Almost sure hedging under permanent price impact

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Based on joint works with
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Motivation

Option pricing with liquidity impact in the literature (part of)

- Super-heding/hedging :
  - Sircar and G. Papanicolaou 1998, Frey 1996, Schönbucher and Wilmot 2000, Liu and Yong 2005 : equilibrium, impact - formal arguments.
  - Cetin, Jarrow and Protter 2004 : illiquidity, no impact, pricing à la B&S.
  - Cetin, Soner and Touzi 2009 : restrictions on strategies.
  - Bank and Dolinsky 2019.
  - Loeper 2014 : impact + illiquidity, verification argument.

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□ Other pricing rules (not replication nor super-replication) : Abergel and Loeper 2013, Almgren and Li 2013, Millot and Abergel 2011, Guéant and Pu 2013, Bank, Soner and Voss 2017, ...

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- What we do :
  - Define a continuous time trading dynamics from a discrete time trading rule.
  - Provide a direct argument for the characterization of the hedging policy.

Chapter 1
Impact rule and continuous time trading dynamics

Impact rule and liquidity cost

□ Basic rule : an order δ moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}),$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta \underbrace{\frac{1}{2}(X_{t-} + X_t)}_{\text{av. price}} = \int_0^\delta \underbrace{(X_{t-} + \iota f(X_{t-}))}_{\text{current price}} \underbrace{d\iota}_{\text{add. quantity}} .$$

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□ We just need model the curve around $\delta = 0$. This should be understood for a "small" order δ as one can split orders, in continuous time. Would obtain the same with

$$X_{t-} \longrightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

and costs

$$\int_0^\delta (X_{t-} + F(X_{t-}, \iota))d\iota$$

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if $\partial_\delta F(x,0) = f(x)$, $\partial^2_{\delta x} F(x,0) = f'(x)$, $F(x,0) = \partial^2_{\delta\delta} F(x,0) = 0$.

□ In particular, would lead to the same results if

$$X_{t-} \longrightarrow X_{t-} + F(X_{t-}, \delta)$$

with

$$F(x, \delta) = \Delta \mathrm{x}(x, \delta) := \mathrm{x}(x, \delta) - x\,,$$

and $\mathrm{x}(x, \cdot)$ defined as the solution of

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□ Interpretation in terms of large order splitting : split δ in δ/n then

$$X_{t-} + \frac{\delta}{n} f(X_{t-}) \simeq \mathrm{x}(X_{t-}, \frac{\delta}{n}) \rightsquigarrow \mathrm{x}(\mathrm{x}(X_{t-}, \frac{\delta}{n}), \frac{\delta}{n})) = \mathrm{x}(X_{t-}, \frac{2\delta}{n}) \rightsquigarrow \ldots$$

- In particular, would lead to the same results if

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and $\mathrm{x}(x, \cdot)$ defined as the solution of

$$\mathrm{x}(x, \cdot) \quad = \quad x + \int_0^{\cdot} f(\mathrm{x}(x, s)) ds.$$

- In this case, the cost would be

$$\int_0^{\delta} \mathrm{x}(X_{t-}, \iota) d\iota.$$

Trading signal and discrete trading dynamics

- A trading signal is an Itô process of the form

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s.$$

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□ Trade at times $t_i^n = iT/n$ the quantity $\delta_{t_i^n}^n = Y_{t_i^n} - Y_{t_{i-1}^n}$.

□ We assume that the stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \mu(X_s) ds + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades.

☐ The corresponding dynamics are

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n} \mathbf{1}_{\{t_i^n \le t < t_{i+1}^n\}} + Y_T \mathbf{1}_{\{t=T\}} \ , \ \delta_{t_i^n}^n = Y_{t_i^n}^n - Y_{t_{i-1}^n}^n$$

$$X^n = X_0 + \int_0^\cdot \mu(X_s^n)ds + \int_0^\cdot \sigma(X_s^n)dW_s + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n),$$

$$V^n = V_0 + \int_0^\cdot Y_{s-}^n dX_s^n + \sum_{i=1}^n \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta_{t_i^n}^n)^2 f(X_{t_i^n -}^n),$$

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$$V^n = \text{ cash part } + Y^n X^n = \text{“portfolio value”}.$$

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Warning : The portfolio is $(V^n - Y^n X^n, Y^n)$ whose liquidation will not lead to V^n in cash !

□ Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$
\begin{aligned}
Y &= Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s \\
X &= X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (\mu + a_s \sigma f')(X_s) ds \\
V &= V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds,
\end{aligned}
$$

at a speed $\sqrt{n}$.

□ More details on the limit... : We have

$$X^n = X_0 + \int_0^{\cdot} \mu(X_s^n)ds + \int_0^{\cdot} \sigma(X_s^n)dW_s + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n),$$

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in which

$$\begin{aligned}
\delta_{t_{i+1}^n}^n f(X_{t_{i+1}^n-}^n) =& (\int_{t_i^n}^{t_{i+1}^n} dY_t) f\left(X_{t_i^n}^n + \int_{t_i^n}^{t_{i+1}^n} dX_{t-}^n\right) \\
=& \int_{t_i^n}^{t_{i+1}^n} f\left(X_{t_i^n} + \int_{t_i^n}^{t} dX_r^{n,c}\right) dY_t \\
& + \int_{t_i^n}^{t_{i+1}^n} d\langle \int_{t_i^n}^{\cdot} dY_r, f\left(X_{t_i^n}^n + \int_{t_i^n}^{\cdot} dX_r^n\right)\rangle_t + \text{neglectable}
\end{aligned}$$

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in which

$$\begin{aligned}\delta^n_{t^n_{i+1}} f(X^n_{t^n_{i+1}-}) =& (\int_{t_i^n}^{t^n_{i+1}} dY_t) f\left(X^n_{t_i^n} + \int_{t_i^n}^{t^n_{i+1}} dX^n_{t-}\right)\\ =& \int_{t_i^n}^{t^n_{i+1}} f\left(X_{t_i^n} + \int_{t_i^n}^{t} dX^{n,c}_r\right) dY_t\\ &+ \int_{t_i^n}^{t^n_{i+1}} d\langle \int_{t_i^n}^{\cdot} dY_r, f\left(X^n_{t_i^n} + \int_{t_i^n}^{\cdot} dX^n_r\right)\rangle_t + \text{neglectable}\end{aligned}$$

so that

$$X = X_0 + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (\mu + a_s\sigma f')(X_s)ds.$$

□ More details on the limit... : We have

$$V^n = V_0 + \int_0^{\cdot} Y^n_{s-} dX^n_s + \sum_{i=1}^{n} \mathbf{1}_{[t^n_i, T]} \frac{1}{2} (\delta^n_{t^n_i})^2 f(X^n_{t^n_i -}),$$

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in which

$$\begin{aligned}(\delta^n_{t^n_{i+1}})^2 f(X^n_{t^n_{i+1}} -) =& (\int_{t^n_i}^{t^n_{i+1}} dY_t)^2 f\left(X^n_{t^n_i} + \int_{t^n_i}^{t^n_{i+1}} dX^{n,c}_t\right)\\ =& \int_{t^n_i}^{t^n_{i+1}} f\left(X^n_{t^n_i} + \int_{t^n_i}^{t} dX^n_r\right) d\langle Y\rangle_t + \text{neglectable}\end{aligned}$$

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so that

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \int_0^{\cdot} a^2_s f(X_s) ds.$$

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta \nu(d\delta, ds)$$

where

$$\nu(A, B) = \sum_{i \geq 1} \mathbf{1}_{(\delta_i, \tau_i) \in A \times B}$$

in which τ_i is a stopping time and δ_i is $\mathcal{F}_{\tau_i}$-measurable.

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□ Approximation : Jump δ_i at time τ_i is passed on $[\tau_i, \tau_i + \varepsilon]$ at a rate δ_i/ε.

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$$Y^{\varepsilon} = Y_{0-} + \int_0^{\cdot} (b_s + \sum_{i \geq 1} \mathbf{1}_{[\tau_i, \tau_i + \varepsilon)}(s) \frac{\delta_i}{\varepsilon}) ds + \int_0^{\cdot} a_s dW_s.$$

□ The limit dynamics when $\varepsilon \to 0$ is

$$
\begin{aligned}
X &= X_{0-} + \int_0^{\cdot} \sigma(X_s)dW_s + \int_0^{\cdot} f(X_s)dY_s^c + \int_0^{\cdot} (\mu + a_s \sigma f')(X_s)ds \\
&\quad + \int_0^{\cdot} \int \Delta \mathrm{x}(X_{s-}, \delta)\nu(d\delta, ds) \\
V &= V_{0-} + \int_0^{\cdot} Y_s dX_s^c + \frac{1}{2}\int_0^{\cdot} a_s^2 f(X_s)ds \\
&\quad + \int_0^{\cdot} \int \left(Y_{s-}\Delta \mathrm{x}(X_{s-}, \delta) + \mathfrak{I}(X_{s-}, \delta)\right) \nu(d\delta, ds)
\end{aligned}
$$

in which Y^c is the continuous part of Y, and

$$
\mathrm{x}(x, \delta) = x + \int_0^{\delta} f(\mathrm{x}(x, s))ds \quad , \quad \Delta \mathrm{x}(x, \delta) := \mathrm{x}(x, \delta) - x
$$

$$
\mathfrak{I}(x, \delta) := \int_0^{\delta} s f(\mathrm{x}(x, s))ds.
$$

Adding resilince

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_t + R$$
$$R = R_0 + \int_0^{\cdot} f(X_t) dY_t + \int_0^{\cdot} (a_t (f'\sigma)(X_t) - \rho R_t) dt$$
$$Y = y + \int_0^{\cdot} a_t dW_t + \int_0^{\cdot} b_t dt$$
$$V = V_0 + \int_0^{\cdot} Y_t dX_t + \int_0^{\cdot} \frac{1}{2} a_t^2 f_t(X_t) dt.$$

See D. Becherer and T. Bilarev. Hedging with transient price impact for non-covered and covered options. arXiv preprint arXiv :1807.05917, 2018.

Zero cost immediate round trips

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Similarly, the impact on the portfolio value is

$$y\Delta\mathrm{x}(x, \delta) + \mathfrak{I}(x, \delta)$$

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- There is no hidden cost : this is why perfect hedging will be possible ! !

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- Warning : be careful with barrier-like options !

Other possible specifications

- Multiplicative formulation

$$X = X^{\circ}\ell(Y)$$

cf D. Becherer and T. Bilarev. Hedging with transient price impact for non-covered and covered options. *arXiv :1807.05917*, 2018.

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☐ Immediate partial resilience
cf B. Bouchard, G. Loeper, M. Soner and C. Zhou. Second order stochastic target problems with generalized market impact. *SIAM Journal on Control and Optimization*, 57(6), 4125-4149, 2019.

Chapter 2 - Hedging of un-covered options

Super-hedging problem

- Fix a claim $g = (g_0, g_1)$ with
  - g_0 = cash part
  - g_1 = # of stocks to deliver.

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- $g_1 = \#$ of stocks to deliver.

☐ Super-hedging price $=$ minimal initial cash so that

$$V_T - Y_T X_T \geq g_0(X_T) \;\text{ and }\; Y_T = g_1(X_T).$$

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- g_0 = cash part
- g_1 = # of stocks to deliver.

☐ Super-hedging price = minimal initial cash so that

$$V_T - Y_T X_T \geq g_0(X_T) \text{ and } Y_T = g_1(X_T).$$

$\Rightarrow$ Match perfectly the number of stocks and be above the cash requirement.

Super-hedging price

- $w(0, X_{0-})$ is the inf over V_{0-} such that one super-hedges for some (a, b, ν), starting from $Y_{0-} = 0$.

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$$\begin{aligned} V_{0-} = \hat{w}(t, x, y) - \mathfrak{I}(\mathrm{x}(x, -y), y) &\longrightarrow \hat{w}(t, x, y) \\ X_{0-} = \mathrm{x}(x, -y) &\longrightarrow \mathrm{x}(\mathrm{x}(x, -y), y) = x \\ Y_{0-} = 0 &\longrightarrow y. \end{aligned}$$

Dynamic programming principle for stochastic targets

- Geometric Dynamic Programming Principle : Let θ be a stopping time.
  - GDP1 : if $V_{0-} > \hat{w}(0, X_{0-}, Y_{0-})$ then $V_\theta \geq \hat{w}(\theta, X_\theta, Y_\theta)$ for some (a, b, ν).
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- Geometric dynamic programming transferred from $\hat{w}$ to w by using

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$\Rightarrow$ This will kill the singularity issue !

Pricing equation

- If $v = w(t, x)$ the GDP "implies"

$$d\mathcal{E}_t := dV_t - dw(t, \mathrm{x}(X_t, -Y_t)) - d\mathfrak{I}(\mathrm{x}(X_t, -Y_t), Y_t) = 0,$$

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□ Key property :

$$\begin{aligned} d\mathcal{E} &= [Y - \check{Y}] \left[(\mu - f' f a^2/2)(X)dt + \sigma(X)dW\right] \\ &\quad + \hat{F}[w](\cdot, \mathrm{x}(X, -Y), Y)dt \end{aligned}$$

in which

$$\check{Y} := Y + \frac{\mathrm{x}(X, -Y) - X}{f(X)} + \partial_x w(\cdot, \mathrm{x}(X, -Y)) \frac{f(\mathrm{x}(X, -Y))}{f(X)}$$

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$$w(T-,\cdot) = G(\cdot) := \inf\left\{y\mathrm{x}(x,y) + g_0(\mathrm{x}(x,y)) : \; y = g_1(\mathrm{x}(x,y))\right\}.$$

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□ To be taken in the discontinuous viscosity sense for the relaxed semi-limits associated to problems with bounded controls.

Verification

- Assume that w is a smooth solution of

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  - Make an initial jump of size $Y_0 = \hat{y}(0, X_{0-}) = \mathrm{x}^{-1}(X_{0-}, X_{0-} + f(X_{0-})\partial_x w(0, X_{0-}))$.
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$\Rightarrow$ Jumps only at 0 and T !

Viscosity solution approach

□ **Proposition :** Let σ and μ be adapted, bounded, and a.s. right-continuous at 0. Assume that

$$Z_t := \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s \geq 0$$

a.s., for all $t \leq t_0$. Then, $\sigma_0 = 0$ and $\mu_0 \geq 0$.

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By sending $t \to 0$, we obtain : $\mu_0 - n|\sigma_0|^2 \geq 0$, for all $n \geq 0$. □

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Apply the above to $Z:=V-[\varphi(\cdot,\mathrm{x}(X_\cdot,-Y_\cdot))+\mathfrak{I}(\mathrm{x}(X_\cdot,-Y_\cdot),Y_\cdot)]$.

Then, $\hat{F}[\varphi](t_0,x_0,\hat{y}(t_0,x_0))\geq 0$.

□ **Proposition :** Let σ and μ be adapted, bounded. Assume that there exists a stopping time $\theta > t_0$ such that

$$\sigma \mathbf{1}_{[\![t_0,\theta]\!]} = 0 \ \text{ and } \mu \mathbf{1}_{[\![t_0,\theta]\!]} \geq 0.$$

Then

$$\int_0^\theta \mu_s ds + \int_0^\theta \sigma_s dW_s \geq 0.$$

Take φ such that $\max(w - \varphi) = (w - \varphi)(t_0, x_0) = 0$ with $(w - \varphi)(t, x) < 0$ for $(t, x) \neq (t_0, x_0)$. Assume that

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$$\begin{aligned}
V_\theta &\geq \varphi(\theta,\mathrm{x}(X_\theta,-Y_\theta))+\mathfrak{I}(\mathrm{x}(X_\theta,-Y_\theta),Y_\theta)-\varepsilon\\
&\geq w(\theta,\mathrm{x}(X_\theta,-Y_\theta))+\mathfrak{I}(\mathrm{x}(X_\theta,-Y_\theta),Y_\theta)+2\varepsilon-\varepsilon\\
&> w(\theta,\mathrm{x}(X_\theta,-Y_\theta))+\mathfrak{I}(\mathrm{x}(X_\theta,-Y_\theta),Y_\theta).
\end{aligned}$$

Proposition : Comparison holds.

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This implies uniqueness and convergence of monotone finite difference numerical schemes.

A simple example : Bachelier model

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□ In this case, $\mathrm{x}(x, \delta) = x + f\delta$, $\mathfrak{I}(x, \delta) = \frac{1}{2}\delta^2 f$, and the pde is

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☐ Interpretation :

- We have $\mathrm{x}(X_t, -Y_t) = \mathrm{x}(\mu t + \sigma W_t + Y_t f, -Y_t) = \mu t + \sigma W_t$, i.e. moves on price due to trading will cancel when the position is closed.
- Cost of trading is compensated by the impact on prices :

$$-\delta\, 0 - \frac{1}{2}\delta^2 f + \delta(0 + \mu t + \sigma W_t + \delta f) - \frac{1}{2}\delta^2 f = \delta(\mu t + \sigma W_t).$$

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☐ Call hedging :

- Cash settlement : $G(x) = g_0(x) = [x-K]^+$
- With delivery :

$$\begin{aligned} G(x) &= \min\left\{y(x+yf) - K\mathbf{1}_{\{x+yf\geq K\}} : \; y = \mathbf{1}_{\{x+yf\geq K\}}\right\} \\ &= \; (x+f-K)^+\mathbf{1}_{\{K>x\}} + (x+f-K)\mathbf{1}_{\{x\geq K\}} \end{aligned}$$

Chapter 3 - Hedging of covered options

Super-hedging problem

- Fix a claim g :
  - At 0, the trader asks for receiving an initial amount of stocks Y_0 and of cash such that cash$+Y_0X_0=$premium.
  - At T, the trader delivers Y_T stocks plus some cash such that cash$+Y_TX_T=g(X_T)$.

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We set

$$\mathrm{v}(0, X_0) := \inf\{v = c + Y_0X_0 \ : \ c, Y_0, (a, b) \text{ s.t. } V_T \geq g(X_T)\}.$$

Hedging and pricing - informal derivation

Let us assume that we use the delta-hedging rule :

$$V = \mathrm{v}(\cdot, X) \quad , \quad Y = \partial_x \mathrm{v}(\cdot, X).$$

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$$\gamma^a := \frac{a}{\sigma(X) + f(X)a} = \partial^2_{xx} \mathrm{v}(\cdot, X) \in \mathbb{R} \setminus \{1/f\}$$

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By definition of γ^a and a little bit of algebra :

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} \mathrm{v})} \partial^2_{xx} \mathrm{v} \right] (\cdot, X) = 0.$$

The pricing pde should be

$$
\begin{aligned}
-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v} = 0 \quad \text{on } [0, T) \times \mathbb{R}, \\
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Singular pde :

- Can find smooth solutions s.t. $1 > f\partial^2_{xx}\mathrm{v}$, cf. below.
- In general, needs to take care of $1 \neq f\partial^2_{xx}\mathrm{v}$
- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f\partial^2_{xx}\mathrm{v} > 1$ does not make sense.

Hedging with a gamma contraint

- By a change of variable, we write the dynamics in the form :

$$dY = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt \ \text{ and } \ dX = \sigma^a(X)dW + \mu_X^{a,b}(X)dt.$$

- We now define v with respect to the gamma constraint

$$\gamma^a(X) \leq \bar{\gamma}(X)$$

with

$$f\bar{\gamma} < 1 - \varepsilon, \ \ \varepsilon > 0.$$

Pricing pde :

$$\min\left\{-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v}\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{v}\right\} = 0 \quad \text{on } [0,T)\times\mathbb{R}.$$

Propagation of the gamma contraint at the boundary :

$$\mathrm{v}(T-,\cdot) = \hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

$$\min\left\{\varphi - g\ ,\ \bar{\gamma}-\partial^2_{xx}\varphi\right\} = 0.$$

See Soner and Touzi 00, and Cheridito, Soner and Touzi 05.

Super-solution property

Use a weak formulation approach and results on small time behavior of double stochastic integrals, see Soner and Touzi 00 and Cheridito, Soner and Touzi 05.

It is based on the Geometric DPP (Soner and Touzi) :
if

$$V_0 > \mathrm{v}(0, X_0)$$

then we can find (a, b, Y_0) such that

$$V_\theta \geq \mathrm{v}(\theta, X_\theta)$$

for any stopping time θ with values in $[0, T]$.

Sub-solution property

□ Main difficulty : can not establish the reverse Geometric DPP, i.e.

If (a, b, Y_0) are such that

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- can neither go smoothly to it as it will move X because of the impact, and therefore $\hat{Y}$ (sort of fixed point problem), compare with Cheridito, Soner, and Touzi 05.

The smoothing approach

In place, we use a smoothing/verification approach initiated by B. and Nutz 13 (inspired from Jensen's and Krylov's ideas).

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Conclusion : v is the (unique) viscosity solution.

How to construct the smooth super-solution (in a nutshell)

Consider a viscosity solution to the PDE (with F convexe non-decreasing)

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Then, smooth it out and use the fact that $-F$ is concave and non-increasing

$$\begin{aligned}0 &= \int \left(-\partial_t \mathrm{w}^\iota - F((\partial^2_{xx}\mathrm{w}^\iota)^{\mathrm{abs}})\right)(t',x')\phi_\delta(t'-t,x'-x)dt'dx', \\ &\leq -\partial_t \mathrm{w}^\iota_\delta(t,x) - F(\partial^2_{xx}\mathrm{w}^\iota_\delta)(t,x).\end{aligned}$$

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$$\frac{\sigma^2(x)}{1 - f(x)\partial^2_{xx}\varphi(t,x)}\partial^2_{xx}\varphi(t,x) = \mathbb{E}\left[\frac{\sigma^2(\tilde X_T)}{1 - f(\tilde X_T)\partial^2_{xx} g(\tilde X_T)}\partial^2_{xx} g(\tilde X_T)\right]$$

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$\Rightarrow \partial^2_{xx}\varphi \le 1/f - \varepsilon_g$ with $\varepsilon_g > 0$.

Smooth solution

□ **Proposition :** Assume that $\partial^2_{xx} g \leq 1/f - \varepsilon$ for some $\varepsilon > 0$ (+ smoothness conditions). Then, v is a smooth solution of

$$0 = -\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} \mathrm{v})} \partial^2_{xx} \mathrm{v}$$

and $\partial^2_{xx} \mathrm{v} \leq 1/f - \varepsilon_g$ for some $\varepsilon_g > 0$.

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□ Proposition :

$$\mathrm{v}^\epsilon(0,x) = \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T \left[\sigma^2 f|\partial^2_x \mathrm{v}^0|^2\right](s,\tilde{X}_s)ds\right] + o(\epsilon)$$

where, $\tilde{X}$ is the solution on $[0,T]$ of

$$\tilde{X} = x + \int_t^\cdot \sigma(\tilde{X}_s)dW_s.$$

Proof : Since

$$0 = -\partial_t \mathrm{v}^\epsilon - \frac{1}{2}\frac{\sigma^2}{(1-\epsilon f \partial_{xx}^2 \mathrm{v}^\epsilon)}\partial_{xx}^2 \mathrm{v}^\epsilon,$$

we have

$$\begin{aligned} 0 &= -\partial_t \mathrm{v}^\epsilon - \frac{1}{2}\sigma^2 \partial_{xx}^2 \mathrm{v}^\epsilon - \frac{\epsilon}{2}\sigma^2 f |\partial_{xx}^2 \mathrm{v}^\epsilon|^2 - o(\epsilon) \\ &= -\partial_t \mathrm{v}^0 - \frac{1}{2}\sigma^2 \partial_{xx}^2 \mathrm{v}^0. \end{aligned}$$

There exists a constant $C > 0$ such that

$$|V_T^\epsilon - g(X_T^\epsilon)| \leq C\epsilon^2$$

in which

$$V_0^\epsilon = \mathrm{v}^0(0, X_0) + \epsilon \Delta v(0, X_0)$$
$$Y^\varepsilon = \partial_x \mathrm{v}^0(0, X_0) + \epsilon \partial_x \Delta v(0, X_0),$$

with

$$\Delta v(0, x) := \frac{1}{2}\mathbb{E}\left[\int_0^T \left[\sigma^2 f |\partial_{xx}^2 \mathrm{v}^0|^2\right](s, \tilde{X}_s) ds\right].$$

Numerical illustration

- Constant impact and constraint.
- Bachelier model : $dX_t = 0.2\, dW_t$.
- Butterfly option : $g(x) = (x+1)^+ - 2x^+ + (x-1)^+,\ T = 2.$

Covered option.

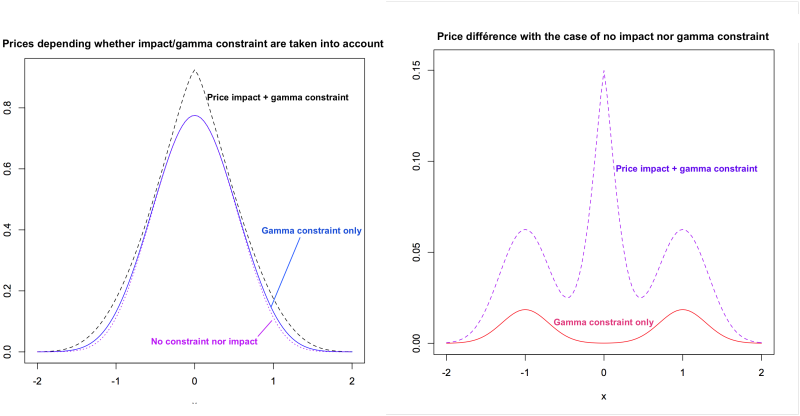

Figure – **Left : Dashed line :** $f = 0.5$, $\bar{\gamma} = 1.75$; **solid line :** $f = 0$, $\bar{\gamma} = 1.75$; **dotted line :** $f = 0$, $\bar{\gamma} = +\infty$.

Towards a duality

Observe that :

$$
\begin{aligned}
0 = & -\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{1 - f\partial^2_{xx}\mathrm{v}}\partial^2_{xx}\mathrm{v} \\
= & \inf_{\mathrm{s}\in\mathbb{R}} \left(-\partial_t \mathrm{v} - \frac{1}{2}\mathrm{s}^2\partial^2_{xx}\mathrm{v} + \frac{\gamma}{2}(\mathrm{s}-\sigma)^2 \right).
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\end{aligned}
$$

□ Then

$$
\mathrm{v}(0, x) = \bar{\mathrm{v}}(0, x) := \sup_{\mathfrak{s} \in \mathcal{A}_2} \mathbb{E} \left[g(\bar{X}^{\mathfrak{s}}_T) - \int_0^T \frac{\gamma(\bar{X}^{\mathfrak{s}}_t)}{2} (\mathfrak{s}_t - \sigma(\bar{X}^{\mathfrak{s}}_t))^2 dt \right]
$$

with

$$
\bar{X}^{\mathfrak{s}} := x + \int_0^{\cdot} \mathfrak{s}_t dW_t.
$$

⇒ Dual formulation !

Chapter 4 - Understanding the dual formulation

Relaxed formulation

□ We now consider the relaxed formulation with path dependent coefficients :

$$
\begin{aligned}
Y^{a,\mathfrak{B}} &= Y_0 + \int_0^{\cdot} a_t dW_t - \mathfrak{B} \\
X^{a,\mathfrak{B}} &= \mathrm{x}_{\wedge 0} + \int_0^{\cdot} (\sigma_t + a_t f_t)(X^{a,\mathfrak{B}}) dW_t, \\
V_T^{a,\mathfrak{B}} &= V_0 + \int_0^T Y_t^{a,\mathfrak{B}} dX_t^{a,\mathfrak{B}} + \int_0^T \frac{1}{2} f_t(X^{a,\mathfrak{B}}) a_t^2 dt = g(X^{a,\mathfrak{B}}).
\end{aligned}
$$

where

- $\mathrm{x} \in C([0,T])$,
- $\sigma, f : [0,T] \times C([0,T]) \mapsto \mathbb{R}$ are non-anticipative,
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The above corresponds to the dynamics of $X^{a,\mathfrak{B}}$ under its "martingale measure".

Assuming hedging holds...

Assume we have a hedging strategy $(\hat{a}, \hat{\mathfrak{B}})$ for a path dependent payoff g, then

$$V_0 = \mathbb{E}^{\mathbb{Q}^{\hat{a},\hat{\mathfrak{B}}}}\left[g(X^{\hat{a},\hat{\mathfrak{B}}}) - \int_0^T \frac{1}{2} f_t(X^{\hat{a},\hat{\mathfrak{B}}})\hat{a}_t^2 dt\right]$$

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&\leq \sup_{(a,\mathfrak{B})} \mathbb{E}^{\mathbb{Q}^{a,\mathfrak{B}}}\left[g(X^{a,\mathfrak{B}}) - \int_0^T \frac{1}{2} f_t(X^{a,\mathfrak{B}})a_t^2 dt\right].
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We need to retrieve

$$
\sup_{\mathfrak{s}} \mathbb{E}\left[g(\bar{X}_T^{\mathfrak{s}}) - \int_0^T \frac{1}{2}\gamma_t(\bar{X}^{\mathfrak{s}})(\mathfrak{s}_t - \sigma_t(\bar{X}^{\mathfrak{s}}))^2 dt\right]
$$

with

$$
\bar{X}^{\mathfrak{s}} := \mathrm{x}_{\wedge 0} + \int_0^{\cdot} \mathfrak{s}_t dW_t \ \text{ while } \ X^{a,\mathfrak{B}} = \mathrm{x}_{\wedge 0} + \int_0^{\cdot} (\sigma_t + a_t f_t)(X^{a,\mathfrak{B}}) dW_t^{a,\mathfrak{B}}.
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Ok, up to change of variable : $\mathfrak{s}_t = \sigma_t(X^{a,\mathfrak{B}}) + a_t f_t(X^{a,\mathfrak{B}})$.

Note that super-hedging does not permit to say anything... :

$$V_0 \geq \mathbb{E}^{\mathbb{Q}^{\hat{a},\hat{\mathfrak{B}}}} \left[g(X^{\hat{a},\hat{\mathfrak{B}}}) - \int_0^T f_t(X^{\hat{a},\hat{\mathfrak{B}}}) a_t^2 dt \right]$$

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\end{aligned}
$$

Fundamental assumption

Set

$$\bar{\mathrm{v}}(0,\mathrm{x}) := \sup_{\mathfrak{s}} \mathbb{E}\left[g(\bar{X}^{\mathfrak{s}}_T) - \int_0^T \frac{1}{2}\gamma_t(\bar{X}^{\mathfrak{s}})(\mathfrak{s}_t - \sigma_t(\bar{X}^{\mathfrak{s}}))^2 dt \right]$$

Assumption : $\bar{\mathrm{v}}(t,\mathrm{x})$ admits a solution $\hat{\mathfrak{s}}[t,\mathrm{x}]$ (need weak...) + smoothness assumptions.

Differentiability of the gain function

- For differentiability, we use the notion of Dupire's derivative.

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□ **Dupire derivative :** A function φ is said to be horizontally differentiable if, for all (t, x), its horizontal derivative

$$\partial_t \varphi(t, \mathrm{x}) \ := \ \lim_{h \searrow 0} \frac{\varphi(t+h, \mathrm{x}_{t\wedge \cdot}) - \varphi(t, \mathrm{x}_{t\wedge \cdot})}{h}$$

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is well-defined.
A function φ is said to be vertically differentiable if, for all (t, x), its vertical derivative

$$\nabla_{\mathrm{x}} \varphi(t, \mathrm{x}) \;:=\; \lim_{y \to 0, y \neq 0} \frac{\varphi(t, \mathrm{x} \oplus_t y) - \varphi(t, \mathrm{x})}{y}$$

is well-defined.

Dupire's derivative of the gain function

Result #1 : The gain function

$$J(t,\mathrm{x};\mathfrak{s}) := \mathbb{E}\left[g(\bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \int_t^T \frac{1}{2}\gamma_r(\bar{X}^{\mathfrak{s}})(\mathfrak{s}_r - \sigma_r(\bar{X}^{\mathfrak{s}}))^2 dr\right],$$

$$\bar{X}^{t,\mathrm{x},\mathfrak{s}} := \mathrm{x}_{\wedge t} + \int_t^{\cdot} \mathfrak{s}_r dW_r,$$

admits a Dupire vertical derivative

$$\nabla_{\mathrm{x}} J(t,\mathrm{x};\mathfrak{s}) := \mathbb{E}\left[\mathfrak{B}_T^{\mathfrak{s}} - \mathfrak{B}_t^{\mathfrak{s}}\right]$$

where $\mathfrak{B}^{\mathfrak{s}}$ is an adapted BV process.

Proof for constant coefficients : Recall

$$\bar{X}^{t,\mathrm{x},\mathfrak{s}} := \mathrm{x}_{\wedge t} + \int_t^{\cdot} \mathfrak{s}_r dW_r.$$

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then

$$\nabla_x J(t,\mathrm{x};\mathfrak{s}) := \mathbb{E}\left[\int_t^T \lambda_g(dr;\bar{X}^{t,\mathrm{x},\mathfrak{s}})\right]$$

where λ_g is the Fréchet derivative of g at $\bar{X}^{t,\mathrm{x},\mathfrak{s}}$:

$$g(\mathrm{x}') - g(\mathrm{x}) = \int_0^T (\mathrm{x}'_t - \mathrm{x}_t)\lambda_g(dt;\mathrm{x}) + \|\mathrm{x} - \mathrm{x}'\|\epsilon(\mathrm{x}',\mathrm{x})$$

with $\epsilon(\mathrm{x}',\mathrm{x}) \to 0$ as $\mathrm{x}' \to \mathrm{x}$

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then

$$\nabla_x J(t,\mathrm{x};\mathfrak{s}) := \mathbb{E}\left[\int_t^T \lambda_g(dr;\bar{X}^{t,\mathrm{x},\mathfrak{s}})\right] = \mathbb{E}\left[\int_t^T \lambda_g^{\circ}(dr;\bar{X}^{t,\mathrm{x},\mathfrak{s}})\right],$$

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with $\epsilon(\mathrm{x}',\mathrm{x}) \to 0$ as $\mathrm{x}' \to \mathrm{x}$, and $\lambda_g^{\circ}(\cdot;\bar{X}^{t,\mathrm{x},\mathfrak{s}})$ is its dual predictable projection.

Calculus of variations

Result #2 : By a simple calculus of variations argument,

$$\gamma(\hat{\mathfrak{s}}[t,\mathrm{x}] - \sigma)(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) = \hat{a}[t,\mathrm{x}]$$

where $(m[t,\mathrm{x}], \hat{a}[t,\mathrm{x}])$ is such that

$$m[t,\mathrm{x}] + \int_t^T \hat{a}[t,\mathrm{x}]_u dW_u = \hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_t.$$

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Recall that

$$\nabla_{\mathrm{x}} J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_t\right].$$

Proof for

$$J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[g(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) - \int_t^T \frac{1}{2}\gamma(\hat{\mathfrak{s}}[t,\mathrm{x}]_r - \sigma)^2 dr\right],$$

the first order condition implies (for all δ adapted bounded) :

$$\begin{aligned} 0 =& \mathbb{E}[\int_t^T (\int_t^r \delta_s dW_s)\lambda_g(dr; \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) \\ & - \int_t^T \delta_r \gamma_r(\hat{\mathfrak{s}}[t,\mathrm{x}]_r - \sigma_r)(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) dr] \end{aligned}$$

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&- \int_t^T \delta_r \gamma_r(\hat{\mathfrak{s}}[t,\mathrm{x}]_r - \sigma_r)(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]})dr]
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Set $\int_t^T \lambda_g^\circ(dr;\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) = m + \int_t^T \hat{a}[t,\mathrm{x}]_r dW_r$.

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Result #2 : By a simple calculus of variations argument,

$$\gamma(\hat{\mathfrak{s}}[t,\mathrm{x}] - \sigma)(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) = \hat{a}[t,\mathrm{x}]$$

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Since, $\nabla_{\mathrm{x}}J(\cdot,\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]};\hat{\mathfrak{s}}[t,\mathrm{x}]):=\mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T-\hat{\mathfrak{B}}[t,\mathrm{x}]_\cdot|\mathcal{F}_\cdot\right]$,

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Since, $\nabla_{\mathrm{x}} J(\cdot, \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}; \hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_\cdot | \mathcal{F}_\cdot\right]$,

$$\hat{Y}[t,\mathrm{x}] := m[t,\mathrm{x}] + \int_t^\cdot \hat{a}[t,\mathrm{x}]_u dW_u - (\hat{\mathfrak{B}}[t,\mathrm{x}] - \hat{\mathfrak{B}}[t,\mathrm{x}]_t)$$

satisfies

$$\begin{aligned}\hat{Y}[t,\mathrm{x}] &= \mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_\cdot | \mathcal{F}_\cdot\right] - (\hat{\mathfrak{B}}[t,\mathrm{x}] - \hat{\mathfrak{B}}[t,\mathrm{x}]_t) \\ &= \nabla_{\mathrm{x}} J(\cdot, \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}; \hat{\mathfrak{s}}[t,\mathrm{x}]).\end{aligned}$$

Concavity of the value function

Result #3 : Set

$$\Gamma(t,\mathrm{x}) = \int_0^{\mathrm{x}_t} \int_0^{y^1} \gamma_t(\mathrm{x}_{\wedge t} + \mathbf{1}_{\{t\}}(y^2 - \mathrm{x}_t))dy^2 dy^1,$$

then $y \mapsto (\bar{\mathrm{v}} - \Gamma)(t, \mathrm{x} + \mathbf{1}_{\{t\}}y)$ is concave ($\bar{\mathrm{v}} - \Gamma$ is Dupire concave).

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Cf constant coefficients + Markov :

$$\bar{\mathrm{v}}(t,\mathrm{x}) = \sup_{\mathfrak{s}} \mathbb{E}[\bar{\mathrm{v}}(t+h, \bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \int_t^{t+h} \frac{\gamma}{2}(\mathfrak{s}_r - \sigma)^2 dr]$$

implies

$$\begin{aligned}
&\bar{\mathrm{v}}(t,\mathrm{x}) - \frac{\gamma}{2}\mathrm{x}_t^2 \\
&= \sup_{\mathfrak{s}} \mathbb{E}[\bar{\mathrm{v}}(t+h, \bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \frac{\gamma}{2}(\bar{X}^{t,\mathrm{x},\mathfrak{s}}_{t+h})^2 - \int_t^{t+h} \gamma(-\mathfrak{s}_r\sigma + \frac{1}{2}|\sigma|^2)dr].
\end{aligned}$$

□ **Proof** in a simpler situation : Assume that, for all $\mathfrak{s}$, $h > 0$,

$$\varphi(t,x) \geq \mathbb{E}[\varphi(t+h, \bar{X}^{t,x,\mathfrak{s}}_{t+h})],$$

where

$$\bar{X}^{t,x,\mathfrak{s}}_{t+h} = x + \int_t^{t+h} \mathfrak{s}_s dW_s.$$

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where

$$\bar{X}^{t,x,\mathfrak{s}}_{t+h} = x + \int_t^{t+h} \mathfrak{s}_s dW_s.$$

Take $x = \lambda x^1 + (1 - \lambda)x^2$ and $\mathfrak{s}$ s.t.

$$\mathbb{P}[\bar{X}^{t,x,\mathfrak{s}}_{t+h} = x^1] = \lambda = 1 - \mathbb{P}[\bar{X}^{t,x,\mathfrak{s}}_{t+h} = x^2].$$

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$\Rightarrow \varphi$ is concave.

Differentiability of the value function

Result #4 : $\bar{\mathrm{v}}$ admits a continuous vertical Dupire derivative given by

$$\nabla_{\mathrm{x}}\bar{\mathrm{v}}(t,\mathrm{x}) = \nabla_{\mathrm{x}} J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) = \mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_t\right] (= \hat{Y}[t,\mathrm{x}]_t)$$

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And (Meyer-Tanaka + martingale property - just need $\mathrm{C}^{0,1}_{\mathrm{r}}$)

$$\begin{aligned}\bar{\mathrm{v}}(t',\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) =& \bar{\mathrm{v}}(t,\mathrm{x}) + \int_t^{t'} \nabla_{\mathrm{x}}\bar{\mathrm{v}}(r,\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]})d\bar{X}_r^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]} \\ &+ \int_t^{t'} \frac{1}{2}\gamma_r(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]})(\mathfrak{s}_r - \sigma_r(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}))^2 dr.\end{aligned}$$

More generally

Let Z be a $(\mathbb{F}, \mathbb{P})$-continuous semi-martingale such that $\mathbb{E}^{\mathbb{P}}[\|Z\|^2] < \infty$. Let ϕ be a non-anticipative map in $\mathrm{C}^{0,1}_{\mathrm{r}}$. Assume that there exists $R \in \mathrm{C}^{1,2}_{\mathrm{r}}$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that :

1. $\phi - R$ is Dupire-concave (i.e. $y \mapsto (\phi - R)(t, \mathrm{x} + \mathbf{1}_{\{t\}} y)$ is concave for all t),
2. $\phi - \ell$ is non-increasing in time ($(\phi - \ell)(t + h, \mathrm{x}_{\wedge t}) \leq (\phi - \ell)(t, \mathrm{x}_{\wedge t})$).

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Then, there exists a non-increasing predictable process A starting at 0 such that

$$\phi_{\cdot}(Z) - \int_0^{\cdot} \frac{1}{2} \nabla_{\mathrm{x}}^2 R_r(Z) d\langle Z\rangle_r = \phi_0(Z) + \int_0^{\cdot} \nabla_{\mathrm{x}} \phi_r(Z) dZ_r + A + \ell(\cdot) - \ell(0).$$

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Moreover, if Z and $\phi_\cdot(Z) - B$ are $(\mathbb{P}, \mathbb{F})$-martingales, for some predictable bounded variation process B, then

$$\phi_\cdot(Z) = \phi_0(Z_0) + \int_0^\cdot \nabla_{\mathrm{x}} \phi_t(Z) dZ_t + B\ , \quad \text{on } [0, T].$$

More generally

Let Z be a $(\mathbb{F},\mathbb{P})$-continuous semi-martingale such that $\mathbb{E}^{\mathbb{P}}[\|Z\|^2]<\infty$. Let ϕ be a non-anticipative map in $\mathrm{C}^{0,1}_{\mathrm{r}}$. Assume that there exists $R\in\mathrm{C}^{1,2}_{\mathrm{r}}$ and a continuous function $\ell:[0,T]\to\mathbb{R}$ such that :

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2. $\phi-\ell$ is non-increasing in time ($(\phi-\ell)(t+h,\mathrm{x}_{\wedge t})\le(\phi-\ell)(t,\mathrm{x}_{\wedge t})$).

Then, there exists a non-increasing predictable process A starting at 0 such that

$$\phi_{\cdot}(Z)-\int_0^{\cdot}\frac{1}{2}\nabla^2_{\mathrm{x}}R_r(Z)d\langle Z\rangle_r=\phi_0(Z)+\int_0^{\cdot}\nabla_{\mathrm{x}}\phi_r(Z)dZ_r+A+\ell(\cdot)-\ell(0).$$

Moreover, if Z and $\phi_{\cdot}(Z)-B$ are $(\mathbb{P},\mathbb{F})$-martingales, for some predictable bounded variation process B, then

$$\phi_{\cdot}(Z)=\phi_0(Z_0)+\int_0^{\cdot}\nabla_{\mathrm{x}}\phi_t(Z)dZ_t+B\ ,\quad\text{on }[0,T].$$

Compare with Cont and Fournier (2013), Saporito (2017) for the Functional Itô-Meyer-Tanaka, Russo and Vallois (1996), and Gozzi and Russo (2006) for C^1 functionals of semi-martingales.

Remark : see also B. Bouchard and X. Tan, A quasi-sure optional decomposition and super-hedging result on the Skorokhod space, arXiv :2004.11105, for the case where ϕ is not C^1 in space.

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$$\begin{aligned}
&\bar{\mathrm{v}}(t, \mathrm{x}) \\
&= \sup_{\mathfrak{s}} \mathbb{E}[\bar{\mathrm{v}}(t+h, \bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \int_t^{t+h} \frac{1}{2}\gamma_r(\bar{X}^{t,\mathrm{x},\mathfrak{s}})(\mathfrak{s}_r - \sigma_r(\bar{X}^{t,\mathrm{x},\mathfrak{s}}))^2 dr] \\
&\geq \mathbb{E}[\bar{\mathrm{v}}(t+h, \mathrm{x}_{\wedge t}) - \int_t^{t+h} \frac{1}{2}\gamma_r(\mathrm{x}_{\wedge t})|\sigma_r(\mathrm{x}_{\wedge t})|^2) dr] \;\; (\mathfrak{s} \equiv 0) \\
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□ Finally, the DPP

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\bar{\mathrm{v}}(t, \mathrm{x}) = \sup_{\mathfrak{s}} \mathbb{E}[\bar{\mathrm{v}}(t+h, \bar{X}^{t,\mathrm{x},\mathfrak{s}}) - \int_t^{t+h} \frac{1}{2}\gamma_r(\bar{X}^{t,\mathrm{x},\mathfrak{s}})(\mathfrak{s}_r - \sigma_r(\bar{X}^{t,\mathrm{x},\mathfrak{s}}))^2 dr]
$$

implies that

$$
\left(\bar{\mathrm{v}}(s, \bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]}) - \int_t^s \frac{1}{2}\gamma_r(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]})(\hat{\mathfrak{s}}[t,x]_r - \sigma_r(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]}))^2 dr\right)_{s \geq t}
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is a martingale.

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By Meyer-Tanaka formula : $\exists$ K^n non-increasing s.t.

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Hence,

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Construction of the hedging strategy

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$$\nabla_{\mathrm{x}}\bar{\mathrm{v}}(t,\mathrm{x}) = \nabla_{\mathrm{x}} J(t,\mathrm{x};\hat{\mathfrak{s}}[t,\mathrm{x}]) := \mathbb{E}\left[\hat{\mathfrak{B}}[t,\mathrm{x}]_T - \hat{\mathfrak{B}}[t,\mathrm{x}]_t\right] = \hat{Y}[t,x]_t.$$

And (Meyer-Tanaka + martingale property - just need $C^{0,1}$)

$$\begin{aligned}\bar{\mathrm{v}}(t',\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}) =& \bar{\mathrm{v}}(t,\mathrm{x}) + \int_t^{t'} \nabla_{\mathrm{x}}\bar{\mathrm{v}}(r,\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]}) d\bar{X}_r^{t,\mathrm{x},\hat{\mathfrak{s}}[t,x]} \\ &+ \int_t^{t'} \frac{1}{2}\gamma_r(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]})(\mathfrak{s}_r - \sigma_r(\bar{X}^{t,\mathrm{x},\hat{\mathfrak{s}}[t,\mathrm{x}]}))^2 dr.\end{aligned}$$

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where

$$\hat{Y}[t,\mathrm{x}] = m[t,\mathrm{x}] + \int_t^{\cdot} \hat{a}[t,\mathrm{x}]_u dW_u - (\hat{\mathfrak{B}}[t,\mathrm{x}] - \hat{\mathfrak{B}}[t,\mathrm{x}]_t).$$

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&+ \int_0^T \frac{1}{2}\gamma_r(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})(\mathfrak{s}_r - \sigma_r(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}))^2 dr, \\
\hat{Y}[\mathrm{x}] =& m[\mathrm{x}] + \int_0^\cdot \hat{a}[\mathrm{x}]_r dW_r - \hat{\mathfrak{B}}[\mathrm{x}].
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Recall that $\hat{\mathfrak{s}}[\mathrm{x}] = \sigma(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}) + \hat{a}[\mathrm{x}]f(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})$

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$$
\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}=\mathrm{x}_{\wedge 0}+\int_0^\cdot\hat{\mathfrak{s}}[\mathrm{x}]_r dW_r=\mathrm{x}_{\wedge 0}+\int_0^\cdot(\sigma_r(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})+\hat{a}[\mathrm{x}]_r f_r(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}))dW_r.
$$

Recall that $\bar{\mathrm{v}}(T,\cdot)=g$ and

$$
\begin{aligned}
g(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})=\bar{\mathrm{v}}(T,\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})=&\bar{\mathrm{v}}(0,\mathrm{x})+\int_0^T \hat{Y}[\mathrm{x}]_r d\bar{X}_r^{\mathrm{x},\hat{\mathfrak{s}}[x]}\\
&+\int_0^T \frac{1}{2}\gamma_r(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})(\mathfrak{s}_r-\sigma_r(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}))^2 dr,\\
\hat{Y}[\mathrm{x}]=&m[\mathrm{x}]+\int_0^\cdot \hat{a}[\mathrm{x}]_r dW_r-\hat{\mathfrak{B}}[\mathrm{x}].
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Recall that $\hat{\mathfrak{s}}[\mathrm{x}]=\sigma(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})+\hat{a}[\mathrm{x}]f(\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]})$ so that

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\bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}=\mathrm{x}_{\wedge 0}+\int_0^\cdot \hat{\mathfrak{s}}[\mathrm{x}]_r dW_r = X^{\mathrm{x},\hat{a}[\mathrm{x}],\hat{\mathfrak{B}}[\mathrm{x}]}\qquad .
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$\Rightarrow$ $\hat{\mathfrak{s}}[\mathrm{x}]$ **provides** $(\hat{a}[\mathrm{x}], -\hat{\mathfrak{B}}[\mathrm{x}])$ **which is the hedging strategy starting from** $V_0 = \bar{\mathrm{v}}(0,\mathrm{x})$ **and** $Y_0 = \nabla_{\mathrm{x}}\bar{\mathrm{v}}(0,\mathrm{x})$.

Absolute continuity of $\hat{\mathfrak{B}}[\mathrm{x}]$?

□ Example of the constant coefficients case :

$$\hat{\mathfrak{B}}[\mathrm{x}] = \int_0^{\cdot} \lambda_g^{\circ}(dr; \bar{X}^{\mathrm{x},\hat{\mathfrak{s}}[\mathrm{x}]}).$$

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In particular, $\hat{\mathfrak{B}}[\mathrm{x}]$ is absolutely continuous.

Sufficient conditions for existence I : strong existence

- From now, we assume for simplicity that all coefficients are bounded.

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If g and $(\mathrm{s},\mathrm{x}) \mapsto -\gamma_r(\mathrm{x})(\mathrm{s} - \sigma_r(\mathrm{x}))^2$ are concave, then existence holds.

Sufficient conditions for existence II : weak existence

- The problem is :

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For using typical results ensuring tightness, one would need a penalty of the form

$$\gamma_r(\bar{X}^{\mathrm{x},\mathfrak{s}})(\mathfrak{s}_r - \sigma_r(\bar{X}^{\mathrm{x},\mathfrak{s}}))^{2+\iota}$$

with $\iota > 0$!

□ Assume that

$$y \in \mathbb{R} \mapsto (\mathrm{v} - \bar{\Gamma}_{\varepsilon_0})(t, \mathrm{x} \oplus_t y) \quad \text{is concave for all } (t, \mathrm{x}) \in [0, T] \times D([0, T]).$$

with

$$\bar{\Gamma}_{\varepsilon_0}(t, \mathrm{x}) \;:=\; \bar{\Gamma}_0(t, \mathrm{x}) - \varepsilon_0 \mathrm{x}_t^2,$$

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□ We claim that (for $\mathfrak{s}$ a maximizing sequence - encoded into $\mathbb{P}_n$)

$$\lim_{\theta \searrow 0} \delta(\theta) = 0, \quad \text{with} \;\; \delta(\theta) := \limsup_{n\to\infty} \sup_{\sigma,\tau\in\mathcal{T},\sigma\le\tau\le\sigma+\theta} \mathbb{E}^{\mathbb{P}_n}\Big[\big|\bar{X}_\tau^{\mathfrak{s}} - \bar{X}_\sigma^{\mathfrak{s}}\big|^2\Big].$$

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If not, $\exists \ \theta_n \to 0$, and $(\sigma_n, \tau_n)_n$ s.t.

$$2c \ := \ \liminf_n \mathbb{E}^{\mathbb{P}^n}\Big[\int_{\sigma_n}^{\tau_n} |\mathfrak{s}_s|^2 ds\Big] \ > \ 0.$$

□ Set

$$\phi := \mathrm{v} - \bar{\Gamma}_{\varepsilon_0} \quad \text{and} \quad \xi_n := \mathbb{E}^{\mathbb{P}^n}_{\sigma_n}\big[\phi(\tau_n, \bar{X}^{\mathfrak{s}}) - \phi(\tau_n, (\bar{X}^{\mathfrak{s}} \oplus_{\sigma_n} (\bar{X}^{\mathfrak{s}}_{\tau_n} - \bar{X}^{\mathfrak{s}}_{\sigma_n}))_{\sigma_n \wedge \cdot})\big].$$

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Then,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^n}_{\sigma_n}\Big[\mathrm{v}(\tau_n, \bar{X}^{\mathfrak{s}}) - \frac{1}{2}\int_{\sigma_n}^{\tau_n} \gamma_s(s, \bar{X}^{\mathfrak{s}}_s)\mathfrak{s}_s^2 ds\Big] \\
=&\mathbb{E}^{\mathbb{P}^n}_{\sigma_n}\Big[\phi(\tau_n, (\bar{X}^{\mathfrak{s}} \oplus_{\sigma_n} (\bar{X}^{\mathfrak{s}}_{\tau_n} - \bar{X}^{\mathfrak{s}}_{\sigma_n}))_{\sigma_n \wedge \cdot}) - \frac{1}{2}\int_{\sigma_n}^{\tau_n} \varepsilon_0 \mathfrak{s}_s^2 ds\Big] + \bar{\Gamma}_{\varepsilon_{\mathbf{0}}}(\sigma_n, \bar{X}^{\mathfrak{s}}) + \xi_n \\
\leq&\phi(\sigma_n, \bar{X}^{\mathfrak{s}}) + C\theta_n - \frac{\varepsilon_0}{2}\mathbb{E}^{\mathbb{P}^n}_{\sigma_n}\Big[\int_{\sigma_n}^{\tau_n} \mathfrak{s}_s^2 ds\Big] + \bar{\Gamma}_{\varepsilon_{\mathbf{0}}}(\sigma_n, \bar{X}^{\mathfrak{s}}) + \xi_n \\
=&\mathrm{v}(\sigma_n, \bar{X}^{\mathfrak{s}}) + C\theta_n - \frac{\varepsilon_0}{2}\mathbb{E}^{\mathbb{P}^n}_{\sigma_n}\Big[\int_{\sigma_n}^{\tau_n} \mathfrak{s}_s^2 ds\Big] + \xi_n.
\end{aligned}$$

Hence,

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$$\leq \ \mathbb{E}^{\mathbb{P}^n}\Big[\mathrm{v}(\sigma_n, \bar{X}^{\mathfrak{s}})\Big] + C(\theta_n)^{\frac{1}{2}} - \varepsilon_0 c + \xi_n.$$

Hence,

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$$\leq \; \mathbb{E}^{\mathbb{P}^n}\Big[\mathrm{v}(\sigma_n, \bar{X}^{\mathfrak{s}})\Big] + C(\theta_n)^{\frac{1}{2}} - \varepsilon_0 c + \xi_n.$$

while the DPP implies that

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^n}\Big[\mathrm{v}(\tau_n, X) - \int_{\sigma_n}^{\tau_n} \gamma_s(s, \bar{X}^{\mathfrak{s}}_s)(\mathfrak{s}_s - \sigma_s(\bar{X}^{\mathfrak{s}}))^2 ds\Big] \; = \; \lim_{n\to\infty} \mathbb{E}^{\mathbb{P}^n}\big[\mathrm{v}(\sigma_n, X)\big].$$

Hence,

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Contradiction of

$$2c \ := \ \liminf_n \mathbb{E}^{\mathbb{P}^n}\Big[\int_{\sigma_n}^{\tau_n}|\mathfrak{s}_s|^2 ds\Big] \ > \ 0.$$

$\Rightarrow$ the optimization sequence is tight !

□ How to prove by a pure probabilistic approach that

$$y \in \mathbb{R} \mapsto (\mathrm{v} - \bar{\Gamma}_{\varepsilon_0})(t, \mathrm{x} \oplus_t y) \quad \text{is concave for all } (t, \mathrm{x}) \in [0, T] \times D([0, T]).$$

with

$$\bar{\Gamma}_{\varepsilon_0}(t, \mathrm{x}) \ := \ \bar{\Gamma}_0(t, \mathrm{x}) - \varepsilon_0 \mathrm{x}_t^2,$$

for some $\varepsilon_0 > 0$, by using just the properties of the terminal data g ?

Open question

□ **Conclusion :** In a fairly general path-dependent setting, solving the dual problem provides <u>one</u> solution to the hedging problem.

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☐ **Conclusion :** In a fairly general path-dependent setting, solving the dual problem provides <u>one</u> solution to the hedging problem.

☐ **Open question :** In the Markovian setting, and under smoothness conditions, the super-hedging price is the only hedging price. How to prove this in the path-dependent case by simply using probabilistic arguments ?

General take away message

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General take away message

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□ Stochastic target technics allows one to derive the associated pde (in the viscosity solution sens).

□ In this model, covered and un-covered options are of very different nature.

□ The question of understanding the non-Markovian case is still quite open !

Thank you !

B. Bouchard, G. Loeper, and Y. Zou. Almost-sure hedging with permanent price impact.
Finance and Stochastics, 20(3), 741-771, 2016.

B. Bouchard, G. Loeper, and Y. Zou. Hedging of covered options with linear market impact and gamma constraint.
SIAM Journal on Control and Optimization, 55(5), 3319-3348, 2017.

B. Bouchard, G. Loeper, M. Soner and C. Zhou. Second order stochastic target problems with generalized market impact.
SIAM Journal on Control and Optimization, 57(6), 4125-4149, 2019.

B. Bouchard, and X. Tan. Understanding the dual formulation for the hedging of path-dependent options with price impact.
arXiv preprint arXiv :1912.03946, 2019.

G. Loeper. Option Pricing with Market Impact and Non-Linear Black and Scholes Equations,
arXiv :1301.6252v3

D. Becherer, T. Bilarev, and P. Frentrup. Stability for gains from large investors' strategies in M1/J1 topologies. *Bernoulli*.
To appear.

D. Becherer and T. Bilarev. Hedging with transient price impact for non-covered and covered options.
arXiv preprint arXiv :1807.05917, 2018.

More references

U. Cetin, R. A. Jarrow, and P. Protter.
Liquidity risk and arbitrage pricing theory.
Finance and stochastics, 8(3) :311–341, 2004.

P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir.
Second-order backward stochastic differential equations and fully nonlinear parabolic pdes.
Communications on Pure and Applied Mathematics : A Journal Issued by the Courant Institute of Mathematical Sciences, 60(7) :1081–1110, 2007.

R. Cont and D.-A. Fournié.
Functional itô calculus and stochastic integral representation of martingales.
The Annals of Probability, 41(1) :109–133, 2013.

B. Dupire.
Functional itô calculus.
Portfolio Research Paper, 04, 2009.

R. Frey.
Perfect option hedging for a large trader.
Finance and Stochastics, 2(2) :115–141, 1998.

H. Liu and J. M. Yong.
Option pricing with an illiquid underlying asset market.
Journal of Economic Dynamics and Control, 29 :2125–2156, 2005.

F. Russo and P. Vallois.
Itô formula for c^1-functions of semimartingales.
Probability theory and related fields, 104(1) :27–41, 1996.

Y. F. Saporito.
The functional meyer–tanaka formula.
Stochastics and Dynamics, 18(04) :1850030, 2018.

P. J. Schönbucher and P. Wilmott.
The feedback effects of hedging in illiquid markets.
SIAM Journal on Applied Mathematics, 61 :232–272, 2000.

K. R. Sircar and G. Papanicolaou.
Generalized black-scholes models accounting for increased market volatility from hedging strategies.
Applied Mathematical Finance, 5(1) :45–82, 1998.

Appendix - Itô's Lemma for $C^{0,1}$ functions.

Preliminaries

□ Given two measurable continuous X and Y,

$$[X, Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) ds, \ t \geq 0,$$

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□ A measurable continuous process A is a weak zero energy process if $[A, N] = 0$ a.s. for all continuous local martingale N.

□ X is a weak Dirichlet process if it admits the decomposition $X = M + A$ in which M is a continuous local martingale and A is a weak zero energy process.

□ Remark : If X is Y-integrable and Y is a semimartingale then

$$\int_0^t X_s dY_s = \lim_{\varepsilon \searrow 0} \int_0^t X_s \frac{Y_{s+\varepsilon} - Y_s}{\varepsilon} ds, \ t \geq 0.$$

Assumptions

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$$\varphi(t, \mathrm{x}) - \varphi(t, \mathrm{x}') =$$

$$O\left(\int_{[0,t)} |\mathrm{x}_s - \mathrm{x}'_s| \mu(ds; t, \mathrm{x}) + \|\mathrm{x}_{t\wedge \cdot} - \mathrm{x}'_{t\wedge \cdot}\|^{1+\eta} (1 + \|\mathrm{x}\|^\beta + \|\mathrm{x}'\|^\beta) \right)$$

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for $(\mathrm{x}, \mathrm{x}')$ s.t. $\mathrm{x}_t = \mathrm{x}'_t$ ($\Rightarrow$ always true in the not path dependent case).

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(i) There exists a weak zero energy process $\mathcal{B}$ such that

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(iii) If X and $\varphi(\cdot,X)$ are both martingales, then (ii) holds with $\mathcal{B}' \equiv 0$.

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We want to show that

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for all (bounded) continuous martingale N, i.e.

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (\mathcal{B}_{s+\varepsilon} - \mathcal{B}_s)(N_{s+\varepsilon} - N_s) ds = 0.$$

Corollary - Clark's formula

- Let X be a continuous martingale with independent increments. Then,

$$\Phi(X) = \mathbb{E}[\Phi(X)] + \int_0^T \mathbb{E}[\lambda_\Phi([t, T]; X)|\mathcal{F}_t]dX_t.$$