Feuille d'exercices n°4 : Ruin theory

In all exercises that use the Cramer-Lundberg model, we denote by $c > 0$ the premimum rate, we denote by $\lambda > 0$ the intensity of the Poisson process that models the number of claims and we denote by $u \geq 0$ the initial wealth of the insurer.

Exercise 1.

1. Show that the following distribution are thin tailed :
   (a) the distribution of a nonnegative bounded random variable.
   Correction : If X is bounded by M, then $\mathbb{P}(X > x) = 0$ for all $x > M$.
   (b) the Gamma distribution.
   Correction : Let $X \sim \Gamma(\alpha, \beta)$. Recall that the density of this law is given by $f(x) = \frac{\beta}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbf{1}_{x>0}$. We observe that $\sup_{x>1} x^{\alpha-1} e^{-\beta x/2} < \infty$. Therefore, $f(x) \lesssim e^{-\beta x/2}$ uniformly over all $x > 1$, so that there exists $M > 0$ such that $\mathbb{P}(X > x) \leq M \int_x^\infty e^{-\beta y/2} dy = M2e^{-\beta x/2}/\beta$.
   (c) the Weibull distribution, with parameters $C > 0, \gamma \geq 1$. The density function of a Weibull distribution with parameters C, γ is
   $$f(x) = C\gamma x^{\gamma-1} \exp(-Cx^\gamma) \mathbf{1}_{\{x>0\}}.$$
   Correction : We observe that $\sup_{x>1} x^{\gamma-1} \exp(-Cx^\gamma/2) < \infty$, therefore $f(x) \lesssim e^{-Cx^\gamma/2}$ uniformly over all $x > 1$. Since $\gamma \geq 1$, $e^{-Cx^\gamma/2} \leq e^{-Cx/2}$. Hence, there exists $M > 0$ such that $\mathbb{P}(X > x) \leq M2e^{-Cx/2}/C$.
2. Show that the following distributions are sub-exponential :
   (a) the Pareto distribution with parameters $\alpha > 0, \beta > 0$ ($f(x) = \alpha\beta^\alpha/(\beta + x)^{\alpha+1}, x > 0$).
   Correction : We have $\mathbb{P}(X > x) = \beta^\alpha(\beta + x)^{-\alpha} \sim \beta^\alpha x^{-\alpha}$ so that X has a sub-exponential tail.
   (b) the Weibull distribution with parameters $C > 0, \gamma < 1$.
   Correction : We apply Pitman's theorem. Let $q(x) = f(x)/\mathbb{P}(X > x)$. A simple calculation yields $\mathbb{P}(X > x) = e^{-Cx^\gamma}$ so that $q(x) = C\gamma x^{\gamma-1}$. Therefore, q is non-increasing and $x \mapsto e^{xq(x)} f(x) = x^{\gamma-1} e^{-C(1-\gamma)x^\gamma}$ is integrable since $0 < \gamma < 1$. By Pitman's theorem, we deduce that X is sub-exponential.

Exercise 2. The parameters $c > 0$, $\lambda > 0$ et $\beta > 0$ are fixed throughout. For every integer $k \in \mathbb{N}^*$, we consider the Cramer-Lundberg model, where the costs of the claims are distributed according to a $\Gamma(k, \beta)$ distribution. Set $\psi^{(k)}(u)$ for the ruin probability of this model. Show that for every $u > 0$ and every $k \in \mathbb{N}^*$,

$$\psi^{(k)}(u) \leq \psi^{(k+1)}(u).$$

Correction : Here

$$S^k(t) = \sum_{i=1}^{N(t)} X_i^k,$$

where

— (X_i^k) are iid $\sim \Gamma(k, \beta)$,

— N is a renewal process, independent of X^k, $\lambda := 1/\mathbb{E}[\tau_1] > 0$.

Prime $p(t) = ct$. Risque process $U^k(t) := u + ct - S^k(t)$.

$$X_i^k \overset{law}{=} \sum_{j=1}^{k} Z_j^i,\ Z_i^j, \text{ iid } \sim \mathcal{E}(\lambda).$$

Set

$$\psi^k(u) := \mathbb{P}[\exists t \geq 0,\ U^k(t) < 0 \,|U(0) = u] \,,$$

since

$$\begin{aligned} S^{k+1}(t) &= \sum_{i=1}^{N(t)} X_i^{k+1} \\ &= \sum_{i=1}^{N(t)} \left(\sum_{j=1}^{k} Z_j^i + Z_{k+1}^i \right) \\ &= S^k(t) + \sum_{i=1}^{N(t)} Z_{k+1}^i \\ &\geq S^k(t), \end{aligned}$$

we have $U^{k+1}(t) \leq U^k(t)$, thus

$$\psi^k(u) \leq \psi^{k+1}(u)$$

Exercise 3. We condider the Cramér-Lundberg model, where the costs of the claims follow an exponential distribution with parameter $\gamma > 0$. The safety loading ρ is positive. We wish to give en explicit formula for the ruin probability $\psi(u)$.

1. Show that the exponential distribution is thin tailed and compute the correspon-ding adjustment coefficient R.

   Correction : Let $X \sim \mathcal{E}(\gamma)$. Then, $\mathbb{P}[X \geq x] = e^{-\gamma x}$ and therefore $e^{\frac{\gamma}{2}x}\mathbb{P}[X \geq x] \to 0$ as $x \to \infty$, which means that it is thin tailed. We now compute $R > 0$ such that

   $$\mathbb{E}[e^{R(X-c\delta)}] = 1$$

   with $\delta \sim \mathcal{E}(\lambda)$ independent of X. This is equivalent to $\mathbb{E}[e^{RX}]\mathbb{E}[e^{-Rc\delta}] = 1$ which leads to $(1 - R/\gamma)^{-1}(1 + Rc/\lambda)^{-1} = 1$ and $R = \gamma - \lambda/c$ which is > 0 under the net profit condition.

2. Derive a “good” upper bound for the ruin probability thanks to Lundberg inequality.

   Correction : Lundberg inequality implies that $\psi(u) \leq e^{-Ru}$.

3. Write the renewal equation satisfied by $u \mapsto e^{Ru}\psi(u)$.

   Correction : Set $f(u) := e^{Ru}\psi(u)$. The renewal equation for ψ is

$$\psi(u) = \frac{1}{1+\rho}(1 - \hat{F}(u)) + \frac{\gamma}{1+\rho}\int_0^u \psi(u-x)\bar{F}(x)dx$$

   with $\rho := c\gamma/\lambda - 1$, $\bar{F}(x) = \mathbb{P}[X > x]$ and $\hat{F}(x) := \gamma \int_0^x \bar{F}(y)dy$. Hence,

$$\begin{aligned} f(u) &= \frac{e^{Ru}}{1+\rho}(1 - \hat{F}(u)) + \frac{\gamma}{1+\rho}e^{Ru}\int_0^u e^{-R(u-x)}f(u-x)\bar{F}(x)dx \\ &= \frac{\lambda e^{(R-\gamma)u}}{c\gamma} + \frac{\lambda}{c}\int_0^u f(u-x)e^{(R-\gamma)x}dx \\ &= \frac{\lambda e^{-\frac{\lambda}{c}u}}{c\gamma} + \frac{\lambda}{c}\int_0^u f(u-x)e^{-\frac{\lambda}{c}x}dx \end{aligned}$$

4. Solve the equation and compute $\psi(u)$ as a function of γ, ρ and u.

   Correction : The solution is given by

$$f(u) = \frac{\lambda e^{-\frac{\lambda}{c}u}}{c\gamma} + \int_0^u \frac{\lambda e^{-\frac{\lambda}{c}(u-x)}}{c\gamma}\frac{\lambda}{c}dx$$

   as this corresponds to a $\mathcal{E}(\lambda/c)$ distribution. Hence,

$$\psi(u) = e^{-Ru}f(u) = \frac{e^{-Ru}}{1+\rho}(< e^{-Ru}),$$

   since $\rho > 0$.

Exercise 4. We consider the setting of the Cramer-Lundberg model, where the costs $X_i, i \geq 1$ follow a Pareto distribution with index $\alpha > 1$, $\beta = 1$, *i.e.*

$$\overline{F}_{X_1}(x) = (1+x)^{-\alpha}, \quad x \geq 0.$$

1. Compute $\mu = \mathbb{E}[X_1]$ and the associated safety loading ρ. For which values c do we have $\rho > 0$?
   Correction : We have

   $$\mu = \int_0^\infty \overline{F}_{X_1}(x)dx = \int_0^\infty (1+x)^{-\alpha}dx = \frac{1}{\alpha-1} \ .$$

   Then, $\rho = \frac{c(\alpha-1)}{\lambda} - 1$ which is strictly positive iff $c > \lambda/(\alpha-1)$.
2. Show that $\int_0^\infty e^{ux}F_{X_1,I}(\mathrm{d}x) = \infty$ for every $u > 0$. Derive that $F_{X_1,I}$ is not thin tailed.
   Correction : By definition

   $$F_{X_1,I}(x) = \frac{1}{\mu}\int_0^x \mathbb{P}(X_1 > y)dy = \frac{1}{\mu}(1-\alpha)^{-1}((1+x)^{1-\alpha} - 1) = (1-(1+x)^{1-\alpha}) \ .$$

   Now, $F_{X_1,I}(dx) = (\alpha-1)(1+x)^{-\alpha}dx$ so that, for any $u > 0$, there exists $C > 0$ such that

   $$\sup_{x\geq 0} e^{ux/2}(1+x)^{-\alpha} > C \ ,$$

   and consequently

   $$\int_0^\infty e^{ux}F_{X_1,I}(\mathrm{d}x) \geq (\alpha-1)C\int_0^\infty e^{ux/2}dx = \infty \ .$$

3. Show that $F_{X_1,I}$ is subexponential. What can we say about the ruin probability $\psi(u)$ as $u \to \infty$?
   Correction : Since $\overline{F}_{X_1,I}(x) \sim x^{1-\alpha}$ as $x \to \infty$, we deduce that $F_{X_1,I}$ is subexponential. As a consequence,

   $$\psi(u) \sim \frac{\mu}{\frac{c}{\lambda} - \mu}\overline{F}_{X_1,I}(u) \ ,$$

   by Theorem 3.19 in Gantert's lecture notes.

Exercise 5. We work in the Cramer-Lundberg setting.
Partie A. The r.v. $X_i, i \geq 1$ that model the cost claims have a density

$$f(x) = \frac{1}{2\sqrt{x}}e^{-\sqrt{x}}\mathbf{1}_{\{x>0\}}.$$

1. Compute $\mu = \mathbb{E}[X_1]$ and $\overline{F}_{X_1}(x), x \geq 0$.

Correction :

$$\begin{aligned}\mu &= \int_0^{+\infty} \frac{\sqrt{x}}{2} e^{-\sqrt{x}} dx \\ &= \int_0^{+\infty} y^2 e^{-y} dy \\ &= \Gamma(3) = 2.\end{aligned}$$

$$\begin{aligned}\overline{F}_{X_1}(x) &= \int_x^{+\infty} \frac{1}{2\sqrt{y}} e^{-\sqrt{y}} dy \\ &= \int_{\sqrt{x}}^{+\infty} e^{-z} dz \\ &= e^{-\sqrt{x}}.\end{aligned}$$

2. For every $x \geq 0$, set $F_{X_1,I}(x) = \mu^{-1} \int_0^x \overline{F}_{X_1}(y)\mathrm{d}y$ and

$$q(x) = \frac{\overline{F}_{X_1}(x)/\mu}{\overline{F}_{X_1,I}(x)}.$$

(a) Show that

$$\int_x^{\infty} e^{-\sqrt{y}} \mathrm{d}y = 2e^{-\sqrt{x}}(\sqrt{x}+1), \quad \forall x \geq 0,$$

and derive a simple expression for $q(x)$.

Correction :

$$\begin{aligned}\int_x^{\infty} e^{-\sqrt{y}} \mathrm{d}y &= \int_{\sqrt{x}}^{+\infty} 2z e^{-z} dz \\ &\overset{IBP}{=} 2(\sqrt{x}+1)e^{-\sqrt{x}}.\end{aligned}$$

Thus :

$$\begin{aligned}q(x) &= \frac{e^{-\sqrt{x}}/2}{1 - \frac{1}{2}\int_0^x \mathbb{P}(X_1 > y)dy} \\ &= \frac{e^{-\sqrt{x}}/2}{1 - \frac{1}{2}\int_0^x e^{-\sqrt{y}}dy} \\ &= \frac{e^{-\sqrt{x}}/2}{1 - \frac{1}{2}\left(\int_0^{+\infty} e^{-\sqrt{y}}dy - \int_x^{+\infty} e^{-\sqrt{y}}dy\right)} \\ &= \frac{e^{-\sqrt{x}}/2}{1 - \frac{1}{2}\left(2 - 2(\sqrt{x}+1)e^{-\sqrt{x}}\right)} \\ &= \frac{1}{2(\sqrt{x}+1)}.\end{aligned}$$

(b) Derive that $F_{X_1,I}$ is the cumulative distribution function of a subexponential distribution.

Correction : Let $\tilde{f}(x) := \overline{F}_{X_1}(x)/\mu$. Notice that

$$\int_0^{+\infty} \overline{F}_{X_1}(x)/\mu dx = 1,$$

and by denoting Y a random variable with density $\tilde{f}$, we have

$$q(x) = \frac{\tilde{f}(x)}{\mathbb{P}(Y > x)}.$$

We apply Pitman Theorem since

— q is descreasing,

— we have

$$e^{xq(x)}\tilde{f}(x) = \frac{1}{2}e^{\frac{x}{2(\sqrt{x}+1)}-\sqrt{x}},$$

which is integrable on $\mathbb{R}^+$.

3. Give an equivalent of the ruin probability $\psi(u)$ as $u \to \infty$. Express this equivalent as a function of f and the parameters c, λ.

Correction Since $F_{X_1,I}(x) = \mu^{-1}\int_0^x \overline{F}_{X_1}(y)\mathrm{d}y$ is the cumulative distribution of a sub-exponential distribution, we have (cf Gantert Theorem 3.19)

$$\psi(u) \sim \frac{\mu}{\frac{c}{\lambda} - \mu}\overline{F}_{X_1,I}(u),$$

Thus

$$\psi(u) \sim \frac{2\lambda}{c - 2\lambda}\overline{F}_{X_1,I}(u).$$

Partie B. We now assume that the $X_i, i \geq 1$ have density

$$g(x) = 2xe^{-x^2}\mathbf{1}_{\{x>0\}}.$$

1. Show that $\mu = \sqrt{\pi}/2$.

Correction

$$\begin{aligned}\mu &= \int_0^{+\infty} 2x^2 e^{-x^2} dx \\ &\overset{y=\sqrt{2}x}{=} \int_0^{+\infty} \frac{1}{\sqrt{2}} y^2 e^{-\frac{y^2}{2}} dy \\ &= \sqrt{\pi}\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} y^2 e^{-\frac{y^2}{2}} dy \\ &= \sqrt{\pi}/2.\end{aligned}$$

2. Show that X_1 is thin tailed.

Correction : Cf Ex. 1 : Weibull distribution with $\gamma = 2$.

3. Prove the existence of the adjustment coefficient R.

Correction : We aim at solving

$$\phi(a) := \mathbb{E}[e^{a(X_1 - c\tau_1)}] = \mathbb{E}[e^{aX_1}]\mathbb{E}[e^{-ac\tau_1}] = \frac{\lambda}{\lambda + ac}\mathbb{E}[e^{aX_1}] = 1.$$

Notice that

$$\begin{aligned}
\mathbb{E}[e^{aX_1}] &= \int_0^{+\infty} 2xe^{-x^2}e^{ax}dx \\
&= [-e^{-x^2}e^{ax}]_0^{+\infty} + \int_0^{+\infty} ae^{ax}e^{-x^2}dx \\
&= 1 + \int_0^{+\infty} ae^{-(x-a/2)^2}e^{a^2/4}dx \\
&= 1 + \int_{-a/2}^{+\infty} ae^{-y^2}e^{a^2/4}dy \\
&= 1 + ae^{a^2/4}\int_{-a/2}^{+\infty} e^{-y^2}dy
\end{aligned}$$

Then,

$$\phi(a) = \frac{\lambda}{\lambda + ac}\mathbb{E}[e^{aX_1}] \longrightarrow +\infty,\ a \to +\infty$$

and

$$\phi'(0) = \mu - \frac{c}{\lambda} < 0,$$

since the net profit condition is satisfied. Thus, there exists a positive solution to $\phi(a) = 1$.

4. Express the integral $\int_0^{\infty} ye^{Ry-y^2}\mathrm{d}y$ as a function of c, λ and R. Derive an expression for $\int_0^{\infty} e^{Ry-y^2}\mathrm{d}y$ as a function of c, λ and R.

Correction :

$$\begin{aligned}
\int_0^{\infty} ye^{Ry-y^2}\mathrm{d}y &= \frac{1}{2}\mathbb{E}[e^{RX_1}] \\
&= \frac{1}{2}\mathbb{E}[e^{-Rc\tau_1}]^{-1} \\
&= \frac{\lambda + Rc}{2\lambda}.
\end{aligned}$$

Notice that

$$\int_0^{+\infty} (R - 2y)e^{Ry-y^2}dy = -1,$$

by setting $I := \int_0^{\infty} e^{Ry-y^2}\mathrm{d}y$, we get

$$RI - \frac{\lambda + Rc}{\lambda} = -1$$

thus, $I = \frac{c}{\lambda}$.

5. Compute $\mathrm{d}F_{X_1,I}$ and give the renewal equation satisfied by the function $u \mapsto e^{Ru}\psi(u)$, and check that the required conditions are satisfied here.

Correction :

$$\begin{aligned} dF_{X_1,I}(x) &:= \frac{1}{\mu}\overline{F}_{X_1}(x)dx \\ &= \frac{2}{\sqrt{\pi}}\int_x^{+\infty} 2ye^{-y^2}dydx \\ &= \frac{2}{\sqrt{\pi}}e^{-x^2}dx. \end{aligned}$$

Assumptions : Cramer-Lundberg, X_1 has a density, net profit condition, thin tailed and the adjustment coefficient exists. Thus :

$$e^{Ru}\psi(u) = \frac{e^{Ru}(1 - F_{X_1,I}(u))}{1+\rho} + \int_0^u \psi(u-y)e^{R(u-y)}dF_R(y),$$

with

$$\rho := \frac{c}{\lambda\mu} - 1, \ F_R(x) = \frac{1}{1+\rho}\int_0^x e^{Ry}dF_{X_1,I}(y).$$

6. Give the asymptotic behaviour of the ruin probability $\psi(u)$ as $u \to \infty$ as a function of c, λ, R and π.

Correction :

$$e^{Ru}\psi(u) \longrightarrow \lambda_R \int_0^{+\infty} \frac{e^{Ry}(1 - F_{X_1,I}(y))}{1+\rho}dy =: K,$$

with

$$\begin{aligned} \lambda_R^{-1} &:= \int_0^{+\infty} xdF_R(x) \\ &= \frac{2}{\sqrt{\pi}}\frac{\lambda\mu}{c}\int_0^{+\infty} xe^{Rx}e^{-x^2}dx \\ &= \frac{\lambda + Rc}{2c}. \end{aligned}$$

Thus

$$\begin{aligned} K &= \frac{2c}{\lambda + Rc}\frac{\lambda\mu}{c}\int_0^{+\infty} e^{Ry}\int_y^{+\infty}\frac{2}{\sqrt{\pi}}e^{-z^2}dzdy \\ &= \frac{2c}{\lambda + Rc}\frac{\lambda\mu}{c}\int_0^{+\infty}\frac{2}{\sqrt{\pi}}e^{-z^2}\int_0^z e^{Ry}dydz \\ &= \frac{2c}{\lambda + Rc}\frac{\lambda}{Rc}\int_0^{+\infty} e^{-z^2}(e^{Rz} - 1)dz \\ &= \frac{2c}{\lambda + Rc}\frac{\lambda}{Rc}\left(I - \frac{\sqrt{\pi}}{2}\right) \\ &= \frac{2c\lambda}{Rc(\lambda + Rc)}\left(\frac{c}{\lambda} - \frac{\sqrt{\pi}}{2}\right). \end{aligned}$$

Exercise 6.

1. **Part 1**

   An insurer has a risky portfolio with risks which are partitioned into two classes : the big claims, denoted by $X_i^1, i \geq 1$ and the small claims, denoted by $X_i^2, i \geq 1$. It is moreover assumed that the two kind of risks are independent. The total claim amount of the insurer at time t is denoted by

   $$S_t = S_t^1 + S_t^2$$

   where $S_t^1 = \sum_{i=1}^{N_t^1} X_i^1$ is the total claim amount of the first kind (big claims) and $S_t^2 = \sum_{i=1}^{N_t^2} X_i^2$ is the total claim amount of the second kind (small claims).
   The processes $(N^i)_{i=1,2}$ are independent Poisson processes with intensities λ^i, and they are independent of the different costs $X_i^1, X_i^2, i \geq 1$. We assume that $(X_i^1, i \geq 1)$ are i.i.d. with distribution F^1 and that the $(X_i^2, i \geq 1)$ are i.i.d. with distribution F^2 .

   (a) Compute the value of the moment generating function $M_{S_t^1}$, of S_t^1, the moment generating function of S_t^2 and derive the moment generating function of S_t.
   Correction :

   $$M_{S_t^1}(u) = \mathbb{E}[M_{X_1^1}(u)^{N_t}] = e^{\lambda^1 t(M_{X_1^1}(u)-1)}$$

   and

   $$M_{S_t}(u) = e^{\lambda^1 t(M_{X_1^1}(u)-1)} e^{\lambda^2 t(M_{X_1^2}(u)-1)}$$

   (b) Check that S is a compound Poisson process that will be written in the form

   $$S_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

   where N is a Poisson process with intensity $\lambda = \lambda^1 + \lambda^2$ and $Y_i, i \geq 1$ are i.i.d. with distribution F being a mixture of F^1 and F^2. Compute the mixture coefficients explicitly.
   Correction : The sum of the two Poisson processes is a Poisson process of parameter $\lambda^1 + \lambda^2$ (compute the Laplace transform). The moment generating function of S_t corresponds to the one of compound Poisson process associated to N and an iid sequence $(Y_i)_{i \geq 1}$ with

   $$M_{Y_1}(u) = \frac{\lambda^1}{\lambda^1 + \lambda^2} M_{X_1^1}(u) + = \frac{\lambda^2}{\lambda^1 + \lambda^2} M_{X_1^2}(u),$$

   i.e. law of X_1^1 with probability $p := \lambda^1/(\lambda^1 + \lambda^2)$ and law of X_1^2 with probability $q := \lambda^2/(\lambda^1 + \lambda^2)$.

   (c) We now assume that $F^1 = \mathcal{E}(\gamma)$ is the exponential distribution with parameter $\gamma > 0$ and $F^2 = \mathcal{P}ar(\alpha, 1)$ is the Pareto distribution with parameters α, 1, with $\alpha > 1$. Compute in that case the function $\hat{F}_{Y_1}(y) := \mathbb{E}[Y_1]^{-1} \int_0^y (1 - F_{Y_1}(t))dt$, the expectation $\mathbb{E}[Y_1]$ and the coefficient $q(y) = \frac{\hat{f}_{Y_1}(y)}{1-\hat{F}_{Y_1}(y)}$ with $\hat{f}_{Y_1}(y) = \partial_y \hat{F}_{Y_1}(y)$.

Correction : Recall that $\mathcal{P}ar(\alpha, x_\circ)$ has cumulated distribution function $(1-(x/x_\circ)^{-\alpha})\mathbf{1}_{x\geq x_\circ}$. Then, (for $y>0$)

$$\begin{aligned}
\hat{f}_{Y_1}(y) &= p\gamma e^{-\gamma y} + q\alpha y^{-\alpha-1} \\
F_{Y_1}(y) &= 1 - pe^{-\gamma y} - qy^{-\alpha} \\
\mathbb{E}[Y_1] &= \frac{p}{\gamma} + \frac{q\alpha}{\alpha-1} \\
\hat{F}_{Y_1}(y) &= \mathbb{E}[Y_1]^{-1}\left(\frac{p}{\gamma} - \frac{p}{\gamma}e^{-\gamma y} - \frac{q}{\alpha-1}y^{-\alpha+1}\right) \\
q(y) &= \frac{pe^{-\gamma y} + qy^{-\alpha}}{1 - \mathbb{E}[Y_1]^{-1}(\frac{p}{\gamma} - \frac{p}{\gamma}e^{-\gamma y} - \frac{q}{\alpha-1}y^{-\alpha+1})}
\end{aligned}$$

(d) Consider the Cramer-Lundberg model

$$U_t = u + ct - S_t, \quad t \geq 0$$

where $u \geq 0$ is the initial wealth of the company. We assume that the safety loading coefficient ρ is the same for each class and we take as premimu rate

$$c := (1+\kappa)\mathbb{E}[Y_1]; \text{ with } \kappa > 0.$$

Under the assumption of Question (c), compute c as a function of the model parameters and compute an asymptotic equivalent $\psi(r)$.

Correction :

$$c = (1+\rho)\left(\frac{p}{\gamma} + \frac{q\alpha}{\alpha-1}\right).$$

Since $\hat{F}_{Y_1}$ is sub-exponential ($y^{\alpha-1}(1-\hat{F}_{Y_1}(y)) \to \frac{q}{\alpha-1}$), we have that

$$\frac{\psi(r)}{\hat{F}_{Y_1}(r)} \sim \frac{\rho}{1-\rho}$$

for r large, with $\rho := c/(\lambda\mathbb{E}[Y_1]) - 1$.

2. **Part 2.**

The insurer decides to mix the two groups adding an insurance excess $a>0$. This means that the insurer only pays for claims with a cost greater than a threshold $a>0$, and for a claim with cost $Z>a$, the insurer only covers the amount $(Z-a)$ We consider the Cramer -Lundberg model

$$U_t = u + ct - S_t \quad \text{where} \quad S_t = \sum_{i=1}^{N_t} Y_i^a \quad \text{and} \quad Y_i^a = (Z_i - a)^+$$

N being a Poisson process with intensity λ.

(a) Compute $\mu = \mathbb{E}[Y_1^a] = \mathbb{E}[(Z_1-a)^+]$. when the claims have a cost Z following a $\mathcal{E}(\gamma)$ distribution.

Correction : $\mu = e^{-\gamma a}$.

(b) Compute $M_{Y_1^a}$, the moment generating function of Y_1^a, and derive the moment generating function of S_t.

Correction : For $u < \lambda$,

$$M_{Y_1^a}(u) = \int_0^a \gamma e^{-\gamma t} dt + \int_a^\infty e^{u(z-a)} \gamma e^{-\gamma z} dt = (1 - e^{-\gamma a}) + \frac{\gamma e^{-\gamma a}}{\gamma - u}.$$
$$M_{S_t}(u) = e^{\lambda t (M_{Y_1^a}(u) - 1)}.$$

(c) Show that $M_{S_t}(u) = M_{S'_t}(u)$ where

$$S'_t = \sum_{i=1}^{N'_t} Z_i$$

N'_t being a Poisson process with intensity $\lambda \exp(-\gamma a)$ independent of the Z_i's.

Correction :

$$M_{S'_t}(u) = e^{\lambda e^{-\gamma a} t (M_{Z_1}(u) - 1)} = e^{\lambda e^{-\gamma a} t (\frac{\gamma}{\gamma - u} - 1)} = M_{S_t}(u).$$

(d) Derive that the processes S and S' have the same distribution.

Correction : Same moment generating function.

(e) Derive that the risk process U has the same distribution as U' defined as

$$U'_t = u + ct - S'_t, \quad t \geq 0.$$

Show that $\psi(u) = \mathbb{P}[\inf_{t \geq 0} U_t < 0] = \mathbb{P}[\inf_{t \geq 0} U'_t < 0]$ and compute an asymptotic equivalent for $\psi(u)$.

Correction : They have the same laws (any n-uplets in time) since the same marginals (by the above) and independent and stationary increments. We are thus in the case of small risks. We can use exercise 3 to obtain an explicit formulation of $\psi(u)$. One can otherwise use that (see lectures)

$$e^{Lu} \psi(u) \to \rho \mathbb{E}[Z_1] / (L \int_0^\infty z e^{Lz} \bar{F}_{Z_1}(z) dz)$$

with L the adjustment coefficient, $\bar{F}_{Z_1}(z) = e^{-\gamma z}$ and $\rho := c / (\mathbb{E}[Z_1] \lambda e^{-\gamma a}) - 1$. We have

$$\mathbb{E}[Z_1] = \lambda^{-1} e^{\gamma a}$$
$$\int_0^\infty z e^{Lz} \bar{F}_{Z_1}(z) dz = \int_0^\infty z e^{(L-\gamma) z} dz = \frac{1}{\gamma - L}$$