Barycenters in the Wasserstein space

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Abstract

In this paper, we introduce a notion of barycenter in the Wasserstein space which generalizes McCann's interpolation to the case of more than two measures. We provide existence, uniqueness, characterizations and regularity of the barycenter, and relate it to the multi-marginal optimal transport problem considered by Gangbo and Święch in [8]. We also consider some examples and in particular rigorously solve the gaussian case. We finally discuss convexity of functionals in the Wasserstein space.

Keywords: Optimal transport, Wasserstein space, convexity, duality.

AMS Subject Classifications:. 49J40, 49K21, 49K30.

1 Introduction

In this paper, we consider a nonlinear interpolation between several probabilty measures on $\mathbb{R}^d$. By analogy with the Euclidean case where the barycenter of points $(x_1, ..., x_p)$ with barycentric coordinates $(\lambda_1, .., \lambda_p)$ is obtained as the minimizer of $x \mapsto \sum_{i=1}^p \lambda_i |x - x_i|^2$, we propose the same procedure in the Wasserstein space by simply replacing the squared euclidean distance with the squared 2-Wasserstein distance.

In the case of two probability measures, such an interpolation is already known as the McCann's interpolation [11] that led to the concept of displacement convexity that has proved to be a very powerful tool in the theory of

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gradient flows in the Wasserstein space as explained in the recent book of Ambrosio, Gigli and Savare [1]. Our interest in interpolating between more than two probability measures by minimizing an averaged sum of squared Wasserstein distances originated from the recent papers in mathematical economics by Carlier and Ekeland [4], and Chiappori, McCann and Nesheim [5] who studied matching and hedonic pricing problems by means of optimal transport techniques. Interestingly, such problems also arise in a very different applied setting in image processing for which we refer to the recent paper of Bernot, Delon, Peyré and Rabin [2].

We will therefore study the following minimization problem

$$\inf_{\nu} \sum_{i=1}^{p} \lambda_i W_2^2(\nu_i, \nu) \tag{1.1}$$

where the ν_i's are probability measures with finite second moments, the λ_i's are positive weights summing to 1, and W_2^2 denotes the squared 2-Wasserstein distance. We will establish existence and uniqueness (provided one of the ν_i's vanishes on small sets) of the solution to (1.1), and we will naturally refer to it as the barycenter of the measures ν_i's with weights λ_i's. We will also provide characterization and regularity results, and finally discuss some examples.

In section 2, we introduce a dual problem to (1.1) from which optimality conditions and uniqueness of the barycenter are derived in section 3. Section 4 relates the barycenter problem to the quadratic multi-marginal optimal transport problem considered by Gangbo and Święch in [8]. In Section 5, we establish an L^∞ regularity results for the barycenter. Section 6 is devoted to some examples. Finally, section 7 discusses various convexity properties of functionals of measures.

2 The primal problem and its dual

The space of continuous functions with at most quadratic growth,

$$Y := (1 + |\ .\ |^2) C_b(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d)\ :\ \frac{f}{1 + |\ .\ |^2} \text{ is bounded } \right\},$$

will be equipped with the norm

$$\|f\|_Y := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + |x|^2}.$$

Let X be the closed subspace of Y given by

$$X := (1 + |\ .\ |^2) C_0(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d)\ :\ \lim_{|x| \to \infty} \frac{f(x)}{1 + |x|^2} = 0 \right\}.$$

We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of bounded Radon measures on $\mathbb{R}^d$, as usual identified with the dual of $C_0(\mathbb{R}^d)$ (space of continuous functions that vanish at infinity) and by $\mathcal{M}_+^1(\mathbb{R}^d)$ the set of Radon probability measures on $\mathbb{R}^d$. We shall naturally identify the dual of X with

$$X' = \{\mu \in \mathcal{M}(\mathbb{R}^d) \,:\, (1+|x|^2)\mu \in \mathcal{M}(\mathbb{R}^d)\}.$$

Given two probability measures with finite second moments μ and ν (i.e. μ and ν in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$), the 2-Wasserstein distance between μ and ν, $W_2(\mu,\nu)$, is defined as the value of the following Monge-Kantorovich optimal transportation problem:

$$W_2^2(\mu,\nu) := \inf\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y),\ \gamma \in \Pi(\mu,\nu)\right\} \tag{2.1}$$

where $\Pi(\mu,\nu)$ denotes the set of transport plans between μ and ν i.e. the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having μ and ν as marginals.

Given an integer $p \geq 2$, p probability measures $\nu_1, ..., \nu_p$ in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ and p real numbers $\lambda_1, ...\lambda_p$ such that $\lambda_i > 0$ and $\sum_{i=1}^p \lambda_i = 1$, we are interested in the following problem:

$$(\mathcal{P}) \quad \inf_{\nu \in \mathcal{M}_+^1(\mathbb{R}^d)\cap X'} J(\nu) = \sum_{i=1}^p \frac{\lambda_i}{2} W_2^2(\nu_i, \nu). \tag{2.2}$$

In analogy with the Euclidean case, a solution of the previous problem will be called the barycenter of the probabilities ν_i with weights λ_i.

To study existence and uniqueness of the barycenter, we introduce the dual (this terminology will be justified later on) of $(\mathcal{P})$,

$$(\mathcal{P}_0^*) \quad \sup\left\{F(f_1,...,f_p) = \sum_{i=1}^p \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i \ : \sum_{i=1}^p f_i = 0,\ f_i \in X\right\} \tag{2.3}$$

where

$$S_\lambda f(x) := \inf_{y\in\mathbb{R}^d} \left\{\frac{\lambda}{2}|x-y|^2 - f(y)\right\},\ \ \forall x \in \mathbb{R}^d,\ f \in X,\ \lambda > 0. \tag{2.4}$$

For $i \in \{1, \cdots, p\}$, we define

$$H_i(f) := -\int_{\mathbb{R}^d} S_{\lambda_i} f(x) d\nu_i(x).$$

Note that $S_\lambda f$ defined as above may take the value $-\infty$. It is easy to check that H_i is convex and l.s.c. on Y. By definition, the Legendre-Fenchel transform of H_i is

$$\begin{aligned} H_i^*(\nu) &:= \sup_{f\in X}\left\{\int_{\mathbb{R}^d} f d\nu - H_i(f)\right\} \\ &= \sup_{f\in X}\left\{\int_{\mathbb{R}^d} f d\nu + \int_{\mathbb{R}^d} S_{\lambda_i} f d\nu_i\right\},\ \forall \nu\in X'. \end{aligned}$$

Since the supremum in $(\mathcal{P}_0^*)$ may not be attained, we consider its "relaxed" problem

$$(\mathcal{P}^*)\quad \sup\left\{F(f_1,...,f_p)\ :\sum_{i=1}^p f_i = 0,\ f_i\in Y\right\}.$$

Now, let $\nu\in\mathcal{M}_+^1(\mathbb{R}^d)\cap X'$, $\gamma_i\in\Pi(\nu_i,\nu)$ and $(f_1,...,f_p)\in X^p$ which sum to 0. Integrating the inequality

$$S_{\lambda_i} f_i(x_i) + f_i(y) \le \frac{\lambda_i}{2}|x_i - y|^2$$

with respect to γ_i and summing over i, we have after using that $\sum_{i=1}^p f_i = 0$,

$$\sum_{i=1}^p \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i \le \sum_{i=1}^p \frac{\lambda_i}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} |x_i - y|^2 d\gamma_i(x_i,y).$$

We then deduce that

$$\inf(\mathcal{P}) \ge \sup(\mathcal{P}^*) \ge \sup(\mathcal{P}_0^*). \tag{2.5}$$

To prove the converse inequality, we first establish the following intermediate result:

Lemma 2.1. *For every $\nu\in X'$, one has*

$$H_i^*(\nu) = \begin{cases} \frac{\lambda_i}{2}W_2^2(\nu_i,\nu) & \textit{if } \nu\in X'\cap\mathcal{M}_+^1(\mathbb{R}^d) \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. We first prove that $H_i^*(\nu) = +\infty$ whenever $\nu\in X'\setminus\mathcal{M}_+^1(\mathbb{R}^d)$. Indeed, if ν is not positive, there exists $f\in X$, $f\le 0$ such that $\int_{\mathbb{R}^d} f d\nu > 0$. Then $S_{\lambda_i}(tf)\ge 0$ for every $t\ge 0$, and we have

$$H_i^*(\nu) \ge \sup_{t\ge 0} t\int_{\mathbb{R}^d} f d\nu = +\infty.$$

But if $\nu \in X'$ is positive and $\nu(\mathbb{R}^d) \neq 1$, say $\nu(\mathbb{R}^d) < 1$, using $f(x) = -t$ with $t \geq 0$ in $H_i^*(\nu)$, we have

$$H_i^*(\nu) \geq \sup_{t\geq 0} t\left(-\int_{\mathbb{R}^d} d\mu + \int_{\mathbb{R}^d} d\nu_i\right) = \big(1-\nu(\mathbb{R}^d)\big)\sup_{t\geq 0} t = +\infty.$$

We conclude that $H_i^*(\nu) = +\infty$ whenever $\nu \in X' \setminus \mathcal{M}_+^1(\mathbb{R}^d)$.

Now let $\nu \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$. The well-known Kantorovich duality formula (see for instance [13]) reads as:

$$\begin{aligned}\frac{\lambda_i}{2}W_2^2(\nu,\nu_i) &= \sup\left\{\int_{\mathbb{R}^d} f d\nu + \int_{\mathbb{R}^d} g d\nu_i,\ f,\ g \in C_b,\ f(x)+g(y) \leq \frac{\lambda_i}{2}|x-y|^2\right\}\\ &= \sup_{f\in X}\left\{\int_{\mathbb{R}^d} f d\nu + \int_{\mathbb{R}^d} S_{\lambda_i} f d\nu_i\right\} = H_i^*(\nu). \qquad (2.6)\end{aligned}$$

This ends the proof.

□

We then easily deduce the following duality result:

Proposition 2.2.

$$\inf(\mathcal{P}) = \sup(\mathcal{P}_0^*) = \sup(\mathcal{P}^*).$$

Proof. Thanks to lemma 2.1, we have on the one hand

$$\inf(\mathcal{P}) = \inf_{X'}\sum_{i=1}^p H_i^* = -\left(\sum_{i=1}^p H_i^*\right)^*(0).$$

Defining H as the inf-convolution of H_1,..., H_p,

$$H(f) = \left(\mathop{\square}_{i=1}^p H_i\right)(f) := \inf\left\{\sum_{i=1}^p H_i(f_i)\ f_i \in X,\ \sum_{i=1}^p f_i = f\right\},\ \forall f \in X,$$

we have on the other hand

$$\sup(\mathcal{P}_0^*) = -H(0).$$

It is a well-known fact of convex analysis (see [7]) that

$$H^* = \left(\mathop{\square}_{i=1}^p H_i\right)^* = \sum_{i=1}^p H_i^*.$$

Then

$$\inf(\mathcal{P}) = -H^{**}(0) \geq -H(0) = \sup(\mathcal{P}_0^*).$$

Hence, using (2.5), the desired result amounts to prove that $H(0) = H^{**}(0)$. And since H is convex, this is equivalent (see [7]) to H being l.s.c. at 0 for the norm topology of X (i.e. the topology induced by that of Y). It is therefore sufficient to show that H is continuous at 0 for the norm topology of Y. To prove that, we rewrite H_i as

$$H_i(f_i) = \int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} \left\{ f_i(y) - \frac{\lambda_i}{2}|x-y|^2 \right\} d\nu_i(x),$$

which yields in particular

$$H_i(f_i) \geq f_i(0) - \frac{\lambda_i}{2} \int_{\mathbb{R}^d} |x|^2 d\nu_i(x)$$

so that

$$H(f) \geq f(0) - \sum_{i=1}^{p} \frac{\lambda_i}{2} \int_{\mathbb{R}^d} |x|^2 d\nu_i(x) > -\infty, \ \forall f \in X.$$

Now, let $f \in Y$ be such that $4\|f\|_Y \leq p \min\{\lambda_1, ..., \lambda_p\}$. Choosing $f_i = f/p$ in $H(f)$, we have

$$\begin{aligned} H(f) &\leq \sum_{i=1}^{p} H_i\left(\frac{f}{p}\right) \leq \sum_{i=1}^{p} \int_{\mathbb{R}^d} \sup_{y \in \mathbb{R}^d} \left\{ \frac{\lambda_i}{4}(1+|y|^2) - \frac{\lambda_i}{2}|x-y|^2 \right\} d\nu_i(x) \\ &= \sum_{i=1}^{p} \int_{\mathbb{R}^d} \left(\frac{\lambda_i}{4} + \frac{\lambda_i}{2}|x|^2 \right) d\nu_i(x) \\ &= \frac{1}{4} + \sum_{i=1}^{p} \frac{\lambda_i}{2} \int_{\mathbb{R}^d} |x|^2 d\nu_i(x). \end{aligned}$$

Note that we use in the above computations that $f_i(y) \leq \frac{p\lambda_i}{4}(1+|y|^2)$ and $\sum_{i=1}^{p} \lambda_i = 1$. We thus proved that the convex function H never takes the value $-\infty$ and is bounded from above in a neighbourhood of 0 in Y. By a standard convex analysis result (see [7]), it is therefore continuous at 0. Hence, $H(0) = H^{**}(0)$ and the result follows.

□

The next proposition gives existence of an optimizer to the primal problem $(\mathcal{P})$ and its dual $(\mathcal{P}^*)$.

Proposition 2.3. *Both problems* $(\mathcal{P})$ *and* $(\mathcal{P}^*)$ *have solutions.*

Proof. Let ν^n be a minimizing sequence of $(\mathcal{P})$. It is easy (for instance, by using (2.6)) to check that $\int_{\mathbb{R}^d} |x|^2 d\nu^n(x)$ is bounded; hence ν^n is tight.

It then follows from Prokhorov's Theorem (see [6]) that a (non relabeled) subsequence converges narrowly to some $\nu \in \mathcal{M}_+^1(\mathbb{R}^d)$, and it is easy to check that $\nu \in X'$. Using (2.6), we then immediately get $J(\nu) \leq \liminf J(\nu^n) = \inf(\mathcal{P})$, which shows that $(\mathcal{P})$ has a minimizer.

Proving the existence of a solution to $(\mathcal{P}^*)$ requires some preliminaries. First we show that in $(\mathcal{P}^*)$, one can choose the f_i such that $(S_{\lambda_i} \circ S_{\lambda_i}) f_i = f_i$ for all $i = 1, \cdots, p-1$. Indeed, if $(f_1, ..., f_p)$ is admissible for $(\mathcal{P}^*)$, then setting $g_i := S_{\lambda_i} f_i$, $h_i := S_{\lambda_i} g_i$ for all i, it is easy to check that $f_i \leq h_i$, $h_i \in X$ and $S_{\lambda_i} f_i = S_{\lambda_i} h_i$. Now, define $(\widetilde{f}_1, ..., \widetilde{f}_{p-1}) := (h_1, ..., h_{p-1})$ and $\widetilde{f}_p := -\sum_{i=1}^{p-1} \widetilde{f}_i$. Since $f_i \leq \widetilde{f}_i$ for $i = 1, ..., p-1$, then $\widetilde{f}_p \leq -\sum_{i=1}^{p-1} f_i = f_p$. For $i = 1, ..., p-1$, we then have $S_{\lambda_i} \widetilde{f}_i = S_{\lambda_i} f_i$, and since S_{λ_p} is order-reversing, then $S_{\lambda_p} \widetilde{f}_p \geq S_{\lambda_p} f_p$ which shows that $F(\widetilde{f}_1, ..., \widetilde{f}_p) \geq F(f_1, ..., f_p)$. This proves that in $(\mathcal{P}^*)$, one may assume that $f_i = S_{\lambda_i} g_i$ with $g_i = S_{\lambda_i} f_i$ for $i = 1, ..., p-1$. The relation $f_i = S_{\lambda_i} g_i$ can equivalently be written as

$$\frac{\lambda_i}{2}|y|^2 - f_i(y) = \sup_{x \in \mathbb{R}^d} \left\{ \lambda_i x \cdot y - \frac{\lambda_i}{2}|x|^2 + g_i(x) \right\}$$

which implies that $\frac{\lambda_i}{2}|.|^2 - f_i$ is convex for $i = 1, ..., p-1$. Finally, we remark that, since the functional F and the constraint of $(\mathcal{P}^*)$ are invariant, when one adds constants that sum to 0 to the f_i's, there is no loss of generality in assuming $f_i(0) = 0$ for all i.

Now let $f^n := (f_1^n, .., f_p^n)$ be a maximizing sequence for $(\mathcal{P}^*)$ which, as explained before, can be chosen such that $f^n(0) = 0$ and $f_i^n = S_{\lambda_i} g_i^n$ for $i = 1, ..., p-1$ with $g_i^n := S_{\lambda_i} f_i^n$ for $i = 1, ..., p$. In particular, $\frac{\lambda_i}{2}|.|^2 - f_i^n$ is convex for $i = 1, ..., p-1$ and every n. Using the fact that $f_i^n(0) = 0$ and $g_i^n := S_{\lambda_i} f_i^n$ we then have

$$g_i^n(x) \leq \frac{\lambda_i}{2}|x|^2. \tag{2.7}$$

Since f^n is a maximizing sequence for $(\mathcal{P}^*)$, there exists a constant C_1 such that

$$\sum_{i=1}^p \int_{\mathbb{R}^d} g_i^n(x) d\nu_i(x) \geq C_1, \ \forall n. \tag{2.8}$$

Then (2.7) and (2.8) imply that for all $j = 1, \cdots, p$,

$$\begin{aligned} \int_{\mathbb{R}^d} g_j^n(x) d\nu_j(x) &\geq C_1 - \sum_{i \neq j} \int_{\mathbb{R}^d} g_i^n(x) d\nu_i(x) \\ &\geq C_1 - \sum_{i \neq j} \frac{\lambda_i}{2} \int_{\mathbb{R}^d} |x|^2 d\nu_i(x). \end{aligned} \tag{2.9}$$

And since the ν_i have finite second moments, we deduce from (2.7) and (2.9) that the integrals $\int_{\mathbb{R}^d} g_i^n(x)d\nu_i(x)$ are bounded. Now integrating the inequality $f_i^n(y) \le \lambda_i|x-y|^2/2 - g_i^n(x)$ with respect to $\nu_i(x)$ and using the bound on $\int_{\mathbb{R}^d} g_i^n(x)d\nu_i(x)$, we see that there exists a constant C_2 such that

$$f_i^n(y) \le C_2(1+|y|^2),\ \forall i,\ \forall n,\ \forall y \in \mathbb{R}^d. \tag{2.10}$$

Applying the above inequality to f_p^n and recalling that $f_p^n = -\sum_{i=1}^{p-1} f_i^n$, we also have

$$\sum_{i=1}^{p-1} f_i^n(y) \ge -C_2(1+|y|^2),\ \forall n,\ \forall y \in \mathbb{R}^d.$$

Then we deduce by using again (2.10) that

$$f_i^n(y) \ge -C_2(1+|y|^2) - \sum_{j\neq i} f_j^n(y) \ge -pC_2(1+|y|^2),\ \forall i,\ \forall n,\ \forall y \in \mathbb{R}^d.$$

The subsequent inequality combined with (2.10) show that f_i^n is bounded in Y for every i. Since $\frac{\lambda_i}{2}|.|^2 - f_i^n$ is convex for every n and $i = 1, ..., p-1$, passing to a subsequence if necessary, we may assume that f^n converges uniformly on compact subsets to some $f = (f_1, ...f_p)$ with each $f_i \in Y$. Finally, the fact that f solves $(\mathcal{P}^*)$ follows from Fatou's Lemma, the positivity of $\frac{\lambda_i}{2}|.|^2 - g_i^n$ and the inequality

$$\begin{aligned}\limsup_n g_i^n(x) &\le \inf_{y\in\mathbb{R}^d}\left\{\limsup_n \left(\frac{\lambda_i}{2}|x-y|^2 - f_i^n(y)\right)\right\}\\ &= \inf_{y\in\mathbb{R}^d}\left\{\frac{\lambda_i}{2}|x-y|^2 - f_i(y)\right\} = S_{\lambda_i} f_i(x).\end{aligned}$$

□

3 Characterization of barycenters

With the results of the previous section at hand, namely,

$$\min(\mathcal{P}) = \max(\mathcal{P}^*),$$

our aim now is to further exploit this duality to characterize the barycenters i.e. the solution of $(\mathcal{P})$.

Let $(f_1, ..., f_p)$ be a solution of $(\mathcal{P}^*)$. It follows from the previous duality relation that there exists $\nu \in \mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ solution of $(\mathcal{P}^*)$ such that

$$\sum_{i=1}^p \frac{\lambda_i}{2} W_2^2(\nu_i, \nu) = \sum_{i=1}^p \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i = \sum_{i=1}^p \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i + \sum_{i=1}^p \int_{\mathbb{R}^d} f_i d\nu. \tag{3.1}$$

And since $\lambda_i W_2^2(\nu_i,\nu)/2 \geq \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i + \int_{\mathbb{R}^d} f_i d\nu$ because of (2.6), then (3.1) is equivalent to

$$\frac{\lambda_i}{2} W_2^2(\nu_i,\nu) = \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i + \int_{\mathbb{R}^d} f_i d\nu,\ \forall i \in \{1,...,p\}. \tag{3.2}$$

Now, let $\gamma_i \in \Pi(\nu_i,\nu)$ be an optimal transportation plan between ν_i and ν (i.e. $\gamma_i \in \Pi(\nu_i,\nu)$ and $W_2^2(\nu_i,\nu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma_i(x,y)$). We deduce from (3.2) and $\gamma_i \in \Pi(\nu_i,\nu)$ that

$$\begin{aligned}\frac{\lambda_i}{2} W_2^2(\nu_i,\nu) = \frac{\lambda_i}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma_i(x,y) &= \int_{\mathbb{R}^d} S_{\lambda_i} f_i d\nu_i + \int_{\mathbb{R}^d} f_i d\nu \\ &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big(S_{\lambda_i} f_i(x) + f_i(y) \Big) d\gamma_i(x,y).\end{aligned}$$

But since by definition of $S_{\lambda_i} f_i(x)$, $\frac{\lambda_i}{2}|x-y|^2 \geq S_{\lambda_i} f_i(x) + f_i(y)$, we get:

$$\frac{\lambda_i}{2}|x-y|^2 = S_{\lambda_i} f_i(x) + f_i(y), \quad \gamma_i\text{-a.e.} \tag{3.3}$$

We have already noticed that $S_{\lambda_i}\Big(S_{\lambda_i} f_i\Big) \geq f_i$. So (3.3) implies that for γ_i-a.e.(x,y), one has $f_i(y) = \frac{\lambda_i}{2}|x-y|^2 - S_{\lambda_i} f(x) \geq S_{\lambda_i}\Big(S_{\lambda_i} f_i\Big)(y)$, so that $f_i = S_{\lambda_i}\Big(S_{\lambda_i} f_i\Big)$ ν-a.e.. Therefore using the constraint $\sum_{i=1}^p f_i = 0$ of $(\mathcal{P}^*)$, we have

$$\sum_{i=1}^p S_{\lambda_i}\Big(S_{\lambda_i} f_i\Big) \geq 0,\ \sum_{i=1}^p S_{\lambda_i}\Big(S_{\lambda_i} f_i\Big) = 0 \quad \nu\text{-a.e.} \tag{3.4}$$

Now, we consider the convex function ϕ_i defined by

$$\lambda_i \phi_i(x) := \frac{\lambda_i}{2}|x|^2 - S_{\lambda_i} f_i(x), \tag{3.5}$$

and we denote by $\partial\phi_i$ the graph of its subdifferential, i.e.

$$\partial\phi_i := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \phi_i(x) + \phi_i^*(y) = x \cdot y\right\},$$

where ϕ_i^* denotes the conjugate of ϕ_i, i.e.

$$\lambda_i \phi_i^*(y) := \frac{\lambda_i}{2}|y|^2 - S_{\lambda_i}\Big(S_{\lambda_i} f_i\Big)(y).$$

Combining (3.3) and (3.4), we have that the support of γ_i is included in $\partial\phi_i$, and

$$\sum_{i=1}^p \lambda_i \phi_i^*(y) \leq \frac{|y|^2}{2},\ \forall y \in \mathbb{R}^d, \text{ with equality } \nu\text{-a.e.} \tag{3.6}$$

Moreover, it is easy to check that ϕ_i solves

$$\sup \left\{ \int_{\mathbb{R}^d} \phi(x) d\nu_i(x) + \int_{\mathbb{R}^d} \phi^*(y) d\nu(y) : \quad \phi \text{ convex l.s.c. } \right\}. \tag{3.7}$$

The previous conditions are of course very related to classical results from optimal transportation theory. To be more precise, let us first recall some definitions.

Definition 3.1. *Let $\mu \in \mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ and σ be a Borel map $\mathbb{R}^d \to \mathbb{R}^d$. The push-forward of μ through σ is the measure denoted $\sigma \sharp \mu$ defined by*

$$\int_{\mathbb{R}^d} f(y) d\Big(\sigma \sharp \mu\Big)(y) = \int_{\mathbb{R}^d} f(\sigma(x)) d\mu(x), \forall f \in C_b(\mathbb{R}^d).$$

Definition 3.2. *A probability measure $\mu \in \mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ is said to vanish on small sets if and only if $\mu(A) = 0$ for every Borel set A of $\mathbb{R}^d$, having Hausdorff dimension less than $d-1$.*

Let us now recall the following classical results from optimal transportation theory which initially are due to Brenier and later generalized by McCann (see [3], [10] or [13]):

Proposition 3.3. *Let μ and ν be in $\mathcal{M}_+^1(\mathbb{R}^d) \cap X'$, and $\gamma \in \Pi(\mu, \nu)$. Then γ is an optimal transportation plan between μ and ν if and only if the support of γ is included in $\partial\phi$ for some convex l.s.c. function ϕ; in this case ϕ solves*

$$\sup \left\{ \int_{\mathbb{R}^d} \phi(x) d\mu(x) + \int_{\mathbb{R}^d} \phi^*(y) d\nu(y) : \quad \phi \textit{ convex l.s.c. } \right\}. \tag{3.8}$$

If in addition μ vanishes on small sets, then there is a unique optimal transportation plan γ which is of the form $\gamma = (\mathrm{id}, \nabla\phi)\sharp\mu$ with ϕ convex. Uniqueness also holds in the sense that if $\nabla\psi\sharp\mu = \nabla\phi\sharp\mu = \nu$ and ψ is convex, then $\nabla\psi = \nabla\phi$ μ-almost everywhere.

Definition 3.4. *Let μ and ν be in $\mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ such that μ vanishes on small sets. Then the Brenier's map transporting μ to ν is the unique (up to μ-a.e. equivalence) map of the form $\nabla\phi$ with ϕ convex such that $\nu = \nabla\phi\sharp\mu$.*

Note that although the Brenier's map $\nabla\phi$ is uniquely defined μ-a.e., the potential ϕ is not (for instance, it is easy to build counter-examples if the support of μ is disconnected).

We then deduce the following uniqueness result and characterization of the barycenter:

Proposition 3.5. *Assume that there is an index $i \in \{1, ...p\}$ such that ν_i vanishes on small sets. Then $(\mathcal{P})$ admits a unique solution ν which is given by $\nu = \nabla\phi_i \sharp \nu_i$ where ϕ_i is the convex potential defined by (3.5).*

This enables us to define unambiguously barycenters as follows:

Definition 3.6. *Given $(\nu_1, ..., \nu_p)$ in $\mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ one of which vanishes on small sets, and given positive reals $(\lambda_1, ..., \lambda_p)$ that sum to 1, the barycenter of $(\nu_1, ..., \nu_p)$ with weights $(\lambda_1, ..., \lambda_p)$ is the unique solution of $(\mathcal{P})$. It will be denoted by $(\mathrm{bar}(\nu_i, \lambda_i)_{i=1,...,p})$.*

Proof. Let $\gamma_i \in \Pi(\nu_i, \nu)$ be an optimal transportation plan between ν_i and ν. Then the support of γ_i is included in $\partial\phi_i$ where the convex potential ϕ_i is defined by (3.5) (in particular it does not depend on ν but only on f_i). Since ϕ_i is convex and ν_i vanishes on small sets, ϕ_i is differentiable ν_i a.e.. It is then easy to see that this implies that $\nu = \nabla\phi_i \sharp \nu_i$, which proves the claim. □

Remark 3.7. If all the ν_i's vanish on small sets, defining ϕ_i by (3.5) (with $(f_1, ..., f_p)$ an arbitrary solution of $(\mathcal{P}^*)$) we see that $\nu = \nabla\phi_i \sharp \nu_i$ for every i. The condition that $\nabla\phi_i \sharp \nu_i$ does not depend on i then appears as an optimality condition for $(\mathcal{P}^*)$.

The following proposition further characterizes the barycenter.

Proposition 3.8. *Assume that ν_i vanishes on small sets for every $i = 1, .., p$, and let $\nu \in \mathcal{M}_+^1(\mathbb{R}^d) \cap X'$. Then the following conditions are equivalent:*

1. *ν solves $(\mathcal{P})$.*
2. *$\nu = \nabla\phi_i \sharp \nu_i$ for every i, where ϕ_i is defined by (3.5).*
3. *There exist convex potentials ψ_i such that $\nabla\psi_i$ is the Brenier's map transporting ν_i to ν, and a constant C such that*

$$\sum_{i=1}^{p} \lambda_i \psi_i^*(y) \leq C + \frac{|y|^2}{2}, \ \forall y \in \mathbb{R}^d, \text{ with equality } \nu\text{-a.e.} \tag{3.9}$$

Proof. The equivalence between 1. and 2. follows from proposition 3.5. Next we prove that 1. is equivalent to 3. Indeed, if 1. holds, then it is enough to take $\psi_i = \phi_i$ where ϕ_i is defined by (3.5); then (3.9) directly follows from (3.6). Finally, if 3. is satisfied (with $C = 0$, say), then (3.9) together with the Kantorovich duality formula give

$$\sum_{i=1}^{p} \frac{\lambda_i}{2} W_2^2(\nu_i, \nu) = \sum_{i=1}^{p} \lambda_i \int_{\mathbb{R}^d} \Big(\frac{|x|^2}{2} - \psi_i^*(x)\Big) d\nu_i(x).$$

Now let $\mu \in \mathcal{M}_+^1(\mathbb{R}^d) \cap X'$. Using the inequality $x \cdot y \leq \psi_i(x) + \psi_i^*(y)$, i.e.

$$\lambda_i \frac{|x|^2}{2} - \lambda_i \psi_i(x) + \lambda_i \frac{|y|^2}{2} - \lambda_i \psi_i^*(y) \leq \frac{\lambda_i}{2}|x-y|^2,$$

we have after integration

$$\frac{\lambda_i}{2} W_2^2(\nu_i, \mu) \geq \lambda_i \int_{\mathbb{R}^d} \Big(\frac{|x|^2}{2} - \psi_i(x)\Big) d\nu_i(x) + \lambda_i \int_{\mathbb{R}^d} \Big(\frac{|y|^2}{2} - \psi_i^*(y)\Big) d\mu(y).$$

Summing these inequalities over i, and using (3.9), we get

$$\begin{aligned}\sum_{i=1}^p \frac{\lambda_i}{2} W_2^2(\nu_i, \mu) &\geq \sum_{i=1}^p \lambda_i \int_{\mathbb{R}^d} \Big(\frac{|x|^2}{2} - \psi_i(x)\Big) d\nu_i(x) + \int_{\mathbb{R}^d} \Big(\frac{|y|^2}{2} - \sum_{i=1}^p \lambda_i \psi_i^*(y)\Big) d\mu(y) \\ &\geq \sum_{i=1}^p \lambda_i \int_{\mathbb{R}^d} \Big(\frac{|x|^2}{2} - \psi_i(x)\Big) d\nu_i(x) = \sum_{i=1}^p \frac{\lambda_i}{2} W_2^2(\nu_i, \nu)\end{aligned}$$

which proves that ν solves $(\mathcal{P})$. □

Remark 3.9. If ν and the potentials ψ_i satisfy the third statement of the previous proposition, then the support of ν, $\mathrm{Supp}(\nu)$, is included in the contact set where the convex function $\varphi := \sum_{i=1}^p \lambda_i \psi_i^*$ agrees with its quadratic majorant $C + \frac{|.|^2}{2}$. Now let us remark that at such a contact point x, we have

$$\sum_{i=1}^p \lambda_i \partial \psi_i^*(x) \subset \partial \varphi(x) \subset \{x\}$$

so that each potential ψ_i^* is differentiable at x. The potentials ψ_i^* are therefore differentiable on $\mathrm{Supp}(\nu)$ and satisfy on this set the relation

$$\sum_{i=1}^p \lambda_i \nabla \psi_i^* = \mathrm{id}\,. \tag{3.10}$$

We also remark that if (3.10) holds everywhere for the Brenier's maps $\nabla \psi_i$ transporting ν_i to ν, then ν is optimal for $(\mathcal{P})$.

4 Multi-marginal formulation

Our aim in this section is to prove that $(\mathcal{P})$ is equivalent to a linear programming problem of multi-marginal optimal transportation type similar to the one solved by Gangbo and Święch in [8].

For every $x := (x_1, ..., x_p) \in (\mathbb{R}^d)^p$, we define

$$T(x) := \sum_{i=1}^{p} \lambda_i x_i. \tag{4.1}$$

Of course the Euclidean barycenter $T(x)$ is characterized by the property

$$\sum_{i=1}^{p} \lambda_i |x_i - T(x)|^2 = \inf_{y\in\mathbb{R}^d} \left\{ \sum_{i=1}^{p} \lambda_i |x_i - y|^2 \right\}. \tag{4.2}$$

Let us now introduce the multi-marginal optimal transportation problem

$$\inf \left\{ \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{p} \frac{\lambda_i}{2} |x_i - T(x)|^2 \Big) d\gamma(x_1, ...x_p),\ \gamma \in \Pi(\nu_1, ..., \nu_p) \right\} \tag{4.3}$$

where $\Pi(\nu_1, ..., \nu_p)$ is the set of probability measures on $(\mathbb{R}^d)^p$ having $\nu_1, ..., \nu_p$ as marginals. Developing the squares in (4.3), it is easy to see that (4.3) is equivalent to

$$(\mathcal{Q})\ \sup \left\{ \int_{(\mathbb{R}^d)^p} \Big(\sum_{1\le i<j\le p} \lambda_i \lambda_j x_i \cdot x_j \Big) d\gamma(x_1, ...x_p),\ \gamma \in \Pi(\nu_1, ..., \nu_p) \right\}. \tag{4.4}$$

The previous multi-marginal problem $(\mathcal{Q})$ has been solved by Gangbo and Święch in [8]. As usual, a key tool is the dual problem

$$(\mathcal{Q}^*)\ \inf \left\{ \sum_{i=1}^{p} \int_{\mathbb{R}^d} g_i d\nu_i,\ \sum_{i=1}^{p} g_i(x_i) \ge \sum_{1\le i<j\le p} \lambda_i \lambda_j x_i \cdot x_j,\ \forall x \in (\mathbb{R}^d)^p \right\}. \tag{4.5}$$

As the classical optimal transportation problem with quadratic cost is solved by Brenier, $(\mathcal{Q})$ and $(\mathcal{Q}^*)$ have a very special structure that we next discuss heuristically. Firstly, in $(\mathcal{Q})$, one can restrict to potentials that satisfy

$$g_i(x_i) = \sup_{(x_j)_{j\neq i}} \Big\{ \frac{1}{2} \sum_{1\le k\neq j\le p} \lambda_k \lambda_j x_k \cdot x_j - \sum_{j\neq i} g_j(x_j) \Big\} \tag{4.6}$$

which, in particular, implies the convexity of the potentials g_i's. Secondly, the duality relation between $(\mathcal{Q})$ and $(\mathcal{Q}^*)$ expresses that, γ solves $(\mathcal{Q})$ and $(g_1, ..., g_p)$ solves $(\mathcal{Q}^*)$ if and only if

$$\sum_{i=1}^{p} g_i(x_i) = \frac{1}{2} \sum_{1\le i\neq j\le p} \lambda_i \lambda_j x_i \cdot x_j,\ \gamma\text{-a.e.}. \tag{4.7}$$

Finally, if, in addition the potentials g_i's are differentiable γ-a.e., then one can deduce from (4.6) and (4.7) that for γ-a.e. $x = (x_1, ..., x_d)$, one has

$$\nabla g_i(x_i) = \lambda_i \sum_{j \neq i} \lambda_j x_j$$

which can be rewritten as

$$\nabla \Big(\frac{\lambda_i}{2} |\ .\ |^2 + \frac{g_i}{\lambda_i} \Big)(x_i) = \sum_{j=1}^{p} \lambda_j x_j = \nabla \Big(\frac{\lambda_1}{2} |\ .\ |^2 + \frac{g_1}{\lambda_1} \Big)(x_1)$$

or in a more explicit way

$$x_i = \nabla \Big(\frac{\lambda_i}{2} |\ .\ |^2 + \frac{g_i}{\lambda_i} \Big)^* \circ \nabla \Big(\frac{\lambda_1}{2} |\ .\ |^2 + \frac{g_1}{\lambda_1} \Big)(x_1).$$

This (formally) yields that the optimal γ is in fact supported by the graph of a map of the form $x_1 \to (x_1, \nabla u_i^*(\nabla u_1(x_1)), ..., \nabla u_p^*(\nabla u_1(x_1)))$ for some potentials u_i such that $u_i - \lambda_i |\ .\ |^2/2$ is convex. The previous discussion being informal, we refer to the paper of Gangbo and Święch [8] for the details. Here, we simply summarize their main results as follows:

Theorem 4.1. *Assume that ν_i vanishes on small sets for $i = 1, .., p$. Then $(\mathcal{Q})$ admits a unique solution $\gamma \in \Pi(\nu_1, ..., \nu_p)$. Moreover, γ is of the form $\gamma = (T_1^1, ..., T_p^1)\sharp\nu_1$ with $T_i^1 = \nabla u_i^* \circ \nabla u_1$ for $i = 1, ..., p$ where the u_i's are strictly convex potentials defined by*

$$u_i(x) := \frac{\lambda_i}{2}|x|^2 + \frac{g_i(x)}{\lambda_i},\ \forall x \in \mathbb{R}^d \tag{4.8}$$

and $(g_1, ..., g_p)$ are convex potentials that solve $(\mathcal{Q}^*)$.

In the sequel, we will refer to the maps T_i^1 of the previous theorem as the Gangbo-Święch maps between ν_1 and ν_i. Note that the Gangbo-Święch maps a priori depend on the whole collections of the ν_i's and the weights λ_i's. These maps are transport maps in the sense that $T_i^1\sharp\nu_1 = \nu_i$. Of course, by permuting the indices, one can similarly define the Gangbo-Święch maps $T_i^j := \nabla u_i^* \circ \nabla u_j$ between a reference measure ν_j and ν_i.

The next result gives the precise relationship between our initial barycenter problem $(\mathcal{P})$ and the multi-marginals problem $(\mathcal{Q})$.

Proposition 4.2. *Assume that ν_i vanishes on small sets for $i = 1, .., p$. Then the solution of $(\mathcal{P})$ is given by $\nu = T\sharp\gamma$, where T is defined by (4.1) and γ is the solution of $(\mathcal{Q})$.*

Proof. For every $i \in \{1, ..., p\}$ denote by π_i the i-th canonical projection from $(\mathbb{R}^d)^p$ to $\mathbb{R}^d$ (i.e. $\pi_i(x) = x_i$) and define $\gamma_i := (\pi_i, T)\sharp\gamma$. By construction, $\gamma_i \in \Pi(\nu_i, \nu)$. Then

$$W_2^2(\nu_i, \nu) \leq \int_{(\mathbb{R}^d)^p} |x_i - T(x)|^2 d\gamma(x),$$

and thus,

$$\sum_{i=1}^{p} \frac{\lambda_i}{2} W_2^2(\nu_i, \nu) \leq \int_{(\mathbb{R}^d)^p} \Big(\sum_{i=1}^{p} \frac{\lambda_i}{2} |x_i - T(x)|^2 \Big) d\gamma(x). \tag{4.9}$$

Now let $\mu \in \mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ and $\eta_i \in \Pi(\nu_i, \mu)$ for $i = 1, ..., p$. By the disintegration theorem (see [6]), we can write $\eta_i = \eta_i^y \otimes \mu$ for a Borel family of (conditional) probability measures $(\eta_i^y)_{y \in \mathbb{R}^d}$ which precisely means that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x_i, y) d\eta_i(x_i, y) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(x_i, y) d\eta_i^y(x_i) \Big) d\mu(y), \ \forall f \in C_b^0(\mathbb{R}^d \times \mathbb{R}^d).$$

Now, let us define $\eta \in \mathcal{M}_+^1((\mathbb{R}^d)^{p+1})$ by

$$\int_{(\mathbb{R}^d)^{p+1}} f(x, y) d\eta(x, y) = \int_{\mathbb{R}^d} \Big(\int_{(\mathbb{R}^d)^p} f(x, y) d\eta_1^y(x_1) ... d\eta_p^y(x_p) \Big) d\mu(y)$$

for every test-function $f \in C_b((\mathbb{R}^d)^{p+1})$. Let $\theta \in \mathcal{M}_+^1((\mathbb{R}^d)^p)$ be the canonical projection of η on $(\mathbb{R}^d)^p$ i.e.

$$\int_{(\mathbb{R}^d)^p} f(x) d\theta(x) = \int_{\mathbb{R}^d} \Big(\int_{(\mathbb{R}^d)^p} f(x) d\eta_1^y(x_1) ... d\eta_p^y(x_p) \Big) d\mu(y), \ \forall f \in C_b^0((\mathbb{R}^d)^p).$$

By construction, $\theta \in \Pi(\nu_1, ..., \nu_p)$. Using (4.2) and the fact that γ solves (4.3), we then obtain

$$\begin{aligned}
\sum_{i=1}^{p} \frac{\lambda_i}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_i - y|^2 d\eta_i(x_i, y) &= \int_{(\mathbb{R}^d)^{p+1}} \Big(\sum_{i=1}^{p} \frac{\lambda_i}{2} |x_i - y|^2 \Big) d\eta(x, y) \\
&\geq \int_{(\mathbb{R}^d)^{p+1}} \Big(\sum_{i=1}^{p} \frac{\lambda_i}{2} |x_i - T(x)|^2 \Big) d\eta(x, y) \\
&= \int_{(\mathbb{R}^d)^p} \Big(\sum_{i=1}^{p} \frac{\lambda_i}{2} |x_i - T(x)|^2 \Big) d\theta(x) \\
&\geq \int_{(\mathbb{R}^d)^p} \Big(\sum_{i=1}^{p} \frac{\lambda_i}{2} |x_i - T(x)|^2 \Big) d\gamma(x).
\end{aligned}$$

Since in the previous inequality, the η_i's are arbitrary transport plans between ν_i and μ, we conclude with (4.9).

$\square$

Combining Theorem 4.1 and Proposition 4.2, we see that the barycenter of the (ν_i, λ_i)'s is also characterized by

$$\nu := \Big(\sum_{i=1}^{p} \lambda_i T_i^1\Big)\sharp\nu_1 = \Big(\sum_{i=1}^{p} \lambda_i T_i^j\Big)\sharp\nu_j \tag{4.10}$$

where the T_i^j are the Gangbo-Święch maps between ν_j and ν_i which are given by $T_i^j = \nabla u_i^* \circ \nabla u_j$, where the u_j's are strictly convex potentials defined by (4.8). As an immediate consequence, we deduce that the support of the barycenter $\mathrm{bar}((\nu_i, \lambda_i)_i)$ is included in $\sum_{i=1}^p \lambda_i \mathrm{Supp}(\nu_i)$. Let us also remark that the center of mass of $\mathrm{bar}((\nu_i, \lambda_i)_i)$ is $\sum_{i=1}^p \lambda_i \int_{\mathbb{R}^d} x d\nu_i(x)$.

5 Regularity of the barycenter

The relation between the barycenter and the Gangbo-Święch maps easily enables us to obtain a regularity result on the barycenter.

Theorem 5.1. *Let $(\nu_1, ..., \nu_p)$ in $\mathcal{M}_+^1(\mathbb{R}^d) \cap X'$ vanish on small sets and let $(\lambda_1, ..., \lambda_p)$ be positive reals that sum to 1. Also, assume that $\nu_1 \in L^\infty$ (i.e. is absolutely continuous and has a bounded density with respect to the Lebesgue measure) and define $\overline{\nu} := \mathrm{bar}((\nu_i, \lambda_i))$. Then $\overline{\nu} \in L^\infty$ and more precisely:*

$$\|\overline{\nu}\|_{L^\infty} \le \frac{1}{\lambda_1^d} \|\nu_1\|_{L^\infty}. \tag{5.1}$$

Proof. By proposition 4.2, $\overline{\nu} = \overline{T}\sharp\nu_1$ where

$$\overline{T} = \sum_{i=1}^{p} \lambda_i \nabla u_i^* \circ \nabla u_1 = \lambda_1 \,\mathrm{id} + \sum_{i=2}^{p} \lambda_i \nabla u_i^* \circ \nabla u_1$$

and the potentials u_i are defined as in the previous section, and $u_i - \frac{\lambda_i}{2}|.|^2$ is convex. Hence $u_i^* \in C^{1,1}$, and

$$D^2 u_i \ge \lambda_i \,\mathrm{id},\ D^2 u_i^* \le \frac{1}{\lambda_i} \,\mathrm{id}\,. \tag{5.2}$$

For $\varepsilon > 0$, we regularize u_1 as follows:

$$u_1^\varepsilon := \rho_\varepsilon \star v_1^\varepsilon$$

where $\rho_\varepsilon := \varepsilon^{-d}\rho(./\varepsilon)$, $\rho \in C_c^\infty$, $\rho \geq 0$, $\int \rho = 1$ and $\mathrm{Supp}(\rho) \subset B_1$, and we define v_1^ε as the usual Moreau-Yosida regularization by infimal-convolution:

$$v_1^\varepsilon := \Big(u_1^* + \frac{\varepsilon}{2}|.|^2\Big)^*.$$

By construction, u_1^ε is smooth and strictly convex, and we precisely have

$$\varepsilon \,\mathrm{id} \leq D^2(u_1^\varepsilon)^* \leq \Big(\frac{1}{\lambda_1} + \varepsilon\Big)\,\mathrm{id},\ \varepsilon^{-1}\,\mathrm{id} \geq D^2 u_1^\varepsilon \geq \Big(\frac{\lambda_1}{1+\lambda_1\varepsilon}\Big)\,\mathrm{id}\,. \tag{5.3}$$

Moreover, it is classical to check that ∇u_1^ε converges to ∇u_1 in L^1_{loc}. We can now define the Lipschitz map

$$\overline{T}^\varepsilon = \lambda_1\,\mathrm{id} + \sum_{i=2}^p \lambda_i \nabla u_i^* \circ \nabla u_1^\varepsilon$$

and the image measure $\overline{\nu}^\varepsilon := \overline{T}^\varepsilon \sharp \nu_1$. We first claim that $\overline{T}^\varepsilon$ is injective. Indeed, if $0 = \overline{T}^\varepsilon(x) - \overline{T}^\varepsilon(y)$, taking the scalar product with $\nabla u_1^\varepsilon(x) - \nabla u_1^\varepsilon(y)$ and using the monotonicity of ∇u_i^*, we get

$$\begin{aligned}
0 &= \lambda_1(x-y)\cdot(\nabla u_1^\varepsilon(x) - \nabla u_1^\varepsilon(y)) \\
&\quad + \sum_{i=2}^p \lambda_i\Big(\nabla u_i^*(\nabla u_1^\varepsilon(x)) - \nabla u_i^*(\nabla u_1^\varepsilon(y))\Big)\cdot(\nabla u_1^\varepsilon(x) - \nabla u_1^\varepsilon(y)) \\
&\geq \frac{\lambda_i^2}{1+\lambda_1\varepsilon}|x-y|^2
\end{aligned}$$

which proves that $x = y$. The same argument applies to show that $\overline{T}$ is also injective when restricted to the set of points where u_1 is differentiable.

Since ∇u_1^ε is a smooth diffeomorphism with Jacobian bounded away from zero, and since the maps ∇u_i^* are Lipschitz, the singular set S of points x such that $\sum_{i=2}^p \lambda_i \nabla u_i^*$ fails to be differentiable at $\nabla u_1^\varepsilon(x)$ is Lebesgue negligible. For $x \notin S$, by the usual chain rule, we have

$$D\overline{T}^\varepsilon(x) = \lambda_1\,\mathrm{id} + \sum_{i=2}^p \lambda_i D^2 u_i^*(\nabla u_1^\varepsilon(x)) D^2 u_1^\varepsilon(x)$$

which we rewrite as $\lambda_1\,\mathrm{id} + AB$ where both matrices $A := D^2 u_i^*(\nabla u_1^\varepsilon(x))$ and $B := D^2 u_1^\varepsilon(x)$ are symmetric and positive definite. Now, we claim that this implies

$$\det D\overline{T}_\varepsilon(x) = \det(\lambda_1\,\mathrm{id} + AB) \geq \lambda_1^d. \tag{5.4}$$

Indeed, if $z \in \mathbb{R}^d$ and $\mu \in \mathbb{R}$, we have $ABz = \mu z$ if and only if $(B^{1/2}AB^{1/2})B^{1/2}z = \mu B^{1/2}z$. So AB has a basis of eigenvectors (because $B^{1/2}AB^{1/2}$ has one and $B^{1/2}$ is invertible) and all its eigenvalues are positive since they are eigenvalues of the definite positive matrix $B^{1/2}AB^{1/2}$. Hence $\lambda_1 \,\mathrm{id} + AB$ has a basis of eigenvectors and all its eigenvalues are larger than λ_1, which proves (5.4). By standard results (see for instance lemma 5.5.3 in [1]), we obtain that $\overline{\nu}^\varepsilon \in L^\infty$ and

$$\|\overline{\nu}^\varepsilon\|_{L^\infty} \leq \frac{1}{\lambda_1^d}\|\nu_1\|_{L^\infty}.$$

We easily deduce (5.1) by remarking that for some sequence ε_n converging to 0, $\overline{\nu}_{\varepsilon_n}$ converges narrowly to $\overline{\nu}$ because $\overline{T}^{\varepsilon_n}$ converges a.e. to $\overline{T}$.

□

Remark 5.2. In the previous L^∞ estimate, it is actually not necessary to require that all the measures ν_i vanish on small sets; this assumption just allowed us to use results of [8]. Indeed, if we simply assume that $\nu_1 \in L^\infty$ vanishes on small sets, it follows from proposition 3.5 that $\overline{\nu} = \mathrm{bar}(\nu_i, \lambda_i)_i$ is uniquely defined. Moreover it is easy to check that if we approximate $(\nu_i)_{i=2,..,p}$ by measures that vanish on small sets, the barycenter of the approximations converges narrowly to $\overline{\nu}$. We thus recover the estimate (5.1) by approximation.

6 Examples

6.1 The case $d = 1$

In the unidimensional space case, $d = 1$, the description of the barycenter is quite simple and this is (roughly speaking) due to the fact that gradient of convex functions are simply nondrecreasing functions, and this property is stable by composition. Let $\nu_1, ..., \nu_p$ be nonatomic probability measures on the real line that have finite second moments, and let $\lambda_1, ..., \lambda_p$ be positive reals that sum to 1. From formula (4.10), the barycenter $\overline{\nu} := \mathrm{bar}(\nu_i, \lambda_i)_i$ is given by

$$\overline{\nu} = \Big(\sum_{i=1} \lambda_i T_i^1\Big)\sharp\nu_1$$

where T_i^1 is the Gangbo-Święch map between ν_1 and ν_i. Therefore, T_i^1 is a nondecreasing map that pushes ν_1 forward to ν_i. There is only one such map, and that is of course the Bernier's map which is given by the usual rearrangement or quantile-like formula $T_i^1 := F_i^{-1} \circ F_1$, where F_i is the cumulative function of F_i i.e. $F_i(\alpha) = \nu_i((-\infty, \alpha])$, and F_i^{-1} denotes the generalized inverse of F_i,

$$F_i^{-1}(t) := \inf\{\alpha \ : \ F_i(\alpha) \geq t\}.$$

Therefore, $\mathrm{bar}(\nu_i, \lambda_i)_i$ is simply obtained as the image of ν_1 by the linearly interpolated transport map $\sum_i \lambda_i T_i^1$. Of course, one also has

$$\overline{\nu} = \Big(\sum_{i=1} \lambda_i T_i^j\Big)\sharp\nu_j$$

where T_i^j is the Brenier's map between ν_j and ν_i. The fact that the resulting measure does not depend on j is very specific to the unidimensional case and does not hold in general in higher dimensions.

6.2 The case $p = 2$

In the case of two mesures ν_0 and ν_1 (regular enough), and $t \in (0,1)$, it is reasonable to expect that the barycenter of $(\nu_0, (1-t))$ and (ν_1, t) is McCann's interpolant [11]

$$\nu_t := ((1-t)\mathrm{id} + t\nabla\phi)\sharp\nu_0 = (t\mathrm{id} + (1-t)\nabla\phi^*)\sharp\nu_1$$

where $\nabla\phi$ is the Brenier's map between ν_0 and ν_1. To see this, it is enough to prove that

$$(1-t)f_t + tg_t = \frac{1}{2}|.|^2 + C \tag{6.1}$$

where C is constant, and ∇f_t and ∇g_t are respectively the Brenier's maps between ν_t and ν_0, and ν_t and ν_1, i.e.,

$$f_t = \Big(\frac{(1-t)}{2}|.|^2 + t\phi\Big)^*, \ g_t = \Big(\frac{t}{2}|.|^2 + (1-t)\phi^*\Big)^*.$$

To prove the identity (6.1) (with $C = 0$ in fact), we first write

$$-f_t(p) = \inf_{x\in\mathbb{R}^d}\{(1-t)\frac{|x|^2}{2} - p\cdot x + t\phi(x)\}$$

which, thanks to the Fenchel-Rockafellar duality theorem, can be rewritten as

$$-f_t(p) = \sup_{z\in\mathbb{R}^d}\{-t\phi^*(-z/t) - \frac{1}{2(1-t)}|p+z|^2\}.$$

Therefore

$$\begin{aligned}
-(1-t)f_t(p) &= \sup_{y\in\mathbb{R}^d}\{-t(1-t)\phi^*(y) - \frac{1}{2}|p-ty|^2\} \\
&= -\frac{1}{2}|p|^2 + t\sup_{y\in\mathbb{R}^d}\{p\cdot y - (\frac{t}{2}|y|^2 + (1-t)\phi^*(y))\} \\
&= -\frac{1}{2}|p|^2 + tg_t(p).
\end{aligned}$$

We thus conclude that the set of barycenters of ν_0 and ν_1 is nothing but the geodesic curve $t\in[0,1]\mapsto \nu_t$ given by McCann's interpolation [11].

6.3 The gaussian case

Let us now consider the case where for $i=1,...,p$, $\nu_i = \mathcal{N}(0,S_i)$ i.e. ν_i is a gaussian measure with mean 0 and covariance matrix S_i. We assume that each S_i is positive definite and, given weights $\lambda_i > 0$ that sum to 1, we consider again the barycenter problem (2.2). This gaussian case was already considered by Knott and Smith [9] who suggested an almost explicit construction for the barycenter (which turns out to be gaussian as well). But the existence and uniqueness of the barycenter was not proved in their paper. The following theorem addresses these issues and also gives an explicit construction of the barycenter in this case.

Theorem 6.1. *In the gaussian framework of this paragraph, there is a unique solution $\overline{\nu}$ to (2.2). Moreover, $\overline{\nu}=\mathcal{N}(0,\overline{S})$ where $\overline{S}$ is the unique positive definite root of the matrix equation*

$$\sum_{i=1}^p \lambda_i\Big(S^{1/2}S_iS^{1/2}\Big)^{1/2} = S. \tag{6.2}$$

Proof. **Step 1:** *existence of a solution to (6.2).* Let α_i and β_i denote respectively the smallest and largest eigenvalue of S_i, and α and β be such that

$$\Big(\sum_{i=1}^p\lambda_i\sqrt{\beta_i}\Big)^2 \geq \beta \geq \alpha \geq \sum_{i=1}^p\Big(\lambda_i\sqrt{\alpha_i}\Big)^2.$$

Let $K_{\alpha,\beta}$ be the (convex and compact) set of symmetric matrices S such that $\beta I\geq S\geq \alpha I$. For $S\in K_{\alpha,\beta}$, define

$$F(S) := \sum_{i=1}^p \lambda_i\Big(S^{1/2}S_iS^{1/2}\Big)^{1/2}.$$

It is easy to see that

$$\beta I \geq \sum_{i=1}^{p} \lambda_i \sqrt{\beta \beta_i} I \geq F(S) \geq \sum_{i=1}^{p} \lambda_i \sqrt{\alpha \alpha_i} I \geq \alpha I, \;\; \forall S \in K_{\alpha,\beta}.$$

Then F is a self-map of $K_{\alpha,\beta}$. It is also continuous on $K_{\alpha,\beta}$. The existence of a solution to (6.2) in $K_{\alpha,\beta}$ then directly follows from Brouwer's fixed-point theorem.

Step 2: *sufficiency.* Set $\overline{\nu} := \mathcal{N}(0, \overline{S})$ where $\overline{S}$ is a positive definite solution of (6.2). The optimal transport between $\overline{\nu}$ and ν_i is then the linear map

$$T_i = S_i^{1/2} \left(S_i^{1/2} \overline{S} S_i^{1/2} \right)^{-1/2} S_i^{1/2}.$$

Let us now prove that $\sum_{i=1}^{p} \lambda_i T_i = I$ that we already know, from proposition 3.8, to be a sufficient condition for $\overline{\nu}$ to solve (2.2). Set $K_i = S_i^{1/2}$ and $\overline{K} := \overline{S}^{1/2}$. Using the identity

$$(\overline{K} K_i^2 \overline{K})^{1/2} = \overline{K} K_i (K_i \overline{K} K_i)^{-1/2} K_i \overline{K}$$

we may rewrite $F(\overline{S}) = \overline{S}$ as

$$\sum_{i=1}^{p} \lambda_i \overline{K} K_i (K_i \overline{K} K_i)^{-1/2} K_i \overline{K} = \overline{K}.$$

And since $\overline{K}$ is invertible, this yields

$$\sum_{i=1}^{p} \lambda_i K_i (K_i \overline{K} K_i)^{-1/2} K_i = \sum_{i=1}^{p} \lambda_i T_i = I$$

which proves that $\overline{\nu}$ is optimal.

Step 3: We already know that (2.2) admits a unique solution and from the previous step, we have for any positive definite solution $\overline{S}$ of (6.2), $\mathcal{N}(0, \overline{S})$ solves (2.2). This proves that (6.2) has a unique positive definite solution.

$\square$

The crucial role played by the nonlinear matrix equation (6.2) and the sufficiency step in the proof above is due to Knott and Smith [9]. But the authors left open the existence and uniqueness issues for (6.2); indeed the fact that barycenters of gaussians are also gaussians was not proved in their paper. Let us point out here that Rüschendorf and Uckelman proved in [12] existence of a solution to (6.2) by a completely different argument than Brouwer's fixed point, but they did not study its uniqueness.

7 Convex functionals

Here we extend the notion of displacement convexity introduced by McCann [11] for two probability measures, to any finite number of probability measures. For simplicity, we assume from now on that all probability measures are absolutely continuous with respect to the Lebesgue measure. Also, the space $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ will be equipped with the Wasserstein metric (2.1); it will then be called the Wasserstein space.

Definition 7.1. *A functional $\mathcal{F} : X' \cap \mathcal{M}_+^1(\mathbb{R}^d) \to \mathbb{R}$ is said to be convex along barycenters in the Wasserstein space $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$, if given any $p \geq 2$ probability measures $\nu_1, \cdots, \nu_p$ in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ and any p positive real numbers $\lambda_1, \cdots, \lambda_p$ that sum to 1, we have*

$$\mathcal{F}\left(\mathrm{bar}(\nu_i, \lambda_i)_{i=1,\cdots,p}\right) \leq \sum_{i=1}^{p} \lambda_i \mathcal{F}(\nu_i), \tag{7.1}$$

where $\mathrm{bar}(\nu_i, \lambda_i)_{i=1,\cdots,p}$ denotes the barycenter of the probability measures ν_i with the weights λ_i.

To compare the convexity along barycenters with McCann's displacement convexity, let us first recall the definition of displacement convexity as introduced in [11].

Definition 7.2. *A functional $\mathcal{F} : X' \cap \mathcal{M}_+^1(\mathbb{R}^d) \to \mathbb{R}$ is said to be displacement in the Wasserstein space $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$, if given any two probability measures ν_0 and ν_1 in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$, the function $[0,1] \ni t \mapsto \mathcal{F}(\nu_t) \in \mathbb{R}$ is convex; here ν_t is McCann's interpolant between ν_0 and ν_1 given by $\nu_t = ((1-t)\mathrm{id} + tT)\,\sharp\nu_0$, where $T = \nabla\phi$ is Brenier's map transporting ν_0 to ν_1.*

The following proposition shows that the convexity along barycenters generalizes McCann's displacement convexity.

Proposition 7.3. *If a functional $\mathcal{F} : X' \cap \mathcal{M}_+^1(\mathbb{R}^d) \to \mathbb{R}$ is convex along barycenters, then it is displacement convex.*

Proof. Let $\nu_0, \nu_1 \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ and $t \in [0,1]$. From section 6.2, we have that the barycenter of ν_0 and ν_1 with weights $\lambda_0 = 1-t$ and $\lambda_1 = t$ respectively, is McCann's interpolant ν_t. Then if $\mathcal{F}$ is convex along barycenters, then (7.1) implies that

$$\mathcal{F}(\nu_t) \leq (1-t)\mathcal{F}(\nu_0) + t\mathcal{F}(\nu_1),$$

which proves the displacement convexity of $\mathcal{F}$.

□

Contrarily to the Euclidean case, definitions 7.1 and 7.2 may not be equivalent, except in dimension $d = 1$ (see proposition below). But they are related to a third notion of convexity, that is the convexity along generalized geodesics joining any two probability measures ν_1 and ν_2 with base a probability measure ν (see [1]). In fact, the convexity along generalized geodesics is the strongest among these three notions of convexity, while McCann's displacement convexity is the weakest. Before justifying this statement, let us first recall the definition of generalized geodesics and its corresponding convexity, as presented in [1].

Definition 7.4. *Let $\nu_1, \nu_2, \nu \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$. Denote by $T_i = \nabla\phi_i$ the Brenier's map transporting ν to ν_i for $i = 1, 2$. Then the generalized geodesic joining ν_1 and ν_2 with base ν is the interpolated curve*

$$[\nu_1, \nu_2]_t^\nu := ((1-t)T_1 + tT_2) \sharp \nu, \quad t \in [0, 1]. \tag{7.2}$$

Moreover, a functional $\mathcal{F} : X' \cap \mathcal{M}_+^1(\mathbb{R}^d) \to \mathbb{R}$ is said to be convex along generalized geodesics, if given any three probability measures $\nu_1, \nu_2, \nu \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$, we have

$$\mathcal{F}\left([\nu_1, \nu_2]_t^\nu\right) \leq (1-t)\mathcal{F}(\nu_1) + t\mathcal{F}(\nu_2), \tag{7.3}$$

where $[\nu_1, \nu_2]_t^\nu$ is defined as above.

The definition of convexity along generalized geodesics could as well be given with any finite number of probability measures. Indeed, we have:

Proposition 7.5. *A functional $\mathcal{F} : X' \cap \mathcal{M}_+^1(\mathbb{R}^d) \to \mathbb{R}$ is convex along generalized geodesics if and only if, given any $p \geq 2$ probability measures $\nu_1, \cdots, \nu_p \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$, any p positive real numbers $\lambda_1, \cdots, \lambda_p$ that sum to 1, and a reference measure $\nu \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$, we have*

$$\mathcal{F}\left([\nu_1, \cdots, \nu_p]_{\lambda_i}^\nu\right) \leq \sum_{i=1}^p \lambda_i \mathcal{F}(\nu_i) \tag{7.4}$$

where

$$[\nu_1, \cdots, \nu_p]_{\lambda_i}^\nu := \left(\sum_{i=1}^p \lambda_i \nabla\Phi_i\right) \sharp \nu \tag{7.5}$$

and $\nabla\Phi_i$ is the Brenier's map transporting ν to ν_i.

Proof. The proof is done by induction on the number p of probability measures. We start with three measures ν_1, ν_2, ν_3 in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ with respective weights $\lambda_1, \lambda_2, \lambda_3$ such that $\sum_{i=1}^3 \lambda_i = 1$. We have that $\sum_{i=1}^3 \lambda_i \nabla\Phi_i =$

$\lambda_{12}\nabla\Phi_{12}+\lambda_3\nabla\Phi_3$, where $\lambda_{12}=\lambda_1+\lambda_2$, and $\Phi_{12}=\frac{\lambda_1}{\lambda_{12}}\Phi_1+\frac{\lambda_2}{\lambda_{12}}\Phi_2$ is convex. Since $\lambda_{12}+\lambda_3=1=(\lambda_1/\lambda_{12})+(\lambda_2/\lambda_{12})$, then using (7.5) and (7.3), we have

$$\begin{aligned}
\mathcal{F}\left([\nu_1,\nu_2,\nu_3]_{\lambda_i}^{\nu}\right) &= \mathcal{F}\left((\lambda_{12}\nabla\Phi_{12}+\lambda_3\nabla\Phi_3)\sharp\nu\right)\\
&\leq \lambda_{12}\mathcal{F}\left(\nabla\Phi_{12}\sharp\nu\right)+\lambda_3\mathcal{F}\left(\nabla\Phi_3\sharp\nu\right)\\
&\leq \lambda_{12}\left(\frac{\lambda_1}{\lambda_{12}}\mathcal{F}(\nabla\Phi_1\sharp\nu)+\frac{\lambda_2}{\lambda_{12}}\mathcal{F}(\nabla\Phi_2\sharp\nu)\right)+\lambda_3\mathcal{F}\left(\nabla\Phi_3\sharp\nu\right)\\
&= \sum_{i=1}^{3}\mathcal{F}\left(\nabla\Phi_i\sharp\nu\right)=\sum_{i=1}^{3}\mathcal{F}(\nu_i).
\end{aligned}$$

(7.4) then readily follows by an induction argument on p.

□

The next proposition shows that convexity along barycenters is intermediate to the other two notions of convexity.

Proposition 7.6. *If $\mathcal{F}: X'\cap\mathcal{M}_+^1(\mathbb{R}^d)\to\mathbb{R}$ is convex along generalized geodesics, then it is convex along barycenters (and therefore displacement convex because of proposition 7.3).*
Moreover if $d=1$, then displacement convexity implies convexity along generalized geodesics; hence in this case, these three notions of convexity are equivalent.

Proof. Let $\nu_1,\cdots,\nu_p\in X'\cap\mathcal{M}_+^1(\mathbb{R}^d)$, and consider the barycenter $\mathrm{bar}(\nu_i,\lambda_i)_{i=1,\cdots,p}$ of these measures with respective weights $\lambda_1,\cdots,\lambda_p$. From the multi-marginal characterization (4.10) of barycenters, we have

$$\mathrm{bar}(\nu_i,\lambda_i)_{i=1,\cdots,p}=\left(\sum_{i=1}^{p}\lambda_i\nabla u_i^*\circ\nabla u_1\right)\sharp\nu_1=\left(\sum_{i=1}^{p}\lambda_i\nabla u_i^*\right)\sharp\tilde{\nu}_1$$

where $\tilde{\nu}_1:=\nabla u_1\sharp\nu_1$, the u_i are strictly convex on $\mathbb{R}^d$, and the $T_i^1=\nabla u_i^*\circ\nabla u_1$ satisfy $T_i^1\sharp\nu_1=\nu_i$. So if $\mathcal{F}$ is convex along generalized geodesics, then using (7.4) and (7.5), we conclude that

$$\mathcal{F}((\mathrm{bar}(\nu_i,\lambda_i)_{i=1,\cdots,p})\leq\sum_{i=1}^{p}\lambda_i\mathcal{F}\left(\nabla u_i^*\sharp\tilde{\nu}_1\right)=\sum_{i=1}^{p}\lambda_i\mathcal{F}\left(T_i^1\sharp\nu_1\right)=\sum_{i=1}^{p}\lambda_i\mathcal{F}(\nu_i),$$

that is the convexity of $\mathcal{F}$ along barycenters.

Next assume that $d=1$, and consider $\nu,\nu_0,\nu_1\in X'\cap\mathcal{M}_+^1(\mathbb{R})$. Then the Brenier's map transporting ν to ν_i is Φ_i', where Φ_i is nondecreasing for $i=0,1$. So $T=\Phi_1'\circ(\Phi_0')^{-1}$ is nondecreasing and $T\sharp\nu_0=\nu_1$, so that T

is Brenier's map transporting ν_0 to ν_1. Therefore, the generalized geodesic joining ν_0 and ν_1 with base ν coincides with McCann's geodesic, since

$$[\nu_0,\nu_1]_t^\nu = [(1-t)\Phi_0' + t\Phi_1']\,\sharp\left((\Phi_0')^{-1}\sharp\nu_0\right) = [(1-t)\mathrm{id} + tT]\,\sharp\nu_0.$$

Hence if $\mathcal{F}$ is displacement convex, we have

$$\begin{aligned}\mathcal{F}\left([\nu_0,\nu_1]_t^\nu\right) &= \mathcal{F}\left([(1-t)\mathrm{id}+tT]\,\sharp\nu_0\right)\\ &\leq (1-t)\mathcal{F}(\nu_0) + t\mathcal{F}\left(T\sharp\nu_0\right)\\ &= (1-t)\mathcal{F}\left(\Phi_0'\sharp\nu\right) + t\mathcal{F}\left(\Phi_1'\sharp\nu\right),\end{aligned}$$

which shows that $\mathcal{F}$ is convex along generalized geodesics.

□

We end this section by showing that the three basic examples of functionals, i.e. the internal energy, the potential energy and the interaction energy, are convex along barycenters, since they are all convex along generalized geodesics in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$. In the sequel, we will identify the probability measures of $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ with their density functions that we denote by ρ.

Proposition 7.7. *Let $F : [0,\infty) \to \mathbb{R}$ be continuous on $[0,\infty)$, and $V, W : \mathbb{R}^d \to \mathbb{R}$ be continuous on $\mathbb{R}^d$, with W being an even function.*

1. *If $F(0) = 0$ and $(0,\infty) \ni t \mapsto t^d F(t^{-d})$ is convex and nonincreasing, then the internal energy $E(\rho) = \int_{\mathbb{R}^d} F(\rho(x))\,dx$ is convex along generalized geodesics in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$.*

2. *If V is convex, then the potential energy $E(\rho) = \int_{\mathbb{R}^d} V(x)\rho(x)\,dx$ is convex along generalized geodesics in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$.*

3. *If W is convex, then the interaction energy*

   $$E(\rho) = \frac{1}{2}\int_{\mathbb{R}^d} (W \star \rho)(x)\rho(x)\,dx = \frac{1}{2}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} W(x-y)\rho(x)\rho(y)\,dx\,dy$$

   is convex along generalized geodesics in $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$.

Therefore, all these functionals are convex along barycenters in the Wasserstein space $X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$.

Proof. Let $\rho, \rho_1, \rho_2 \in X' \cap \mathcal{M}_+^1(\mathbb{R}^d)$ be probability densities. Denote by $\nabla\Phi_i$ the Brenier's map transporting ρ to ρ_i, and by $\rho_t^{(1,2)}$ the generalized geodesic joining ρ_1 and ρ_2 (with base ρ), i.e.,

$$\rho_t^{(1,2)} = \nabla\Phi_t^{(1,2)}\sharp\rho, \quad \Phi_t^{(1,2)} := (1-t)\Phi_1 + t\Phi_2, \quad t \in [0,1]$$

For almost every $x \in \mathbb{R}^d$ w.r.t. the measure ρ, the Monge-Ampère equation holds:

$$\rho(x) = \rho_t^{(1,2)} \left(\nabla \Phi_t^{(1,2)}(x)\right) \det D^2 \Phi_t^{(1,2)}(x).$$

Using the above relation, the substitution $y = \nabla \Phi_t^{(1,2)}(x)$, and the fact that $F(0) = 0$, we can rewrite $E\left(\rho_t^{(1,2)}\right)$ as

$$\begin{aligned} E\left(\rho_t^{(1,2)}\right) &= \int_{\mathbb{R}^d} F\left(\frac{\rho(x)}{\det D^2\Phi_t^{(1,2)}(x)}\right) \det D^2\Phi_t^{(1,2)}(x) \mathrm{d}x \\ &= \int_{[\rho\neq 0]} F\left(\frac{\rho(x)}{\det D^2\Phi_t^{(1,2)}(x)}\right) \det D^2\Phi_t^{(1,2)}(x) \mathrm{d}x \\ &= \int_{[\rho\neq 0]} A\circ B\left(M_t(x)\right)\rho(x)\,\mathrm{d}x, \end{aligned} \tag{7.6}$$

where,

$$M_t(x) = D^2\Phi_t^{(1,2)}(x) = (1-t)D^2\Phi_1(x) + tD^2\Phi_2(x),$$

$A(t) = t^d F(t^{-d})$ and $B(M) = (\det M/\rho(x))^{1/d}$. Since the function $M \mapsto (\det M)^{1/d}$ is concave on the set of symmetric positive $d \times d$ matrices, and A is convex and nonincreasing in $(0, \infty)$, then if $\rho(x) \neq 0$, we have that $M \mapsto A \circ B(M)$ is convex. Then using that the matrices $D^2\Phi_1(x)$ and $D^2\Phi_2(x)$ are diagonalizable with positive eigenvalues for $x \in [\rho \neq 0]$, we have

$$A\circ B\left(M_t(x)\right) \leq (1-t)A\circ B\left(D^2\Phi_1(x)\right) + A\circ B\left(D^2\Phi_2(x)\right). \tag{7.7}$$

Inserting (7.7) into (7.6), and tracing back the equalities above (7.6) where we use in place of $\Phi_t^{(1,2)}$ and $\rho_t^{(1,2)}$ the functions Φ_i and ρ_i, $i = 1, 2$, we obtain

$$E\left(\rho_t^{(1,2)}\right) \leq (1-t)E(\rho_1) + tE(\rho_2),$$

which proves that the internal energy is convex along generalized geodesics.

For the potential energy, using $\rho_t^{(1,2)} = \nabla\Phi_t^{(1,2)}\sharp\rho$, we can write

$$E\left(\rho_t^{(1,2)}\right) = \int_{\mathbb{R}^d} V\left(\nabla\Phi_t^{(1,2)}(x)\right)\rho(x)\,\mathrm{d}x. \tag{7.8}$$

By the convexity of V, we have

$$V\left(\nabla\Phi_t^{(1,2)}(x)\right) \leq (1-t)V\left(\nabla\Phi_1(x)\right) + tV\left(\nabla\Phi_1(x)\right). \tag{7.9}$$

Combining (7.8) and (7.9), we conclude that the potential energy is also convex along generalized geodesics.

As for the interaction energy, using again $\rho_t^{(1,2)} = \nabla\Phi_t^{(1,2)} \sharp \rho$, we have

$$E\left(\rho_t^{(1,2)}\right) = \frac{1}{2} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} W\left(\nabla\Phi_t^{(1,2)}(x) - \nabla\Phi_t^{(1,2)}(y)\right) \rho(x)\rho(y)\, \mathrm{d}x\, \mathrm{d}y.$$

The convexity of W then concludes the proof.

□

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