Feuille d'exercices n°1 : Poisson processes

Exercise 1. Let $(\tau_n, n \geq 1)$ be a sequence of IID (independent and identically distributed) non-negative random variables. Set

$$T_n = \tau_1 + ... + \tau_n, \quad n \geq 1,$$

$(T_0 = 0)$ and

$$N_t = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0.$$

1. Give a necessary and sufficient condition for having

$$\mathbb{P}(N \text{ only makes jumps of size } 1) > 0.$$

2. Show that under this condition

$$\mathbb{P}(N \text{ only makes jumps of size } 1) = 1.$$

3. Is it possible that $(T_n, n \geq 0)$ converges to a finite limit with positive [1] probability ?
4. Compute the probability of the event $\{\exists t \geq 0 : N(t) = \infty\}$.

Exercise 2. Let N be a Poisson process with intensity $\lambda > 0$. Prove and give an interpretation of the following properties

1. $\mathbb{P}(N_h = 1) = \lambda h + o(h)$ $(h \to 0)$
2. $\mathbb{P}(N_h \geq 2) = o(h)$ $(h \to 0)$
3. $\mathbb{P}(N_h = 0) = 1 - \lambda h + o(h)$ $(h \to 0)$.
4. $\forall t \geq 0$, $\mathbb{P}(N \text{ jumps at time } t) = 0$.
5. Compute $\mathrm{Cov}(N_s, N_t)$, $\forall s, t \geq 0$.

Exercise 3. Let N be a counting process with stationary and independent increments. Assume that there exists $\lambda > 0$ such that

$$\mathbb{P}(N_h = 1) = \lambda h + o(h), \;\; \mathbb{P}(N_h \geq 2) = o(h).$$

For $u \in \mathbb{R}$, let $g_t(u) = \mathbb{E}[e^{iuN_t}]$.

1. Prove that $g_{t+h}(u) = g_t(u)g_h(u)$ for every $t, h \geq 0$.

1. En anglais le mot *positive* signifie strictement positif. Pour dire positif au sens large on dit *non-negative*. De même les termes *negative, bigger, smaller* sont à prendre au sens strict.

2. Prove that
$$\tfrac{d}{dt}g_t(u) = \lambda(e^{iu} - 1)g_t, \;\; g_0(u) = 1.$$
3. Conclude.

Exercise 4. Let N be a Poisson process with intensity $\lambda > 0$, modelling the arrival times of the claims for an insurance company. Let T_1 denote the arrival time of the first claim. Show that the conditional law of T_1 given $N_t = 1$ is uniformly distributed over $[0, t]$.

Exercise 5. Let $(T_n, n \geq 0)$ $(T_0 = 0)$ be a renewal process and N its associated counting process. Assume that N has independent and stationary increments.

1. Show that
$$\mathbb{P}(T_1 > s + t) = \mathbb{P}(T_1 > t)\mathbb{P}(T_1 > s), \quad \forall s, t \geq 0.$$
2. Derive that N is a Poisson process.

Exercise 6.

1. Show that two independent Poisson processes cannot jump simultaneously a.s.
2. Let N^1 and N^2 be two independent Poisson processes with parameters $\lambda_1 > 0$ and λ_2 respectively. Show that the process
$$N_t = N_t^1 + N_t^2, \;\; t \geq 0$$
is a Poisson process and give its intensity.
3. Derive that the sum of n independent Poisson processes with respective intensities $\lambda_1 > 0, ..., \lambda_n > 0$ is a Poisson process and give its intensity.

Exercise 7. Insects fall into a soup bowl according to a Poisson process N with intensity $\lambda > 0$ (the event $\{N_t = n\}$ means that there are n insects in the bowl at time t). Assume that every insect is green with probability $p \in (0, 1)$ and that its colour is independent of the colour of the other insects. Show that the number of green insects that fall into the bowl, as a function of time, is a Poisson process with intensity λp.

Exercise 8. Liver transplants arrive at an operating block following a Poisson process N with intensity $\lambda > 0$. Two patients wait for a transplant. The first patient has lifetime T (before the transplant) according to an exponential distribution with parameter μ_1. The second one has lifetime T'(before the transplant) according to an exponential distribution wity parameter μ_2. The rule is that the first transplant arrival to the hospital is given to the first patient if still alive, and to the second patient otherwise. Assume that T, T' and N are independent.

1. Compute the probability that the second patient is transplanted.
2. Compute the probability that the first patient is transplanted.
3. Let X denote the number of transplants arrived at the hospital during $[0, T]$. Compute the law of X.

Exercise 9. The Bus paradox. Buses arrive at a given bus stop according to a Poisson process with intensity $\lambda > 0$. You arrive at the bus stop at time t.

1. Give a first guess for the value of the average waiting time before the following bus arrives ?
2. Let $A_t = T_{N_t+1} - t$ be the waiting time before the next bus, and let $B_t = t - T_{N_t}$ denote the elapsed time since the last bus arrival. Compute the joint distribution of (A_t, B_t) (hint : compute first $\mathbb{P}(A_t \geq x_1, B_t \geq x_2)$ for $x_1, x_2 \geq 0$).
3. Derive that the random variables A_t and B_t are independent. What are their distributions ?
4. In particular, compute $\mathbb{E}[A_t]$. Compare with your initial first guess.

Exercise 10. Law of large numbers and central limit theorem.

1. Recall and prove a law of large numbers for a Poisson process with intensity $\lambda > 0$.
2. Prove that N satisfies the following central limit theorem

$$\frac{N_t - \lambda t}{\sqrt{\lambda t}} \xrightarrow{\text{law}} \mathcal{N}(0,1) \text{ as } t \to \infty,$$

   (a) by using characteristic functions
   (b) by showing first that $(N_n - \lambda n)/\sqrt{\lambda n}$ converges in distribution as $n \to \infty$ and then $\max_{t\in[n,n+1)} (N_t - N_n)/\sqrt{n} \to 0$ in probability.

Exercise 11.

1. Give an expression for the density function of the conditional distribution of

$$(T_1, ..., T_n) \text{ given } N_t = n$$

   when N is a Poisson process with intensity λ and $0 < T_1 < ... < T_n < ...$ are its jump times.
2. Derive an expression for the density of T_i given $N_t = n$, $\forall 1 \leq i \leq n$ and similarly for (T_i, T_j) given $N_t = n$, $\forall 1 \leq i < j \leq n$.
3. Set $U_{i,j} = T_j - T_i$, $1 \leq i < j \leq n$. Give an expression for the density of $U_{i,j}$ given $N_t = n$. Derive an expression for the density of $T_n - T_{n-1}$ given $N_t = n$.

Exercise 12. Martingales. Let $X = (X_t, t \geq 0)$ be a continuous time process and $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ a *filtration*, i.e. a nested family of sigma-fields $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{A}$ $\forall s \leq t$, where $\mathcal{A}$ is the sigma-field on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ over which X is defined. The process X is a *martingale* with respect to the filtration $\mathcal{F}$ if X_t is $\mathcal{F}_t$-measurable and integrable $\forall t$ and

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad \forall 0 \leq s \leq t.$$

Let $N = (N_t, t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. Show that the three processes

1. $(N_t - \lambda t, t \geq 0)$;
2. $\left((N_t - \lambda t)^2 - \lambda t\right), t \geq 0)$;
3. $(\exp(uN_t + \lambda t(1 - e^u)), t \geq 0)$ (for a given real number u) ;

are martingales with respect to the filtration generated by N, *i.e.* $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$.

Exercise 13. Let N be a Poisson process with intensity $\lambda > 0$ and let $0 < T_1 < ... < T_n < ...$ denote its jump times.

1. Show that T_n/n converges almost surely as $n \to \infty$ and identify its limit.
2. Show that $\sum_{i \geq 1} T_i^{-2}$ converges almost surely. Let X denote its limit.
3. Show that $X_{N_t} = \sum_{i=1}^{N(t)} T_i^{-2} \to X$ a.s. as $t \to \infty$.
4. Let $(U_i, i \geq 1)$ denote a sequence of independent uniform random variables on $[0, 1]$. We admit the following result

   $$n^{-2} \sum_{i=1}^{n} U_i^{-2} \underset{n \longrightarrow \infty}{\overset{\text{law}}{\to}} Z,$$

   where Z is a positive random variable, whose Laplace transform is given by $\mathbb{E}[\exp(-sZ)] = \exp(-c\sqrt{s}), \forall s \geq 0$, for some $c > 0$. **The goal is to show that X and $c'Z$ have same law for some c' that we will explicitly compute**.

   We asssume moreover that $(U_i, i \geq 1)$ is independent of N.

   (a) Show that for every $n \geq 1$ and every $t > 0$, the law of X_{N_t} given $N_t = n$ is the same as the law of $t^{-2} \sum_{i=1}^{n} U_i^{-2}$.

   (b) Derive that $X_{N(t)}$ has same distribution as $t^{-2} \sum_{i=1}^{N(t)} U_i^{-2}$.

   (c) Prove that

   $$N(t)^{-2} \sum_{i=1}^{N(t)} U_i^{-2} \xrightarrow{\text{law}} Z \quad \text{as } t \to \infty.$$

   (d) Recall the law of large numbers for Poisson processes and conclude.
5. Derive $\mathbb{E}[X] = \infty$.

Feuille d'exercices n°2 : Mixed Poisson process, total claim amount, renewal theory.

Exercise 1. Let $\tilde{N}$ be a mixed Poisson process and denote by

$$0 < \tilde{T}_1 < ... < \tilde{T}_n < ...$$

its jumps times. Prove that the conditional distribution of $(\tilde{T}_1, ..., \tilde{T}_n)$ given $\tilde{N}(t) = n$ ($n \in \mathbb{N} \setminus \{0\}$) coincides with the distribution of the order statistic of n independent random variables with common uniform distribution on $[0, t]$.

Exercise 2. A random variable X follows a negative binomial distribution on $\{0, 1, 2, ...\}$ with parameters $r > 0$ and $p \in (0, 1)$ if

$$\mathbb{P}(X = k) = \frac{\Gamma(r+k)}{\Gamma(r)k!} p^r (1-p)^k, \quad \forall k \geq 0.$$

Let $\tilde{N}$ be a mixed Poisson process with mixture distribution $\Theta \sim \Gamma(\gamma, \beta)$. What is the distribution of $\tilde{N}(t)$? The process $\tilde{N}$ is called a *negative binomial process*. The negative binomial law is also called mixed Poisson or mixed Gamma-Poisson distribution.

Exercise 3. Let $N = (N_t, t \geq 0)$ be a standard Poisson process with intensity $\lambda > 0$. Let $f : [0, \infty) \to [0, \infty)$ be a locally bounded Borel function. Set

$$N(f)_t = \sum_{i \geq 1} f(T_i) \mathbf{1}_{\{T_i \leq t\}} \text{ for } t \geq 0,$$

where the $(T_i)_{i \geq 1}$ are the jump times of N.

1. Show that for all $t \geq 0$, we have $N(f)_t < \infty$ almost-surely.
2. If $f(s) = \mathbf{1}_{(a,b]}(s)$ where $[a, b] \subset [0, t]$, what is the distribution of $N(\mathbf{1}_{(a,b]})_t$?
3. Show that for $u \geq 0$, we have

$$\mathbb{E}\left[e^{-uN(f)_t} \middle| N_t = n\right] = \frac{1}{t^n} \left(\int_0^t e^{-uf(s)} ds \right)^n.$$

4. Derive $\mathbb{E}\left[e^{-uN(f)_t}\right]$ ad find back the result of Question 2.
5. Compute $\mathbb{E}\left[N(f)_t\right]$ and $\mathrm{Var}[N(f)_t]$.
6. Prove that $N(f)_t - \lambda \int_0^t f(s) ds$ is a martingale.

Exercise 4. The total claim amount of a portfolio for a year is modelled by

$$X = \sum_{j=1}^{N} C_j$$

where N is the number of claims in the year and C_j is the cost of the j-th claim. Assume that N follows a mixed Poisson distribution with random parameter Λ, *i.e.* the conditional distribution of N given $\Lambda = \lambda$ is Poisson(λ). Assume moreover that Λ is distributed according to a $\Gamma(b, b)$ distribution, for some $b > 0$. Assume that the cost of the claims $(C_j)_{j\geq 1}$ are independent and identically distributed random variables, independent of N.

1. Compute $\mathbb{E}(\Lambda)$ and $\mathrm{Var}(\Lambda)$.
2. Compute $\mathbb{E}(N)$ and $\mathrm{Var}(N)$.
3. We assume that $C_1 \sim$ Exponential (α) for some $\alpha > 0$. Show that the conditional law of X given N is a Gamma distribution and identify its parameters. What is the pure premium ?
4. Show that the conditional law of Λ given (X, N) is independent of X and that it is a Gamma distribution. Identify its parameters.

Exercise 5. Let $(\xi_i, i \geq 1)$ be a sequence of i.i.d. real-valued random variables, with second-order moments, independent of the Poisson process $N = (N_t, t \geq 0)$ with intensity $\lambda > 0$. For $t \geq 0$, we set [1]

$$X_t = \sum_{i=1}^{N_t} \xi_i.$$

1. Show that $t^{-1}X_t$ converges almost surely as $t \to \infty$ and identify its limit.

We set [2], for $t \geq 0$

$$M_t = \sqrt{N_t}\Big(\frac{X_t}{N_t} - \mu\Big).$$

3. Compute $\mathbb{E}\left[e^{iuM_t}\right]$. and derive that M_t converges in distribution as $t \to \infty$. Identify its limit.
5. Show that

$$\sqrt{t}\big(\frac{X_t}{t} - \mu\frac{N_t}{t}\big)$$

converges in distribution as $t \to \infty$ and identify its limit.

Exercise 6. Let $(T_n)_{n\geq 1}$ be a renewal process and let N_t denote its counting function. We assume that the common law of the interarrival times admits a density function f on $\mathbb{R}$ (with value 0 on $(-\infty, 0]$). For $n \geq 1$, we denote by f_n the density function of T_n and F_n its cumulative distribution function.

1. Putting $X_t = 0$ on $\{N_t = 0\}$.
2. Putting $M_t = 0$ on $\{N_t = 0\}$.

1. Check that $f_1(t) = f(t)$ and show that

$$f_{n+1}(t) = \int_{\mathbb{R}} f_n(t-s) f(s) ds.$$

Define $r(t) = \mathbb{E}[N_t]$ if $t \geq 0$ and $r(t) = 0$ for $t < 0$.

2. Show that $r(t) = \sum_{n \geq 1} F_n(t)$ and that for $n \geq 2$, we have $\mathbb{P}(T_n \leq t \,|\, T_1) = F_{n-1}(t - T_1)$.
3. Show that r satisfies the renewal equation

$$r(t) = F(t) + \int_0^t r(t-s) f(s) ds, \quad t \geq 0$$

wher F denotes the cumulative distribution fcuntion of the common interarrival times.

Feuille d'exercices n°3 : Compound Poisson processes, renewal processes.

Exercise 1. Give a necessary and sufficient condition for a compound Poisson process to be a standard Poisson process.

Exercise 2. Find the renewal function and the renewal measure of a renewal process when the interarrival times are distributed according to a Gamma distribution with parameters 2, $\beta > 0$. Find back the renewal theorem and the key renewal theorem in that case.

Exercise 3. An appliance has repeated failures in time and is repaired after each failure. Denote by $(X_i, i \geq 1)$ the successive durations when the appliance is functioning, and $(Y_i, i \geq 1)$ the successive durations when the appliance is being repaired. In other words, the appliance is in order in the time interval $[0, X_1)$, it is being repaired in the time interval $[X_1, X_1 + Y_1)$, then it is in order again in the time interval $[X_1 + Y_1, X_1 + Y_1 + X_2)$ and so on. Assume that the $X_i's$ are i.i.d. and that their common law has no atom and satisfies $\mathbb{P}(X_1 > 0) = 1$ and $\mathbb{E}[X_1] < \infty$. Assume that the $Y_i's$ have the same properties and that they are moreover independent of the $X_i's$.

Denote by $p(t)$ the probability that the appliance is functioning at time t.

1. Let $Z_1 = X_1 + Y_1$ and let F be the cumulative distribution function of Z_1. Show that $p(t)$ satisfies the renewal equation
$$p(t) = \mathbb{P}(X_1 > t) + \int_0^t p(t-s)\mathrm{d}F(s), \quad \forall t \geq 0.$$
2. Derive
$$\lim_{t\to\infty} p(t) = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1 + Y_1]}.$$

Exercise 4. Let N be a renewal process and let $F(t) = T_{N(t)+1} - t$ denote the time elapsed between t and the time of the $N(t) + 1$-th renewal, for $t \geq 0$. We denote by $\mathbb{P}_{T_1}$ the duration of the interarrival times and F_{T_1} its cumulative distribution function.

1. Show that for every $x \geq 0$, the function $\mathbb{P}(F(t) > x)$ satifies the renewal equation
$$\mathbb{P}(F(t) > x) = 1 - F_{T_1}(t + x) + \int_0^t \mathbb{P}(F(t-u) > x)\mathrm{d}\mathbb{P}_{T_1}(u), \quad t \geq 0.$$
(Hint : start by decomposing $\mathbb{P}(F(t) > x)$ in two terms, according to the event $\{T_1 > t\}$ or $\{T_1 \leq t\}$.)

2. Solve this equation when the interarrival times follow an exponential distribution.

Exercise 5. In the same setting as Exercise 4, we consider a renewal process with interarrival times distributed according to a Pareto distribution

$$\mathbb{P}(\tau_1 > x) = \frac{1}{(1+x)^\alpha}, \quad x \geq 0.$$

1. Let X be a nonnegative random variable. Show that for all $r > 0$,

$$\int_0^\infty r x^{r-1} \mathbb{P}(X > x) dx = \mathbb{E}[X^r].$$

2. Use the previous identity and the renewal equation satisfied by $F(t)$ to show that

$$\mathbb{E}[F(t)^2] = \int_0^t \Big(\int_0^\infty \frac{2x}{(1+t-u+x)^\alpha} dx \Big) dm(u),$$

where $m(t) = \mathbb{E}[N(t)]$ is the renewal measure of N.

3. Derive that for $\alpha > 3$, we have

$$\mathbb{E}[F(t)^2] \longrightarrow 2 \int_0^\infty x(1+x)^{1-\alpha} dx \quad \text{as} \quad t \to \infty$$

and compute this limit explicitly.

Feuille d'exercices n°4 : Ruin theory

In all exercises that use the Cramer-Lundberg model, we denote by $c > 0$ the premimum rate, we denote by $\lambda > 0$ the intensity of the Poisson process that models the number of claims and we denote by $u \geq 0$ the initial wealth of the insurer.

Exercise 1.

1. Show that the following distribution are thin tailed :
   (a) the distribution of a nonnegative bounded random variable.
   (b) the Gamma distribution.
   (c) the Weibull distribution, with parameters $C > 0, \gamma \geq 1$. The density function of a Weibull distribution with parameters C, γ is
   $$f(x) = C\gamma x^{\gamma-1} \exp(-Cx^\gamma)\mathbf{1}_{\{x>0\}}.$$
2. Show that the following distributions are sub-exponential :
   (a) the Pareto distribution with parameters $\alpha > 0, \beta > 0$ ($f(x) = \alpha\beta^\alpha/(\beta + x)^{\alpha+1}, x > 0$).
   (b) the Weibull distribution with parameters $C > 0, \gamma < 1$.

Exercise 2. The parameters $c > 0$, $\lambda > 0$ et $\beta > 0$ are fixed throughout. For every integer $k \in \mathbb{N}^*$, we consider the Cramer-Lundberg model, where the costs of the claims are distributed according to a $\Gamma(k, \beta)$ distribution. Set $\psi^{(k)}(u)$ for the ruin probability of this model. Show that for every $u > 0$ and every $k \in \mathbb{N}^*$,

$$\psi^{(k)}(u) \leq \psi^{(k+1)}(u).$$

Exercise 3. We condider the Cramér-Lundberg model, where the costs of the claims follow an exponential distribution with parameter $\gamma > 0$. The safety loading ρ is positive. We wish to give en explicit formula for the ruin probability $\psi(u)$.

1. Show that the exponential distribution is thin tailed and compute the corresponding adjustment coefficient R.
2. Derive a "good" upper bound for the ruin probability thanks to Lundberg inequality.
3. Write the renewal equation satisfied by $u \mapsto e^{Ru}\psi(u)$.
4. Using the renewal theorem, solve the equation and compute $\psi(u)$ as a function of γ, ρ and u.

Exercise 4. We consider the setting of the Cramer-Lundberg model, where the costs $X_i, i \geq 1$ follow a Pareto distribution with index $\alpha > 1$, $\beta = 1$, *i.e.*

$$\overline{F}_{X_1}(x) = (1+x)^{-\alpha}, \quad x \geq 0.$$

1. Compute $\mu = \mathbb{E}[X_1]$ and the associated safety loading ρ. For which values c do we have $\rho > 0$?
2. Show that $\int_0^\infty e^{ux} F_{X_1,I}(\mathrm{d}x) = \infty$ for every $u > 0$. Derive that $F_{X_1,I}$ is not thin tailed.
3. Show that $F_{X_1,I}$ is subexponential. What can we say about the ruin probability $\psi(u)$ as $u \to \infty$?

Exercise 5. We work in the Cramer-Lundberg setting.

Partie A. The r.v. $X_i, i \geq 1$ that model the cost claims have a density

$$f(x) = \frac{1}{2\sqrt{x}} e^{-\sqrt{x}} \mathbf{1}_{\{x>0\}}.$$

1. Compute $\mu = \mathbb{E}[X_1]$ and $\overline{F}_{X_1}(x), x \geq 0$.
2. For every $x \geq 0$, set $F_{X_1,I}(x) = \mu^{-1} \int_0^x \overline{F}_{X_1}(y)\mathrm{d}y$ and

$$q(x) = \frac{\overline{F}_{X_1}(x)/\mu}{\overline{F}_{X_1,I}(x)}.$$

   (a) Show that

$$\int_x^\infty e^{-\sqrt{y}} \mathrm{d}y = 2e^{-\sqrt{x}}(\sqrt{x}+1), \quad \forall x \geq 0,$$

   and derive a simple expression for $q(x)$.

   (b) Derive that $F_{X_1,I}$ is the cumulative distribution function of a subexponential distribution.
3. Give an equivalent of the ruin probability $\psi(u)$ as $u \to \infty$. Express this equivalent as a function of f and the parameters c, λ.

Partie B. We now assume that the $X_i, i \geq 1$ have density

$$g(x) = 2xe^{-x^2} \mathbf{1}_{\{x>0\}}.$$

1. Show that $\mu = \sqrt{\pi}/2$.
2. Show that X_1 is thin tailed.
3. Prove the existence of the adjustment coefficient R.
4. Express the integral $\int_0^\infty y e^{Ry-y^2} \mathrm{d}y$ as a function of c, λ and R. Derive an expression for $\int_0^\infty e^{Ry-y^2} \mathrm{d}y$ as a function of c, λ and R.
5. Compute $\mathrm{d}F_{X_1,I}$ and give the renewal equation satisfied by the function $u \mapsto e^{Ru}\psi(u)$, and check that the required conditions are satisfied here.

6. Give the asymptotic behaviour of the ruin probability $\psi(u)$ as $u \to \infty$ as a function of c, λ, R and π.

Exercise 6.

1. **Part 1**

   An insurer has a risky portfolio with risks which are partitioned into two classes : the big claims, denoted by $X_i^1, i \geq 1$ and the small claims, denoted by $X_i^2, i \geq 1$. It is moreover assumed that the two kind of risks are independent. The total claim amount of the insurer at time t is denoted by

   $$S_t = S_t^1 + S_t^2$$

   where $S_t^1 = \sum_{i=1}^{N_t^1} X_i^1$ is the total claim amount of the first kind (big claims) and $S_t^2 = \sum_{i=1}^{N_t^2} X_i^2$ is the total claim amount of the second kind (small claims).
   The processes $(N^i)_{i=1,2}$ are independent Poisson processes with intensities λ^i, and they are independent of the different costs $X_i^1, X_i^2, i \geq 1$. We assume that $(X_i^1, i \geq 1)$ are i.i.d. with distribution F^1 and that the $(X_i^2, i \geq 1)$ are i.i.d. with distribution F^2 .

   (a) Compute the value of the moment generating function $M_{S_t^1}$, of S_t^1, the moment generating function of S_t^2 and derive the moment generating function of S_t.

   (b) Check that S is a compound Poisson process that will be written in the form

   $$S_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

   where N is a Poisson process with intensity $\lambda = \lambda^1 + \lambda^2$ and $Y_i, i \geq 1$ are i.i.d. with distribution F being a mixture of of F^1 and F^2. Compute the mixture coefficients explicitly.

   (c) We now assume that $F^1 = \mathcal{E}(\gamma)$ is the exponential distribution with parameter $\gamma > 0$ and $F^2 = \mathcal{P}ar(\alpha, 1)$ is the Pareto distribution with parameters α, 1, with $\alpha > 1$. Compute in that case the density function $f_{Y_1,I}(y)$, the function $\bar{F}_{Y_1,I}(y)$, the expectation $\mathbb{E}[Y_1]$ and the coefficient $q(y) = \frac{f_{Y_1,I}(y)}{\bar{F}_{Y_1,I}(y)}$.

   (d) Consider the Cramer-Lundberg model

   $$U_t = u + ct - S_t, \quad t \geq 0$$

   where $u \geq 0$ is the initial wealth of the company. We assume that the safety loading coefficient ρ is the same for each class and we take as premimu rate

   $$c := (1 + \rho)\mathbb{E}[Y_1]; \text{ with } \rho > 0.$$

   Under the assumption of Question (c), compute c as a function of the model parameters and compute an asymptotic equivalent $\psi(u)$.

2. **Part 2.**

The insurer decides to mix the two groups adding an insurance excess $a > 0$. This means that the insurer only pays for claims with a cost greater than a threshold $a > 0$, and for a claim with cost $Z > a$, the insurer only covers the amount $(Z - a)$ We consider the Cramer -Lundberg model

$$U_t = u + ct - S_t \quad \text{where} \quad S_t = \sum_{i=1}^{N_t} Y_i^a \quad \text{and} \quad Y_i^a = (Z_i - a)^+$$

N being a Poisson process with intensity λ.

(a) Compute $\mu = \mathbb{E}[Y_1^a] = \mathbb{E}[(Z_1 - a)^+]$. when the claims have a cost Z following a $\mathcal{E}(\gamma)$ distribution.

(b) Compute $M_{Y_1^a}$, the moment generating function of Y_1^a, and derive the moment generating function of S_t.

(c) Show that $M_{S_t}(u) = M_{S'_t}(u)$ where

$$S'_t = \sum_{i=1}^{N'_t} Z_i$$

N'_t being a Poisson process with intensity $\lambda \exp(-\gamma a)$ indepenent of the Z_i's.

(d) Derive that the processes S and S' have the same distribution.

(e) Derive that the risk process U has the same distribution as U' defined as

$$U'_t = u + ct - S'_t, \quad t \geq 0.$$

Show that $\psi(u) = \mathbb{P}[\inf_{t \geq 0} U_t < 0] = \mathbb{P}[\inf_{t \geq 0} U'_t < 0]$ and compute an asymptotic equivalent for $\psi(u)$.