A numerical approach for a class of risk-sharing problems

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Abstract

This paper deals with risk-sharing problems between many agents, each of whom having a strictly concave law invariant utility. In the special case where every agent's utility is given by a concave integral functional of the quantile of her individual endowment, we fully characterize the optimal risk-sharing rules. When there are many agents, these rules cannot be computed analytically. We therefore give a simple convergent algorithm and illustrate it on several examples.

Keywords: risk-sharing, comonotonicity, sup-convolution, calculus of variations, numerical approximation.

1 Introduction

Risk-sharing problems have their roots in the seminal works of Arrow [1], [2] and Borch [3] in insurance and have received a lot of attention since. Starting from the case of two-agents having preferences given by expected utilities, the theory has developed in recent years in particular to incorporate more general law invariant preferences such as *rank-dependent utilities* or *monetary risk measures* in the financial literature (see Dana [9], Jouini, Schachermayer and Touzi, [13], Carlier and Dana [6], [7], [8] as well as the book by Föllmer and Schied [11] and the references therein).

For suitable law invariant (or quantile-based) and concave utilities, the abstract risk-sharing problem may be brought down to the maximization of

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some concave functional over the set of *comonotone* allocations. When utilities are integral functionals of the quantile (which covers the rank-dependent utility case and more generally the so-called rank-linear utility case), this enables one to rewrite the problem as a tractable variational problem subject to a comonotonicity constraint. This reformulation is a key point in the papers mentioned above to analyze and actually solve many instances of the two-agents case.

The theoretical analysis of the many agents-case is not significantly harder than in the two-agents case. However, the analysis of optimality conditions is more involved since many more cases may arise in the many agents case. Typically, optimal risk-sharing rules exhibit ranges of aggregated risk for which a subset of agents is fully insured by the others. In the two-agents case, optimal solutions are combinations of three regimes: agent 1 insures agent 2, agent 2 insures agent 1 or the solution is interior and thus given by some first-order condition. If there are more agents, there are many more possibilities for the comonotonicity constraint to be binding. We thus claim that the difficulty of the many agents case is in fact a matter of combinatorics and that the search for an efficient computational scheme is therefore natural.

The paper is organized as follows. In section 2, we reformulate a class of risk-sharing problems as tractable variational problems subject to a comonotonicity constraint. More precisely, some notations and preliminaries are given in paragraph 2.1, the two-agents case is then addressed in paragraph 2.2, we finally show a stability by sup-convolution result in paragraph 2.3 which enables to reformulate the risk-sharing problem as a comonotonicity constrained variational problem. Optimality conditions for such problems are established in section 3. Finally, a simple and easy to implement algorithm is described in section 4 in which convergence is established and various numerical simulations are presented.

2 Reformulation of a class of risk-sharing problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a nonatomic probability space i.e. a probability space such that there is no $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) \in \{0, \mathbb{P}(A)\}$ for every $B \in \mathcal{F}$ with $B \subset A$. The nonatomicity of $(\Omega, \mathcal{F}, \mathbb{P})$ is well-known to be equivalent to the existence of a uniformly distributed random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Given a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$, the law of X is denoted $\mathcal{L}(X)$. If X and Y are two random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$, we shall denote by $X \sim Y$ the fact that $\mathcal{L}(X) = \mathcal{L}(Y)$.

In the sequel, we will only consider essentially bounded random variables -or *risks*- on $(\Omega, \mathcal{F}, \mathbb{P})$. We will denote for short by L^∞_+ the nonnegative cone of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathcal{C}$ be the class of utility functions $V : L^\infty_+ \to \mathbb{R}$ that :

- are strictly concave,
- are monotone, i.e. $V(X) \geq V(Y)$ whenever $(X, Y) \in L^\infty_+ \times L^\infty_+$ and $X \geq Y$ a.s.,
- law invariant, i.e. $V(X) = V(Y)$ whenever $(X, Y) \in L^\infty_+ \times L^\infty_+$ and $\mathcal{L}(X) = \mathcal{L}(Y)$,
- satisfy the following Fatou property:

$$V(X) \geq \limsup_n V(X_n)$$

  whenever $(X_n)_n$ is bounded in L^∞_+ and X_n converges a.s. to X.

Given an aggregate risk $X_0 \in L^\infty_+$ and $d+1$ agents, $i = 1, ..., d+1$, each of whom having a utility V_i in the class $\mathcal{C}$, the optimal risk-sharing of X_0 among those agents is determined by solving the *sup-convolution* problem:

$$\left(\mathop{\square}_{i=1}^{d+1} V_i \right)(X_0) := \sup \left\{ \sum_{i=1}^{d+1} V_i(X_i),\ X_i \in L^\infty_+,\ \sum_{i=1}^{d+1} X_i = X_0 \right\}. \tag{2.1}$$

The sup-convolution $\mathop{\square}_{i=1}^{d+1} V_i$ is said to be *exact* if the previous supremum is attained (and then it is attained at a unique point by strict concavity). Our aim is to reformulate the previous risk-sharing problem in a more tractable way using the notion of comonotone allocations. Before we do so, we shall need some preliminaries.

2.1 Preliminaries

Let X be a bounded random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $F_X(t) = \mathbb{P}(X \leq t)$, $t \in \mathbb{R}$ denote its distribution function. The generalized inverse of F_X (or *quantile* function of X) is defined by:

$$F_X^{-1}(t) = \inf\{z \in \mathbb{R} : F_X(z) > t\}, \text{ for all } t \in (0,1). \tag{2.2}$$

The random variable X and the quantile function F_X^{-1} are then *equimeasurable* in the sense that for every continuous function φ one has:

$$\mathbb{E}(\varphi(X)) = \int_0^1 \varphi(F_X^{-1}(t))dt.$$

Let us now recall the well known Hardy-Littlewood inequality (see [12], [15], [4]).

Proposition 2.1. *Let X and Y be in L_+^∞ then one has*

$$\mathbb{E}(XY) \leq \int_0^1 F_X^{-1}(t)F_Y^{-1}(t)dt = \sup_{Z\sim Y} \mathbb{E}(XZ)$$

and for every concave $u : \mathbb{R} \to \mathbb{R}$ one has

$$\mathbb{E}(u(X-Y)) \leq \int_0^1 u(F_X^{-1}(t) - F_Y^{-1}(t))dt.$$

Definition 2.2. *Let X and Y be in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, then X dominates Y in the sense of second order stochastic dominance (which we denote by $X \succeq Y$) if $\mathbb{E}(u(X)) \geq \mathbb{E}(u(Y))$ for every concave increasing function $u : \mathbb{R} \to \mathbb{R}$.*

Various characterizations of stochastic dominance are well-known (see [16]) among which:

$$X \succeq Y \Leftrightarrow \int_0^t F_X^{-1}(t)dt \geq \int_0^t F_Y^{-1}(t)dt, \ \forall t \in [0,1],$$

which is equivalent to

$$\int_0^1 g(t)F_X^{-1}(t)dt \leq \int_0^1 g(t)F_Y^{-1}(t)dt, \ \forall g \text{ bounded nondecreasing.} \quad (2.3)$$

The next statement is originally due to Ryff [17] in the case $(\Omega, \mathcal{F}, \mathbb{P})$ is $[0,1]$ endowed with its Borel field and the Lebesgue measure, we believe it is well-known but give a short proof for the sake of completeness:

Lemma 2.3. *Let X and Y be in L_+^∞, then $X \succeq Y$ if and only if there is a sequence Z_n of the form $Z_n = \sum_{i=1}^{N_n} \lambda_i^n Y_i^n$ with $\lambda_i^n \geq 0$, $\sum_{i=1}^{N_n} \lambda_i^n = 1$ and each $Y_i^n \sim Y$, such that Z_n converges a.s. to X.*

Proof. Assume first that Z_n is of the form mentioned above and Z_n converges a.s. to X, then for each concave u, one has $\mathbb{E}(u(Z_n)) \geq \mathbb{E}(u(Y))$ and one concludes with Lebesgue's dominated convergence theorem (the Z_n's being uniformly bounded).

Conversely, assume $X \succeq Y$ and let us prove that $X \in K$ where K is the closed convex hull in L^1 of the set $\{Z \in L^1,\ Z \sim Y\}$ (clearly $K \subset L^\infty_+$ since $Y \in L^\infty_+$). If $X \notin K$ it follows from Hahn's-Banach theorem that there exists $P \in L^\infty$ and $\varepsilon > 0$ such that

$$\mathbb{E}(PX) \geq \sup_{Z \sim Y} \mathbb{E}(PZ) + \varepsilon.$$

With Hardy-Littlewood's inequality, this yields

$$\int_0^1 F_P^{-1} F_X^{-1} \geq \int_0^1 F_P^{-1} F_Y^{-1} + \varepsilon$$

which contradicts $X \succeq Y$ because of inequality (2.3) recalled above. We thus deduce that X is the L^1 (and thus also a.s. taking a subsequence if necessary) limit of a sequence Z_n of the form mentioned in the statement of the lemma. □

As a consequence of the previous lemma, we have the following compatibility result which is originally due to Dana [9]:

Lemma 2.4. *If a utility function V is in the class $\mathcal{C}$ then it is compatible with second order stochastic dominance that is $V(X) \geq V(Y)$ whenever $(X, Y) \in L^\infty_+ \times L^\infty_+$ and $X \succeq Y$.*

Proof. Assume $X \succeq Y$, using lemma 2.3, X is an a.s. limit of a sequence Z_n as in lemma 2.3. Since V is concave and law invariant, $V(Z_n) \geq V(Y)$ and by the Fatou property, we get $V(X) \geq V(Y)$. □

A notion that will play an important role in the sequel is that of *comonotonicity* that we now recall,

Definition 2.5. *Let X_1 and X_2 be two real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, then the pair (X_1, X_2) is called comonotone if*

$$(X_1(\omega') - X_1(\omega))(X_2(\omega') - X_2(\omega)) \geq 0 \text{ for } \mathbb{P} \otimes \mathbb{P}\text{-a.e. } (\omega, \omega') \in \Omega^2.$$

A family of random variable $(X_1, ..., X_{d+1})$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be comonotone if (X_i, X_j) is comonotone for every $(i, j) \in \{1, ..., d+1\}^2$.

Roughly speaking $(X_1, ..., X_{d+1})$ is comonotone if all the X_i's evolve in the same direction (or *coevolve*) and we recall that it is well-known that it is equivalent to the fact that each X_i can be written as a nondecreasing function of the sum $\sum_i X_i$ (see for instance [10]). Given $X \in L^\infty_+$, the set of allocations $(X_1, ..., X_{d+1}) \in L^\infty_+$ that sum to X and are comonotone can then be parametrized by $X_i := x_i(X)$ where the x_i's are nondecreasing function that sum to the identity map (on the range of X but this can be extended to the whole line if necessary). Obviously, each map x_i is 1-Lipschitz and therefore, by Ascoli's theorem, this parametrization of comonotone allocations with a fixed sum is compact.

Comonotonicity is well-known to be a key property in risk-sharing problems as soon as agents have preferences that are compatible with second order stochastic dominance. Indeed, an important result of Landsberger and Meilijson [14] states that any allocation is dominated (for second order stochastic dominance) by a comonotone one.

2.2 The two-agents case

In this paragraph, we study in details the case of two agents ($d = 1$).

Proposition 2.6. *Let V_1 and V_2 be two utilities in the class $\mathcal{C}$ and $X_0 \in L^\infty_+$, there exists a unique $(\overline{X}_1, \overline{X}_2) \in L^\infty_+ \times L^\infty_+$ such that $\overline{X}_1 + \overline{X}_2 = X_0$ and*

$$V_1 \square V_2(X_0) = V_1(\overline{X}_1) + V_2(\overline{X}_2).$$

Moreover $(\overline{X}_1, \overline{X}_2)$ is comonotone hence there exist two nondecreasing functions $\overline{x}_1$ and $\overline{x}_2 : [F^{-1}_{X_0}(0), F^{-1}_{X_0}(1)] \to \mathbb{R}$ such that $\overline{x}_1(t) + \overline{x}_2(t) = t$ for all $t \in [F^{-1}_{X_0}(0), F^{-1}_{X_0}(1)]$ and $(\overline{X}_1, \overline{X}_2) = (\overline{x}_1(X_0), \overline{x}_2(X_0))$.

Proof. Let us first suppose that X_0 has no atom (i.e. F_{X_0} is continuous, or equivalently $F^{-1}_{X_0}$ increasing). Let X_1 and X_2 be in L^∞_+ such that $X_1 + X_2 = X_0$, it follows from Jensen's inequality that $\mathbb{E}(X_i \mid X_0) \succeq X_i$ so that in the sup-convolution, it is enough to maximize over pairs (X_1, X_2) that are measurable functions of X_0. Let $X_1 = f(X_0)$ and $X_2 := X_0 - f(X_0)$ be such a pair. Since X_0 is nonatomic there exists a nondecreasing map (or monotone *rearrangement*) $\tilde{f}$ such that $\tilde{f}(X_0) \sim f(X_0)$ with $0 \leq \tilde{f}(X_0) \leq X_0$ (see [18]). Then set $Y_1 := \tilde{f}(X_0)$ and $Y_2 := X_0 - \tilde{f}(X_0)$. Since $Y_1 \sim X_1$, $V(Y_1) := V_1(X_1)$, we now claim that $V_2(Y_2) \geq V_2(X_2)$ and to prove it, thanks to lemma 2.4, it is enough to prove that $Y_2 \succeq X_2$. Let u be a concave function, we have

$$\mathbb{E}(u(Y_2)) = \mathbb{E}(u(X_0 - \tilde{f}(X_0)) = \int_0^1 u(F^{-1}_{X_0}(t) - \tilde{f}(F^{-1}_{X_0}(t)))dt$$

but since $\tilde{f}(F^{-1}_{X_0}) = F^{-1}_{X_1}$, by proposition 2.1, we get

$$\mathbb{E}(u(X_2)) = \mathbb{E}(u(X_0 - X_1)) \leq \int_0^1 u(F^{-1}_{X_0}(t) - F^{-1}_{X_1}(t))dt = \mathbb{E}(u(Y_2)).$$

All this proves that given an admissible pair (X_1, X_2) one may find a better one of the form $(f(X_0), X_0 - f(X_0))$ where f is a non decreasing function on $[F^{-1}_{X_0}, F^{-1}_{X_0}] \to \mathbb{R}$ such that $0 \leq f(t) \leq t$ for every $t \in [F^{-1}_{X_0}(0), F^{-1}_{X_0}(1)] \to \mathbb{R}$. In particular, one can find a sequence of such maps f_n such that

$$\lim_n V_1(f_n(X_0)) + V_2(X_0 - f_n(X_0)) = V_1 \square V_2(X_0).$$

Since the functions f_n are all nondecreasing and bounded by $\|X_0\|_\infty$, it follows from Helly's theorem that there is some (not relabeled) sequence that converges pointwise to some function f, hence $(f_n(X_0), X_0 - f_n(X_0))$ is bounded in L^∞ and converges a.s. to $(f(X_0), X_0 - f(X_0))$. The Fatou property guarantees then that

$$V_1(f(X_0)) + V_2(X_0 - f(X_0)) = V_1 \square V_2(X_0).$$

This proves existence of a maximizer in the sup-convolution problem (that is the sup-convolution is exact at X_0), uniqueness follows from the strict concavity of V_1 and V_2. Finally, it remains to show that $(f(X_0), X_0 - f(X_0))$ is comonotone i.e. $X_0 - f(X_0)$ is nondecreasing in X_0. To prove this last claim, we apply the same trick as before : let g be nondecreasing such that $0 \leq g(t) \leq t$ and $g(X_0) \sim X_0 - f(X_0)$, then $X_0 - g(X_0) \succeq f(X_0)$ so that

$$V_1(X_0 - g(X_0)) + V_2(g(X_0)) \geq V_1(f(X_0)) + V_2(X_0 - f(X_0))$$

and therefore $(X_0 - g(X_0), g(X_0))$ is optimal as well and by uniqueness this yields $X_0 - f(X_0) = g(X_0)$. The supremum is thus uniquely attained at some comonotone pair.

Let us now treat the case where X_0 is arbitrary in L^∞_+. A theorem of Ryff (cite) enables to write $X_0 = F^{-1}_{X_0}(U_0)$ with U_0 uniformly distributed. For $n \in \mathbb{N}^*$ then define $X_0^n := F^{-1}_{X_0}(U_0) + n^{-1}U_0$, since X_0^n is nonatomic, we deduce from the previous step that there exist, for every $n \geq 0$, a pair of nondecreasing 1-Lipschitz functions x_1^n and x_2^n summing to the identity such that

$$V_1 \square V_2(X_0^n) = V_1(x_1^n(X_0^n)) + V_2(x_2^n(X_0^n)).$$

By Ascoli's Theorem, we may extract some converging (and not relabeled) subsequence from (x_1^n, x_2^n) and denote $(\overline{x}_1, \overline{x}_2)$ its limit. Since $(x_1^n(X_0^n), x_2^n(X_0^n))$ is bounded and converges a.s. to $(\overline{X}_1, \overline{X}_2) = (\overline{x}_1(X_0), \overline{x}_2(X_0))$ we deduce from the Fatou property that this comonotone pair is the unique maximizer in the sup-convolution problem for X_0.

□

2.3 Stability by sup-convolution and existence for $d+1$ agents

In this section, our aim is to generalize the existence result 2.6 to the case of $d+1$ agents. To do so, we need previously to prove a stability by sup-convolution result.

Proposition 2.7. *Let V_1 and V_2 be two utilities in the class $\mathcal{C}$, then $V_1 \Box V_2$ also belongs to $\mathcal{C}$.*

Proof. Let $X_0, Y_0 \in L^\infty_+ \times L^\infty_+$, then by Proposition 2.6, $V_1 \Box V_2(X_0)$ and $V_1 \Box V_2(Y_0)$ are exact that is, there exist nondecreasing functions $\overline{x}_1, \overline{y}_1$, and $\overline{x}_2, \overline{y}_2$ such that $V_1 \Box V_2(X_0) = V_1(\overline{x}_1(X_0)) + V_2(\overline{x}_2(X_0))$ and $V_1 \Box V_2(Y_0) = V_1(\overline{y}_1(Y_0)) + V_2(\overline{y}_2(Y_0))$. Let us prove step by step that $V_1 \Box V_2$ belongs to $\mathcal{C}$.

- **Strict concavity:** Let $\lambda \in [0,1]$, $X_0, Y_0 \in L^\infty_+ \times L^\infty_+$, since $V_1 \Box V_2$ is a supremum and using the concavity of V_1 and V_2, one gets

$$\begin{aligned} V_1 \Box V_2(\lambda X_0 + (1-\lambda)Y_0) &\geqslant V_1(\lambda \overline{x}_1(X_0) + (1-\lambda)\overline{y}_1(Y_0)) \\ &\quad + V_2(\lambda \overline{x}_2(X_0) + (1-\lambda)\overline{y}_2(Y_0)) \\ &> \lambda[V_1(\overline{x}_1(X_0)) + V_2(\overline{x}_2(X_0))] \\ &\quad + (1-\lambda)[V_1(\overline{y}_1(Y_0)) + V_2(\overline{y}_2(Y_0))]. \end{aligned}$$

  We then easily deduce that $V_1 \Box V_2$ is strictly concave.

- **Monotonicity:** Let $X_0, Y_0 \in L^\infty_+ \times L^\infty_+$ such that $X_0 \geqslant Y_0$. Since V_1 and V_2 are monotone and $\overline{y}_1, \overline{y}_2$ are nondecreasing, one has

$$\begin{aligned} V_1 \Box V_2(X_0) &\geqslant V_1(\overline{y}_1(X_0)) + V_2(\overline{y}_2(X_0)) \\ &\geqslant V_1(\overline{y}_1(Y_0)) + V_2(\overline{y}_2(Y_0)) = V_1 \Box V_2(Y_0). \end{aligned}$$

- **Law invariance property:** Let $X_0, Y_0 \in L^\infty_+ \times L^\infty_+$ such that $X_0 \sim Y_0$. We then have by law invariance of V_1 and V_2

$$V_1 \Box V_2(X_0) = V_1(\overline{x}_1(Y_0)) + V_2(\overline{x}_2(Y_0)) \leqslant V_1 \Box V_2(Y_0),$$

  reversing the role of X_0 and Y_0 then yields $V_1 \Box V_2(X_0) = V_1 \Box V_2(Y_0)$.

- **Fatou property:** Let $(X_n)_n$ be a bounded sequence in L^∞_+ that converges a.s. to a limit X. Then for every $n \geq 1$, there exist nondecreasing and $1-$Lipschitz nonnegative functions $\overline{x}_1^n, \overline{x}_2^n$ such that

$$V_1 \Box V_2(X_n) = V_1(\overline{x}_1^n(X_n)) + V_2(\overline{x}_2^n(X_n)),$$

and, by Ascoli's Theorem, up to a not relabeled subsequence, $\overline{x}_1^n$ and $\overline{x}_2^n$ converge uniformly to some $\overline{x}_1$ and $\overline{x}_2$ so that $\overline{x}_i^n(X_n)$ converges a.s. to $\overline{x}_i(X)$. We thus have

$$\begin{aligned}\limsup_n V_1\Box V_2(X_n) &\leqslant \limsup_n V_1(\overline{x}_1^n(X_n)) + \limsup_n V_2(\overline{x}_2^n(X_n))\\ &\le V_1(\overline{x}_1(X)) + V_2(\overline{x}_2(X))\\ &\le V_1\Box V_2(X)\end{aligned}$$

from which we deduce the Fatou property for $V_1\Box V_2$.

□

Inductively, one immediately deduces from the previous proposition that if $V_1, ..., V_{d+1}$ all belong to the class $\mathcal{C}$ then so does $\mathop{\Box}\limits_{i=1}^{d+1} V_i$.

Theorem 2.8. *Let $X_0 \in L_+^\infty$ and $V_1, ..., V_{d+1}$ be in the class $\mathcal{C}$. Then there exists a unique $\overline{X} = (\overline{X}_1, ..., \overline{X}_{d+1}) \in (L_+^\infty)^{d+1}$ such that $\sum_{i=1}^{d+1} \overline{X}_i = X_0$ and*

$$\mathop{\Box}\limits_{i=1}^{d+1} V_i(X_0) = \sum_{i=1}^{d+1} V_i(\overline{X}_i).$$

Moreover $\overline{X}$ is comonotone hence there exist $d+1$ nondecreasing functions $\overline{x}_i$: $[F_{X_0}^{-1}(0), F_{X_0}^{-1}(1)] \to \mathbb{R}$ such that $\sum_{i=1}^{d+1} \overline{x}_i(t) = t$ for all $t \in [F_{X_0}^{-1}(0), F_{X_0}^{-1}(1)]$ and $\overline{X} = (\overline{x}_1(X_0), ..., \overline{x}_{d+1}(X_0))$.

Proof. For the sake of simplicity, let us prove the Theorem for $d + 1 = 3$. Introducing the notation $W_2 = V_2\Box V_3$, the maximization problem reads as:

$$\sup\{V_1(X_1) + W_2(Y_2); X_1 \in L_+^\infty, Y_2 \in L_+^\infty, X_1 + Y_2 = X_0\}.$$

By Proposition 2.7, we know that $W_2 \in \mathcal{C}$. Then Proposition 2.6 say that there exists a unique solution $(\overline{X}_1 = \overline{x}_1(X_0), \overline{Y}_2 = \overline{y}_2(X_0)) \in L_+^\infty \times L_+^\infty$ such that $\overline{x}_1$,$\overline{y}_2$ are nondecreasing, $\overline{x}_1(X_0) + \overline{y}_2(X_0) = X_0$, and

$$V_1\Box W_2(X_0) = V_1(\overline{x}_1(X_0)) + W_2(\overline{y}_2(X_0)).$$

We then solve the sub-problem

$$\sup\{V_2(X_2) + V_3(X_3); X_2 \in L_+^\infty, X_3 \in L_+^\infty, X_2 + X_3 = \overline{y}_2(X_0)\}.$$

By the same arguments, there exists a unique comonotone solution $(\overline{X}_2 = \overline{z}_2(\overline{y}_2(X_0)), \overline{X}_3 = \overline{z}_3(\overline{y}_2(X_0)) \in L_+^\infty \times L_+^\infty$ satisfying the same properties

as in Proposition 2.6. Finally, it is easy to see that $(\overline{X}_1 = \overline{x}_1(X_0), \overline{X}_2 = \overline{x}_2(X_0), \overline{X}_3 = \overline{x}_3(X_0))$, where $\overline{x}_2 := \overline{z}_2 \circ \overline{y}_2$ and $\overline{x}_3 := \overline{z}_3 \circ \overline{y}_2$, is the unique solution of the sup-convolution $\mathop{\square}_{i=1}^{3} V_i(X_0)$, with the desired properties.
Inductively, one can prove the Theorem for every d. □

Remark 2.9. One may wonder if it is really necessary to proceed inductively to prove the previous result. On the one hand, we believe that proving stability of the class $\mathcal{C}$ by supremal convolution (that is by aggregation) as we did in proposition 2.7 has its own interest. On the other hand, the rearrangement trick we used in the two-agents case, does not generalize to more agents.

Remark 2.10. In the previous statements, we have used strict concavity of the elements of $\mathcal{C}$ for uniqueness purpose only. Strict concavity is in fact quite restrictive since it rules out the case of monetary risk measures. We now claim that strict concavity, though convenient, is in fact not necessary to obtain the existence of at least one optimal comonotone solution in the supremal convolution problem. To see this, it is enough to proceed by approximation and use the compactness of comonotone allocations.

2.4 Reformulation

It follows from the previous results that the risk-sharing problem (2.1) between $d+1$ agents, each of whom has a utility in $\mathcal{C}$, and for aggregate risk X_0, can be simply reformulated as

$$\sup_{x \in \mathcal{A}} J(x), \quad \text{with } J(x) := \sum_{i=1}^{d} V_i\left(x_i(X_0)\right) + V_{d+1}\left(X_0 - \sum_{i=1}^{d} x_i(X_0)\right) \tag{2.4}$$

where

$$\mathcal{A} := \{x \in W^{1,\infty}([a,b], \mathbb{R}^d) \ : \ \dot{x} \in \Delta \text{ a.e.}, \ x(a) \in \mathcal{S}\},$$

$$a := F_{X_0}^{-1}(0), \ b := F_{X_0}^{-1}(1),$$

and Δ and $\mathcal{S}$ are the two simplices:

$$\Delta := \{u \in \mathbb{R}_+^d, \ \sum_{i=1}^{d} u_i \le 1\}, \ \mathcal{S} := \{x \in \mathbb{R}_+^d, \ \sum_{i=1}^{d} x_i \le a\}.$$

This reformulation is more tractable than the initial one since maximization is now performed over a more concrete set of functions over an interval, namely the convex and compact set $\mathcal{A}$. As we shall see in the next paragraph, the fact that constraints are given by simplices will be very convenient to express necessary conditions.

Of course this reformulation is not very helpful if we keep the previous level of generality on utility functions. This is why we shall now further specify the utilities and restrict them to a subclass for which they have a simple expression in terms of quantiles. From now on, we shall restrict utilities to belong to the class of Rank-Linear Utilities (RLU for short), such utilities are of the form

$$V_L(X) := \int_0^1 L(t, F_X^{-1}(t))dt. \tag{2.5}$$

This class of utilities is already quite large since it contains expected utilities as well as the Rank-Dependent Utilities defined by Choquet expectations. Utilities in this class are obviously law invariant and satisfy the Fatou property as soon as L is continuous (say). RLU have been studied in details in [7], where it is proved in particular that for V_L defined by (2.5) (with a C^2 function L to simplify), the following statements are equivalent:

1. V_L is compatible with second order stochastic dominance and monotone,
2. $\partial_x L \geq 0$, $\partial_{xx} L \leq 0$ and $\partial_{tx} L \leq 0$ on $[0, 1] \times \mathbb{R}$
3. V_L is concave, monotone and $\sigma(L^\infty(\Omega), L^1(\Omega))$ upper semi-continuous.

Now for a utility of the form (2.5), if x is a nondecreasing function and X_0 is nonatomic with an increasing distribution function (say), one has

$$V_L(X_0) = \int_0^1 L(t, x(F_{X_0}^{-1}(t))dt = \int_a^b L(F_{X_0}(s), x(s))d\mu_0(s),$$

where μ_0 is the law of X_0. In this framework, (2.4) takes the form of a simple variational problem with

$$J(x) := \int_a^b \Big(\sum_{i=1}^d L_i(F_{X_0}(s), x_i(s)) + L_{d+1}(F_{X_0}(s), s - \sum_{i=1}^d x_i(s))\Big)d\mu_0(s). \tag{2.6}$$

The next paragraphs are precisely devoted to the theoretical and numerical study of such problems (under appropriate regularity and concavity assumptions).

3 Optimality conditions

In this section, our aim is to give necessary and sufficient conditions for the problem

$$\sup_{x\in\mathcal{A}} J(x) \;\text{ with }\; J(x) := \int_a^b F(t, x(t))dt \tag{3.1}$$

where we assume that $F \in C([a,b] \times \mathbb{R}^d, \mathbb{R})$ is such that $F(t,.)$ is strictly concave and differentiable for every $t \in [a,b]$ and $\nabla_x F$ is continuous in its two arguments.

Under these assumptions, problem (3.1) admits a unique solution $\overline{x}$ that is characterized as follows:

Theorem 3.1. *Problem (3.1) admits a unique solution $\overline{x}$ that is characterized by the following:*

- $\overline{x} \in \mathcal{A}$,
- *for a.e. t, $\dot{\overline{x}}_i(t) = 0$ for every $i \in \{1, ..., d\}$ with $i \notin I(t)$ where*

$$I(t) := \{j \in \{1, ..., d\} \;:\; \overline{p}_j(t) = \min_{i=1,...,d} \overline{p}_i(t)\}$$

and $\overline{p}$ is the adjoint variable:

$$\overline{p}(t) = -\int_t^b \nabla_x F(s, \overline{x}(s))ds. \tag{3.2}$$

- $\overline{x}_i(a) = 0$, $\forall i \in \{1, ...d\}$ *with $i \notin I(a)$.*

Proof. Existence and uniqueness follow from standard arguments. Let us denote by $\overline{x}$ the solution of (3.1), by concavity of J and convexity of $\mathcal{A}$, $\overline{x}$ is characterized by the variational inequalities

$$J'(\overline{x}) \cdot (y - \overline{x}) = \int_a^b \nabla_x F(t, \overline{x}(t)) \cdot (y(t) - \overline{x}(t))dt \le 0, \; \forall y \in \mathcal{A}. \tag{3.3}$$

Now for $y \in \mathcal{A}$, defining $\overline{p}$ by (3.2) and integrating by parts yields

$$J'(\overline{x}) \cdot y = -\int_a^b \overline{p} \cdot \dot{y} - \overline{p}(a) \cdot y(a)$$

so that (3.3) becomes

$$\int_a^b \overline{p} \cdot \dot{\overline{x}} + \overline{p}(a) \cdot \overline{x}(a) = \inf_{y \in \mathcal{A}} \left\{ \int_a^b \overline{p} \cdot \dot{y} + \overline{p}(a) \cdot y(a) \right\}, \tag{3.4}$$

now the rightmost member of the previous identity is easy to compute by pointwise minimization over the simplex and is achieved exactly at those y's in $\mathcal{A}$ such that for a.e. $t \in [a, b]$, $\dot{y}_i = 0$ as soon as $i \notin I(t)$ and $y_i(a) = 0$ for $i \notin I(a)$. □

In the algorithm of next section, we shall take full advantage of the fact that (3.4) amounts to linear programming over the simplex and is therefore explicit. In particular, for $x \in \mathcal{A}$ and p the associated adjoint variable:

$$p(t) = -\int_t^b \nabla_x F(s, x(s)) ds, \tag{3.5}$$

this characterizes the set of solutions of the linear programming problem

$$\sup_{y \in \mathcal{A}} J'(x) \cdot y. \tag{3.6}$$

In particular, a solution of (3.6) is given explicitly by:

$$y(t) = y(a) + \int_a^t u(s) ds, \ t \in [a, b], \tag{3.7}$$

with

$$y_i(a) = \begin{cases} \frac{a}{\#I_p(a)} & \text{if } i \in I_p(a) \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

$$u_i(t) = \begin{cases} \frac{1}{\#I_p(t)} & \text{if } i \in I_p(t) \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

and

$$I_p(t) := \{j \in \{1, ..., d\} \ : \ p_j(t) = \min_{i=1,...,d} p_i(t)\}.$$

For every $x \in \mathcal{A}$, set

$$G(x) := \{x + \rho(y - x), \ y \in \text{argmax}_{h \in \mathcal{A}} J'(x) \cdot h, \ \rho \in \text{argmax}_{\lambda \in [0,1]}, \ J(x + \lambda(y - x))\}.$$

In the next section, we shall need the following elementary result:

Lemma 3.2. *The set valued map $x \in \mathcal{A} \mapsto G(x)$ has a closed graph and $\overline{x}$, the solution of (3.1), is characterized by the condition $\overline{x} \in G(\overline{x})$.*

Proof. Let $(x_n)_n \in \mathcal{A}$ converge to some x and $z_n = x_n + \rho_n(y_n - x_n)$ converge to some z and let us prove that $z \in G(x)$. Taking subsequences if necessary, we may assume that ρ_n converges to some $\rho \in [0,1]$ and y_n converges uniformly to some $y \in \mathcal{A}$ so that $z = x + \rho(y - x)$. Using the fact that J is of class C^1 enables us to pass to the limit in

$$J'(x_n)\cdot(y_n - h) \geq 0,\ \forall h \in \mathcal{A},\ J(z_n) \geq J(x + \rho(y_n - x)),\ \forall \rho \in [0,1]$$

and therefore to deduce that $z \in G(x)$.

It is obvious that $\overline{x}$, the solution of (3.1), is such that $\overline{x} \in G(\overline{x})$. Now if $x \in G(x)$ then one has either $\rho = 0$ or $x = y$ for some $\rho \in [0,1]$ and some $y \in \mathcal{A}$ as in the definition of $G(x)$. If $x = y$ then $J'(x)\cdot h$ is maximized over $\mathcal{A}$ for $h = x$ which is exactly the variational inequality characterizing the solution of (3.1). If $\rho = 0$ and $y \neq x$ then $J(x + \lambda(y - x))$ is maximized over $[0,1]$ for $\lambda = 0$ so that $J'(x)\cdot(y - x) \leq 0$ which again means that $J'(x)\cdot h$ is maximized over $\mathcal{A}$ for $h = x$.

□

4 Algorithm and numerical results

4.1 Algorithm and convergence

The algorithm we propose to solve (3.1) is a simple optimal step gradient ascent-like method. The fact that the constraints are easy to handle relies again on the fact that linear problems over the simplex are very simple. Our algorithm is defined as follows :

- start from $x^0 \in \mathcal{A}$ and define p^0 as the corresponding adjoint:

$$p^0(t) = -\int_t^b \nabla_x F(s, x^0(s))ds$$

- given (x^k, p^k) define y^k by

$$y^k(t) = y^k(a) + \int_a^t u^k(s)ds,\ t \in [a,b],$$

with

$$y_i^k(a) = \begin{cases} \frac{a}{\#I_{p^k(t)}} & \text{if } i \in I_{p^k}(a) \\ 0 & \text{otherwise} \end{cases}$$

and

$$u_i(t) = \begin{cases} \frac{1}{\#I_{p^k}(t)} & \text{if } i \in I_{p^k}(t) \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Then set

$$x^{k+1} := x^k + \rho^k(y^k - x^k)$$

where

$$\rho^k = \text{argmax}_{\rho\in[0,1]} J(x^k + \rho(y^k - x^k))$$

is computed by a dichotomous method.
Finally let p^{k+1} be the adjoint state associated to x^{k+1}:

$$p^{k+1}(t) = -\int_t^b \nabla_x F(s, x^{k+1}(s))ds.$$

The convergence of this simple ascent algorithm is then given by:

Theorem 4.1. *The sequence x^k generated by the previous algorithm converges to $\overline{x}$, the solution of (3.1) as $k \to \infty$.*

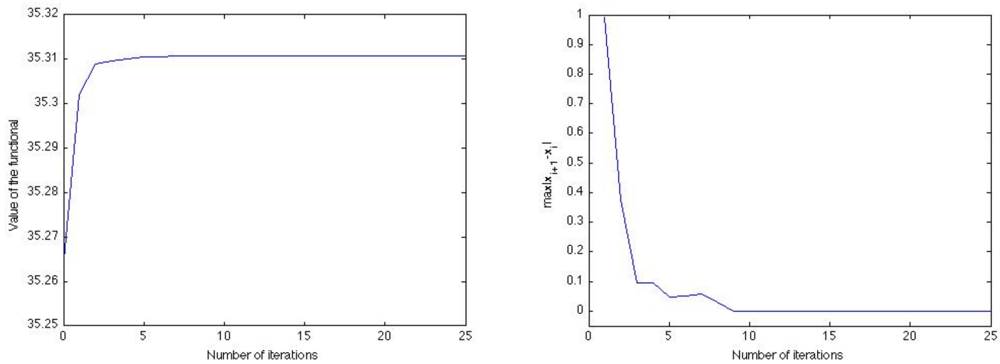
Proof. Since $\mathcal{A}$ is compact (for the uniform topology) and the sequence $(x^k)_k$ has values in $\mathcal{A}$, it is enough to show that $\overline{x}$ is the unique cluster point of $(x^k)_k$. Let x be such a cluster point, and let x^{k_n} be a subsequence converging to x, taking a subsequence if necessary we may also assume that x^{k_n+1} converges to some z. Since G has a closed graph, $z \in G(x)$ i.e. z has of the form $z = x+\rho(y-x)$ with y maximizing $J'(x)\cdot y$ over $\mathcal{A}$ and ρ maximizing $J(x+\rho(y-x))$ over $[0,1]$. By construction $J(x^k)$ is nondecreasing, consequently one has $J(x) = J(z)$. Now, if $x \neq \overline{x}$, then by lemma 3.2, one has $x \notin G(x)$ so that $\rho > 0$ and $J'(x)\cdot(y-x) > 0$ which implies that $J(z) = J(x+\rho(y-x)) > J(x)$, giving the desired contradiction. This proves that $\overline{x}$ is the only cluster point of (x^k) and thus that the whole sequence (x^k) converges to $\overline{x}$. □

Let us now show some numerical results using the previous algorithm.

4.2 Numerical experiments

In this last part, we implement the algorithm introduced in section 4.1. More precisely we study three particular cases for an uniform distribution of the total risk X_0 on $[0,1]$. In a fourth case we compare the solutions for different distributions of X_0 and the last test aims at showing that the numerical procedure is robust for approaching solutions of risk sharing problems with linear utilities. But before doing so we would like to mention that the algorithm

gives very good results with a very quick convergence. Indeed, it is shown on Figure 1-(a) how fast is the convergence of the value of the discretized version of (3.1). We can also see on Figure 1-(b) that the infinite norm of the difference between the discrete solutions of two successive iterations (i and $i+1$, with $i = 1, .., N-1$) converges to zero with about the same speed. In all our tests, the algorithm converges in very few iterations (between 5 and 15).
In the following examples, we always take $a = 0$ and $b = 1$.



(a) Value of (3.1) at each step of the procedure

(b) Convergence of the solution: $||\overline{x}_{i+1} - \overline{x}_i||_\infty$

Figure 1: Numerical convergence for a 20 agents risk sharing

First case: product utilities In the reformulation (2.6), we take functions of the form

$$L_i(t,x) = (1-t)^{\alpha_i}(1+x)^{\beta_i},$$

for $i = 1, ..., d+1$. It is easy to see that for $\alpha_i \geq 0$ and $0 < \beta_i < 1$, such functions lead to utilities that belong to the class $\mathcal{C}$. In this first numerical experiment, we look at the case $d+1 = 5$ and we choose $\alpha := (2.8, 1.6, 2.2, 1.05, 1.3)$ and $\beta := (0.58, 0.52, 0.59, 0.53, 0.57)$. Figure 2 shows the optimal sharing for these data and, more particularly, we paint the cumulative sharing in the sense that the size of the risk supported by an agent is the difference between her labeled line (the lines are labeled on the right hand side of the picture) and the nearest sub-line. Looking at Figure 2, one can for instance notice that agent 4 fully insures the others for any risk with value smaller than 0.34. Then, for risk values smaller than about 0.7, agent 3 and 4 insure the others and finally, for big risk values, every agent support a part of the risk.

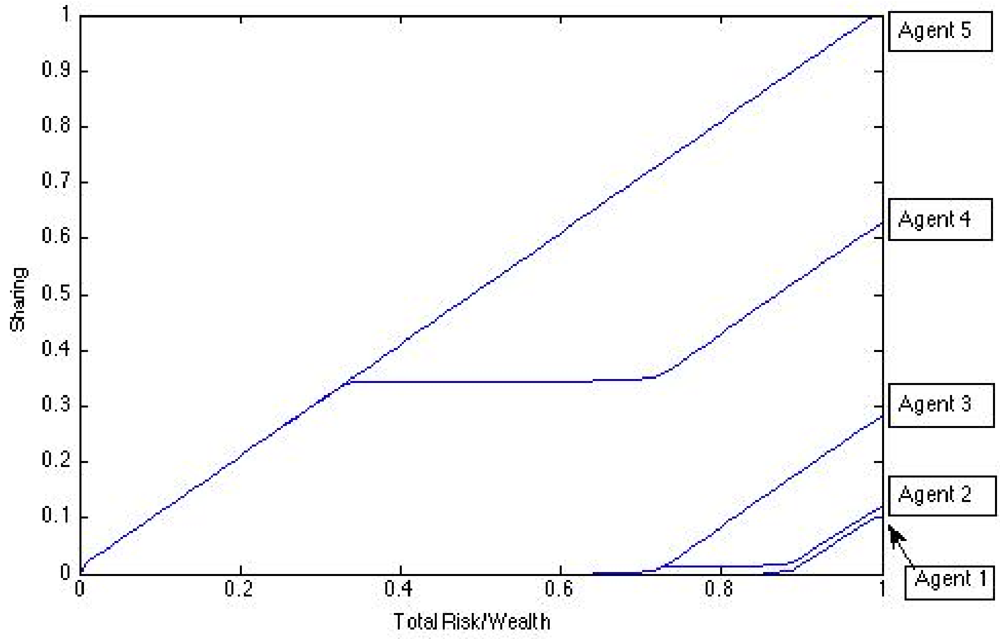


Figure 2: Risk Sharing for 5 agents

Second case: logarithmic utilities In this example, we make a different choice for the functions L_i. We take two logarithmic functions $h_1(t,x) := \log(t^{1/2} + x + 1)$ and $h_2(t,x) := \log(t^{2/3} + x + 1)$. We look at the case of 8 agents, five of them being of type $L_i = h_1$ $(i = 3, 4, 5, 6, 7)$ and three of them of type $L_i = h_2$ $(i = 1, 2, 8)$. We paint the graph of the optimal sharing on Figure 3. The continuous lines correspond to agents of type h_1 and the discontinuous ones to those of type h_2. We can see that for risk values between 0 and 0.3, the three agents of type h_2 fully insure the others. Moreover and without any surprise, we observe that agents of same type support the same proportion of risk.

Third case: logarithmic and product utilities We now turn to a case involving more agents than in the previous ones: $d + 1 = 20$. The functions L_i are chosen between $L_i(t,x) = (1-t)^{\alpha_i}(1+x)^{\beta_i}$, for some α_i and β_i satisfying the same assumptions as before, and $L_i(t,x) = \log(t^{\gamma_i} + x + 1)$, $\gamma_i \in (0,1)$. We do not give more details about functions L_i, the aim only is to show a solution for a many agents case. Indeed, we can see the graph of the solution on Figure 4. Let us remark for instance that actually, the solution is comonotone, and that at most 12 agents share the risk.

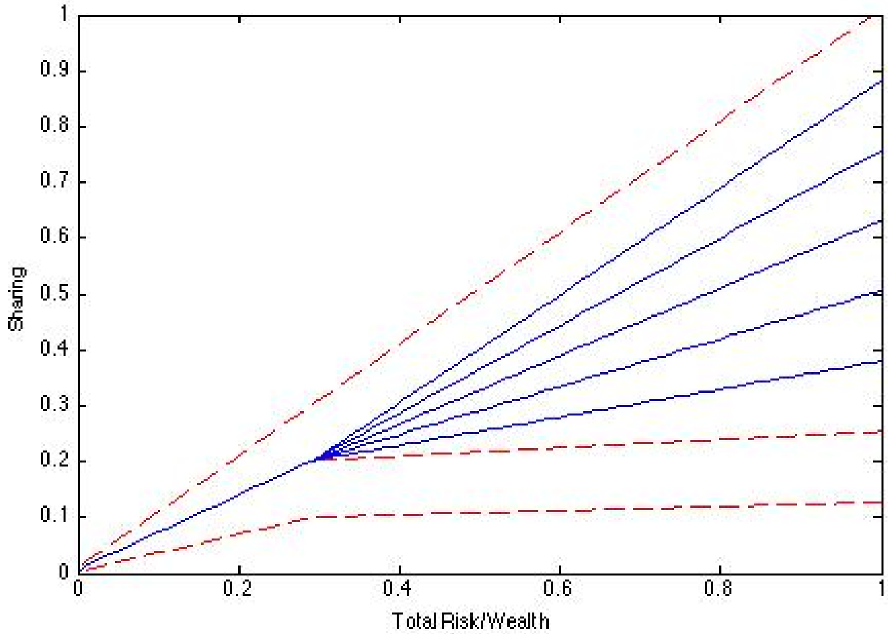


Figure 3: Risk Sharing for 8 agents

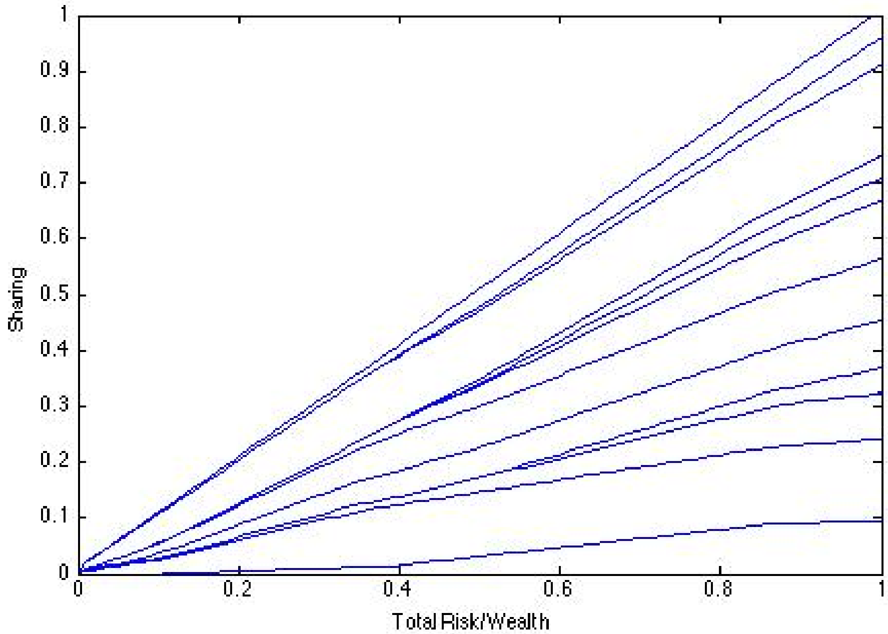
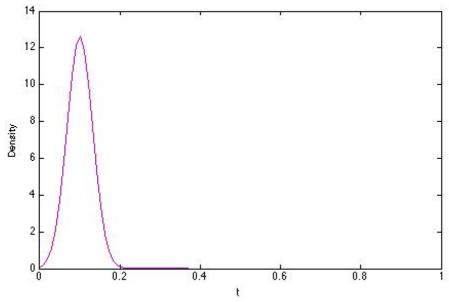
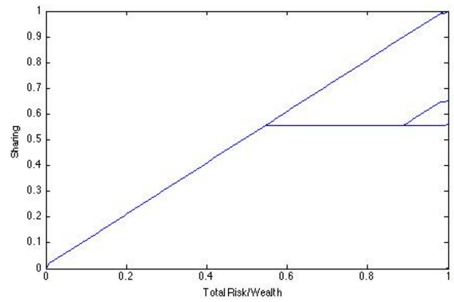


Figure 4: Risk Sharing for 20 agents

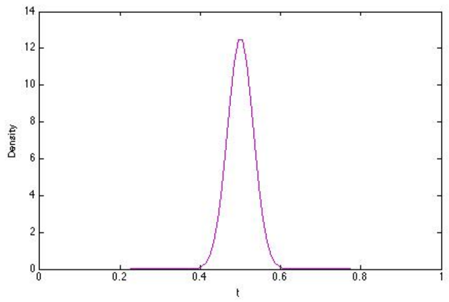
Fourth case: comparative statics for different densities of X_0 We also want to show the impact of the law of the total amount of risk X_0 on the solutions of the problem.



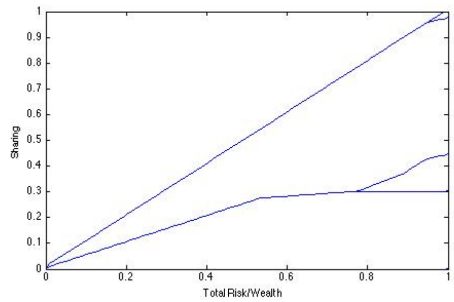
(a) Gaussian density with $\mu = 0.1$



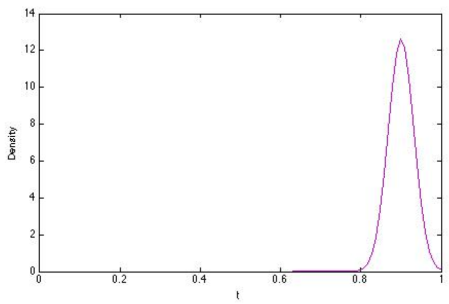
(b) Optimal Risk Sharing for $\mu = 0.1$



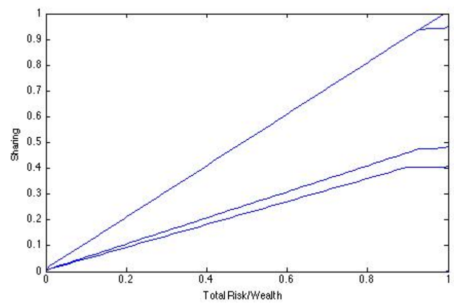
(c) Gaussian density with $\mu = 0.5$



(d) Optimal Risk Sharing for $\mu = 0.5$



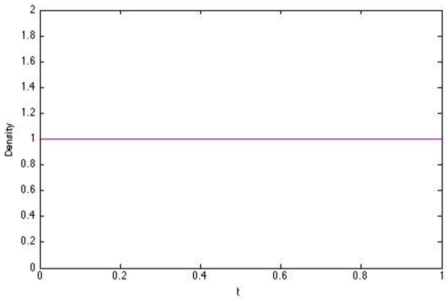
(e) Gaussian density with $\mu = 0.9$



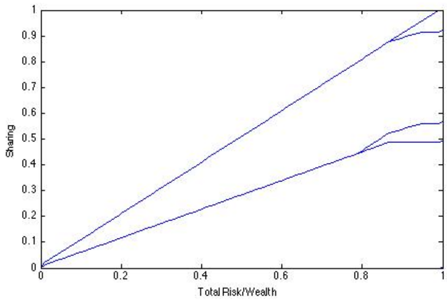
(f) Optimal Risk Sharing for $\mu = 0.9$

Figure 5: Plots of the solutions for three averages of a Gaussian distributions of the total risk X_0.

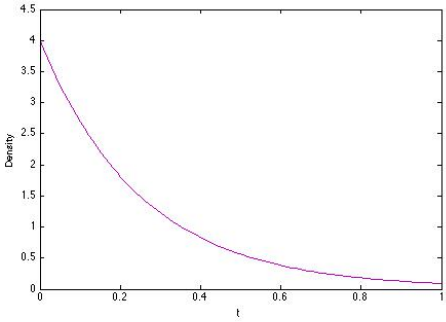
To do so, we take $d+1 = 4$ and functions $L_i(t, x) = (1-t)^{\alpha_i}(1+x)^{\beta_i}$, with the particular choice $\alpha = (1.16, 1.89, 1.21, 2.92)$ and $\beta = (0.52, 0.58, 0.54, 0.59)$. We start with a comparison for three truncated and normalized gaussians



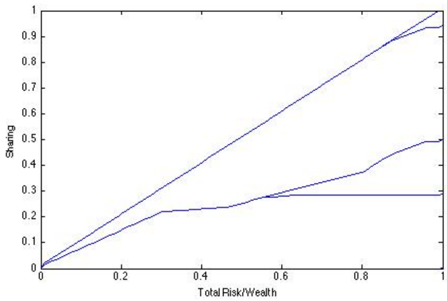
(a) Uniform density



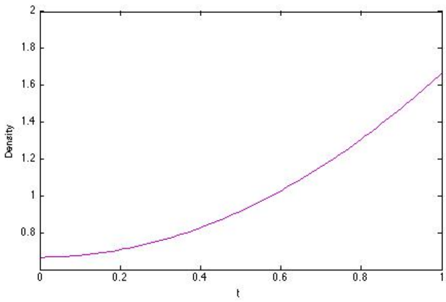
(b) Optimal Risk Sharing for the Uniform density



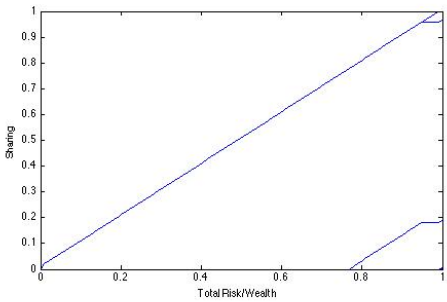
(c) Exponential density with $\lambda = 4$



(d) Optimal Risk Sharing for the Exponential density



(e) Quadratic density



(f) Optimal Risk Sharing for the Quadratic density

Figure 6: Plots of the solutions for three different distributions of the total risk X_0.

with variance $\sigma^2 = 0.01$. More precisely, we plot on Figure 5 the graphs of the density for three different values of the average $\mu = 0.1, 0.5, 0.9$, and the corresponding solutions. Let us note for instance that Agent 1 fully insures the others for values lower than 0.57 when the average is small (0.1). This

phenomenon disappears for bigger values of μ. We can also remark that Agent 5 support a significant proportion of risk only for high values of X_0 and μ. To sum up, we figure out that, in the Gaussian case, the higher is the average of the risk, the most the agents share it (particularly for its big values).
Nevertheless, it appears that this conclusion is not true in general (one should not only consider the first moment of X_0). To see that, we paint on Figure 6 the graphs of three other densities with the corresponding optimal risk-sharing. The densities we study are, for $t \in [0, 1]$:

- $(a) - (b)$: $f_{X_0}(t) = 1$ (Uniform density),
- $(c) - (d)$: $f_{X_0}(t) = \lambda e^{-\lambda t} + e^{-\lambda}$ (Exponential-like density),
- $(e) - (f)$: $f_{X_0}(t) = t^2 + \frac{2}{3}$ (Quadratic density, say).

We easily see that, for the Exponential density (with $\lambda = 4$), the risk is "more shared" (for all values in $[0, 1]$) than in the Uniform case. Moreover, in the Quadratic case, we observe that the risk is shared only by three of the four agents for its big values.

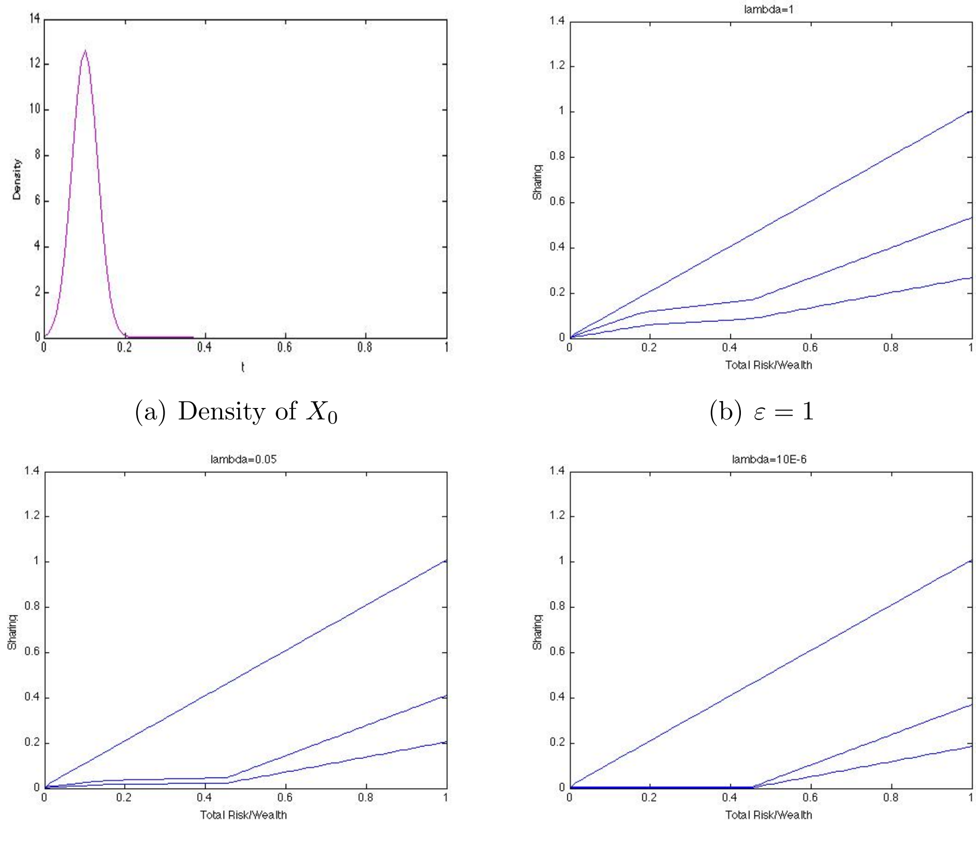
Fifth case: linear utilities approximation To close the numerical tests, we want to point out the robustness of the algorithm for the approximation of the solutions of the problem with linear utilities. In this example, we look at a 5 agents case and a Gaussian distribution of the aggregate risk X_0 (its density is painted on Figure 4.2-(a)). We then take utilities of the form

$$L_i^{\varepsilon}(t, x) = (1 + \varepsilon x^{\beta_i} + (1 - \varepsilon)x)(1 - t)^{\alpha_i},$$

with $0 < \beta_i < 1$ and $\alpha_i > 0$. The idea is to let $\varepsilon \to 0$. We show the numerical results for $\varepsilon = 1$ (Figure 4.2-(b)) , $\varepsilon = 0.05$ (Figure 4.2-(c)) and $\varepsilon = 10^{-5}$ (Figure 4.2-(d)).
We finally give the distance (for the infinite norm) of the approximations to the numerical limit. If $\overline{x}_\varepsilon$ denotes the numerical solution for a fixed ε, we define this quantity by $||\overline{x}_{10^{-5}} - \overline{x}_\varepsilon||_\infty$, and we give the results on Figure 8, for some values of ε in $[0.02, 0.05]$. We then observe a numerical convergence.

Acknowledgment

The authors wishes to express their gratitude to Rose-Anne Dana for several helpful suggestions and her active interest in the topic.



(a) Density of X_0

(b) $\varepsilon = 1$

(c) $\varepsilon = 0.05$

(d) $\varepsilon = 10^{-5}$

Figure 7: Approximation of the solution of the problem for a linear utility

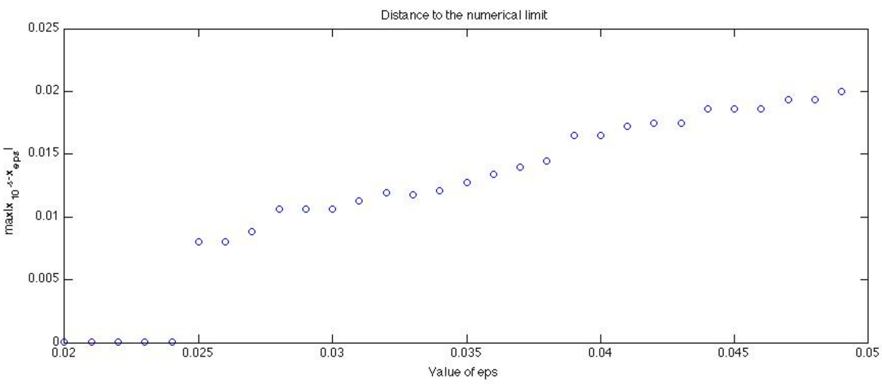


Figure 8: Distance of the approximations to the numerical limit

References

[1] K.J. Arrow, Uncertainty and the welfare of medical care, Amer. Econom. Rev. 53 (1963), 941–973.

[2] K.J. Arrow, Optimal insurance and generalized deductibles, Scand. Actuar. J. (1974), 1–42.

[3] K. Borch, Equilibrium in a reinsurance market, Econometrica, 30 (1962), 424–444.

[4] S. Cambanis, G. Simons and W. Stout Inequalities for $Ek(X, Y)$ when the marginals are fixed, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 36, (1976), 285–294.

[5] G. Carlier, R.-A. Dana, Core of convex distortions of a probability, Journal of Economic Theory, 113 (2003), 199-222 (2003).

[6] G. Carlier, R.-A. Dana, Law invariant concave utility functions and optimization problems with monotonicity and comonotonicity constraints, Statistics and Decisions, 24 (2006) 127-152.

[7] G. Carlier, R.-A. Dana, Two-Persons Efficient Risk-Sharing and Equilibria for Concave Law-Invariant Utilities, Economic Theory, 36 (2008), 189–223.

[8] G. Carlier, R.-A. Dana, Optimal demand for contingent claims when agents have law invariant utilities, to appear in Mathematical Finance.

[9] R.-A Dana, A representation result for concave Schur concave functions, Math. Finance 15 (2005), no. 4, 613–634.

[10] D. Denneberg, Non-additive Measures and Integral, Kluwer Academic Publishers, Holland, 1994.

[11] H. Föllmer, A. Schied, Stochastic finance. An introduction in discrete time , De Gruyter editor, Berlin 2004.

[12] G. H. Hardy, J. E. Littlewood and G. Pòlya, Inequalities, Reprint of the 1952 edition, Cambridge, University Press, Cambridge, 1988.

[13] E. Jouini, W. Schachermayer, N. Touzi, Optimal risk sharing for law invariant monetary utility functions, Math. Finance 18 (2008), 269–292.

[14] M. Landsberger, I.I. Meilijson, Comonotone allocations, Bickel Lehmann dispersion and the Arrow-Pratt measure of risk aversion, Annals of Operation Research 52 (1994) 97–106.

[15] G.G. Lorentz, An inequality for rearrangements, Amer. Math. Monthly, 60 (1953), 176–179.

[16] M. Rothschild, J. E. Stiglitz,. Increasing Risk, I. A Definition, Journal of Economic Theory 2 (1970) 225–243.

[17] J.V. Ryff, Extreme points of some convex subsets of L^1, Proc. AMS 116 (1967), 1026–1034.

[18] J.V. Ryff, Measure preserving Transformations and Rearrangements, Journal of Mathematical Analysis and Applications 31 (1970) 449- 458.