Restrictions and identification in a multidimensional risk-sharing problem

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Abstract

We consider H expected utility maximizers that have to share a risky aggregate multivariate endowment $X \in \mathbb{R}^N$ and address the following two questions: does efficient risk-sharing imply restrictions on the form of individual consumptions as a function of X? Can one identify the individual utility functions from the observation of the risk-sharing? We show that when $H \geq \frac{2N}{N-1}$ efficient risk sharings have to satisfy a system of nonlinear PDEs. Under an additional rank condition, we prove an identification theorem.

Keywords: multidimensional risk-sharing, restrictions, identification.
JEL classification: C10, D61, D81.

1 Introduction

In [9], Townsend tested restrictions of efficient risk-sharing in a pure exchange economy on data from three villages in Southern India. In Townsend's model, the risk to be shared between the different agents is unidimensional and Townsend's test was based on the mutuality principle, that is, the idea that an agent's individual consumption should depend only on aggregate resources and not on her idiosyncratic shocks. As a consequence, if a risk-sharing is efficient then it should be comonotone in the sense that the consumption of each agent should be nondecreasing in the total resource. The idea of

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comonotonicity and its connections with Pareto efficiency have been developed further in a series of papers. It has been shown to be relevant to the case of utilities which are not of von Neumann-Morgenstern type, notably strictly risk-averse rank-dependent expected utility (see [2]) as well as to the multivariate setting (see [3]).

In the present work, we want to address the multivariate case where the resource to be shared has several dimensions (wheat and meat production for instance). We shall see that in this case there are some sharp restrictions on efficient risk sharings between expected utility maximizers that take the form of systems of nonlinear PDEs. We shall also prove an identification theorem i.e. that under some rank condition the knowledge of an efficient risk sharing enables us to reconstruct some sharp information on individual preferences and Pareto weights. The framework of the present work is that of the efficient risk-sharing of some multidimensional risky resource X among several expected utility maximizers with strictly concave and smooth utility functions that are not known to the econometrician. As observed in [3], the first-order condition gives that the consumption of agent h takes the form $X_h = \nabla V_h^*(\nabla V(X))$ (where V_h^* is related to the Legendre transform of agent h's utility and her Pareto weight, see section 2 for details). A first question is whether such forms entail sharp restrictions on the consumptions X_h as functions of X, for instance in the form of a system of PDEs. The second issue we shall address is whether the knowledge of the X_h's as functions of X enable us to identify the individual preferences. To be complete, one should also take into account the economic integration issue i.e. the further requirement that the functions V_h^* and V should be concave, however, this problem will not be addressed here.

We make no assumption about risk-sharing within the group, except that the result is efficient. So our paper is part of the growing literature on formal models of efficient group behavior (see [5] for a survey). This literature considers each group as a black box: inputs (prices, initial endowments) and outputs (consumption) can be observed but individual allocations cannot. One can observe aggregate consumption of the group but not the individual consumption of its members. The problem then is to recover individual consumptions with minimal assumptions on the allocation mechanism within the box. This minimal assumption is that the allocation mechanism is efficient i.e. Pareto-optimal. Browning and Chiappori [1] have shown that this is enough to derive restrictions on aggregate demand, analogous to (but different from) the classical Slutsky conditions of consumer theory and they have tested these conditions on microeconomic data.

Another issue to bear in mind is the so-called identifiability problem (see [5], p.7 for a full discussion): we will not assume that the demand functions have a particular form (so our model is non-parametric) but we will assume that they are smooth functions and that we can observe them. Of course, in any practical situation, one can only observe finitely many values. Proceeding as if one could observe the full demand function is an intermediate step for the econometrician. If we can recover the individual demands in that case, it will be up to him to find the adequate tools to recover the collective demand functions from finitely many points. If he cannot recover the preferences, even he knew the full demand functions, then clearly he will not be able to do so either if he knows only finitely many points

The paper is organized as follows. The model is introduced in section 2. Necessary conditions for a risk-sharing to be efficient are given in section 3 in the form of systems of nonlinear PDEs. Section 4 is devoted to identification issues. Section 5 is devoted to concluding remarks and a discussion of our results.

2 The model

Consider $H \geq 2$ expected utility maximizing[1] agents that have to share ex ante a risky multivariate aggregate endowment X that is some (essentially bounded say) $\mathbb{R}^N$-valued random vector with $N \geq 2$. Ex-ante, the agents have to decide on how to share the total resource X between the H agents in an efficient way. This leads to the following program[2]

$$\sup\Big\{\mathbb{E}\Big(\sum_{h=1}^{H}\lambda_h U_h(X_h)\Big)\ :\ \sum_{h=1}^{H} X_h = X\Big\} \tag{1}$$

where the $\lambda_h > 0$ are the Pareto weights and U_h are agents' von Neumann-Morgenstern utility functions. Assume that the U_h are C^2, that D^2U_h is negative definite everywhere and set $V_h = \lambda_h U_h$. The solution $\overline{X} = (\overline{X}_1, \cdots, \overline{X}_H)$ of (1) can be obtained as $\overline{X}_h = X_h(X)$ where for every $x \in \mathbb{R}^N$, $(X_1(x), \cdots, X_H(x))$ solves the sup-convolution problem:

$$V(x) = \sup\Big\{\sum_{h=1}^{H} V_h(x_h)\ :\ \sum_{h=1}^{H} x_h = x\Big\}. \tag{2}$$

[1] Extensions of the subsequent analysis to a non expected utility framewok, for instance to the case of maximin expected utility (see [4] or [6]) is an interesting issue but it is out of the scope of this paper.

[2] The fact that the Pareto weights are fixed and do not depend on X is precisely justified by the fact that the agents ex ante make a commitment on an allocation on the contract curve before the risk is realized.

The first-order optimality conditions of (2) read as

$$\nabla V_h(X_h(x)) = p(x) \text{ i.e. } X_h(x) = \nabla V_h^*(p(x)).$$

In the previous formula, V_h^* is the Legendre Transform of V_h, we then have $\nabla V_h^* = \nabla V_h^{-1}$ and $p(x)$ is the vector of shadow prices i.e. the multiplier associated to the scarcity constraint $\sum_{h=1}^H x_h = x$. Using that constraint, one can compute $p(x)$ as

$$p(x) = \Big(\sum_{h=1}^H \nabla V_h^*\Big)^{-1}(x) = \nabla V(x).$$

We thus obtain the following form for the risk-sharing:

$$X_h(x) = \nabla V_h^*\Big(\nabla V(x)\Big), \quad \text{for } h = 1, \cdots, H. \tag{3}$$

The issues we shall investigate in the sequel are the following:

- **Necessary conditions/restrictions:** Given maps

$$x \in \mathbb{R}^N \mapsto (X_1(x), \cdots, X_H(x)) \in \mathbb{R}^{N\times H}$$

  that sum to the identity, what conditions should they satisfy if in addition, they come from a risk-sharing problem of the form (2) without knowing neither the utility functions U_h nor the Pareto weights λ_h? As seen in (3), each X_h should be the composition of two gradient maps, the second one being independent of h, We shall see that when H is large enough, $H \geq \frac{2N}{N-1}$, this imposes that the vector fields X_h's solve a system of nonlinear PDEs.

- **Identification:** When the X_h's are obtained from an efficient risk-sharing process, can one recover information about the individual preferences i.e. about the functions $V_h = \lambda_h U_h$? We shall see that under some rank condition, there is identification i.e. the knowledge of individual consumptions as functions of the aggregate consumption enables one to reconstruct the functions V_h.

Finding sufficient conditions for the X_h's to be an efficient risk-sharing seems to be more delicate as will be discussed in the concluding section. The economic integration issue (i.e. the further requirement that the primitives V_h should be concave, or at least quasiconcave) also seems more involved and will not be discussed here.

3 Necessary conditions

Before going further, let us set some notations. We denote by $\mathcal{M}_N$ the space of $N \times N$ real matrices, by A^* the transpose of $A \in \mathcal{M}_N$, by $\mathcal{S}_N$ (respectively $\mathcal{AS}_N$) the subspace of $\mathcal{M}_N$ consisting of symmetric (respectively antisymetric) matrices and by GL_N the linear group of nonsingular matrices. We shall denote by $\langle A, B \rangle := \mathrm{tr}(A^*B)$ the usual inner product on $\mathcal{M}_N$ matrices and recall that $\mathcal{S}_N$ and $\mathcal{AS}_N$ are orthogonal supplementary subspaces for this inner product. For $A \in \mathcal{M}_N$ we denote by $\mathrm{sym}(A)$ its symmetric part i.e. $\mathrm{sym}(A) = \frac{1}{2}(A + A^*)$. Finally, given a linear map Q we denote respectively by $\mathrm{R}(Q)$ and $\mathrm{N}(Q)$ its range and nullspace.

3.1 General case

We are given H vector fields $X_1, \cdots, X_H$ that sum to the identity i.e.

$$\sum_{h=1}^{H} X_h(x) = x, \ \forall x \in \mathbb{R}^N \tag{4}$$

and we wonder whether these X_h can be obtained as a solution of a nondegenerate risk-sharing problem as in section 2 i.e. can be written as in (3) for some functions V_h^* and V with a nonsingular Hessian. Taking $x \in \mathbb{R}^N$ (fixed for the moment), differentiating (3) we get

$$F_h := DX_h(x) = D^2V_h^*(\nabla V(x))D^2V(x). \tag{5}$$

This in particular implies that each F_h is nonsingular,

$$\sum_{h=1}^{H} F_h = I_N \tag{6}$$

and one can find nonsingular and symmetric matrices S_h and S such that

$$F_h = S_h S, \ \forall h \in \{1, \cdots, H\}$$

which, defining $\sigma := S^{-1}$ and $\Phi_h(\sigma) := F_h\sigma$ for $h = 1, \cdots, H$ and $\Phi(\sigma) := (F_1\sigma, \cdots, F_H\sigma)$ we may rewrite as $\Phi(\sigma) = (S_1, \cdots, S_H)$. A necessary condition for the $F_h = DX_h$'s to satisfy (5) for some V_h and V is then:

$$\text{there exists } \sigma \in \mathcal{S}_N \cap GL_N \text{ such that } \Phi(\sigma) \in \mathcal{S}_N^H. \tag{7}$$

As we shall see in the next lemma, it is convenient to express (7) in terms of the linear map $L \in \mathcal{L}(\mathcal{AS}_N^{H-1}, \mathcal{S}_N)$ defined by:

$$L(A_1, \cdots, A_{H-1}) := \mathrm{sym}\left(\sum_{h=1}^{H-1} A_h F_h\right), \ \forall (A_1, \cdots, A_{H-1}) \in \mathcal{AS}_N^{H-1}. \quad (8)$$

Note that if the matrices F_h are observed, the maps Φ and L are known. In the sequel, we will derive restrictions on these maps.

Lemma 1 *If $\sigma \in \mathcal{S}_N$ the following assertions are equivalent:*

1. $\Phi(\sigma) \in \mathcal{S}_N^H$,
2. $\sigma \in \mathrm{R}(L)^\perp$.

   Condition (7) is thus equivalent to the fact that $\mathrm{R}(L)^\perp \cap \mathrm{GL}_N \neq \emptyset$ which in particular implies that L is not surjective.

Proof. Let $\sigma \in \mathcal{S}_N$, $\sigma \in \mathrm{R}(L)^\perp$ means that for every $(A_1, \cdots, A_{H-1}) \in \mathcal{AS}_N^{H-1}$ one has

$$0 = \mathrm{tr}(\sigma \sum_{h=1}^{H-1} A_h F_h) = \sum_{h=1}^{H-1} \mathrm{tr}(A_h F_h \sigma) = -\sum_{h=1}^{H-1} \mathrm{tr}(A_h^* F_h \sigma) = -\sum_{h=1}^{H-1} \langle A_h, \Phi_h(\sigma) \rangle .$$

This is equivalent to the fact that $\Phi_h(\sigma) \in \mathcal{S}_N$ for $h = 1, \cdots, H-1$ but recalling (6) we also have

$$\Phi_H(\sigma) = (I_N - \sum_{h=1}^{H-1} F_h)\sigma = \sigma - \sum_{h=1}^{H-1} \Phi_h(\sigma) \in \mathcal{S}_N.$$

This proves the desired equivalence. $\square$

We deduce the following restrictions on nondegenerate efficient risk-sharings:

Theorem 1 *If $H \geq \frac{2N}{N-1}$ and $x \mapsto (X_1(x), \cdots, X_H(x))$ is a nondegenerate efficient risk-sharing then it solves a system of nonlinear PDEs expressing the fact that the map L defined by (8) is nonsurjective.*

Proof. Since

$$\dim\left(\mathcal{AS}_N^{H-1}\right) = \frac{(H-1)N(N-1)}{2} \text{ and } \dim\left(\mathcal{S}_N\right) = \frac{N(N+1)}{2}$$

the fact that L is nonsurjective entails restrictions on the Jacobian matrices $F_h = DX_h$ as soon as $(H-1)(N-1) \geq N+1$ i.e. $H \geq \frac{2N}{N-1}$. More precisely, in this case, (7) implies that all $\frac{N(N+1)}{2} \times \frac{N(N+1)}{2}$ minors of L should identically vanish: since L depends linearly on the DX_h's this gives a system of $\binom{(H-1)N(N-1)/2}{N(N+1)/2}$ equations that are homogeneous of degree $\frac{N(N+1)}{2}$ in the derivatives $(DX_1, \cdots, DX_{H-1})$. $\square$

At this point, a few remarks are in order:

- The previous theorem asserts that there are restrictions on efficient risk-sharings as soon as the number of agents is large enough: for instance H needs to be larger than 4 when $N = 2$ and larger than 3 if $N \geq 3$. This contrasts with the literature on *aggregate* demand (in the Debreu-Mantel-Sonnenschein line) which typically finds that there are restrictions only when the number of agents is small enough. This is by no means a contradiction since here we are observing *individual* consumptions so that more agents increase the available information.

- In fact (7) is stronger than the condition that L is not surjective since it requires $\mathrm{R}(L)^{\perp} \cap GL_N \neq \emptyset$.

- To obtain restrictions as above, it is important to consider the whole system $F_h = S_h S$, $h = 1, \cdots, H$. Indeed, each equation $F_h = S_h S$ taken separately only implies that F_h is the product of two symmetric matrices and according to a theorem of Frobenius (see for instance [7]), any matrix can be written in such a way.

- The proportional risk-sharing rule corresponds to the most degenerate case where $L \equiv 0$, indeed in this case $F_h(x) = \alpha_h I_N$ for every x (where the α_h's sum to 1), so, for every $(A_1, \cdots, A_{H-1}) \in \mathcal{AS}_N^{H-1}$ one has

$$L(A_1, \cdots, A_{H-1}) := \mathrm{sym}\left(\sum_{h=1}^{H-1} \alpha_h A_h\right) = 0.$$

3.2 Special cases

We now consider some special cases and write explicitly the system of PDEs that nondegenerate risk-sharings should solve in these cases. These two cases are the first ones for which efficient risk-sharing implies some nontrivial restrictions namely:

- the case of 4 agents and 2 goods, in this case L can be identified with an endomorphism of $\mathbb{R}^3$ and (X_1, X_2, X_3) should solvea single PDE,

- the case of 5 agents and 2 goods, in this case L can be identified with an element of $\mathcal{L}(\mathbb{R}^4, \mathbb{R}^3)$ and (X_1, X_2, X_3, X_4) should solve a system of 4 nonlinear PDEs.

These two cases also illustrate the general case. In fact, the computations and arguments below can easily be generalized to larger values of H and N for which $H \geq \frac{2N}{N-1}$. Indeed, $(H,N)=(4,2)$ serves as a prototype for the case $H=\frac{2N}{N-1}$ whereas $(H,N)=(5,2)$ serves as a prototype for the case $H>\frac{2N}{N-1}$. Before studying the examples in details, let us remark that the case $H=\frac{2N}{N-1}$ is rather rare[3], more precisely it consists only of the two cases $(H,N)=(4,2)$ and $(H,N)=(3,3)$.

The case $\mathbf{H=4,\ N=2}$

Writing $X_h=(X_h^1,X_h^2)$, we have

$$F_h=\begin{pmatrix} \partial_1 X_h^1 & \partial_2 X_h^1 \\ \partial_1 X_h^2 & \partial_2 X_h^2 \end{pmatrix},$$

let then

$$A_h=\begin{pmatrix} 0 & x_h \\ -x_h & 0 \end{pmatrix},\ h=1,\cdots,3,$$

a direct computation gives

$$L(A_1,A_2,A_3)=\begin{pmatrix} \sum_{h=1}^{3}\partial_1 X_h^2 x_h & \frac{1}{2}\sum_{h=1}^{3}(\partial_2 X_h^2-\partial_1 X_h^1)x_h \\ \frac{1}{2}\sum_{h=1}^{3}(\partial_2 X_h^2-\partial_1 X_h^1)x_h & -\sum_{h=1}^{3}\partial_2 X_h^1 x_h \end{pmatrix}.$$

Identifying L with the endomorphism of $\mathbb{R}^3$

$$(x_1,x_2,x_3)\mapsto \Big(\sum_{h=1}^{3}\partial_1 X_h^2 x_h, \sum_{h=1}^{3}\partial_2 X_h^1 x_h, \sum_{h=1}^{3}(\partial_2 X_h^2-\partial_1 X_h^1)x_h\Big),$$

we see that a necessary condition for (X_1,X_2,X_3) to be an efficient risk-sharing reads:

$$\det\begin{pmatrix} \partial_1 X_1^2 & \partial_2 X_1^1 & (\partial_2 X_1^2-\partial_1 X_1^1) \\ \partial_1 X_2^2 & \partial_2 X_2^1 & (\partial_2 X_2^2-\partial_1 X_2^1) \\ \partial_1 X_3^2 & \partial_2 X_3^1 & (\partial_2 X_3^2-\partial_1 X_3^1) \end{pmatrix}=0$$

The case $\mathbf{H=5,\ N=2}$

[3] Indeed, assume $N\geq 2$ and that $N-1$ divides $2N$. If N is odd, write $N=2k+1$ and then $\frac{2N}{N-1}=\frac{4k+2}{2k}=2+\frac{1}{k}$ so that $k=1$ and then $N=3$ and $H=3$. If N is even, $N-1$ being odd, it follows from Gauss Lemma that $N-1$ divides N so that $N=2$ and then $H=4$.

Denoting for $h = 1, \cdots, 4$, $X_h = (X_h^1, X_h^2)$ and performing similar computations as before, we find that a necessary condition for (X_1, X_2, X_3, X_4) to be an efficient risk sharing reads:

$$
\begin{aligned}
0 &= \det \begin{pmatrix} \partial_1 X_1^2 & \partial_2 X_1^1 & (\partial_2 X_1^2 - \partial_1 X_1^1) \\ \partial_1 X_2^2 & \partial_2 X_2^1 & (\partial_2 X_2^2 - \partial_1 X_2^1) \\ \partial_1 X_3^2 & \partial_2 X_3^1 & (\partial_2 X_3^2 - \partial_1 X_3^1) \end{pmatrix} \\
&= \det \begin{pmatrix} \partial_1 X_1^2 & \partial_2 X_1^1 & (\partial_2 X_1^2 - \partial_1 X_1^1) \\ \partial_1 X_2^2 & \partial_2 X_2^1 & (\partial_2 X_2^2 - \partial_1 X_2^1) \\ \partial_1 X_4^2 & \partial_2 X_4^1 & (\partial_2 X_4^2 - \partial_1 X_4^1) \end{pmatrix} \\
&= \det \begin{pmatrix} \partial_1 X_1^2 & \partial_2 X_1^1 & (\partial_2 X_1^2 - \partial_1 X_1^1) \\ \partial_1 X_3^2 & \partial_2 X_3^1 & (\partial_2 X_3^2 - \partial_1 X_3^1) \\ \partial_1 X_4^2 & \partial_2 X_4^1 & (\partial_2 X_4^2 - \partial_1 X_4^1) \end{pmatrix} \\
&= \det \begin{pmatrix} \partial_1 X_2^2 & \partial_2 X_2^1 & (\partial_2 X_2^2 - \partial_1 X_2^1) \\ \partial_1 X_3^2 & \partial_2 X_3^1 & (\partial_2 X_3^2 - \partial_1 X_3^1) \\ \partial_1 X_4^2 & \partial_2 X_4^1 & (\partial_2 X_4^2 - \partial_1 X_4^1) \end{pmatrix}.
\end{aligned}
$$

4 Identification

In the previous section, we have found necessary conditions on the Jacobian matrices $F_h(x) = DX_h(x)$ for $(X_1, \cdots, X_H)$ to be a nondegenerate efficient risk-sharing. In this section, we address the identification issue: we assume that $x \mapsto (X_1(x), \cdots, X_H(x))$ is a nondegenerate efficient risk-sharing and we wonder what information on the individual preferences and on the shadow price can be deduced from this risk-sharing.

Given a nondegenerate efficient risk-sharing $(X_1, \cdots, X_H)$ we wish to find functions (maybe locally) V_h and V smooth and with nonsingular Hessians such that

$$X_h = \nabla V_h^* \circ \nabla V, \ h = 1, \cdots, H.$$

By assumption, X_h can be written in such way, and the identification problem consists in reconstructing the functions ∇V_h and ∇V from the knowledge of X_h; this essentially is a uniqueness problem. The best one can hope for is to identify ∇V_h and ∇V up to a common translation (adding the same linear function to the V_h's does not affect the corresponding risk-sharing) and up to a common multiplicative factor. In other words, what one can expect to identify at best is the collection of Hessian matrices $D^2 V$ and $D^2 V_h$ up to a multiplicative constant.

In general, one cannot expect an identification result, even for linear efficient risk-sharing rules. In the linear risk-sharing case, X_h can be identified with a nonsingular matrix and the identification problem consists in studying the uniqueness (up to a multiplicative constant) of the decompostion $X_h = S_h S$ with S_h and S symmetric. If $X_h = \alpha_h I_N$ (proportional risk sharing) the decomposition is highly nonunique since S can be any symmetric nonsingular matrix and $S_h = \alpha_h S^{-1}$. We do not have identification in this case and this is related to the fact that under proportional risk-sharing, the map L defined by (8) is identically 0. More generally, thanks to Lemma 1, we know that when $\mathrm{R}(L)$ has a codimension larger than 2 then there is nonuniqueness of the decomposition. Indeed, by assumption, one can write $DX_h\sigma = S_h$ where σ and S_h are symmetric and nonsingular but since $\mathrm{R}(L)$ has codimension 2, by Lemma 1 there is a symmetric matrix $\tilde{\sigma}$ such σ and $\tilde{\sigma}$ are linearly independent and $DX_h\tilde{\sigma} = \tilde{S}_h \in \mathcal{S}_N$. For small enough ε, $\sigma + \varepsilon\tilde{\sigma}$ is nonsingular and $DX_h = (S_h + \varepsilon\tilde{S}_h)(\sigma + \varepsilon\tilde{\sigma})^{-1}$ which proves that the decomposition is highly nonunique. We will see however that when $\mathrm{R}(L)$ has codimension 1, there is identification even in the nonlinear case.

In the previous section, the value of aggregate endowment x was somehow frozen, it is now essential to let x vary and in particular to emphasize the x-dependence of the map L defined in (8), from now on, we will therefore denote this map by L_x.

4.1 Identification when $\mathrm{R}(L_x)$ has codimension 1

For all $x \in \mathbb{R}^N$, we of course assume the rank condition

$$\mathrm{rank}(L_x) \leq \frac{N(N+1)}{2} - 1 \tag{9}$$

which we already know to be necessary for $(X_h)_h$ to be an efficient risk-sharing. Our aim is to identify the shadow price ∇V (and then the preferences) near a point $\overline{x} \in \mathbb{R}^N$ such that

$$\mathrm{rank}(L_{\overline{x}}) = \frac{N(N+1)}{2} - 1 \tag{10}$$

which implies that for every x in a neighbourhood $\mathcal{U}$ of $\overline{x}$, the subspace $\mathrm{R}(L_x)$ of $\mathcal{S}_N$ has codimension one[4] and thus an orthogonal of dimension 1. For all $x \in \mathcal{U}$, we may therefore find a symmetric (and nonsingular since $(X_1, \cdots, X_H)$ is nondegenerate) matrix $\sigma(x)$ such that:

$$\mathrm{R}(L_x)^{\perp} = \mathbb{R}\sigma(x),\ \forall x \in \mathcal{U}. \tag{11}$$

[4]We already noticed that this rank condition is necessary for identication.

Moreover, thanks to condition (10), it is easy to see that we may choose $x \mapsto \sigma(x)$ in such way that σ is C^1 with respect to x.

Again denoting $F_h = DX_h$, we know that there are smooth functions V_h^* and V with nonsingular Hessians such that

$$X_h = \nabla V_h^* \circ \nabla V \text{ hence } F_h(x) = D^2 V_h^*(\nabla V(x)) D^2 V(x)$$

for every x and we want to deduce as much information as we can from the X_h's to reconstruct ∇V and ∇V_h. It follows from Lemma 1 that $D^2V(x)^{-1}$ should belong to $\mathrm{R}(L_x)^\perp = \mathbb{R}\sigma(x)$ so that setting $T(x) := \sigma(x)^{-1}$, $D^2V(x)$ should be of the form

$$D^2V(x) = \lambda(x)T(x),\ x \in \mathcal{U}$$

for some nonvanishing scalar function λ. In particular, by Schwarz's symmetry theorem, in addition to the symmetry of T, one should have[5]

$$\partial_k(\lambda(x)T_{ij}(x)) = \partial_i(\lambda(x)T_{kj}(x)),\ \forall (i,j,k) \in \{1, \cdots, N\}^3$$

that is

$$\partial_k\lambda(x)T_{ij}(x) - \partial_i\lambda(x)T_{kj}(x) = \lambda(x)(\partial_i T_{kj}(x) - \partial_k T_{ij}(x)). \tag{12}$$

To see that these equations enable us to recover λ (hence D^2V) in a neighbourhood of $\overline{x}$ up to a multiplicative constant, we shall use the following:

Lemma 2 *Let T be an $N \times N$ symmetric and nonsingular matrix and let $(e_1, \cdots, e_N)$ be the canonical basis of $\mathbb{R}^N$ then the family $\{T_{ij}e_k - T_{kj}e_i,\ i,j,k\}$ spans $\mathbb{R}^N$.*

Proof. It is easy to see that the desired statement amounts to proving that the linear map $\Pi \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^{N^3})$ defined by $(\Pi(x))_{ijk} = T_{ij}x_k - T_{kj}x_i$ for all $x \in \mathbb{R}^N$ and all $(i,j,k) \in \{1, \cdots, N\}^3$ is injective. Let x be in the null space of Π i.e.

$$T_{ij}x_k = T_{kj}x_i,\ \forall i,j,k$$

Multiply the previous by arbitrary reals α_i and β_j and sum over i and j to get

$$\langle T\alpha, \beta\rangle\, x = \langle \alpha, x\rangle\, T\beta,\ \forall (\alpha,\beta) \in \mathbb{R}^N \times \mathbb{R}^N$$

Taking $\alpha = x$ yields:

$$\langle Tx, \beta\rangle\, x = |x|^2 T\beta,\ \forall \beta \in \mathbb{R}^N$$

Choosing $\beta \neq 0$ orthogonal to Tx, since $T\beta \neq 0$ we deduce that $x = 0$. $\square$

[5] Since we have differentiated D^2V here, we are assuming that V is at least C^3.

The following identification theorem follows:

Theorem 2 *Let $(X_1, \cdots X_H)$ be a nondegenerate efficient risk-sharing such that the rank condition (10) holds in a neighbourhood of $\overline{x} \in \mathbb{R}^N$, then there is local identification of shadow prices and preferences: one can deduce from $(X_1, \cdots, X_H)$ the shadow price $\nabla V(x)$ (up to a multiplicative factor and an additive constant) in a neighbourhood of $\overline{x}$ as well as the marginal utilities ∇V_h in a neighbourhood of $X_h(\overline{x})$ (up to the same multiplicative and additive constants).*

Proof. Assume that $X_h = \nabla V_h^* \circ \nabla V$. Then as already noted $D^2V(x) = \lambda(x)T(x)$, where $T(x)$ is a given $\mathcal{S}_N$-valued map and λ does not vanish and should satisfy the system of linear PDEs (12) in $\mathcal{U}$, which we simply rewrite as

$$b_\alpha \cdot \frac{\nabla \lambda}{\lambda} = a_\alpha, \alpha = (i, j, k), b_\alpha(x) = T_{ij}(x)e_k - T_{kj}(x)e_i.$$

It follows from Lemma 2 that the family $\{b_\alpha(x)\}_\alpha$ spans $\mathbb{R}^N$ for every $x \in \mathcal{U}$ hence the system (12) contains as subsystem a system of the form

$$B(x)\nabla(\log(\lambda)(x) = a(x)$$

for some $B(x) \in \mathrm{GL}_N$ so that $\nabla(\log(\lambda)(x) = B(x)^{-1}a(x)$ which means that λ, and hence $D^2V(x)$, is determined up to a multiplicative constant, It follows that $\nabla V = \alpha_0 \nabla V_0 + p_0$ where V_0 is totally determined (and has a nonsingular Hessian) by the risk sharing and $\alpha_0 \in \mathbb{R} \setminus \{0\}$ and $p_0 \in \mathbb{R}^N$ are two constants. Once one knows ∇V one easily obtains the desired identification of ∇V_h by observing that $X_h = \nabla V_h^* \circ \nabla V$ can be rewritten as $\nabla V_h = \nabla V \circ X_h^{-1} = \alpha_0 \nabla V_0 \circ X_h^{-1} + p_0$. $\square$

The previous result is optimal: we already explained why the rank condition is important and why the quantities that may be identified are ∇V_h and ∇V up to multiplicative and additive constants.

4.2 The particular case $H = 4$, $N = 2$

We now restrict ourselves again to the simplest case $H = 4$, $N = 2$.

The linear case

Let us first consider the case of a linear risk sharing where $X_h(x) = F_h \times x$ ($x \in \mathbb{R}^2$, $h = 1, \cdots, 3$) and denote by f_h^{ij} the entries of the matrix $F_h \in \mathcal{M}_2$.

We have seen in section 3 that a necessary condition for the X_h to be an efficient risk-sharing is that:

$$\det \begin{pmatrix} f_1^{21} & f_2^{21} & f_3^{21} \\ -f_1^{12} & -f_2^{12} & -f_3^{12} \\ (f_1^{22}-f_1^{11}) & (f_2^{22}-f_2^{11}) & (f_3^{22}-f_3^{11}) \end{pmatrix} = 0. \tag{13}$$

Our goal is to find symmetric matrices σ and S_h such that $F_h\sigma = S_h$ and we have seen that to identify the matrix σ up to a constant we further need that this matrix has rank 2, for instance, we assume that its first two columns are linearly independent[6]. The computation of σ is explicit[7] and gives

$$\sigma = \begin{pmatrix} -f_1^{12}(f_2^{22}-f_2^{11}) + f_2^{12}(f_1^{22}-f_1^{11}) & -f_1^{21}f_2^{12} + f_1^{12}f_2^{21} \\ -f_1^{21}f_2^{12} + f_1^{12}f_2^{21} & -f_1^{21}(f_2^{22}-f_2^{11}) + f_2^{21}(f_1^{22}-f_1^{11}) \end{pmatrix} \tag{14}$$

and this matrix is invertible as soon as the X_h is a nondegenerate efficient risk-sharing.

The nonlinear case

In the nonlinear case, denote by $F_h(x) := DX_h(x)$, the same computations as before give a matrix $\sigma(x)$ which spans $\mathrm{R}(L_x)^\perp$, it is the same as in (14) except that we now understand the entries as $f_h^{ij}(x) := \partial_j X_h^i(x)$. An explicit matrix $T(x)$ that is proportional to $\sigma(x)^{-1}$ is then given by

$$\begin{aligned} T_{11} &= -\partial_1 X_1^2(\partial_2 X_2^2 - \partial_1 X_2^1) + \partial_1 X_2^2(\partial_2 X_1^2 - \partial_1 X_1^1) \\ T_{12} &= \partial_1 X_1^2 \partial_2 X_2^1 - \partial_2 X_1^1 \partial_1 X_2^2 \\ T_{22} &= -\partial_2 X_1^1(\partial_2 X_2^2 - \partial_1 X_2^1) + \partial_2 X_2^1(\partial_2 X_1^2 - \partial_1 X_1^1). \end{aligned}$$

We wish now to identify D^2V which is of the form $\lambda(x)T(x)$ and the fact that λT is a Hessian field gives the system of two PDEs:

$$\begin{aligned} T_{12}\partial_1\lambda - T_{11}\partial_2\lambda &= \lambda(\partial_2 T_{11} - \partial_1 T_{12}) \\ T_{22}\partial_1\lambda - T_{12}\partial_2\lambda &= \lambda(\partial_2 T_{12} - \partial_1 T_{22}) \end{aligned}$$

[6] At least in this example, we see that the rank condition (10) is generic: among non-singular matrices F_1, F_2, F_3 for which (13) holds, it is generic that one of the 2×2 minors in (13) is nonzero.

[7] It is convenient to identify $\mathcal{S}_2$ with $\mathbb{R}^3$ by identifying the vector (a,b,c) with the symmetric matrix $\begin{pmatrix} a & \frac{b}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} & c \end{pmatrix}$, this isomorphism has the advantage to preserve the inner product hence orthogonality, to find a matrix in $\mathrm{R}(L)^\perp$ we simply take the wedge product of the first two columns of its matrix in the canonical basis of $\mathbb{R}^3$.

which we can rewrite as

$$\frac{\nabla\lambda}{\lambda} = \begin{pmatrix} F_1(DX, D^2X) \\ F_2(DX, D^2X) \end{pmatrix}$$

where

$$\begin{aligned}\begin{pmatrix} F_1(DX, D^2X) \\ F_2(DX, D^2X) \end{pmatrix} &:= \begin{pmatrix} T_{12} & -T_{11} \\ T_{22} & -T_{12} \end{pmatrix}^{-1} \begin{pmatrix} \partial_2 T_{11} - \partial_1 T_{12} \\ \partial_2 T_{12} - \partial_1 T_{22} \end{pmatrix} \\ &= \frac{1}{\det(T)} \begin{pmatrix} -T_{12}(\partial_2 T_{11} - \partial_1 T_{12}) + T_{11}(\partial_2 T_{12} - \partial_1 T_{22}) \\ -T_{22}(\partial_2 T_{11} - \partial_1 T_{12}) + T_{12}(\partial_2 T_{12} - \partial_1 T_{22}) \end{pmatrix}\end{aligned}$$

We now wish to emphasize the fact that since the previous vector field is a gradient, we have an additional third-order nonlinear PDE for (X_1, X_2), namely

$$\partial_2 F_1(DX, D^2X) = \partial_1 F_2(DX, D^2X)$$

this supplementary equation is another necessary condition for efficient risk-sharing.

5 Discussion and concluding remarks

In this paper, we found that efficient risk-sharing entails sharp restrictions on individual consumptions as soon as the number of agents is large enough. We have also shown that under a certain rank condition, one can identify individual preferences.

5.1 Empirical implications

Although the previous considerations are rather theoretical, we believe that they may have some practical or empirical implications if one wishes to test efficiency on real data. A typical econometric approach would consist in regressing individual consumption X_h on the aggregate resource X so as to obtain an estimation of, say, a linear (or linearized) model of the form $X_h = F_h X$ for certain matrices F_h. To test efficiency, one could then directly use the approach of section 3 on the nonsurjectivity of the map L given by (8), i.e. test as null hypothesis the fact that certain minors of a matrix that depends linearly on the F_h's identically vanish. In the case $H = 4$, $N = 2$, this amounts to test the single equation (13). It seems important to emphasize that our approach is actually robust to the aggregation of groups of agents, that is, our results remain valid if agents h are replaced by groups of agents (households or even much larger groups). The necessary conditions of section 3 can therefore in principle be tested on macro data which makes our results applicable to international trade data, agricultural data in developing countries...

5.2 On limited commitment

Our results rely very much on the assumption of ex ante efficiency or full commitment i.e. the fact that the Pareto weights are given. In an intertemporal framework, Mazzocco [8] developed a test for intra-household ex ante efficiency and actually rejected this assumption. This suggests that one should also consider the possibility of limited commitment in our analysis. Under limited commitment, it would be more realistic to allow the Pareto weights to depend (at least) on the realization x of the total resource. The analysis of optimality conditions for efficient risk-sharings is of course much more complex in this extended setting. Whether one can derive tractable restrictions on multi-dimensional risk-sharings under limited commitment is an interesting question that can probably be attacked with the tools of exterior differential calculus but this is out of the scope of the present paper.

5.3 Towards sufficient conditions

In this paper, we found a number of necessary conditions for maps $x \mapsto (X_h(x))_{h=1,\dots,H}$ to be an efficient risk-sharing based on the simple observation that, by first order optimality conditions, the X_h's should be of the form $X_h = \nabla V_h^* \circ \nabla V$ where V_h and V are smooth and strictly concave functions (see (3)). Differentiating (3), one sees that for every x, the Jacobian matrix $DX_h(x)$ should be the product of two (semidefinite negative) symmetric matrices, the second one being independent of h. In theorem 1, we have shown that when $H \geq 2N/(N-1)$ this implies that the X_h's satisfy a system of first-order nonlinear PDEs. These equations are however far from being sufficient for the X_h's to be an efficient risk-sharing. Let us briefly explain why there are additional restrictions, a more detailed study of these conditions being left for future research:

- arguing as in section 4, if the X_h's form an efficient risk-sharing and if the rank condition (10) is satisfied, there is a nonsingular symmetric matrix valued function T (which can be computed as we explicitly did in the case $H = 4$, $N = 2$) such that $D^2V(x) = \lambda(x)T(x)$ for some scalar function λ. Expressing the fact that λT is a field of Hessian matrices leads to the system of linear PDE's (12) for λ which can be written as

$$C(x)\nabla\log(\lambda) = b(x)$$

where C is an $N^3 \times N$ matrix and $b \in \mathbb{R}^{N^3}$. In this system C is an explicit functions of T (hence of the first derivatives of $(X_h)_h$) whereas b is an explicit function of DT (hence of the second derivatives of $(X_h)_h$). The linear system above being overdetermined (more equations than unknowns), we should have $b(x) \in \mathrm{R}(C(x))$ for every x which is a system of nonlinear *second-order* PDEs for $(X_h)_h$. These clearly are additional restrictions on the X_h's not captured by Theorem 1.

- on top of this, as proved in Lemma 2, the linear system (12) contains an invertible $N \times N$ subsystem $B(x)\nabla\log(\lambda) = a(x)$, which gives that $B^{-1}(x)a(x)$ is a gradient so that it should have a symmetric Jacobian. This imposes additional *third-order* PDEs for $(X_h)_h$ (exactly as in the case $H = 4$, $N = 2$ detailed above).

We have not detailed all the conditions listed above because, on the one hand, they lead to complicated expressions in general (we just gave some details for the case $H = 4$, $N = 2$) and, on the other hand, they are not sufficient yet. Indeed, the considerations above on the system (12) eventually ensure that one can write $DX_h(x) = S_h(x)D^2V(x)$ where the S_h are symmetric, but this does not guarantee that S_h is of the form $D^2V_h^*(\nabla V)$... Finally, we have ignored economic integration issues i.e. the requirement that V_h^* and V should be concave so that DF_h should be the product of semidefinite negative matrices, in particular DX_h should have a positive trace and a positive determinant.

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