BERNARD DACOROGNA

JÜRGEN MOSER

On a partial differential equation involving the Jacobian determinant

On a partial differential equation involving the Jacobian determinant

by

Bernard DACOROGNA

École Polytechnique, Lausanne, Suisse

and

Jürgen MOSER

Eidg. Techn. Hochschule, Zürich, Suisse

ABSTRACT. — Let $\Omega \subset \mathbb{R}^n$ a bounded open set and $f > 0$ in $\bar{\Omega}$ satisfying $\int_\Omega f(x)\,dx = \text{meas}\,\Omega$. We study existence and regularity of diffeomorphisms $u : \bar{\Omega} \to \bar{\Omega}$ such that

$$\begin{cases} \det \nabla u(x) = f(x), & x \in \Omega \\ u(x) = x, & x \in \partial\Omega. \end{cases}$$

. *Key words :* Volume preserving diffeomorphism, nonlinear PDE, elliptic equations, Jacobian determinant.

RÉSUMÉ. — Soit $\Omega \subset \mathbb{R}^n$ un ouvert borné et soit $f > 0$ dans $\bar{\Omega}$ satisfaisant $\int_\Omega f(x)\,dx = \text{mes}\,\Omega$. On étudie l'existence et la régularité de difféomorphismes $u : \bar{\Omega} \to \bar{\Omega}$ tels que

$$\begin{cases} \det \nabla u(x) = f(x), & x \in \Omega \\ u(x) = x, & x \in \partial\Omega. \end{cases}$$

I. INTRODUCTION

Consider a bounded connected open set Ω of $\mathbb{R}^n$ and two n-forms α, β

$$\alpha = f(x)\,dx_1\ldots dx_n, \qquad \beta = g(x)\,dx_1\ldots dx_n,$$

with f, $g > 0$. We shall prove that, under some regularity assumptions on Ω, f and g, there exists a diffeomorphism $\varphi : \bar{\Omega} \to \bar{\Omega}$, keeping the boundary pointwise fixed and such that

$$\varphi^* \beta = \lambda \alpha$$

where $\lambda = \int \beta \Big/ \int \alpha$.

In analytical form, the above result is equivalent to

THEOREM 1. — *Let $k \geq 0$ be an integer, $0 < \alpha < 1$, Ω have a $C^{k+3,\alpha}$ boundary $\partial\Omega$ ($C^{k,\alpha}$ denoting the usual Hölder spaces). Let f, $g \in C^{k,\alpha}(\bar{\Omega})$ with f, $g > 0$ in $\bar{\Omega}$. Then there exists a diffeomorphism φ with φ, $\varphi^{-1} \in C^{k+1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ and satisfying*

(1.1)
$$\begin{cases} g(\varphi(x))\det\nabla\varphi(x) = \lambda f(x), & x \in \Omega \\ \qquad\qquad \varphi(x) = x, & x \in \partial\Omega \end{cases}$$

where $\lambda = \int g \Big/ \int f$.

Remarks. — (i) This scalar equation is clearly underdetermined and uniqueness does not hold as the following trivial example shows. Let Ω be the unit disk of $\mathbb{R}^2$, $f = g = 1$, $a \in C^\infty([0,1])$ with $a(1) = 2\,\mathrm{N}\,\pi$, N an integer, and $\varphi(x) = (r\cos(\theta + a(r^2)),\ r\sin(\theta + a(r^2)))$ where (r, θ) denote the polar coordinates. It is clear that such a φ is a solution of (1.1).

(ii) For notational convenience we shall denote for $k \geq 0$ an integer, $0 \leq \alpha \leq 1$ and Ω a bounded open set of $\mathbb{R}^n$, the set of diffeomorphisms (homeomorphisms if $k = 0$) $\varphi : \bar{\Omega} \to \bar{\Omega}$ with φ, $\varphi^{-1} \in C^{k,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ by $\mathrm{Diff}^{k,\alpha}(\bar{\Omega})$ [if $\alpha = 0$ we just set $\mathrm{Diff}^k(\bar{\Omega})$].

This theorem is a stronger version of a known result. For two volume forms on a smooth compact manifold without boundary it was established in [M] (under stronger smoothness assumptions) and by Banyaga [B] for manifolds with boundaries (in the C^∞ case). For the special case of the ball in dimension 2 or 3, *see* [T], [D] respectively. Our purpose here is to present a simple proof of this theorem, using standard properties of the Laplacian for domains in $\mathbb{R}^n$. This allows us to avoid the use of differential forms and the notion of manifolds. At the same time we obtain precise regularity results. As one would expect the solution φ is one differentiability class smoother than f and g if one works with Hölder norms. In our proof, given in Section 2, this point requires special attention. For

manifolds without boundaries this gain in smoothness was established by Zehnder [Z], under the additional assumptions that f, g are sufficiently close in $C^{0,\alpha}$ norm and g is in C^4. It goes without saying that our proof can be carried over without difficulty to manifolds Ω with boundaries.

As an application of Theorem 1 we can construct a volume preserving diffeomorphism with given boundary data. In other words we claim that if Ω is as in Theorem 1 and $\psi_0 \in \text{Diff}^{k+1,\alpha}(\bar{\Omega})$, then there exists $\psi \in \text{Diff}^{k+1,\alpha}(\bar{\Omega})$ such that

$$\begin{cases} \det \nabla \psi \equiv 1 & \text{in } \Omega \\ \psi = \psi_0 & \text{on } \partial\Omega. \end{cases}$$

Indeed if we use Theorem 1 with $g=1$, $f=\det \nabla \psi_0^{-1}$, then $\lambda=1$ (since $\psi_0 : \bar{\Omega} \to \bar{\Omega}$ is a diffeomorphism) and we can therefore find φ satisfying (1.1). We obtain a solution of the above problem by setting $\psi = \varphi \circ \psi_0$.

This type of problem plays a role in the construction of volume preserving mappings with prescribed periodic orbits and ergodic mappings, as worked out by Alpern [A], Anosov-Katok [AK]. In [D] it was shown how to apply the result to minimization problem in the calculus of variations with further applications to nonlinear elasticity.

Finally it is interesting that the corresponding problem for non compact manifolds leads to additional topological obstructions as shown by Greene-Shiohama [GS].

In Section 3 we present an alternate proof under different regularity assumptions. Here we avoid the use of elliptic partial differential equations and work only with the implicit function theorem. For this reason we use the C^k norm, $k \geqq 0$ an integer, instead of the Hölder norms $C^{k,\alpha}$. This approach allows even the treatment of continuous functions f and g in which case φ turns out to be a homeomorphism and not a diffeomorphism since this approach fails to give the expected derivative gain which we have for Hölder norms in Theorem 1. For this reason the differential equation has to be interpreted in a weaker form, namely

$$(1.2) \qquad \int_E f\,dx = \int_{\varphi(E)} g\,dx$$

for all open sets $E \subset \Omega$. It is clear that if φ is C^1, then (1.2) is equivalent to (1.1) with $\lambda=1$.

This second approach is also used to study a question related to Theorem 1, where we ask for regularity only in the interior, i.e., $\varphi \in \text{Diff}^k(\Omega) \cap \text{Diff}^0(\bar{\Omega})$, satisfying (1.1) (i.e. φ is a diffeomorphism of Ω and it extends as a homeomorphism of $\bar{\Omega}$, keeping the boundary pointwise fixed). For this purpose one needs much weaker regularity requirements for $\partial\Omega$, we shall allow, for example, Lipschitz boundaries or isolated

boundary points as in a punctured disk. These cases cannot be treated by the potential theoretical methods used in Section 2.

The proof of Section 3 is related to an argument given in [M]. However as A. Katok pointed out to one of us, this argument is incorrect (on p. 291 it was not provided that $v=w$ on the boundary, as was required). Our purpose here is to rectify this argument and at the same time to take care of domains with boundary. As before this approach applies to general manifolds and not only to Ω which are embedded in $\mathbb{R}^n$.

We conclude the introduction with an open problem which we were not able to resolve: le Ω be a connected open set with smooth boundary, say C^∞ smooth. Let $f \in C(\bar{\Omega})$, $f > 0$ and $\int f\, dx = \text{meas}\,\Omega$. Does there exist a diffeomorphism $u : \bar{\Omega} \to \bar{\Omega}$ with $u(x) = x$ on $\partial\Omega$ solving $\det \nabla u = f$ in Ω?

II. A DEFORMATION APPROACH

We now state the main result.

THEOREM 1'. $-$ Let $k \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{k+3,\,\alpha}$ boundary $\partial\Omega$. Let $f \in C^{k,\,\alpha}(\bar{\Omega})$, $f > 0$ in $\bar{\Omega}$ and

$$(2.1) \qquad \int_\Omega f(x)\, dx = \text{meas}\,\Omega.$$

Then there exists $u \in \text{Diff}^{k+1,\,\alpha}(\bar{\Omega})$ satisfying

$$(2.2) \qquad \begin{cases} \det \nabla u(x) = f(x), & x \in \Omega \\ u(x) = x, & x \in \partial\Omega. \end{cases}$$

Remark. $-$ Note that Theorem 1 stated in the introduction follows at once from Theorem 1'. Indeed we have the theorem by setting $\varphi = v^{-1} \circ u$ where u and v satisfy

$$\begin{cases} \det \nabla u(x) = \dfrac{f(x)\,\text{meas}\,\Omega}{\displaystyle\int_\Omega f(x)\,dx}, & x \in \Omega \\[4mm] \det \nabla v(x) = \dfrac{g(x)\,\text{meas}\,\Omega}{\displaystyle\int_\Omega g(x)\,dx}, & x \in \Omega \\[4mm] u(x) = v(x) = x, & x \in \partial\Omega. \end{cases}$$

We now describe roughly the idea of the proof

STEP 1 (Theorem 2). − We write $u(x) = x + v(x)$. The linearized problem is then

$$\begin{cases} \operatorname{div} v = f - 1 & \text{in } \Omega \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

We solve the above problem by setting $v = \operatorname{grad} a + c$, where c takes into account the boundary condition and is divergence free. Using standard existence theory and Schauder estimates for elliptic equations we obtain the result. Although the solution of this problem is clearly not unique, our construction provides a well defined solution.

STEP 2 (Lemma 3). − We then find a $C^{k,\alpha}$ solution by a deformation argument, *i.e.* by solving the ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\Phi_t(x) = \dfrac{v(\Phi_t(x))}{t + (1-t)f(\Phi_t(x))} \\ \qquad\quad \Phi_0(x) = x \end{cases}$$

where v is as in Step 1. Standard properties of ordinary differential equations give that $u(x) = \Phi_1(x)$ is a solution of (2.2), but it is only in $C^{k,\alpha}$.

The two last steps are used to obtain the $C^{k+1,\alpha}$ regularity.

STEP 3 (Lemma 4). − Using Step 1 and a smallness assumption on the $C^{0,\beta}$ norm, $0 < \beta \leq \alpha < 1$, of $f - 1$, we obtain a $C^{k+1,\alpha}$ solution by linearizing the equation around the identity.

STEP 4. − We remove the smallness assumption in Step 3 on $C^{0,\beta}$ norm of $f - 1$, by composing two deformations. The first one (using Step 3) which allows to pass from $f \in C^{k,\alpha}$ to $g \in C^{k+1,\alpha}$ with $f - g$ small in the $C^{0,\beta}$ norm. The second one by applying Step 2 to g.

We start with Step 1 and give a theorem concerning the existence and regularity of solutions of the linearized problem.

THEOREM 2. − *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with* $C^{k+3,\alpha}$ *boundary,* $k \geq 0$ *being an integer and* $0 < \alpha < 1$. *Let* $g \in C^{k,\alpha}(\bar{\Omega})$ *with*

(2.3) $$\int_\Omega g(x)\,dx = 0.$$

Then there exists $v \in C^{k+1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ *satisfying*

(2.4) $$\begin{cases} \operatorname{div} v(x) = g(x), & x \in \Omega \\ \quad v(x) = 0, & x \in \partial\Omega. \end{cases}$$

Furthermore there exists $K = K(\alpha, k, \Omega) > 0$ *such that*

(2.5) $$\|v\|_{k+1,\alpha} \leq K \|g\|_{k,\alpha}.$$

Remarks. — (i) We have denoted by $\|.\|_{k,\alpha}$ the $C^{k,\alpha}$ norm.
(ii) In fact the proof of the theorem will show that if

$$X = \left\{ g \in C^{k,\alpha}(\bar{\Omega}) : \int_{\Omega} g(x)\,dx = 0 \right\}$$
$$Y = \left\{ v \in C^{k+1,\alpha}(\bar{\Omega}; \mathbb{R}^n) : v = 0 \text{ on } \partial\Omega \right\}$$

then our construction will provide a bounded linear operator $L : X \to Y$ which associates to every $g \in X$, a unique $v = Lg \in Y$ satisfying (2.4).

Before proving Theorem 2 we introduce some notations

NOTATIONS. — Let $w \in C^1(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2})$ with $w = (w_{ij})_{1 \le i < j \le n}$. For notational convenience we define w_{ij} for $i \ge j$, by letting $w_{ij} = -w_{ji}$. We then define

$$\mathrm{curl}^* w = ((\mathrm{curl}^* w)_j)_{1 \le j \le n} \in \mathbb{R}^n$$

by

$$(\mathrm{curl}^* w)_j = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial w_{ij}}{\partial x_i}.$$

Remarks. — (i) If w_{ij} are the components of an $(n-2)$ form α over $\mathbb{R}^n$ then $(\mathrm{curl}^* w)_j$ are the components of $d\alpha$.
(ii) For every $w \in C^2(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2})$ we have

$$\mathrm{div}\,(\mathrm{curl}^* w) \equiv 0$$

which corresponds to the relation $dd\alpha = 0$.

Proof of Theorem 2. — We decompose the proof into two steps. In both steps we shall use standard results of elliptic operators with Neumann boundary condition and we refer, for example, to Ladyzhenskaya-Uraltseva [LU] (Section 3 of Chapter 3) for details.

STEP 1. — Let $a \in C^{k+2,\alpha}(\bar{\Omega})$ be the unique solution with $\int a\,dx = 0$, of the Neumann problem

(1)
$$\begin{cases} \Delta a = g & \text{in } \Omega \\ \dfrac{\partial a}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where ν is the outward unit normal. Furthermore there exists $K > 0$ with

(2) $$\|a\|_{k+2,\alpha} \le K \|g\|_{k,\alpha}.$$

We then let $c \in C^{k+1,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ be defined by

$$c(x) = -\mathrm{grad}\,a(x).$$

Observe that

(3) $$\langle c; v \rangle = 0 \quad \text{on } \partial\Omega$$

where $\langle .;. \rangle$ denotes the scalar product in $\mathbb{R}^n$.

Suppose (cf. Step 2) that we can find $b \in C^{k+2, \alpha}(\bar{\Omega}; \mathbb{R}^{n(n-1)/2})$ and $K > 0$ such that

(4) $$\text{curl}^* b = c \quad \text{on } \partial\Omega$$
(5) $$\|b\|_{k+2, \alpha} \le K \|c\|_{k+1, \alpha} = K \|\text{grad } a\|_{k+1, \alpha}.$$

If we then set

(6) $$v = \text{grad } a + \text{curl}^* b$$

we have indeed solved the problem (2.4) and (2.5), by combining (2) and (5). (Note that to write v as in (6) is not unusual in magnetism or in elasticity, cf. for example Abraham-Becker [AB] or Love [L].)

STEP 2. — We now consider for $c \in C^{k+1, \alpha}(\bar{\Omega}; \mathbb{R}^n)$ the problem

(4) $$\text{curl}^* b = c \quad \text{on } \partial\Omega$$

with the additional assumption that c is tangential to $\partial\Omega$, i.e.,

(3) $$\langle c; v \rangle = 0 \quad \text{on } \partial\Omega.$$

Equation (4) is again underdetermined and we can assume that $b = (b_{ij})_{1 \le i < j \le n} = 0$ on $\partial\Omega$ so that $\text{grad } b_{ij}$ has the direction of the normal. More precisely, under the condition (3) we can assume this vector in the form

$$\text{grad } b_{ij} = (-1)^{i+j} (c_j v_i - c_i v_j) v, \qquad x \in \partial\Omega.$$

Indeed if the above holds we have on $\partial\Omega$, with the convention that $b_{ji} = -b_{ij}$,

$$(\text{curl}^* b)_j = \sum_{i=1}^{n} (-1)^{i+j} \frac{\partial b_{ij}}{\partial x_i} = \sum_{i=1}^{n} (c_j v_i - c_i v_j) v_i = c_j |v|^2 - \langle c; v \rangle v_j = c_j$$

since (3) holds and $|v| = 1$.

We therefore have reduced the problem to finding $b_{ij} \in C^{k+2, \alpha}(\bar{\Omega})$ satisfying

(7) $$\text{grad } b_{ij} = c_{ij} v, \qquad x \in \partial\Omega$$

where $c_{ij} \in C^{k+1, \alpha}(\bar{\Omega})$ [c_{ij} stands here for $(-1)^{i+j} (c_j v_i - c_i v_j)$].

Note that (7) is not a differential equation for b_{ij} but merely the prescription of the normal derivative of b_{ij} on $\partial\Omega$. Since the gradient of the distance function $d(x, \partial\Omega)$ is $-v$ whenever $x \in \partial\Omega$, one obtains a solution of our problem in the form

$$b_{ij} = -c_{ij} \zeta (d(x, \partial\Omega))$$

with $\zeta(0) = 0$, $\zeta'(0) = 1$ and $\zeta \in C^\infty$ with $\zeta \equiv 0$ outside a small neighbourhood of 0. However this solution lies only in $C^{k+1, \alpha}$.

In order to find a smoother solution in $C^{k+2, \alpha}$ we solve the following Neumann problem $\left(\text{with } \int d_{ij} \, dx = 0 \right)$

$$\begin{cases} \Delta d_{ij} = \dfrac{1}{\text{meas}\,\Omega} \displaystyle\int_\Omega c_{ij} \, d\sigma & \text{in } \Omega \\[2mm] \dfrac{\partial d_{ij}}{\partial \nu} = c_{ij} & \text{on } \partial\Omega. \end{cases}$$

Using the standard results (cf. [LU]) we find $d_{ij} \in C^{k+2, \alpha}(\bar\Omega)$ and $K > 0$ with

(8) $\| d_{ij} \|_{k+2, \alpha} \leq K \| c_{ij} \|_{k+1, \alpha}$.

We let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi'(0) = 0$, $\chi(0) = 1$ and $\chi \equiv 0$ outside a small neighbourhood of 0. We finally let

(9) $b_{ij}(x) = d_{ij}(x) - \chi(d(x, \partial\Omega)) d_{ij}(\psi(x))$

where $\psi(x) = x - d(x, \partial\Omega) \, \text{grad}\,[d(x, \partial\Omega)]$. Observe that since $\partial\Omega$ is $C^{k+3, \alpha}$ and $d_{ij} \in C^{k+2, \alpha}$, then $b_{ij} \in C^{k+2, \alpha}$. From (8) it also follows that

$$\| b_{ij} \|_{k+2, \alpha} \leq K \| c_{ij} \|_{k+1, \alpha}$$

obtaining therefore immediately (5). We then only need to show (7) [and hence (4)] and this follows from the fact that if $x \in \partial\Omega$ (denoting by δ_{kl} the Kronecker symbol)

$$\begin{aligned} \frac{\partial b_{ij}}{\partial x_k} &= \frac{\partial d_{ij}}{\partial x_k} - \sum_{l=1}^n \frac{\partial d_{ij}}{\partial x_l} \frac{\partial \psi_l}{\partial x_k} \\ &= \frac{\partial d_{ij}}{\partial x_k} - \sum_{l=1}^n \frac{\partial d_{ij}}{\partial x_l} (\delta_{kl} - \nu_k \nu_l) = \nu_k \sum_{l=1}^n \frac{\partial d_{ij}}{\partial x_l} \nu_l \\ &= \frac{\partial d_{ij}}{\partial \nu} \nu_k = c_{ij} \nu_k. \end{aligned}$$

Note finally that if we fix the above function χ we have constructed a definite solution v of (2.4), thus defining a linear operator $L : X \to Y$ with the properties given in Remark (ii) above. ∎

The next step in proving Theorem 1, is to establish it with a weaker regularity than stated.

LEMMA 3. − Let $k \geq 1$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{k+3, \alpha}$ boundary $\partial\Omega$. Let $f \in C^{k, \alpha}(\bar\Omega)$ with $f > 0$ in $\bar\Omega$

and

(2.1) $\displaystyle\int_\Omega f(x)\,dx = \text{meas }\Omega.$

Then there exists $u \in \text{Diff}^{k,\,\alpha}(\bar{\Omega})$ *satisfying*

(2.2) $\begin{cases} \det \nabla u(x) = f(x), & x \in \Omega \\ \quad u(x) = x, & x \in \partial\Omega. \end{cases}$

Remark. − The following proof is based on a deformation argument described in [M], which automatically ensures that the solution is a diffeomorphism.

Proof of Lemma 3. − We decompose the proof into two steps.

STEP 1. − Let for $t \in [0, 1]$, $z \in \Omega$

(1) $\displaystyle v_t(z) = \frac{v(z)}{t + (1-t)f(z)}$

where $v \in C^{k+1,\,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ (but $v_t \in C^{k,\,\alpha}(\bar{\Omega}; \mathbb{R}^n)$) satisfies

(2) $\begin{cases} \text{div } v = f - 1 & \text{in } \Omega \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$

(Such a v exists by Theorem 2.)

We then define $\Phi_t(x) : [0, 1] \times \Omega \to \mathbb{R}^n$ as the solution of

(3) $\begin{cases} \dfrac{d}{dt}[\Phi_t(x)] = v_t(\Phi_t(x)), & t > 0 \\[2mm] \qquad \Phi_0(x) = x. \end{cases}$

First note that $\Phi_t \in C^{k,\,\alpha}(\bar{\Omega}; \mathbb{R}^n)$ for every t and that Φ_t is uniquely defined on $[0, 1]$. Observe also that for every $t \in [0, 1]$ we have

$$\Phi_t(x) \equiv x, \qquad \text{if} \quad x \in \partial\Omega.$$

[This follows from the observation that if $x \in \partial\Omega$, then x is a solution of (3), since $v = 0$ on $\partial\Omega$; the uniqueness implies then that $\Phi_t(x) \equiv x$ for every $x \in \partial\Omega$.]

We now show that $u(x) \equiv \Phi_1(x)$ is a solution of (2.2). The boundary condition has already been verified so we need only to check that $\det \nabla \Phi_1(x) = f(x)$. To prove this we let

(4) $h(t, x) \equiv \det \nabla \Phi_t(x) . [t + (1-t)f(\Phi_t(x))].$

If we show (*cf.* Step 2) that

(5) $\dfrac{\partial}{\partial t} h(t, x) \equiv 0$

we shall have the result from the fact that $h(1, x) = h(0, x)$.

STEP 2. — We therefore only need to show (5). Let A be an $n \times n$ matrix, then it is a well known fact (cf. Coddington-Levinson [CL], p. 28) that if ψ satisfies $\psi'(t) = A(t) \psi(t)$ then

$$(\det \psi)' = \text{tr}(A) . \det \psi$$

where $\text{tr}(A)$ stands for the trace of A. We therefore get that

(6) $$\frac{\partial}{\partial t} [\det \nabla \Phi_t(x)] = \det \nabla \Phi_t(x) . \text{div } v_t(\Phi_t(x)).$$

We now differentiate (4) to get

$$\frac{\partial}{\partial t} h(t, x) = \frac{\partial}{\partial t} [\det \nabla \Phi_t] . [t + (1-t) f(\Phi_t)]$$

$$+ [\det \nabla \Phi_t] \left[1 - f(\Phi_t) + (1-t) \left\langle \nabla f(\Phi_t); \frac{d}{dt} \Phi_t \right\rangle \right].$$

Using (3) and (6) we obtain

$$\frac{\partial}{\partial t} h(t, x) = [\det \nabla \Phi_t] [(t + (1-t) f(\Phi_t)) \text{div } v_t(\Phi_t)$$

$$+ (1-t) \left\langle \nabla f(\Phi_t); v_t(\Phi_t) \right\rangle + (1 - f(\Phi_t))].$$

Using the definition of v_t [cf. (1)] we deduce that

$$\text{div } v(y) = (t + (1-t) f(y)) \text{div } v_t(y) + (1-t) \left\langle \nabla f(y); v_t(y) \right\rangle.$$

Combining the two identities we have

$$\frac{\partial}{\partial t} h(t, x) = [\det \nabla \Phi_t] . [\text{div } v(\Phi_t) + (1 - f(\Phi_t))].$$

The definition of v [cf. (2)] gives immediately (5) and thus the lemma. ∎

The third step in proving Theorem 1 is to establish the result with a smallness assumption on the $C^{0, \beta}$ norm of $f - 1$.

LEMMA 4. — Let Ω, k, α and $f \in C^{k, \alpha}(\bar{\Omega})$ be as in Theorem 1 [in particular (2.1) is satisfied]. Let $0 < \beta \leq \alpha < 1$. Then there exists $\varepsilon = \varepsilon(\alpha, \beta, k, \Omega) > 0$ such that if $\|f - 1\|_{0, \beta} \leq \varepsilon$, then there exists $u \in \text{Diff}^{k+1, \alpha}(\bar{\Omega})$ such that

(2.2) $$\begin{cases} \det \nabla u(x) = f(x), & x \in \Omega \\ u(x) = x, & x \in \partial \Omega. \end{cases}$$

Remarks. — (i) A similar result can be found in Zehnder [Z].

(ii) We shall use below some elementary properties of Hölder continuous functions and we refer for a proof of such facts to Hörmander [H]. In particular we shall use the fact that if $f, g \in C^{k, \alpha}$, then there exists a constant $C > 0$ such that

$$\|fg\|_{k, \alpha} \leq C(\|f\|_{k, \alpha} \|g\|_0 + \|f\|_0 \|g\|_{k, \alpha})$$

where $\| . \|_0$ denotes the C^0 norm.

Proof of Lemma 4. – We start by defining two constants K_1, K_2 as follows.

(i) Let

$$X = \left\{ b \in C^{k, \alpha}(\bar{\Omega}) : \int_\Omega b(x)\, dx = 0 \right\},$$
$$Y = \left\{ a \in C^{k+1, \alpha}(\bar{\Omega}; \mathbb{R}^n) : a = 0 \text{ on } \partial\Omega \right\}.$$

As seen in Theorem 2 we can then define a bounded linear operator $L : X \rightarrow Y$ which associates to every $b \in X$ an $a \in Y$ such that

$$\begin{cases} \operatorname{div} a = b & \text{in } \Omega \\ a = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore there exists $K_1 > 0$ such that

(1) $$\| L\, b \|_{1, \beta} \leq K_1 \| b \|_{0, \beta}$$

(2) $$\| L\, b \|_{k+1, \alpha} \leq K_1 \| b \|_{k, \alpha}.$$

(ii) Let for ξ, any $n \times n$ matrix,

(3) $$Q(\xi) \equiv \det(I + \xi) - 1 - \operatorname{tr}(\xi)$$

where I stands for the identity matrix and $\operatorname{tr}(\xi)$ for the trace of ξ. Note that Q is a sum of monomials of degree t, $2 \leq t \leq n$. We therefore can find $K_2 > 0$, such that if w_1, $w_2 \in C^{k, \alpha}$ with $\| w_1 \|_0, \| w_2 \|_0 \leq 1$, then

(4) $$\| Q(w_1) - Q(w_2) \|_{k, \alpha} \leq K_2 (\| w_1 \|_0 + \| w_2 \|_0) \| w_1 - w_2 \|_{k, \alpha}.$$

In order to solve (2.2) we set $v(x) \equiv u(x) - x$ and we rewrite (2.2) as

(5) $$\begin{cases} \operatorname{div} v = f - 1 - Q(\nabla v) & \text{in } \Omega \\ \qquad v = 0 & \text{on } \partial\Omega. \end{cases}$$

If we set

(6) $$N(v) = f - 1 - Q(\nabla v)$$

then (5) is satisfied for any $v \in Y$ with

(7) $$v = LN(v).$$

Note first that the equation is well defined (*i.e.,* $N : Y \rightarrow X$), since if $v = 0$ on $\partial\Omega$ then $\int N(v(x))\, dx = 0$. Indeed from (3) it follows that

$$\int_\Omega N(v(x))\, dx = \int_\Omega [f(x) - 1 - Q(\nabla v(x))]\, dx$$
$$= \int_\Omega [f(x) + \operatorname{div} v(x) - \det(I + \nabla v(x))]\, dx;$$

since $v = 0$ on $\partial \Omega$ and $\int f = \mathrm{meas}\,\Omega$, it follows immediately that the right hand side of the above identity is 0.

We now solve (7) by the contraction principle. We first let for $r > 0$

$$\mathbf{B}_r = \left\{ u \in C^{k+1,\alpha}(\bar{\Omega}; \mathbb{R}^n) : u = 0 \text{ on } \partial\Omega, \| u \|_{1,\beta} \leq r \right\}.$$

We shall show that by choosing $\| f - 1 \|_{0,\beta}$ and r small enough then $\mathrm{LN} : \mathbf{B}_r \to \mathbf{B}_r$ is a contraction mapping (with respect to the $C^{k+1,\alpha}$ norm). The contraction principle will then immediately lead to a solution $v \in C^{k+1,\alpha}$ of (7).

Indeed if we let

$$\text{(8)} \qquad \| f - 1 \|_{0,\beta} < \min\left\{ \frac{1}{8\,\mathrm{K}_1^2\,\mathrm{K}_2} ; \frac{1}{2\,\mathrm{K}_1} \right\}$$

$$\text{(9)} \qquad r = 2\,\mathrm{K}_1 \| f - 1 \|_{0,\beta}$$

and if $v,\, w \in \mathbf{B}_r$ we then have

$$\text{(10)} \qquad \| \mathrm{LN}(v) - \mathrm{LN}(w) \|_{k+1,\alpha} \leq \frac{1}{2} \| v - w \|_{k+1,\alpha}$$

$$\text{(11)} \qquad \| \mathrm{LN}(v) \|_{1,\beta} \leq r.$$

The first inequality, which is also valid for $k = 0$ and $\alpha = \beta$, follows from

$$\| \mathrm{LN}(v) - \mathrm{LN}(w) \|_{k+1,\alpha}$$
$$\leq \mathrm{K}_1 \| \mathrm{N}(v) - \mathrm{N}(w) \|_{k,\alpha} \leq \mathrm{K}_1\,\mathrm{K}_2\,(\| v \|_1 + \| w \|_1)\,\| v - w \|_{k+1,\alpha}$$
$$\leq \mathrm{K}_1\,\mathrm{K}_2\,(\| v \|_{1,\beta} + \| w \|_{1,\beta})\,\| v - w \|_{k+1,\alpha}$$
$$\leq 4\,\mathrm{K}_1^2\,\mathrm{K}_2\,\| f - 1 \|_{0,\beta}\,\| v - w \|_{k+1,\alpha} \leq \frac{1}{2} \| v - w \|_{k+1,\alpha}.$$

To obtain (11) we observe that

$$\| \mathrm{LN}(0) \|_{1,\beta} \leq \mathrm{K}_1 \| \mathrm{N}(0) \|_{0,\beta} = \mathrm{K}_1 \| f - 1 \|_{0,\beta} = \frac{r}{2}$$

and hence combining (10) for $k = 0$ and $\alpha = \beta$ with the above inequality we have immediately (11).

It now remains to show that $u(x) = x + v(x)$ is a diffeomorphism, this is an easy consequence of the fact that $\det \nabla u = f > 0$ and $u(x) = x$ on $\partial \Omega$ (*see* for example Corollary 2, p. 79 in [MO]). ∎

We may now conclude the proof of Theorem 1′.

Proof of Theorem 1′ (Step 4). — By density of C^∞ functions in $C^{k,\alpha}$ with the $C^{0,\beta}$ norm ($0 < \beta < \alpha < 1$) we can find $g \in C^\infty(\bar{\Omega})$, $g > 0$ in $\bar{\Omega}$ such that

$$\text{(1)} \qquad \left\| \frac{f}{g} - 1 \right\|_{0,\beta} \leq \varepsilon$$

(2)
$$\int_\Omega \frac{f(x)}{g(x)}\, dx = \text{meas}\, \Omega$$

where ε is as in Lemma 4.

We then define $b \in \text{Diff}^{k+1,\,\alpha}(\bar\Omega)$ to be a solution [which exists by (1), (2) and Lemma 4] of

$$\det \nabla b(x) = \frac{f(x)}{g(x)}, \qquad x \in \Omega$$
$$b(x) = x, \qquad x \in \partial\Omega.$$

We further let $a \in \text{Diff}^{k+1,\,\alpha}(\bar\Omega)$ to be a solution of

$$\begin{cases} \det \nabla a(y) = g(b^{-1}(y)), & y \in \Omega \\ \quad a(y) = y, & y \in \partial\Omega. \end{cases}$$

Such a solution exists by Lemma 3 since $g \circ b^{-1} \in C^{k+1,\,\alpha}(\bar\Omega)$ and

$$\int_\Omega g(b^{-1}(y))\, dy = \int_\Omega g(x) \det \nabla b(x)\, dx = \int_\Omega f(x)\, dx = \text{meas}\, \Omega.$$

Finally observe that the function $u \equiv a \circ b$ has all the claimed properties. ∎

III. ANOTHER APPROACH

We now present a second approach for solving (1.1) which is more elementary, in the sense that it does not require the existence theory and Schauder estimates for elliptic partial differential equations. It will use, as a main tool, the implicit function theorem.

We can now state the first theorem of this section.

THEOREM 5. – *Let $k \geq 0$ be an integer, Ω be a bounded connected open set of $\mathbb{R}^n$ with $C^k \cap C^1$ boundary $\partial\Omega$. Let $f, g \in C^k(\bar\Omega)$, $f, g > 0$ in $\bar\Omega$ with*

(3.1)
$$\int_\Omega f(x)\, dx = \int_\Omega g(x)\, dx.$$

Then there exists $\varphi \in \text{Diff}^k(\bar\Omega)$ with $\varphi(x) = x$ on $\partial\Omega$ and such that

(3.2)
$$\int_E f(x)\, dx = \int_{\varphi(E)} g(x)\, dx.$$

for every open set $E \subset \Omega$.

Moreover if $\text{supp}(f-g) \subset \Omega$, then $\text{supp}\{\varphi - id\} \subset \Omega$ where id stands for the identity map.

Remarks. − (i) Recall first that if $\varphi \in \text{Diff}^0(\bar{\Omega})$, then it is understood that φ is a homeomorphism from $\bar{\Omega}$ onto $\bar{\Omega}$.

(ii) As already observed in the introduction if $k \geq 1$, (3.2) is equivalent to (1.1) by a change of variables

$$\int_{\varphi(E)} g(x)\,dx = \int_E g(\varphi(x)) \det \nabla \varphi(x)\,dx = \int_E f(x)\,dx.$$

Since E is arbitrary, it follows that $g(\varphi) \det \nabla \varphi = f$ in Ω.

(iii) We can also rewrite (3.2) in the following equivalent form

$$\int_\Omega f(x)\zeta(x)\,dx = \int_\Omega g(x)\zeta(\varphi^{-1}(x))\,dx,$$

for every $\zeta \in C^\infty(\Omega)$ with compact support. Approximating the characteristic function χ_E by ζ we obtain (3.2). Therefore the above identity can be viewed as the "weak form" of the equation $g(\varphi(x)) \det \nabla \varphi(x) = f(x)$.

(iv) Aside from the weaker boundary smoothness requirement, the above theorem is for $k \geq 1$ weaker than Theorem 1, since it does not provide any smoothness gain for φ. The main points of this theorem are firstly that $k = 0$ is admitted and secondly that φ can be chosen so as to be the identity near $\partial\Omega$ if $f = g$ near $\partial\Omega$. These results cannot be obtained by the method presented in Section 2. For the equivalence problem of measures under homeomorphisms one has, of course, the stronger theorem of Oxtoby and Ulam (*see* [A] for references).

We now turn to the second result where we search for $\varphi \in \text{Diff}^k(\Omega) \cap \text{Diff}^0(\bar{\Omega})$ $(k \geq 1)$ with $\varphi(x) = x$ on $\partial\Omega$, *i.e.*, φ is a diffeomorphism of Ω which extends as a homeomorphism of $\bar{\Omega}$ keeping the boundary pointwise fixed. This requires much weaker regularity assumptions on $\partial\Omega$. For the following we shall require that $\Omega \subset \mathbb{R}^n$ has, with respect to the volume form $\tau = dx$, the property (H_k) defined below.

DEFINITION. − Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set and $k \geq 1$ be an integer. Ω is said to satisfy (H_k) if it can be covered by finitely many open domains Ω_j such that for every j (*see* the figure below),

(i) there exists a C^k diffeomorphism

$$\psi_j: \quad \bar{\Omega}_j - \omega_j \to P^n$$

where $\omega_j = \bar{\Omega}_j \cap \partial\Omega$ and $P^n = \bar{Q}^n - q$ where $Q^n = (0, 1)^n$ is the unit cube of $\mathbb{R}^n$ and $q = \{ x \in \bar{Q}^n : x_1 = 0 \}$ if $\omega_j \neq \varnothing$ and $q = \varnothing$ if $\omega_j = \varnothing$. Moreover $\det \nabla \psi_j \in C^k$ and there exists $A \geq 1$ with

$$\frac{1}{A} < \det \nabla \psi_j < A.$$

(ii) The map ψ_j is proper, *i.e.* if $K \subset P^n$ is compact so is $\psi_j^{-1}(K)$ and ψ_j^{-1} extends to a continuous map of P^n with $\psi_j^{-1}(x) \in \omega_j$ if $x \in q$.

(iii) $\psi_k^{-1} \circ \psi_j \in C^k$ in $\Omega_k \cap \Omega_j$.

Remark. — The property (H_k) is of local character. In the appendix we shall show that the following domains have this property:

(i) if $\partial\Omega$ is locally given as a graph of a Lipschitz function, e. g. any open convex polyhedron (*see* Propositon A.2);

(ii) domains with isolated boundary points (*see* Proposition A.3), or a combination of both.

Before stating our second result, we give a proposition which explains why the above definition is required.
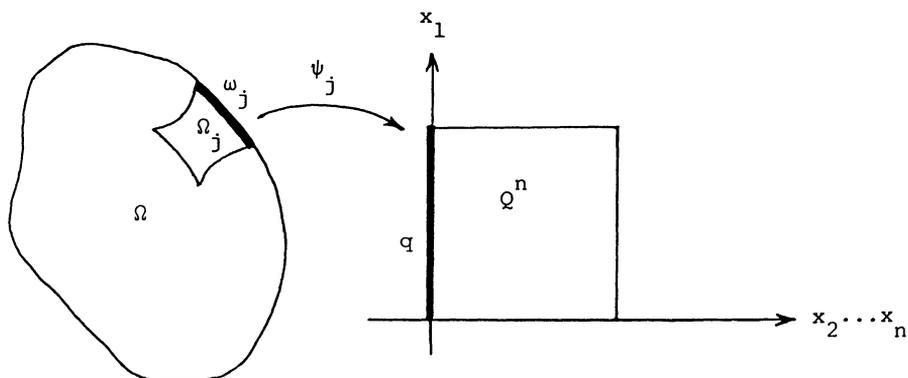
PROPOSITION 6. — *With the above notations if* $\varphi : Q^n \cup q \to Q^n \cup q$ *is continuous with* $\varphi(x) = x$ *for* $x \in q$, *then*

$$\tilde{\varphi} = \psi_j^{-1} \circ \varphi \circ \psi_j : \Omega_j \to \Omega_j$$

extends continuously to ω_j *with* $\tilde{\varphi}(x) = x$ *for* $x \in \omega_j$.

Proof. — We drop for simplicity the index j in Ω_j, ω_j and ψ_j. We let $p^\nu \in \Omega$ be a sequence of points such that $p^\nu \to p^* \in \omega$. We have to show that

(1) $$d(\tilde{\varphi}(p^\nu); p^\nu) \to 0$$

so that $\tilde{\varphi}(p^*) = p^*$.

Since ψ is a proper map we conclude that

(2) $$d(\psi(p^\nu); q) \to 0$$

i. e., if $\psi(p^\nu) = r^\nu$ we must have $r_1^\nu \to 0$ where $r^\nu = (r_1^\nu, \ldots, r_n^\nu)$ otherwise for a subsequence, still denoted r^ν, we would have $r^\nu \to r^*$ with $r_1^* > 0$. Therefore $K = \{r^\nu, r^*\}$ would be compact in P and thus $\psi^{-1}(K) = \{p^\nu, \psi^{-1}(r^*)\}$ would be compact in $\bar{\Omega} - \omega$, which contradicts the fact that $p^\nu \to p^* \in \omega$.

Since φ is continuous on q with $\varphi(x)=x$, it follows from the fact that $r_1^\nu \to 0$, that

$$d(\varphi(r^\nu), r^\nu) \to 0$$

and hence by the continuity of ψ^{-1} that

(3) $d(\psi^{-1}(\varphi(r^\nu)), \psi^{-1}(r^\nu)) \to 0.$

(3) implies then immediately (1). ∎

We now have the following

THEOREM 7. — *Let $k \geq 0$ and Ω satisfy $(H_{k'})$ with $k' = \max\{1, k\}$. Let f, $g \in C^k(\Omega)$ with $f, g > 0, f + \dfrac{1}{f}, g + \dfrac{1}{g}$ bounded and satisfying*

(3.1) $\displaystyle\int_\Omega f(x)\,dx = \int_\Omega (g(x)\,dx.$

Then there exists $\varphi \in \mathrm{Diff}^k(\Omega) \cap \mathrm{Diff}^0(\bar\Omega)$ with $\varphi(x)=x$ on $\partial\Omega$ such that

(3.2) $\displaystyle\int_E f(x)\,dx = \int_{\varphi(E)} g(x)\,dx,$

for every open set $E \subset \Omega$.

Moreover if $\mathrm{supp}\{f-g\} \subset \Omega$, then $\mathrm{supp}\{\varphi - \mathrm{id}\} \subset \Omega$.

Remark. — With the help of Theorem 7 it is easy to construct a volume preserving mapping of a convex polyhedron Ω in $\mathbb{R}^n$ which permutes finitely many of its points, say $P_1, \ldots, P_N$, in a prescribed manner and keeps the boundary pointwise fixed (*see* Alpern [A]). To do so one first constructs any diffeomorphism ψ of Ω permuting the P_j in the desired manner. This diffeomorphism takes then the volume form $\tau = dx_1 \ldots dx_n$ into $\psi^*\tau \equiv (\det \nabla \psi)\tau$. We then apply the above theorem to $\Omega - \bigcup\limits_{j=1}^{N} P_j$ and find a diffeomorphism φ with $\varphi^*\tau \equiv (\det \nabla \varphi)\tau = f\tau$ where $f = \det \nabla \psi^{-1}$, which keeps the boundary fixed. The desired volume preserving mapping is then $\varphi \circ \psi$.

We next turn to the proof of Theorems 5 and 7. It can be reduced to the case of a cube, using a covering of Ω by sets Ω_j as described in the above definition. Using Lemma 1 of [M] one can construct a sequence of functions $f_j > 0$ such that $f_0 = f, f_N = g$ and $f_{j+1} - f_j$ has support in Ω_j or in $\Omega_j \cup \omega_j$ $(\omega_j = \bar\Omega_j \cap \partial\Omega)$ and satisfying $\displaystyle\int_\Omega f_j\,dx = \int_\Omega f_{j+1}\,dx$. Thus using the mapping $\psi_j : \bar\Omega_j - \omega_j \to P^n$ $(P^n = \overline{Q^n} - q$ with $q = \{x \in Q^n : x_1 = 0\}$ if $\omega_j \neq \varnothing$ and $q = \varnothing$ if $\omega_j = \varnothing)$ we can map f_j, f_{j+1} into $f_j^* = f_j(\psi_j^{-1})\det \nabla \psi_j^{-1}$, $f_{j+1}^* = f_{j+1}(\psi_j^{-1})\det \nabla \psi_j^{-1}$ corresponding to the volume forms $f_j^*\,dx$, $f_{j+1}^*\,dx$ defined in P^n. Therefore it will suffice to construct $\tilde\varphi_j \in \mathrm{Diff}^k(P^n)$

with supp $\{\tilde{\varphi}_j - \mathrm{id}\} \subset \mathrm{P}^n$ such that

$$\int_E f_j^* \, dx = \int_{\tilde{\varphi}_j(E)} f_{j+1}^* \, dx$$

for every open set E in Q^n. Applying this argument to Ω_j, $j = 1, 2, \ldots, \mathrm{N}$ we obtain Theorems 5 and 7; Proposition 6 ensuring that $\psi_j^{-1} \circ \tilde{\varphi}_j \circ \psi_j$ keeps the boundary ω_j pointwise fixed.

Without further elaboration of this patching argument we formulate, for the case of the cube, the precise conditions and statements needed for the proof of Theorems 5 and 7. We have to distinguish between two cases depending on whether $\bar{\Omega}_j$ meets the boundary $\partial\Omega$ or not. This corresponds to the cases $q = \{x \in \overline{\mathrm{Q}^n} : x_1 = 0\}$ or $q = \varnothing$ respectively. We also denote by $\overset{\circ}{q} = \{x_1 = 0, (x_2, \ldots, x_n) \in (0, 1)^{n-1}\}$ if $q \neq \varnothing$ and $\overset{\circ}{q} = \varnothing$ if $q = \varnothing$. We also drop for simplicity the index n in P^n.

PROPOSITION 8. — *Let* $k \geq 0$, $\mathrm{Q}^n = (0, 1)^n$, $\mathrm{P} = \overline{\mathrm{Q}^n} - q$ *with either* $q = \varnothing$ *or* $q = \{x \in \overline{\mathrm{Q}^n}, x_1 = 0\}$. *Assume that*
   (i) $0 < f$, $g \in \mathrm{C}^k(\mathrm{P})$, f, f^{-1}, g, g^{-1} *are bounded in* P.

   (ii) $\displaystyle\int_{\mathrm{Q}^n} [f(x) - g(x)] \, dx = 0.$
   (iii) supp $\{f - g\} \subset \mathrm{Q}^n \cup \overset{\circ}{q}$.
Then there exists $\varphi \in \mathrm{Diff}^k(\mathrm{P})$ *which extends as a homeomorphism of* $\overline{\mathrm{Q}^n}$ *with* $\varphi(x) = x$ *for* $x \in q$ *with* supp $\{\varphi - \mathrm{id}\} \subset \mathrm{Q}^n \cup \overset{\circ}{q}$ *and*

(3.3) $$\int_{\varphi(E)} g(x) \, dx = \int_E f(x) \, dx$$

for every open set $\mathrm{E} \subset \overline{\mathrm{Q}^n}$.

If, in addition, f, $g \in \overline{\mathrm{C}^k}(\mathrm{Q}^n)$ *then the above conclusions hold and* $\varphi \in \mathrm{Diff}^k(\overline{\mathrm{Q}^n})$.

Remark. — The additional statement is appropriate for the proof of Theorem 5, where φ is C^k up to the boundary, while the first statement fits the proof of Theorem 7.

IV. AN ELEMENTARY PROOF OF PROPOSITION 8

In this section we proceed with the proof of Proposition 8, thus completing the proof of Theorems 5 and 7. The argument can be considered as an analogue of the separation of variables for the differential equation $\det \nabla u = f$ where at each stage an ordinary differential equation has to be solved. Moreover this differential equation admits an integral and can be solved via the implicit function theorem. For this reason even the case of

continuous f and g can be handled. In this sense the argument is quite elementary though a bit tricky.

To give an idea of the proof of Proposition 8, we first try to find φ as a mapping preserving the line segments $x_1 = a$, *i.e.* we take φ of the form

$$\varphi : \quad (x_1, x_2, \ldots, x_n) \to (v(x), x_2, \ldots, x_n)$$

with a function v monotone increasing in x_1. The relation (3.3), when applied to $E = [0, a] \times \prod_{j=2}^{n} [a_j, b_j]$, leads for $b_j - a_j \to 0$ to the equivalent requirement

$$\int_0^a f(x_1, x') \, dx_1 = \int_0^{v(a, x')} g(x_1, x') \, dx_1,$$

for every $x' = (x_2, \ldots, x_n) \in Q^{n-1}$. Since $g > 0$ this equation defines $v(x)$ uniquely with v monotone in x_1, $v = 0$ for $x_1 = 0$ and $v = x_1$ for x' near ∂Q^{n-1} (since $f - g = 0$ for x' near ∂Q^{n-1}). However in order to achieve $v = x_1$ for $x_1 = 1$ we need the condition

$$(4.1) \qquad \int_0^1 f(x_1, x') \, dx_1 = \int_0^1 g(x_1, x') \, dx_1$$

for every $x' \in Q^{n-1}$. We note that v, $\dfrac{\partial v}{\partial x_1} \in C^k(P)$, but no such assertion holds for the other derivatives. Observe also that $\dfrac{\partial v}{\partial x_1}, \left(\dfrac{\partial v}{\partial x_1} \right)^{-1}$ are bounded, since f, f^{-1}, g, g^{-1} are bounded.

We have therefore proved.

PROPOSITION 9. − *If f and g satisfies the hypotheses of Proposition 8 as well as (4.1), then there exists φ satisfying all the assertions of Proposition 8. Moreover φ preserves all line segments parallel to the x_1-axis in Q^n.*

In order to conclude the proof of Proposition 8, it therefore suffices to transform g by an appropriate diffeomorphism ψ such that the condition (4.1) is realized. This will be achieved in the following.

PROPOSITION 10. − *If $0 < f$, $g \in C^k(Q^n)$ with supp $\{f - g\} \subset Q^n \cup \overset{\circ}{q}$ and*

$$(4.2) \qquad \int_{Q^n} [f(x) - g(x)] \, dx = 0,$$

then there exists $\psi \in \mathrm{Diff}^k(Q^n)$ with supp $\{\psi - \mathrm{id}\} \subset Q^n$, and $g_1 \in C^k(P)$ such that

$$(4.3) \qquad \int_E g_1(x) \, dx = \int_{\psi(E)} g(x) \, dx$$

for every open set $E \subset Q^n$ *and*

$$(4.4) \qquad \int_0^1 g_1(x_1, x') \, dx_1 = \int_0^1 f(x_1, x') \, dx_1$$

for every $x' \in Q^{n-1}$.

Remarks. – (i) Since $\psi(x) = x$ near ∂Q^n, it follows from (4.3) that $g_1(x) = g(x)$ near ∂Q^n.

(ii) In this proposition no restriction about the boundary behaviour of f and g near $\overset{\circ}{q}$ are required (such restrictions are only needed in Proposition 9).

(iii) The combination of Propositions 9 and 10 give immediately Proposition 8.

Proof of Proposition 10. – We shall construct ψ as

$$\psi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_2$$

and define $g_n = g$ and for $s = 2, 3, \ldots, n$

$$(1) \qquad g_{s-1}(x) = g_s(\varphi_s(x)) \det \nabla \varphi_s(x)$$

if $k \geq 1$, or equivalently

$$(1') \qquad \int_E g_{s-1}(x) \, dx = \int_{\varphi_s(E)} g_s(x) \, dx$$

for every open set $E \subset Q^n$. (The latter definition holds also for $k = 0$.)

We shall construct $\varphi_n, \ldots, \varphi_{s+1}$ inductively in such a way that

$$(2) \qquad \int_{Q^s} g_s(x^s, x') \, dx^s = \int_{Q^s} f(x^s, x') \, dx^s$$

where $x^s = (x_1, \ldots, x_s)$ and $x' = (x_{s+1}, \ldots, x_n)$, *i.e.* such that integrals over s-dimensional cubes $x_j = c_j$ for $j > s$ match. For $s = n$ this corresponds exactly to (4.2) and for $s = 1$ this is our desired assertion (4.4).

We proceed by induction and assume that $\varphi_n, \ldots, \varphi_{s-1}$ are already constructed so that (1) and (2) hold and that they agree with the identity near the boundary. We therefore have

$$g_s = \ldots = g_n = g, \quad \text{near } \partial Q^n.$$

To complete the induction we construct φ_s as follows

$$(3) \quad \varphi_s(x_1, \ldots, x_n) = (x^{s-1}, v(x), x') = (x_1, \ldots, x_{s-1}, v(x), x_{s+1}, \ldots, x_n)$$

with

$$(4) \qquad v(x) = x_s + \zeta(x^{s-1}) u(x_s, x'),$$

where the functions u and ζ will be defined below. Observe that φ_s preserves all line segments $x_j = c_j$ ($j \neq s$) parallel to the x_s-axis.

We now define ζ as a cut-off function, $\zeta \in C^\infty (Q^{s-1})$ with compact support and satisfying

(5)
$$\left\{ \begin{array}{l} 0 \leq \zeta \leq 1 + \varepsilon \quad \text{in } Q^{s-1} \\[2mm] \displaystyle\int_{Q^{s-1}} \zeta(x^{s-1}) \, dx^{s-1} = 1 \\[2mm] \displaystyle\int_{Q^{s-1}} |\zeta(x^{s-1}) - 1| \, dx^{s-1} < \varepsilon \end{array} \right.$$

where $\varepsilon > 0$ is chosen so that

(6)
$$\varepsilon \max g_s < \min g_s, \ \frac{1}{2} \min f.$$

We next construct u. To derive the condition u has to satisfy, we first set $u(0, x') = 0$. We then would like (2) to be satisfied for $(s-1)$ in place of s, i.e.

(7)
$$\int_{Q^{s-1}} [g_{s-1}(x^{s-1}, x_s, x') - f(x^{s-1}, x_s, x')] \, dx^{s-1} = 0$$

where g_{s-1} satisfies (1'). Integrating (7) over $0 < x_s < a$ and denoting by $Q_a^s = \{ x^s \in Q^s : 0 < x_s < a \}$ we have

(8)
$$\int_{Q_a^s} [g_{s-1}(x^s, x') - f(x^s, x')] \, dx^s = 0.$$

Using (3), (4) and the fact that $u(0, x') = 0$, we have

$$R_a^s \equiv \varphi_s(Q_a^s) = \{ x^s \in Q^s \quad \text{with } 0 < x_s < v(x^{s-1}, a, x') = a + \zeta(x^{s-1}) u(x_s, x') \}.$$

Combining (8) and (1') we have

(9)
$$\int_{R_a^s} g_s(x^s, x') \, dx^s = \int_{Q_a^s} f(x^s, x') \, dx^s.$$

(From now on we drop, for convenience, the dependence on x', since x' appears only as a parameter.)

We can rewrite (9) in the following form. Letting $b \in I_a \equiv \left[\dfrac{-a}{1+\varepsilon}, \dfrac{1-a}{1+\varepsilon} \right]$, $R_{ab}^s = \{ x^s \in Q^s : 0 < x_s < a + \zeta(x^{s-1}) b \}$ and defining

$$G(a, b) = \int_{R_{ab}^s} g_s(x^s) \, dx^s$$

$$F(a) = \int_{Q_a^s} f(x^s) \, dx^s,$$

we have then that (9) is equivalent to the equation

(10) $$G(x_s, u(x_s)) = F(x_s).$$

[Note that since $g_s > 0$ it follows from (10) that $u(0) = 0$ is the unique solution of (10) for $x_s = 0$.]

We now claim that (10) has a unique solution $u \in I_{x_s}$ with u and $\dfrac{\partial u}{\partial x_s} \in C^k$. Moreover, we have

(11) $$u(1) = 0$$

(12) $$\frac{\partial v}{\partial x_s} = 1 + \zeta(x^{s-1}) \frac{\partial u}{\partial x_s} > 0.$$

If we can achieve this, we shall have with the help of (3) and (4) constructed φ_s with the appropriate conditions. Indeed the only thing which remains to be checked is that $\varphi_s(x) = x$ for every x near ∂Q^n. For x^{s-1} near Q^{s-1} this is ensured by the cut-off function ζ. For x_j near 0 or 1, $j \geqq s$, this follows from the assumption that $f = g$ for x_j near 0 or 1 if $j \neq 1$, so that by the uniqueness of u we have $u = 0$ if x_j is near 0 or 1. Thus φ_s has all the desired properties and the induction is completed. The proposition therefore is proved.

It remains to show that we can find u solving (10) with the claimed properties. To see that (10) has at most one solution $u \in I_{x_s}$, it is sufficient to observe that G is monotone in b, indeed

(13) $$\frac{\partial G}{\partial b} = \int_{Q^{s-1}} \zeta(x^{s-1}) g_s(x^{s-1}, a + b\zeta(x^{s-1})) \, dx^{s-1} > 0.$$

The existence follows from the fact that for $a \in (0, 1)$, the function $b \to G(a, b) - F(a)$ has opposite signs at the end points of I_a. At the left end point we have

(14) $$G\left(a, \frac{-a}{1+\varepsilon}\right) \leqq \max\{g_s\} a \int_{Q^{s-1}} \left|1 - \frac{\zeta(x^{s-1})}{1+\varepsilon}\right| dx^{s-1}$$
$$\leqq \frac{2\varepsilon a \max\{g_s\}}{1+\varepsilon} < 2\varepsilon a \max\{g_s\} < a \min\{f\} \leqq F(a),$$

where we have used (5) and (6). Similarly one shows that

(15) $$G\left(a, \frac{1-a}{1+\varepsilon}\right) > F(a).$$

Indeed, using the induction hypothesis (2) we have

$$G(1, 0) = F(1)$$

and therefore (15) is equivalent to

$$G(1,0) - G\left(a, \frac{1-a}{1+\varepsilon}\right) - F(1) + F(a) < 0$$

which is straightforward [as in (14)], therefore (15) is established.

Collecting (13), (14) and (15) we have indeed shown the existence of a unique u satisfying (10). Moreover $u(0)=0$ and $u(1)=0$ by construction and from (2) respectively. Thus the mapping $x_s \to v(x) = x_s + \zeta(x^{s-1}) u(x_s)$ takes the interval [0, 1] into itself.

We now show (12) and that $u, \dfrac{\partial u}{x_s} \in C^k$. Indeed $\dfrac{\partial u}{\partial x_s}$ exists as is seen by differentiation of the relation (10), $i.e.$

(16) $$\frac{\partial G}{\partial a} + \frac{\partial G}{\partial b} \frac{\partial u}{\partial x_s} = \frac{\partial F}{\partial a}$$

where $\dfrac{\partial G}{\partial b} = G_b$ is given by (13) and

(17) $$\begin{cases} \dfrac{\partial G}{\partial a} = G_a = \displaystyle\int_{Q^{s-1}} g_s(x^{s-1}, v)\, dx^{s-1}, \qquad v = v(x^{s-1}, a) \\[2mm] \dfrac{\partial F}{\partial a} = F_a = \displaystyle\int_{Q^{s-1}} f(x^{s-1}, a)\, dx^{s-1}. \end{cases}$$

Thus u and $\dfrac{\partial u}{\partial x_s} \in C^k$. To conclude the proof it remains therefore to show (12). By (16) it suffices to prove that

(18) $$E \equiv G_b + \zeta(F_a - G_a) = G_b - G_a + \zeta F_a + (1-\zeta) G_a > 0.$$

For this purpose we need (5). From (13) and (17) we obtain

$$G_b - G_a \geq -\max g_s \int_{Q^{s-1}} |\zeta - 1|\, dx^{s-1} \geq -\varepsilon \max\{g_s\}$$

and if $\zeta \in [0, 1]$ we obviously have

$$\zeta F_a + (1-\zeta) G_a \geq \min\{F_a, G_a\} \geq \min\{\min f, \min g_s\} > \varepsilon \max g_s.$$

The combination of the two inequalities proves (18) if $\zeta \in [0, 1]$. If $\zeta \in (1, 1+\varepsilon]$ we obtain

$$\zeta F_a + (1-\zeta) G_a \geq F_a - \varepsilon G_a \geq \min f - \varepsilon \max g_s \geq \min f - 2\varepsilon \max g_s > 0$$

where we have used (6) in the last inequality. Therefore (18) is established and this concludes the proof of Proposition 10. ∎

We conclude with some comments. The diffeomorphism ψ of Proposition 10 is given in the form $\psi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_2$. The diffeomorphism φ

of Proposition 9, which we rename φ_1, is constructed by the same process as φ_s for $s \geq 2$. The diffeomorphism of Proposition 8 is then

$$\varphi = \psi \circ \varphi_1 = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_2 \circ \varphi_1.$$

In the above exposition we prefered to separate φ_1 from ψ since $\psi(x) = x$ near the boundary and the boundary behaviour of φ is controlled by φ_1 alone, which was obtained by integration along the lines parallel to the x_1-axis.

Incidentally, if we replace ζ by $\zeta \equiv 1$, the above construction of φ_s agrees with that of [M], but then the condition $\varphi_s(x) = x$ near ∂Q^n cannot be ensured.

Finally we point out that the Theorems 5 and 7 are, of course, valid for arbitrary compact $C^{k'}$-manifolds, $k' = \max(1, k)$ with boundaries satisfying the condition $(H_{k'})$.

APPENDIX

Here we want to verify the remarks about Condition (H_k) stated in Section 3. We first start with

PROPOSITION A.1. — *Let Ω be a domain with C^k boundary, $k \geq 1$. Then Ω can be covered by open domains Ω_j such that there exists a diffeomorphism $\psi_j \in \mathrm{Diff}^k(\bar{\Omega}_j; \overline{Q^n})$ with $\det \nabla \psi_j \in C^k(\bar{\Omega}_j)$.*

Proof. — It suffices to consider the neighbourhood point, which we may take as the origin in $\mathbb{R}^n$. Without loss of generality we may assume that the boundary $\partial \Omega$ near $y = 0$ is locally given as the graph $y_1 = b(y')$, $y' = (y_2, \ldots, y_n)$ of a C^k function b with $b(0) = 0$. Let $\varepsilon, \delta > 0$, we let $\bar{\Omega}_j$ be

(A.1) $\qquad \bar{\Omega}_j = \{ y \in \mathbb{R}^n : b(y') \leq y_1 \leq \delta, \ 0 \leq y_j \leq \varepsilon, j = 2, \ldots, n \}.$

Denoting the Lipschitz constant of b by L, we have $|b(y')| \leq L\varepsilon$ and we choose ε so small that $0 < 4\varepsilon L < \delta$, hence

$$|b(y')| \leq L\varepsilon < \frac{\delta}{4} \quad \text{for} \quad y' \in Q^{n-1}.$$

We then define the mapping $\psi_j^{-1} : \overline{Q^n} \to \bar{\Omega}_j$ by

(A.2) $\qquad \psi_j^{-1}: \quad x \to y = (y_1, y') = (\delta x_1 + b(\varepsilon x') \zeta(x_1), \varepsilon x')$

where $\zeta \in C^{\infty}(\mathbb{R})$ is a cut-off function satisfying $\zeta(t) = 1$ for $t \leq 0$, $\zeta(t) = 0$ for $t \geq 1$ and $|\zeta'| \leq 2$. It is clear that $\psi_j^{-1} \in C^k$ and maps $\overline{Q^n}$ onto $\bar{\Omega}_j$. Moreover

$$\det(\nabla \psi_j^{-1}) = (\delta + b(\varepsilon x') \zeta'(x_1)) \varepsilon^{n-1}$$

which is $C^k(\overline{Q^n})$ and is positive since $|b\zeta'| < 2\varepsilon L < \delta/2$. ∎

PROPOSITION A.2. — *Let Ω be a domain with Lipschitz boundary, then Ω satisfies (H_k) for every $k \geq 1$.*

Proof. — We adopt the same notations as the above proposition. Again we can assume that the boundary is locally near the boundary point 0 represented by $y_1 = b(y')$, $b(0) = 0$ and $\bar{\Omega}_j$ is given by (A.1), but b is now a Lipschitz function defined for $y' \in \varepsilon Q^{n-1}$. For convenience we extend b as a Lipschitz function of $\mathbb{R}^{n-1} \to \mathbb{R}$, with the same Lipschitz constant L.

To define ψ_j^{-1} we have to modify (A.2) since b is not C^k. Therefore we replace b by a mollified function $c = c(y) \in C^\infty(\mathbb{R}_+^n)$ (where $\mathbb{R}_+^n = \{ y \in \mathbb{R}^n : y_1 > 0 \}$) with $c(0, y') = b(y')$ and $\left| \dfrac{\partial c}{\partial y_1} \right| \leq L$. (The construction of such a c is standard and is done below.) We then define

$$(A.3) \qquad \begin{cases} \psi_j^{-1} : \quad Q^n - q \to \bar{\Omega}_j - \omega_j \\ \psi_j^{-1}(x) = y = (y_1, y') = (\delta x + c(\delta x_1, \varepsilon x') \zeta(x_1), \varepsilon x'). \end{cases}$$

By construction this is a C^∞-diffeomorphism from $Q^n - q$ onto $\bar{\Omega}_j - \omega_j$ with the property that

$$\frac{\delta}{2} \varepsilon^{n-1} \leq \det \nabla \psi_j^{-1} \leq \frac{3\delta}{2} \varepsilon^{n-1}.$$

Thus Ω satisfies (H_k) for every k.

It therefore remains to construct c. Let $\eta \in C^\infty(\mathbb{R}^{n-1})$ with compact support in $|z'| < 1$, $\eta \geq 0$ and $\displaystyle\int_{\mathbb{R}^{n-1}} \eta(y') \, dy' = 1$. Define

$$c(y) = \int_{\mathbb{R}^{n-1}} \eta(z') \, b(y' - y_1 z') \, dz',$$

so that $c \in C^\infty(\mathbb{R}_+^n)$, $c(0, y') = b(y')$ and

$$\left| c(y_1, y') - c(\bar{y}_1, y') \right| \leq L \, |y_1 - \bar{y}_1| \int_{\mathbb{R}^{n-1}} |z'| \, \eta(z') \, dz' \leq L \, |y_1 - \bar{y}_1|.$$

Hence for $y_1 > 0$ we have $\left| \dfrac{\partial c}{\partial y_1} \right| \leq L$.   ∎

PROPOSITION A.3. — *A domain obtained by removing a finite number of isolated points from a domain satisfying (H_k) satisfies also (H_k).*

Proof. — Since the question is a local one it suffices to consider a punctured disk in Ω, *i.e.*,

$$\Omega_j = \{ y \in \mathbb{R}^n : 0 < |y| \leq 1 \},$$

with $\bar{\Omega}_j \cap \partial\Omega = \{ 0 \}$.

Instead of verifying (H$_k$) directly it is more convenient to use a "blow up" map with the property that Ω_j is mapped smoothly into a domain with smooth boundary and by a mapping whose Jacobian is bounded away from 0 and ∞. This is obviously sufficient for the Property (H$_k$) to hold near $y=0$. Let $\rho>0$, the mapping in question is defined by

$$\psi_j: \quad \Omega_j \to D_\rho = \left\{ z \in \mathbb{R}^n : \rho < |z| \leq (1+\rho^n)^{1/n} \right\}$$

with

$$\psi_j(y) = z = \lambda y, \qquad \lambda = \left(\frac{\rho^n}{|y|^n} + 1 \right)^{1/n}.$$

This mapping has a Jacobian 1, since the volume form

$$dy = dy_1 \ldots dy_n = r^{n-1} \, dr \, d\omega = \frac{1}{n} d(r^n) \, d\omega$$

(with $r = |y|$ and $d\omega$ the area on the $(n-1)$-dimensional sphere S^{n-1}) is mapped into $dz = \frac{1}{n} d(s^n) \, d\omega$ with $s = |z|$ since the mapping ψ_j is radial with $|z|^n = \lambda^n |y|^n = |y|^n + \rho^n$ and hence $ds^n = dr^n$. ∎

Remark. − Clearly, the class of domains Ω satisfying (H$_k$) is much larger than indicated by these cases. We can also admit for example lower dimensional boundaries as well as cusps.

REFERENCES

[AB] M. ABRAHAM and R. BECKER, *Classical Theory of Electricity and Magnetism,* Glasgow, 1950.

[A] S. ALPERN, New Proofs that Weak Mixing is Generic, *Inventiones Math.,* Vol. **32**, 1976, pp. 263-279.

[AK] D. V. ANOSOV and A. B. KATOK, New Examples in Smooth Ergodic Theory. Ergodic Diffeomorphisms, *Trudy Moskov Mat. Obsc. Tom,* Vol. **23**, 1970, *Trans. Moscow Math. Soc.,* Vol. **23**, 1970, pp. 1-35.

[B] A. BANYAGA, Formes volume sur les variétés à bord, *Enseignement Math.,* Vol. **20**, 1974, pp. 127-131.

[CL] E. A. CODDINGTON and N. LEVINSON, *Theory of Ordinary Differential Equations,* MacGraw-Hill, New York, 1955.

[D] B. DACOROGNA, A Relaxation Theorem and its Application to the Equilibrium of Gases, *Arch. Ration. Mech. Anal.,* Vol. **77**, 1981, pp. 359-385.

[GS]   R. E. GREENE and K. SHIOHAMA, Diffeomorphisms and Volume Preserving Embed-
        dings of Non Compact Manifolds, *Trans. Am. Math. Soc.*, Vol. **255**, 1979, pp. 403-
        414.
[H]   L. HÖRMANDER, The Boundary Problems of Physical Geodesy, *Arch. Ration. Mech.
        Anal.*, Vol. **62**, 1976, pp. 1-52.
[LU]   O. A. LADYZHENSKAYA and N. N. URALTSEVA, *Linear and Quasilinear Elliptic
        Equations*, Academic Press, New York, 1968.
[L]   A. E. H. LOVE, *Treatise on the Mathematical Theory of Elasticity*, New York, 1944.
[MO]   G. H. MEISTERS and C. OLECH, Locally One to One Mappings and a Classical
        Theorem on Schlicht Functions, *Duke Math. J.*, Vol. **30**, 1963, pp. 63-80.
[M]   J. MOSER, On the Volume Elements on a Manifold, *Trans. Am. Math. Soc.*, Vol. **120**,
        1965, pp. 286-294.
[T]   L. TARTAR, private communication, 1979.
[Z]   E. ZEHNDER, Note on Smoothing Symplectic and Volume Preserving Diffeomor-
        phisms, Springer, *Lect. Notes Math.*, Vol. **597**, 1976, pp. 828-855.

(Manuscript received December 6, 1988.)