\!\!1_{\frac{\partial^2\varphi}{\partial s^2}\leq 0}\right)(t,s)\ .$$

Assume further that $\bar{G}_{t_1}$ and $\bar{G}_T$ are finite, then $\underline{G}_{t_1}$ and $\bar{G}_{t_1}$ are viscosity super and sub-solutions on $(0,\infty)$ of

$$\min\left\{\varphi(s) - \sup_{x\in(0,\infty)}\tilde{v}_*(t_1,s(1-\delta(x)))\ ;\ \chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} \;=\; 0\ , \tag{4.5.3}$$

$$\min\left\{\varphi(s) - \sup_{x\in(0,\infty)}\tilde{v}^*(t_1,s(1-\delta(x)))\ ;\chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} \;=\; 0\ , \tag{4.5.4}$$

and $\underline{G}_T$, $\bar{G}_T$ are viscosity super and subsolutions on $(0,\infty)$ of

$$\min\left\{\varphi(s) - \psi(s)\ ;\ \chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} = 0\ . \tag{4.5.5}$$

Remark 4.5.27 Assume that we can prove a comparison theorem for (4.5.2)-(4.5.5), then v is continuous on (t_1,T) and we can replace $\tilde{v}_*$ and $\tilde{v}^*$ by v in (4.5.3) and (4.5.4). We may even expect to have a comparison theorem for (4.5.2)-(4.5.3)-(4.5.4). In this case, we may be able to estimate v numerically. It suffices to compute v on $[t_1,T]$ and then use its value in t_1 to approximate it on $[0,t_1)$ by using the boundary conditions (4.5.3)-(4.5.4).

Proof. First notice that, by Remark 4.5.26, v_*, $\underline{G}_{t_1}$ and $\underline{G}_T$ are finite. We only prove that v_* is a viscosity super-solution of (4.5.2) on (t_1,T). The other results are proved similarly by using Theorems 4.3.1, 4.3.2 and Lemma 4.5.10.

<u>1st Step .</u> We first prove that v_* is independent of x. Fix $(t_0,s_0,x_0) \in (t_1,T) \times (0,\infty)^2$ and a $C^2((t_1,T)\times(0,\infty)^2)$ function φ such that (t_0,s_0,x_0) is a strict local minimum for $v_* - \varphi$.

Assume that φ is locally strictly increasing in x at (t_0,s_0,x_0). Then, for all $C \geq 0$, (t_0,s_0,x_0) is a strict local minimum for $v_* - \tilde{\varphi}$ where $\tilde{\varphi}$ is defined on $(t_1,T) \times (0,\infty)^2$ by $\tilde{\varphi}(t,s,x) := \varphi(t,s,x-C(x-x_0)^2)$.

By Theorem 4.3.1, this proves that $\tilde{\varphi}$ satisfies :

$$\inf_{\sigma\in\mathbb{R}^d}\tilde{\varphi}(t_0,s_0,x_0) - \tilde{\varphi}(t_0,s_0,x_0+x_0 b(s_0,x_0,\sigma)) \geq 0\ .$$

Hence,

$$\varphi(t_0, s_0, x_0) \geq \sup_{\sigma \in \mathbb{R}^d} \varphi(t_0, s_0, x_0 + x_0 b(s_0, x_0, \sigma) - C(x_0 b(s_0, x_0, \sigma))^2) . \tag{4.5.6}$$

From assumption (ii) there exists some $\tilde{\sigma} \in \mathbb{R}^d$ such that $b(s_0, x_0, \tilde{\sigma}) > 0$. Since φ is C^1 and locally strictly increasing in x at (s_0, x_0), we can find some sufficiently small $C > 0$ such that

$$\varphi(s_0, x_0 + x_0 b(s_0, x_0, \tilde{\sigma}) - C(x_0 b(s_0, x_0, \tilde{\sigma}))^2) > \varphi(s_0, x_0)$$

which contradicts (4.5.6). Hence, $(\partial\varphi/\partial x)(t_0, s_0, x_0) \leq 0$.
We can prove similarly that $(\partial\varphi/\partial x)(t_0, s_0, x_0) \geq 0$. Hence, v_* is a viscosity super-solution of $\partial\varphi/\partial x \geq 0$ and $-\partial\varphi/\partial x \geq 0$. By Remark 4.5.26 and Lemmas 5.3 and 5.4 in [6], this proves that v_* is independent of x.

2nd Step . We now prove that v_* is a viscosity super-solution of (4.5.2) on (t_1, T). Recall that v_* is independent of x. Fix $(t_0, s_0) \in (t_1, T) \times (0, \infty)$ and a $C^2((t_1, T) \times (0, \infty))$ function φ such that (t_0, s_0) is a local minimum for $v_* - \varphi$. By Theorem 4.3.1, for all $x \in (0, \infty)$, φ satisfies :

$$\min\left\{-\frac{\partial\varphi}{\partial t}(t_0, s_0) - \frac{1}{2}s_0^2\alpha^2(s_0, x)\frac{\partial^2\varphi}{\partial s^2}(t_0, s_0)\ ;\ \chi_U\left(\frac{s_0}{\varphi(t_0, s_0)}\frac{\partial\varphi}{\partial s}(t_0, s_0)\right)\right\} \geq 0.$$

Consider a maximizing sequence $(x_n)_n$ of $\alpha^2(t_0, s_0, \cdot)(\partial^2\varphi/\partial s^2)(t_0, s_0)$. Then, the previous inequality also holds at x_n for all n and the desired result is obtained by sending n to ∞ and using the continuity of α with respect to x. □
Notice that in the case where $\delta = 0$, the model reduces to a stochastic volatility one where the volatility is driven by a pure jump process. In this last case we have the

Theorem 4.5.3 *Assume that $\delta = 0$. Assume further that v^* is finite. Then, the value function v is a discontinuous viscosity solution on $(0, T) \times (0, \infty)$ of :*

$$\min\left\{-\frac{\partial\varphi}{\partial t}(t, s) - \frac{1}{2}s^2\hat{\alpha}\frac{\partial^2\varphi}{\partial s^2}(t, s)\ ;\ \chi_U\left(\frac{s}{\varphi(t, s)}\frac{\partial\varphi}{\partial s}(t, s)\right)\right\} = 0$$

where

$$\hat{\alpha}(t, s) := \left(\bar{\alpha}^2 \mathbb{1}_{\frac{\partial^2\varphi}{\partial s^2} \geq 0} + \underline{\alpha}^2 \mathbb{1}_{\frac{\partial^2\varphi}{\partial s^2} \leq 0}\right)(t, s) .$$

Assume further that $\bar{G}_T$ is finite, then $\underline{G}_T$ and $\bar{G}_T$ are viscosity super and subsolutions on $(0, \infty)$ of

$$\min\left\{\varphi(s) - \psi(s)\ ;\ \chi_U\left(\frac{s}{\varphi(s)}\frac{\partial\varphi}{\partial s}(s)\right)\right\} = 0 .$$

Proof. The result is obtained by the same arguments as in the previous proof. □

Chapter 5

Optimal control of non-Markovian diffusions: the stochastic maximum principle

In this chapter, we come back to the control problem studied in Section 1 but we now allow f, b and σ to be random maps such that $(t,\omega) \mapsto (f_t(\omega,\cdot), b_t(\omega,\cdot), \sigma_t(\omega,\cdot))$ is predictable (we omit the ω argument in the following).
Due to the randomness of b and σ, the PDE approach of the previous sections can not be used any more. In particular, we can not find the optimal control by using a verification argument. However, there exists a stochastic counterpart of this approach based on the so-called *stochastic maximum principle*. It corresponds to the *Pontryagin principle* in deterministic control. We explain here how to use it.

We thus consider here the problem of maximizing an expected gain of the form

$$J(\nu) := \mathbb{E}\left[g(X_T^\nu) + \int_0^T f_t(X_s^\nu, \nu_t)dt\right],$$

in which X^ν is the solution of the one dimensional sde

$$dX_t^\nu = b_t(X_t^\nu, \nu_t)dt + \sigma_t(X_t^\nu, \nu_t)dW_t$$

with ν in the set $\mathcal{U}$ of predictable processes with values in $\mathbb{R}$.
In the above, the random maps f, b and σ are such that $(t,\omega) \mapsto (f_t(\omega,x,u), b_t(\omega,x,u), \sigma_t(\omega,x,u))$ is predictable for any $(x,u) \in \mathbb{R}^2$ (we omit the ω argument in the following). We also assume that they are $dt \times d\mathbb{P}$-a.e. bounded, C^1 in their argument (x,u), and that themselves as well as there first derivatives are Lipschitz. The function g maps

$\Omega \times \mathbb{R} \to \mathbb{R}$, $g(0)$ is uniformly bounded, and g is a.s. C^1 with bounded first derivative in x.

In the following, we shall show how BSDEs permits to provide necessary and sufficient conditions for optimality. We refer to Peng [19, 20] for further references.

1 Necessary condition

Let us start with a necessary condition for a control $\hat{\nu}$ to be optimal. The general idea is to used a spike variation of the form $\nu^{\varepsilon,\tau} := \hat{\nu}\mathbf{1}_{[0,\tau)\cup[\tau+\varepsilon,T]} + \nu\mathbf{1}_{[\tau,\tau+\varepsilon)}$ with $\varepsilon \in (0, T-\tau)$ and ν a $\mathcal{F}_\tau$-measurable random variable, $\tau \in \mathcal{T}$.
By optimality of $\hat{\nu}$, we must have

$$J(\hat{\nu}) \geq J(\nu^{\varepsilon,\tau}),$$

and therefore, if $\varepsilon \mapsto J(\nu^{\varepsilon,\tau})$ is smooth,

$$\partial_\varepsilon J(\nu^{\varepsilon,\tau})_{|\varepsilon=0} \leq 0 \; . \tag{5.1.1}$$

The first problem is therefore to show that this map is smooth. From now on, we write $\hat{X}$ for $X^{\hat{\nu}}$ and $X^{\nu^{\tau,\varepsilon}}$ for $X^{\nu^{\tau,\varepsilon}}$, and we assume that σ does not depend on ν for sake of simplicity, see [20] for the general case.

Under this additional condition, we can first show that $X^{\nu^{\tau,\varepsilon}}$ is smooth with respect to ε.

Proposition 5.1.24 *Let us consider the process $\hat{Y}^{\tau,\nu}$ defined as the solution of*

$$\begin{aligned} Y_t &= \mathbf{1}_{t\geq\tau}\left(b_\tau(\hat{X}_\tau,\nu) - b_\tau(\hat{X}_\tau,\hat{\nu}_\tau)\right) \\ &+ \int_\tau^t \partial_x b_s\left(\hat{X}_s,\hat{\nu}_s\right)Y_s ds + \int_\tau^t \partial_x\sigma_s\left(\hat{X}_s\right)Y_s dW_s \; . \end{aligned} \tag{5.1.2}$$

Assume that $\hat{\nu}$ has $\mathbb{P}-a.s.$ right-continuous paths. Then, $\hat{Y}^{\nu,\tau} = \frac{\partial}{\partial\varepsilon}X^{\nu^{\tau,\varepsilon}}{}_{|\varepsilon=0}$ on $[0,T]$ $\mathbb{P}-a.s.$ Moreover,

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}J(\nu^{\varepsilon,\tau})_{|\varepsilon=0} &= \mathbb{E}\left[\partial_x g(\hat{X}_T)\hat{Y}_T^{\nu,\tau} + \int_\tau^T \partial_x f_s(\hat{X}_s,\hat{\nu}_s)\hat{Y}_s^{\nu,\tau}ds\right] \\ &+ \mathbb{E}\left[f_\tau(\hat{X}_\tau,\nu) - f_\tau(\hat{X}_\tau,\hat{\nu}_\tau)\right] \; . \end{aligned} \tag{5.1.3}$$

The idea of the stochastic maximum principle is to introduce a set of dual variables in order to exploit (5.1.3). Let us first define the Hamiltonian:

$$\mathcal{H}_t(x, u, p, q) := b_t(x, u)p + \sigma_t(x)q + f_t(x, u).$$

Then, we assume that there exists a couple $(\hat{P}, \hat{Q})$ of square integrable adapted processes satisfying the BSDE

$$\hat{P}_t \quad = \quad \partial_x g(\hat{X}_T) + \int_t^T \partial_x \mathcal{H}_s(\hat{X}_s, \hat{\nu}_s, \hat{P}_s, \hat{Q}_s) ds - \int_t^T \hat{Q}_s dW_s \; . \tag{5.1.4}$$

This equation is called the adjoint equation and $(\hat{P}, \hat{Q})$ the adjoint process.

The reason for introducing this process becomes clear once we apply Itô's Lemma to $\hat{P}\hat{Y}^{\tau,\nu}$. Indeed, assuming that the local martingale part of $\hat{P}\hat{Y}^{\tau,\nu}$ is a true martingale, we obtain that $\partial_x g(\hat{X}_T)\hat{Y}_T^{\tau,\nu} = \hat{P}_T \hat{Y}_T^{\tau,\nu}$ is equal in expectation to

$$\begin{aligned}
&\hat{P}_\tau(b_\tau(\hat{X}_\tau, \nu) - b_\tau(\hat{X}_\tau, \hat{\nu}_\tau)) - \int_\tau^T \hat{Y}_s^{\tau,\nu} \partial_x \mathcal{H}_s(\hat{X}_s, \hat{\nu}_s, \hat{P}_s, \hat{Q}_s) ds \\
&+ \int_\tau^T \partial_x b_s \left(\hat{X}_s, \hat{\nu}_s\right) \hat{Y}_s^{\tau,\nu} \hat{P}_s ds + \int_\tau^t \partial_x \sigma_s \left(\hat{X}_s\right) \hat{Y}_s^{\tau,\nu} \hat{Q}_s ds,
\end{aligned}$$

which, by definition of $\mathcal{H}$, is equal to

$$\hat{P}_\tau(b_\tau(\hat{X}_\tau, \nu_\tau) - b_\tau(\hat{X}_\tau, \hat{\nu})) - \int_\tau^T Y_s^{\tau,\nu} \partial_x f_s \left(\hat{X}_s, \hat{\nu}_s\right) ds \; .$$

It follows that

$$\partial_\varepsilon J(\nu^{\varepsilon,\tau})_{|\varepsilon=0} \quad = \quad \mathbb{E}\left[\mathcal{H}_\tau(\hat{X}_\tau, \nu_\tau, \hat{P}_\tau, \hat{Q}_\tau) - \mathcal{H}_\tau(\hat{X}_\tau, \hat{\nu}, \hat{P}_\tau, \hat{Q}_\tau)\right] \; .$$

By arbitrariness of ν, this implies the *necessary condition*

$$\mathcal{H}_\tau(\hat{X}_\tau, \hat{\nu}_\tau, \hat{P}_\tau, \hat{Q}_\tau) = \max_{u \in \mathbb{R}} \mathcal{H}_\tau(\hat{X}_\tau, u, \hat{P}_\tau, \hat{Q}_\tau) \quad \mathbb{P} - \text{a.s.} \tag{5.1.5}$$

for all $\tau \in \mathcal{T}$.
A similar analysis can be carried out when σ does depend on the control ν but it requires a second order expansion in the definition of Y above. See Peng [19, 20].

2 Sufficient condition

We work within the same framework as above, except that we now allow σ to depend on the control process ν.

We assume here that the maps

$$x \mapsto g(x) \;\text{ and }\; x \mapsto \hat{\mathcal{H}}_t(x, \hat{P}_t, \hat{Q}_t) := \sup_{u \in \mathbb{R}} \mathcal{H}_t(x, u, \hat{P}_t, \hat{Q}_t) \quad \text{are } \mathbb{P} - \text{a.s. concave} \tag{5.2.1}$$

for almost every $t \in [0, T]$, and that

$$\partial_x \mathcal{H}_\tau(\hat{X}_\tau, \hat{\nu}_\tau, \hat{P}_\tau, \hat{Q}_\tau) = \partial_x \hat{\mathcal{H}}_\tau(\hat{X}_\tau, \hat{P}_\tau, \hat{Q}_\tau) \tag{5.2.2}$$

for all stopping times τ. Note that the latter corresponds to the enveloppe principle along the path of $(\hat{X}, \hat{P}, \hat{Q})$.

Under the above assumptions, the condition

$$\mathcal{H}_\tau(\hat{X}_\tau, \hat{\nu}_\tau, \hat{P}_\tau, \hat{Q}_\tau) = \max_{u \in \mathbb{R}} \mathcal{H}_\tau(\hat{X}_\tau, u, \hat{P}_\tau, \hat{Q}_\tau) \;\; \forall\, \tau \in [0, T] \tag{5.2.3}$$

is actually a sufficient condition for optimality.

Indeed, we first note that, by concavity of g,

$$\mathbb{E}\left[g(\hat{X}_T) - g(X_T^\nu)\right] \;\;\geq\;\; \mathbb{E}\left[\partial_x g(\hat{X}_T)(\hat{X}_T - X_T^\nu)\right] = \mathbb{E}\left[\hat{P}_T(\hat{X}_T - X_T^\nu)\right],$$

which, by Itô's Lemma and (5.2.2), implies

$$\begin{aligned}
\mathbb{E}\left[g(\hat{X}_T) - g(X_T^\nu)\right] \;\;&\geq\;\; \mathbb{E}\left[\int_0^T \hat{P}_s(b_s(\hat{X}_s, \hat{\nu}_s) - b_s(X_s^\nu, \nu_s))ds\right] \\
&-\;\; \mathbb{E}\left[\int_0^T \partial_x \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s)(\hat{X}_s - X_s^\nu)ds\right] \\
&+\;\; \mathbb{E}\left[\int_0^T \left(\sigma_s(\hat{X}_s) - \sigma_s(X_s^\nu)\right)\hat{Q}_s ds\right].
\end{aligned}$$

By definition of $\mathcal{H}$, $\hat{\mathcal{H}}$ and (5.2.1)-(5.2.3), this leads to

$$\begin{aligned}
J(\hat{\nu}) - J(\nu) \;\;&\geq\;\; \mathbb{E}\left[\int_0^T (\mathcal{H}_s(\hat{X}_s, \hat{\nu}_s, \hat{P}_s, \hat{Q}_s) - \mathcal{H}_s(X_s^\nu, \nu_s, \hat{P}_s, \hat{Q}_s))ds\right] \\
&-\;\; \mathbb{E}\left[\int_0^T \partial_x \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s)(\hat{X}_s - X_s^\nu)ds\right] \\
&\geq\;\; \mathbb{E}\left[\int_0^T \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s) - \hat{\mathcal{H}}_s(X_s^\nu, \hat{P}_s, \hat{Q}_s)ds\right] \\
&-\;\; \mathbb{E}\left[\int_0^T \partial_x \hat{\mathcal{H}}_s(\hat{X}_s, \hat{P}_s, \hat{Q}_s)(\hat{X}_s - X_s^\nu)ds\right] \\
&\geq\;\; 0\,.
\end{aligned}$$

Remark 5.2.28 *Let us now assume that μ, σ and f are non-random and assume that there exists a smooth solution φ to the Hamilton-Jacobi-Bellman equation:*

$$0 = \sup_{u\in\mathbb{R}} \left(\frac{\partial}{\partial t}\varphi(t,x) + b_t(x,u)\partial_x\varphi(t,x) + \frac{1}{2}(\sigma_t(x,u))^2\partial^2_{xx}\varphi(t,x) + f_t(x,u) \right)$$

with terminal condition $\varphi(T,\cdot) = g$. Assume that the sup is attained by some $\hat{u}(t,x)$. Set $p := \partial_x\varphi$ and $q := \partial^2_{xx}\varphi\sigma$. It follows from the envelope theorem, that (p,q) formally solves (take the derivative with respect to x in the above equation)

$$0 = \mathcal{L}^{\hat{u}(t,x)}p(t,x) + \partial_x\hat{\mathcal{H}}_t(x,p(t,x),q(t,x,\hat{u}(t,x)))$$

with the terminal condition $p(T,\cdot) = \partial_x g$. Let now $\hat{X}$ be the controlled process associated to the Markov control $\hat{\nu} = \hat{u}(\cdot,\hat{X}_\cdot)$ (assuming that it is well defined). Then, Itô's Lemma implies that

$$\begin{aligned} p(t,\hat{X}_t) &= \partial_x g(\hat{X}_T) + \int_t^T \partial_x\mathcal{H}_s(\hat{X}_s,\hat{\nu}_s,p(s,\hat{X}_s),q(s,\hat{X}_s,\hat{\nu}_s))ds \\ &- \int_t^T q(s,\hat{X}_s,\hat{\nu}_s)dW_s\ . \end{aligned}$$

Under mild assumptions ensuring that there is only one solution to the above BSDE, this shows that

$$\hat{P}_t = p(t,\hat{X}_t) = \partial_x\varphi(t,\hat{X}_t) \quad \textit{and} \quad \hat{Q}_t = q(t,\hat{X}_t,\hat{\nu}_t) = \partial^2_{xx}\varphi(t,\hat{X}_t)\sigma_t(\hat{X}_t,\hat{\nu}_t)\ .$$

Otherwise stated, the adjoint process $\hat{P}$ can be seen as the derivative of the value function with respect to the initial condition in space, while $\hat{Q}$ is intimately related to the second derivative.

3 Examples

3.1 Logarithmic utility

Let us first consider the problem

$$\max \mathbb{E}\left[\ln(X^\nu_T)\right]$$

where X^ν is defined as

$$X^\nu_t = x_0 + \int_0^t X^\nu_s\nu_s\frac{dS_s}{S_s} = x_0 + \int_0^t X^\nu_s\nu_s\mu_s ds + \int_0^t X^\nu_s\nu_s\sigma_s dW_s \tag{5.3.1}$$

for some $x_0 > 0$ and where

$$S_t = S_0 e^{\int_0^t (\mu_s - \sigma_s^2/2)ds + \int_0^t \sigma_s dW_s}$$

for some bounded predictable processes μ and $\sigma > 0$ with $1/\sigma$ bounded as well.

This corresponds to the problem of maximizing the expected logarithmic utility of the discounted terminal wealth in a one dimensional Black-Scholes type model with random coefficients. Here, ν stands for the proportion of the wealth X^ν which is invested in the risky asset S.

It is equivalent to maximizing $\mathbb{E}\left[X_T^\nu\right]$ with X^ν now defined as

$$X_t^\nu = \int_0^t (\nu_s \mu_s - \nu_s^2 \sigma_s^2/2) ds \ .$$

The associated Hamiltonian is

$$\mathcal{H}_t(x, u, p, q) = (u\mu_t - (u^2\sigma_t^2/2))p \ .$$

Thus $\hat{\mathcal{H}}_t(x, p, q) = \frac{1}{2}\frac{\mu_t^2}{\sigma_t^2}p$ and the argmax is $\hat{u}(t, x, p, q) := \frac{\mu_t}{\sigma_t^2}$. It follows that the dynamics of the adjoint process $(\hat{P}, \hat{Q})$ is given by

$$\hat{P}_t = 1 - \int_t^T \hat{Q}_s dW_s \ .$$

This implies that $\hat{P} = 1$ and $\hat{Q} = 0$ $dt \times d\mathbb{P}$ a.e. In particular, for $\hat{X} := X^{\hat{\nu}}$ with $\hat{\nu} := \mu/\sigma^2$ the optimality conditions of the previous section are satisfied. This implies that $\hat{\nu}$ is an optimal strategy. Since the optimization problem is clearly strictly concave in ν, this is the only optimal strategy.
Observe that the solution is trivial since it only coincides with taking the max inside the expectation and the integral in $\mathbb{E}\left[X_T^\nu\right] = \mathbb{E}\left[\int_0^T (\nu_s\mu_s - \nu_s^2\sigma_s^2/2)ds\right]$.

3.2 General utility

We consider a similar problem as in the previous section except that we now take a general utility function U which is assumed to be C^1, strictly concave and increasing. We also assume that it satisfies the so-called *Inada conditions*: $\partial_x U(\infty) = 0$ and $\partial_x U(0+) = \infty$.

We want to maximize $\mathbb{E}\left[U(X_T^\nu)\right]$ where X^ν is given by (5.3.1). We write $\hat{X}$ for $X^{\hat{\nu}}$.

In this case, the condition (5.2.3) reads

$$\mathcal{H}_t(\hat{X}_t, \hat{\nu}_t, \hat{P}_t, \hat{Q}_t) \quad = \quad \sup_{u \in \mathbb{R}} \left(u \ \mu_t \hat{X}_t \hat{P}_t + u \ \sigma_t \hat{X}_t \hat{Q}_t \right) \ .$$

But, it is clear that it can be satisfied only if

$$\hat{Q}_t = -\lambda_t \hat{P}_t \quad \text{with } \lambda = \mu/\sigma .$$

Thus, by (5.1.4), $\hat{P}$ should have the dynamics

$$\hat{P}_t = \partial_x U(\hat{X}_T) + \int_t^T \lambda_s \hat{P}_s dW_s .$$

This implies that we have to find a real $\hat{P}_0 > 0$ such that

$$\hat{P}_t = \hat{P}_0 e^{-\frac{1}{2}\int_0^t \lambda_s^2 ds - \int_0^t \lambda_s dW_s}$$

and $\hat{P}_T = \partial_x U(\hat{X}_T)$. Hence, the optimal control, if it exists, should satisfy

$$\hat{X}_T = (\partial_x U)^{-1}\left(\hat{P}_0 e^{-\frac{1}{2}\int_0^T \lambda_s^2 ds + \int_0^T \lambda_s dW_s}\right) . \tag{5.3.2}$$

Now, let $\mathbb{Q} \sim \mathbb{P}$ be defined by $d\mathbb{Q} = \hat{P}_T/\hat{P}_0$ so that $W^{\mathbb{Q}} = W + \int_0^{\cdot} \lambda_s ds$ is a $\mathbb{Q}$-Brownian motion, and that X^ν is a supermatingale under $\mathbb{Q}$ for all $\nu \in \mathcal{U}$. If $\hat{X}$ is actually a true $\mathbb{Q}$-martingale, then we must have

$$x_0 = \mathbb{E}^{\mathbb{Q}}\left[(\partial_x U)^{-1}\left(\hat{P}_0 e^{-\frac{1}{2}\int_0^T \lambda_s^2 ds + \int_0^T \lambda_s dW_s}\right)\right] . \tag{5.3.3}$$

Using the Inada conditions imposed above, it is clear that we can find $\hat{P}_0$ such that the above identity holds. The representation theorem then implies the existence of an admissible control $\hat{\nu}$ such that (5.3.2) is satisfied. Since the sufficient conditions of Section 2 hold, this shows that $\hat{\nu}$ is optimal.

We can also check this by using the concavity of U which implies

$$U(X_T^\nu) \le U(\hat{X}_T) + \partial_x U(\hat{X}_T)\left(X_T^\nu - X_T\right) = U(\hat{X}_T) + \hat{P}_T\left(X_T^\nu - \hat{X}_T\right) .$$

Since, by the above discussion, the last term is non positive in expectation, this shows that the optimal terminal wealth is actually given by (5.3.2).

Part A.

Appendix: A reminder on stochastic processes with jumps

We recall here some basic properties of Itô integrals and marked point processes. We refer to [5], [10] and [17] for more details.

A.1 Itô integral

Let W a d-dimensional Brownian motion and a be an adapted process with values in $\mathbb{M}^d$ such that

$$\int_0^t \|a_s\|^2 ds < \infty \quad \text{for all} \;\; t \geq 0 \,.$$

Then the stochastic integral $\int_0^t a_s dW_s$ is well defined as an Itô integral, see e.g. [10]. Moreover, the process $(\int_0^t a_s dW_s)_{t\geq 0}$ is a local martingale and

$$(\int_0^t a_s dW_s)_{t\geq 0} \quad \text{is a martingale if} \quad \mathbb{E}\left[\int_0^\infty \|a_s\|^2 ds\right] < \infty \,. \tag{A.1.1}$$

Recall that, in this case, we have

$$\mathbb{E}\left[\left\|\int_0^\infty a_s dW_s\right\|^2\right] \;=\; \mathbb{E}\left[\int_0^\infty \|a_s\|^2 ds\right] \tag{A.1.2}$$

as a consequence of the Itô isometry and the so-called *Burkholder-Davis-Gundy inequality* (or Doob's maximal inequality) reads as follows:

Proposition A.1.1 *Let a be an adapted process with values in $\mathbb{M}^d$ such that*

$$\mathbb{E}\left[\int_0^T \|a_s\|^2 ds\right] < \infty \,.$$

Then, for each $p \geq 1$, there is $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\leq T} \| \int_0^t a_s dW_s\|^{2p}\right] \;\leq\; C\,\mathbb{E}\left[\left(\int_0^T \|a_s\|^2 ds\right)^p\right] \,.$$

A.2 $\mathbb{R}^d$-marked point process

A $\mathbb{R}^d$*-market point process* is a sequence of jump times $(T_n)_{n\geq 1}$ and sizes of jumps $(Z_n)_{n\geq 1}$ with values in $\mathbb{R}^d$. Here, T_n is the time of the n-th jump while Z_n is its size. Such a process can be represented in terms of a random measure (counting measure) μ on $\mathbb{R}_+ \times \mathbb{R}^d$ defined by

$$\mu(A, B) := \sum_{n\geq 1} \mathbf{1}_{(Z_n,T_n)\in A\times B} \quad , \; \forall\, (A, B) \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}_+) \,.$$

We shall always identify $(T_n, Z_n)_n$ with μ.

Let $\tilde{\mu}_t(\omega, dz)$ be a transition measure from $\Omega \times [0, \infty)$ into $\mathbb{R}^d$ such that, for each $A \in \mathcal{B}(\mathbb{R}^d)$, $\tilde{\mu}(A)$ is non-negative, predictable, $\mathbb{P}-$ a.s. locally integrable and satisfies:

$$\mathbb{E}\left[\int_0^\infty \xi_s \mu(A, ds)\right] = \mathbb{E}\left[\int_0^\infty \xi_s \tilde{\mu}_s(A) ds\right]$$

for all non-negative predictable process ξ. Then, $\tilde{\mu}$ is called the *predictable intensity kernel* of μ, *compensator*. We note

$$\bar{\mu}(dz, ds) := \mu(dz, ds) - \tilde{\mu}_s(dz)ds$$

the so-called *compensated market point process*.

Let $\mathcal{P}$ the σ-algebra of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$. An indexed process $(\xi_t(z))_{t \geq 0, z \in \mathbb{R}^d}$ such that the map $(t, \omega, z) \mapsto \xi_t(z)(\omega)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable is called a *predictable* $\mathbb{R}^d$*-indexed process*.

Proposition A.2.2 *If μ admits the predictable intensity kernel $\tilde{\mu}$, then for each non-negative predictable $\mathbb{R}^d$-indexed process ξ:*

$$\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \xi_s(z) \mu(dz, ds)\right] \quad = \quad \mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \xi_s(z) \tilde{\mu}_s(dz) ds\right] .$$

Proof. See [5]. □

Corollary A.2.1 *Assume that μ admits the predictable intensity kernel $\tilde{\mu}$. Let ξ be predictable $\mathbb{R}^d$-indexed process such that*

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \|\xi_s(z)\| \tilde{\mu}_s(dz) ds\right] < \infty \ \ \forall\, t \geq 0 .$$

Then, the process $\left(\int_0^t \int_{\mathbb{R}^d} \xi_s(z) \bar{\mu}(dz, ds)\right)_{t \geq 0}$ *is a martingale.*

In the case of pure jump processes of the above form, the *Burkholder-Davis-Gundy inequality* reads as follows:

Proposition A.2.3 *Let the conditions of Corollary A.2.1 hold. Then, for each $p \geq 1$, there is $C > 0$ such that*

$$\mathbb{E}\left[\sup_{t \leq T} \| \int_0^t \int_{\mathbb{R}^d} \xi_s(z) \bar{\mu}(dz, ds) \|^p\right] \quad \leq \quad C\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \|\xi_s(z)\|^p \tilde{\mu}_s(dz) ds\right] .$$

A.3 Mixed diffusion processes and Itô's Lemma

Let (a, b) be an adapted process with values in $\mathbb{M}^d \times \mathbb{R}^d$ and ξ be predictable $\mathbb{R}^d$-indexed process with values in $\mathbb{R}^d$ such that

$$\int_0^t (\|a_s\|^2 + \|b_s\|)ds + \int_0^t \int_{\mathbb{R}^d} \|\xi_s(z)\|\tilde{\mu}_s(dz)ds < \infty \;\; \forall\, t \geq 0 \;.$$

Then, we can define the process X by

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t a_s dW_s + \int_0^t \int_{\mathbb{R}^d} \xi_s(z)\mu(dz, ds)$$

with $X_0 \in \mathbb{R}^d$.

We recall that, in this case, *Itô's Lemma* for processes with jumps implies that, for all $C^{1,2}([0,T] \times \mathbb{R}^d)$ function f, we have

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left(\frac{\partial}{\partial s} f(s, X_s) + \langle b_s, Df(s, X_s) \rangle + \frac{1}{2} \mathrm{Tr}\left[a_s a_s^* D^2 f(s, X_s) \right] \right) ds \\ &+ \int_0^t Df(s, X_s) a_s dW_s + \int_0^t \int_{\mathbb{R}^d} \left(f(s, X_{s-} + \xi_s(z)) - f(s, X_{s-}) \right) \mu(dz, ds) \;. \end{aligned}$$

In the case where the dynamics of X has an additional component L which is an adapted bounded variation process, i.e.

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t a_s dW_s + \int_0^t \int_{\mathbb{R}^d} \xi_s(z)\mu(dz, ds) + L_t \;,$$

then Itô's formula reads

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left(\frac{\partial}{\partial s} f(s, X_s) + \langle b_s, Df(s, X_s) \rangle + \frac{1}{2} \mathrm{Tr}\left[a_s a_s^* D^2 f(s, X_s) \right] \right) ds \\ &+ \int_0^t Df(s, X_s) a_s dW_s + \int_0^t \int_{\mathbb{R}^d} \left(f(s, X_{s-} + \xi_s(z)) - f(s, X_{s-}) \right) \mu(dz, ds) \\ &+ \int_0^t Df(s, X_s) dL_s^c + \sum_{s \leq t} f(s, X_{s-} + \Delta L_s) - f(s, X_{s-}) \end{aligned}$$

where L^c stands for the continuous part of L and ΔL_s for the jump of L at time s, if any.

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