

No-arbitrage with Incomplete Preferences

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Abstract

This article reconsiders the theory of existence of efficient allocations and equilibria when consumption sets are unbounded below under the assumption that agents have incomplete preferences. It is motivated by an example in the theory of assets with short-selling where there is risk and ambiguity. Ambiguity is described by a set of priors. Two types of preferences are considered : Gilboa-Schmeidler's maxmin complete preferences and Bewley's incomplete preferences. An inertia principle is assumed in markets, hence equilibria are individually rational. It is shown that a necessary and sufficient for existence of an individually rational efficient allocation or of an equilibrium is that the relative interiors of the risk adjusted sets of probabilities intersect whether agents have Gilboa-Schmeidler or Bewley's preferences. The paper then turns to incomplete preferences with concave multi-utility representations. Several definitions of efficiency and of equilibrium with inertia are considered. Sufficient conditions and necessary and sufficient conditions are given for existence of efficient allocations and equilibria with inertia.

Keywords: Uncertainty, risk, risk adjusted prior, no arbitrage, equilibrium with short-selling, incomplete preferences, equilibrium with inertia.

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1 Introduction

This paper reconsiders the theory of existence of efficient allocations and equilibria when consumption sets are unbounded below under the assumption that agents have incomplete preferences.

Following a long tradition (e.g. Hart [6], Page [10]) this work is motivated by an example in the theory of assets with short-selling. Markets are complete and a standard Arrow-Debreu model of exchange of contingent claims is considered. Agents are assumed not to have enough information to quantify uncertainty by a single probability. Ambiguity is described by sets of priors. Agents are further assumed to be risk averse. As axiomatized by Gilboa, Marinacci, Maccheroni and Schmeidler [4], two classical types of preferences with same priors and utility indices are considered : Gilboa-Schmeidler's [5] maxmin complete preferences and Bewley's [1] incomplete preferences modeled by a unanimity rule (a contingent claim is better than another if its expected value under any prior is higher). Furthermore as Bewley [1] suggested, an inertia principle is assumed in markets : agents never trade to a contingent claim whose expected utility is not strictly higher under every prior than their status quo. Under standard conditions on utility indices (strict concavity and increasingness) and sets of priors (convexity and compactness), it is shown that a necessary and sufficient for existence of an individually rational efficient allocation or of an equilibrium is that the relative interiors of the risk adjusted sets of probabilities intersect whether agents have Gilboa-Schmeidler preferences or Bewley's preferences. This result may seem at odds with the literature on efficiency or equilibria with Bewley's incomplete preferences and inertia (see Bewley's [1], Rigotti and Shannon [11] or Dana and Riedel [3]) which highlights the differences with using maxmin complete preferences. However it is not : this paper discusses existence issues with shortselling and not properties of efficient allocations.

The paper then turns to incomplete preferences with concave multi-utility representations which generalize Bewley's [1] and Rigotti and Shannon [11] ambiguity models with incomplete preferences. Several definitions of efficiency and equilibrium with inertia are considered. Sufficient conditions are given for existence of efficient allocations and equilibrium with inertia as well as necessary and sufficient conditions.

The paper is organized as follows: section 2 deals with the example and provides an existence theorem while section 3 deals with its generalization.

2 An example

2.1 Bewley preferences

We consider a standard Arrow-Debreu model of complete contingent security markets. There are two dates, 0 and 1. At date 0, there is uncertainty about which state s from a state space $\Omega = \{1, \dots, k\}$ will occur at date 1. At date 0, agents who are uncertain about their future endowments trade contingent claims for date 1. The space of contingent claims is the set of random variables from $\Omega \rightarrow \mathbb{R}$. The random variable X which equals x_1 in state 1, x_2 in state 2 and x_k in state k , is identified with the vector in $X \in \mathbb{R}^k$, $X = (x_1, \dots, x_k)$. Let $\Delta = \{\pi \in \mathbb{R}_+^k : \sum_{s=1}^k \pi_s = 1\}$ be the probability simplex in \mathbb{R}^k . Let $\text{int}\Delta = \{\pi \in \Delta, \pi_s > 0, \forall s\}$. For $A \subseteq \Delta$, $\text{int}A = \{p \in A \mid \exists \text{ a ball } B(p, \varepsilon) \text{ s.t. } B(p, \varepsilon) \cap \text{int}\Delta \subseteq A\}$. For a given $\pi \in \Delta$, we denote by $E_\pi(X) := \sum_{l=1}^k \pi_l x_l$ the expectation of X . Finally, for a given price $p \in \mathbb{R}^k$, $p \cdot X := \sum_{l=1}^k p_l x_l$, the price of X .

There are m agents indexed by $i = 1, \dots, m$. Agent i has an endowment $E^i \in \mathbb{R}^k$ of contingent claims. Let $(E^i)_{i=1}^m$ be the m -tuple of endowments and $E = \sum_{i=1}^m E^i$ be the aggregate endowment. We assume that agent i has a convex compact set of priors $P^i \subseteq \text{int}\Delta$ and an incomplete Bewley preference relation \succeq over \mathbb{R}^k defined by

$$X \succeq^i Y \iff E_\pi(u^i(X)) \geq E_\pi(u^i(Y)), \forall \pi \in P^i \quad (1)$$

where $u^i : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly concave, increasing differentiable utility index fulfilling $u^i(0) = 0$. The associated strict preference is $X \succ^i Y$ if $X \succeq^i Y$ and $E_\pi(u^i(X)) > E_\pi(u^i(Y))$ for some $\pi \in P^i$.

2.2 Individual and collective absence of arbitrage

In this subsection, we define and characterize the useful trading directions of a Bewley preference relation of type (1). Our main result is that they coincide with those of a Gilboa-Schmeidler utility defined by (u, P) of type (2) and with those of a variational utility introduced by Dana and Riedel [3], axiomatized by Mihm [8] and called reference-dependent ambiguity averse (RAA) utility. This implies that the concepts of no-arbitrage prices and of collective absence of arbitrage also coincide.

2.2.1 Useful vectors

To simplify notations, in this subsection, the agent's index is omitted. We consider an agent described by a pair (u, P) of a utility index and a set of priors. For $\pi \in P$, let

$$\widehat{P}_\pi(X) = \{Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X))\}$$

be the set of contingent claims preferred to X for the utility $E_\pi(u(\cdot))$ and $R_\pi(X)$ be its asymptotic cone. As $E_\pi(u(\cdot))$ is concave, $R_\pi(X)$ is independent of X and will be denoted R_π . Let

$$\widehat{P}(X) = \{Y \in \mathbb{R}^k \mid E_\pi(u(Y)) \geq E_\pi(u(X)), \forall \pi \in P\}$$

be the set of contingent claims preferred to X for the preference defined by (1) and $R(X)$ be its asymptotic cone. From Rockafellar's [12] corollary 8.3.3, $R(X) = \bigcap_{\pi \in P} R_\pi(X) = \bigcap_{\pi \in P} R_\pi$. Hence it is independent of X and is denoted R . It is called the set of *useful vectors* for \succeq . As $R_\pi = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq u(0) = 0, \forall \lambda \in \mathbb{R}\}$,

$$R = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0, \forall \lambda \in \mathbb{R}_+, \pi \in P\}.$$

For a given pair (u, P) , let

$$V(X) = \min_{\pi \in P} E_\pi(u(X)) \quad (2)$$

be the Gilboa-Schmeidler's utility. Dana and Le Van [2] characterize the set of useful vectors for variational preferences defined by Maccheroni Marinacci and Rustichini [7]. Since MMR variational preferences generalize Gilboa-Schmeidler's utilities, their results in particular apply to Gilboa-Schmeidler's utility. We recall that the set of useful vectors is

$$\{W \in \mathbb{R}^k \mid V(\lambda W) \geq 0, \forall \lambda \in \mathbb{R}\} = \{W \in \mathbb{R}^k \mid E_\pi(u(\lambda W)) \geq 0, \forall \lambda \in \mathbb{R}, \pi \in P\}$$

Hence it coincides with R . Furthermore given $C \in \mathbb{R}^k$ a reference point, the C -reference dependent ambiguity averse (RAA) utility is defined by

$$V_C(X) = \min_{\pi \in P} [E_\pi(u(X)) - E_\pi(u(C))] \quad (3)$$

It is a variational utility. As $V_C(C) = 0$, the set R_{V_C} of useful vectors for V_C is $R_{V_C} = \{W \in \mathbb{R}^k \mid V_C(C + \lambda W) \geq 0, \forall \lambda \in \mathbb{R}\}$ and therefore

$$R_{V_C} = \{W \in \mathbb{R}^k \mid E_\pi(u(C + \lambda W)) \geq E_\pi(u(C)), \forall \lambda \in \mathbb{R}, \pi \in P\} = R$$

These results are summarized in the following lemma:

Lemma 1 *The set of useful vectors for a Bewley preference defined by (u, P) of type (1) coincides with the set of useful vectors for a Gilboa-Schmeidler preference (2) or of an RAA utility (3) for any reference point $C \in \mathbb{R}^k$.*

We may therefore use a number of results already proven in Dana and Le Van [2]. To this end, let

$$\tilde{P} = \left\{ p \in \Delta \mid \exists \pi \in P, Z \in \mathbb{R}^k \text{ s. t. } p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}, \forall s = 1, \dots, k \right\} \quad (4)$$

be the set of risk adjusted priors. The properties of \tilde{P} may be found in [2] who show the following property which implies that all arbitrage concepts may be expressed in terms of \tilde{P} .

Lemma 2 $R = \{W \in \mathbb{R}^k \mid E_p(W) \geq 0, \text{ for all } p \in \tilde{P}\}$.

In the next two subsections, we introduce two concepts of absence of arbitrage, a concept of individual no-arbitrage and a concept of collective absence arbitrage. These concepts only depend on agents' useful vectors.

2.2.2 No-arbitrage price

We first turn to the concept of no-arbitrage price.

Definition 1 A price vector $p \in \mathbb{R}^k$ is a "no-arbitrage price" for agent i if $p \cdot W > 0$, for all $W \in R^i \setminus \{0\}$. A price vector $p \in \mathbb{R}^k$ is a "no-arbitrage price" for the economy if it is a no-arbitrage price for each agent.

Let S^i denote the set of non arbitrage prices for i . Dana and Le Van [2] characterize S^i :

Lemma 3 1. The set of no-arbitrage prices for agent i is $S^i = \text{cone int } \tilde{P}^i$ where $p \in \text{int } \tilde{P}^i$ if and only if $\exists \pi \in P^i \cap \text{int } \Delta$, $Z \in \mathbb{R}^k$, $\forall s$, $a < u'(z_s) < b$ and $p_s = \frac{\pi_s u'(z_s)}{E_\pi u'(Z)}$. Hence $S^i \neq \emptyset$ if and only if, $\text{int } P^i \neq \emptyset$.

2. The set of no-arbitrage prices for the economy is $\bigcap_i S^i = \text{cone } \bigcap_i \text{int } \tilde{P}^i$.

2.2.3 Collective absence of arbitrage

From now on, a *feasible trade* is an m -tuple W^1, \dots, W^m with $W^i \in \mathbb{R}^k$ for all i and $\sum_i W^i = 0$. We recall the *no-unbounded-arbitrage* condition (NUBA) introduced by Page [9] which requires inexistence of unbounded utility nondecreasing feasible trades.

Definition 2 The economy satisfies the NUBA condition if $\sum_i W^i = 0$ and $W^i \in R^i$ for all i , implies $W^i = 0$ for all i .

From Lemma 2, we may characterize the NUBA condition .

Corollary 1 NUBA is equivalent to: there exists no feasible trade W^1, \dots, W^m with $W^i \neq 0$ for some i that fulfills $E_\pi(W^i) \geq 0$ for all i and $\pi \in \tilde{P}^i$.

2.3 Existence of efficient allocations and equilibria

2.3.1 Concepts in equilibrium theory

We next recall standard concepts in equilibrium theory.

Given $(E^i)_{i=1}^m$, an allocation $(X^i)_{i=1}^m \in (\mathbb{R}^k)^m$ is attainable (or feasible) if $\sum_{i=1}^m X^i = E$. The set of individually rational attainable allocations $A((E^i)_{i=1}^m)$ is defined by

$$A((E^i)_{i=1}^m) = \left\{ (X^i)_{i=1}^m \in (\mathbb{R}^k)^m \mid \sum_{i=1}^m X^i = E \text{ and } X^i \succeq E^i, \forall i \right\}.$$

Definition 3 Given $(E^i)_{i=1}^m$, an attainable allocation $(X^i)_{i=1}^m$ is B-efficient if there does not exist $(X^i)_{i=1}^m$ attainable such that $X_i^i \succeq X_i$ for all i with a strict inequality for some i . It is weakly B-efficient if there does not exist $(X^i)_{i=1}^m$ attainable such that $E_\pi(u^i(X^i)) > E_\pi(u^i(X^i))$, $\forall \pi \in \mathcal{P}^i$, $\forall i$. It is individually rational (weakly) B-efficient if it is (weakly) B-efficient and $X^i \succeq E^i$ for all i .

Since u^i is strictly concave and strictly increasing and P^i is compact for all i , $(X^i)_{i=1}^m$ is B-efficient if and only if $(X^i)_{i=1}^m$ is weakly B-efficient (see Lemma 4 and Remark 3 below).

Definition 4 A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^k \setminus \{0\}$ is a contingent (weak) B-equilibrium with inertia if

1. for each agent i and $X^i \in \mathbb{R}^k$, $X^i \succ X^{i*}$ ($E_\pi(u^i(X^i)) > E_\pi(u^i(X^{i*}))$), $\forall \pi \in \mathcal{P}^i$, $\forall i$) implies $p^* \cdot X^i > p^* \cdot X^{i*}$,
2. for each agent i , $p^* \cdot X^{i*} = p^* \cdot E^i$.

Since u^i is strictly concave for all i , it may easily be verified that (X^*, p^*) is a B-equilibrium with inertia if and only if it is a weak B-equilibrium with inertia. Note that the concept of equilibrium with inertia was introduced by Bewley since with incomplete preferences agents cannot necessarily compare their equilibrium contingent claims with their endowments. In the definition above, the equilibrium is individually rational.

2.3.2 Existence of efficient allocations and equilibria

The following theorem fully characterizes existence of individually rational efficient allocations as well as equilibria with inertia. The economy with Gilboa-Schmeidler utilities is the economy where agent i has utility $V^i(X) = \min_{\pi \in \mathcal{P}^i} E_\pi(u^i(X))$ while in the economy with Bewley preferences, agent i has preference (1).

Theorem 1 *The following assertions are equivalent:*

1. there exists a no arbitrage price or equivalently $\bigcap_i \text{int } \tilde{P}^i \neq \emptyset$,
2. there exists no feasible trade W^1, \dots, W^m with $W^i \neq 0$ for some i and $E_\pi(W^i) \geq 0$ for all $\pi \in \tilde{P}^i$ and for all i ,
3. the set of individually rational attainable allocations for the economy with Gilboa-Schmeidler utilities is compact,
4. the set of individually rational attainable allocations for the economy with Bewley preferences is compact
5. there exists an individually rational efficient allocation for the economy with Gilboa-Schmeidler utilities
6. there exists an individually rational efficient allocation for the economy with Bewley's preferences
7. there exists an equilibrium for the economy with Gilboa-Schmeidler utilities.
8. there exists an equilibrium with inertia for the economy with Bewley's preferences.

Furthermore any equilibrium price is a no-arbitrage price.

Proof: The equivalence between 1, 2, 3, 5 and 7 follows from Dana and Le Van [2] applied to Gilboa-Schmeidler utilities. Since $X^i \succeq E^i$ is equivalent to $V_{E^i}(X) \geq V_{E^i}(E^i) = 0$, the set of individually rational attainable allocations for the economy with RAA utilities with reference points $(E^i)_{i=1}^m$ coincide with the set of individually rational attainable allocations for the economy with Bewley's preferences. Applying Dana and Le Van [2] to RAA utilities with reference points $(E^i)_{i=1}^m$, we obtain the equivalence between 1, 2 and 4 follows. Let us prove that 5 implies 6. Let $(\bar{X}^i)_{i=1}^m$ be an individually rational efficient allocation for the economy with Gilboa-Schmeidler utilities. Let us show that it is Bewley weakly efficient. Suppose not. Then there exists $(X^i)_{i=1}^m$ attainable such that $E_\pi(u^i(X^i)) > E_\pi(u^i(\bar{X}^i))$ for all $\pi \in \mathcal{P}^i$ for all i but then $V^i(X^i) > V^i(\bar{X}^i)$ for all i contradicting the efficiency of $(\bar{X}^i)_{i=1}^m$ for Gilboa-Schmeidler utilities. Therefore $(\bar{X}^i)_{i=1}^m$ is an individually rational Bewley weakly efficient allocation, hence efficient allocation. Let us next show that 6 implies 2. Let $(\bar{X}^i)_{i=1}^m$ be a Bewley efficient allocation. Suppose that there exists a feasible trade W^1, \dots, W^m with $W^i \in R^i$ for all i and $W^i \neq 0$ for some i . We then have $E_\pi(u^i(\bar{X}^i + tW^i)) \geq E_\pi(u^i(\bar{X}^i))$, for all $\pi \in \mathcal{P}^i$ for all i and as u^i is strictly concave, $E_\pi(u^i(\bar{X}^i + tW^i)) > E_\pi(u^i(\bar{X}^i))$ for all $\pi \in \mathcal{P}^i$ for any i such that $W^i \neq 0$. The allocation $(\bar{X}^i + tW^i)_{i \in I}$ being feasible, this contradicts the

Bewley-efficiency of $(\bar{X}^i)_{i=1}^m$. Hence 1-7 are equivalent. Finally let us show that 1 is equivalent to 8. Let us first remark that if (X^*, p^*) is an equilibrium, then p^* is a no arbitrage price. Indeed as u^i is strictly concave, if W^i is useful and $W^i \neq 0$, then $E_\pi(u^i(X^{*i} + tW^i)) > E_\pi(u^i(X^{*i}))$, $\forall \pi \in \mathcal{P}^i$, hence $p^*W^i > 0$ which proves that p^* is a no-arbitrage price for the economy proving that 8 implies 1. Conversely if 1 holds true, then from Dana and Le Van [2], the economy with variational utilities $V_{E^i}(X) = \min_{\pi \in \mathcal{P}^i} [E_\pi(u^i(X)) - E_\pi(u^i(E^i))]$ has an equilibrium (X^*, p^*) . Let us show that (X^*, p^*) is a weak Bewley equilibrium with inertia. Indeed if $E_\pi(u^i(X^i)) > E_\pi(u^i(X^{i*}))$, $\forall \pi \in \mathcal{P}^i$, then $V_{E^i}(X) > V_{E^i}(X^{*i})$, hence $p^* \cdot X^i > p^* \cdot X^{i*}$. Furthermore as the family of utilities $(V_{E^i})_{i \in I}$ is strictly concave, if $X^{*i} \neq E^i$, then $V_{E^i}(X^{*i}) > V_{E^i}(E^i)$, hence $X^{*i} \succ^i E^i$ proving that (X^*, p^*) is a weak Bewley equilibrium with inertia, hence a Bewley equilibrium with inertia. ■

Remark 1 *When stating theorem 1, we have assumed that either all agents had Gilboa-Schmeidler utilities or Bewley's incomplete preference. However a much more general result is true : agent i may either have complete or incomplete preferences with useful vectors R^i . In particular, if she has complete preferences, she may either have a Gilboa-Schmeidler utility or an RAA utility with any reference point or any utility with useful vectors R^i .*

Remark 2 *The strict concavity of the utility functions plays an important and subtle role. First, for most purposes, when $X \succ Y$, we may assume that $E_\pi(u(X)) > E_\pi(u(Y))$, $\forall \pi \in \mathcal{P}$. Second, strict concavity is used to prove the equivalence between weak B-equilibrium and B-equilibrium and weak B-efficiency and B-efficiency. Third, as u is strictly concave, for any useful vector $W \neq 0$ and any $\pi \in \mathcal{P}$, the map $t \rightarrow E_\pi(u(X + tW))$ is strictly increasing. This is used in 6 implies 2 in theorem 1.*

3 An abstract economy with incomplete preferences

3.1 The economy

We consider an economy with m agents and d goods. Agent i has consumption space \mathbb{R}^d , endowment $E^i \in \mathbb{R}^d$ and incomplete (or complete) convex preferences over \mathbb{R}^d , defined by a family $\mathcal{U}^i : \mathbb{R}^d \rightarrow \mathbb{R}$ of concave utility functions as follows: $X^i \in \mathbb{R}^d$ is preferred to $Y^i \in E$ by agent i , denoted $X^i \succeq^i Y^i$ if $u_i(X^i) \geq u_i(Y^i)$ for every $u_i \in \mathcal{U}^i$. The associated strict preference is $X \succ^i Y$ if $X \succeq^i Y$ and $u^i(X) > u^i(Y)$ for some $u^i \in \mathcal{U}^i$. Let $E = \sum_{i=1}^m E^i$ be the aggregate endowment.

We assume that for every i , the utilities in \mathcal{U}^i are everywhere finite, concave, hence superdifferentiable and there is a topology on \mathcal{U}^i which makes it compact and such that the evaluation map $u \in \mathcal{U}^i \mapsto u(X)$ is continuous for every $X \in \mathbb{R}^d$. We assume that $u(0) = 0$ for all $u \in \mathcal{U}^i$ and all i .

An allocation $(X^i)_{i \in I} \in \mathbb{R}^{dm}$ is feasible if $\sum_i X^i = E$. The set of individually rational attainable allocations $A((E^i)_{i=1}^m)$ is defined as in 2.3

3.2 Arbitrage concepts

We briefly redefine for abstract incomplete preferences, the concepts which were defined in section 2.2

$$\widehat{P}^i(X) = \{Y \in \mathbb{R}^k \mid u(Y) \geq u(X), \forall u \in \mathcal{U}^i\} = \bigcap_{u \in \mathcal{U}^i} \{Y \in \mathbb{R}^k \mid u(Y) \geq u(X)\}$$

be the preferred set at X by i and $R^i(X)$ be its asymptotic cone. As the utilities $u \in \mathcal{U}^i$ are concave, $R^i(X)$ is independent of X and denoted R^i . It is called the set of *useful vectors* for \succeq^i . From Rockafellar's [12] corollary 8.3.3,

$$R^i = \{W \in \mathbb{R}^d \mid u(\lambda W) \geq 0, \forall \lambda \in \mathbb{R}, u \in \mathcal{U}^i\}.$$

As discussed in the previous section, for any $C \in \mathbb{R}^d$, R^i is also the set of useful vectors for any complete preference represented by a utility of the form $V_{E^i}(X) = \min_{u \in \mathcal{U}^i} [u(X) - u(E^i)]$. Note that this utility is well defined under our assumptions.

A price vector $p \in \mathbb{R}^d$ is a "no-arbitrage price" for agent i if $p \cdot W > 0$, for all $W \in R^i \setminus \{0\}$. Let S^i denote the set of non arbitrage prices for i . Then $S^i = -\text{int}(R^i)^0$ (where $(R^i)^0 = \{p \in \mathbb{R}^d \mid p \cdot X \leq 0, \text{ for all } X \in R^i\}$). A price vector $p \in \mathbb{R}^k$ is a "no-arbitrage price" for the economy if it is a no-arbitrage price for each agent. A price vector $p \in \mathbb{R}^k$ is a no-arbitrage price for the economy if and only if $p \in \bigcap_i S^i = -\bigcap_i \text{int}(R^i)^0$.

An m -tuple $(W^1, \dots, W^m) \in \mathbb{R}^{dm}$ is a feasible trade if $\sum_i W^i = 0$.

A trade $W \in \mathbb{R}^d \setminus \{0\}$ is a *half-line* for a utility $u : \mathbb{R}^d \rightarrow \mathbb{R}$ if there exists $X \in \mathbb{R}^d$ such that $u(X + \lambda W) = u(X)$ for all $\lambda \geq 0$. As u is concave, when it has no half-line, then if $W \neq 0$, $u(X + \lambda W) > u(X)$ for every X and $\lambda > 0$.

3.3 Efficiency and equilibrium concepts

Definition 5 1. A feasible allocation $(X^i)_{i=1}^m$ is efficient if there does not exist $(X^i)_{i=1}^m$ feasible such that $X^i \succeq X^i$ for all i with a strict inequality for some i . It is weakly efficient if there does not exist $(X^i)_{i=1}^m$ feasible

such that $u^i(X^i) > u^i(X^i)$, $\forall u_i \in \mathcal{U}^i$, $\forall i$ such that $X^i \neq X^i$. It is individually rational (weakly) efficient if it is (weakly) efficient and $X^i \succeq E^i$ for all i .

2. A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^d \setminus \{0\}$ is an (weak) equilibrium with inertia if

(a) for any i and $X^i \in \mathbb{R}^d$, $X^i \succ X^{*i}$ ($u^i(X^i) > u^i(X^{*i})$ for every $u^i \in \mathcal{U}^i$) implies $p^* \cdot X^i > p^* \cdot X^{*i}$,

(b) for any i , $p^* \cdot X^{*i} = p^* \cdot E^i$.

Lemma 4 Let u be strictly concave for every $u \in \mathcal{U}^i$ and i . Then an attainable allocation $(X^i)_{i=1}^m$ is efficient if and only if it is weakly efficient. A pair $(X^*, p^*) \in A((E^i)_{i=1}^m) \times \mathbb{R}^d \setminus \{0\}$ is an equilibrium with inertia if and only if it is a weak equilibrium with inertia

The proof of lemma 4 is omitted.

Remark 3 The definition of weak Pareto-optimality that we have used in definition 5 is stronger than that used in definition 3 that defines weak Pareto-optimality by: a feasible allocation $(X^i)_{i=1}^m$ is weakly efficient if there does not exist $(X^i)_{i=1}^m$ feasible such that $u^i(X^i) > u^i(X^i)$, $\forall u_i \in \mathcal{U}^i$, $\forall i$.

A common increasing direction, is a trade $e \in \mathbb{R}^d$ such that $e \cdot p > 0$ for every $(X, u) \in \mathbb{R}^d \times \mathcal{U}^i$, $p \in \partial u(X)$ and every i . When e is a common increasing direction, $u(X - te) < u(X)$, $u(X + te) > u(X)$ for every $t \geq 0$, $X \in \mathbb{R}^d$, $u \in \mathcal{U}^i$ and i .

When agents have a common increasing direction and utilities are strictly concave, then the concepts of weak efficiency of definitions 3 and 5 coincide with efficiency. Indeed, clearly, if $(X^i)_{i=1}^m$ is efficient, it is weakly efficient in the sense of definition 5. Clearly weak efficiency in the sense of definition 5 implies weak efficiency in the sense of definition 3. Let us show that weak efficiency in the sense of definition 3 implies efficiency. Assume that $(X^i)_{i=1}^m$ is a weakly efficient allocation in the sense of definition 3 and that it is not efficient. W.l.o.g. assume that there exists a feasible allocation $(Y^i)_{i=1}^m$ such that $Y^1 \succ X^1$ and $Y^i \succeq X^i$, $i \neq 1$. By considering $\frac{(Y^1 + X^1)}{2}$ instead of Y^1 , we may assume that $u(Y^1) > u(X^1)$, $\forall u \in \mathcal{U}^1$. Since the map $X \rightarrow u(X)$ is continuous for every $X \in \mathbb{R}^d$, for any $u \in \mathcal{U}^1$, there exists ε_u such that $u(Y^1 - e\varepsilon) > u(X^1)$ for every $\varepsilon \leq \varepsilon_u$. For a given $\varepsilon > 0$, let

$$\mathcal{V}_\varepsilon = \{u \in \mathcal{U}^1 \mid u(Y^1 - e\varepsilon) > u(X^1)\}$$

\mathcal{V}_ε is open and from the previous argument $\cup_\varepsilon \mathcal{V}_\varepsilon = \mathcal{U}^1$. Since \mathcal{U}^1 is compact, there exists a finite subcovering of $u \in \mathcal{U}^1$ by $(\mathcal{V}_{\varepsilon_i})$. Let $\varepsilon = \min_i \varepsilon_i$ and $\varepsilon' = \frac{\varepsilon}{m-1}$. We then have

$$Y^1 - e\varepsilon \succ X^1 \text{ and } Y^i + e\varepsilon' \succ X^i, i \neq 1$$

contradicting the weak efficiency of X in the sense of definition 3.

3.4 An existence result

Theorem 2 *The following assertions are equivalent:*

1. *there exists a no arbitrage price for the economy $(\cap_i \text{int}(R^i))^0 \neq \emptyset$,*
2. *NUBA: $\sum_i W^i = 0$ and $W^i \in R^i$ for all i implies $W^i = 0$ for all i ,*
3. *the set of individually rational attainable allocations is compact.*

Any of the previous assertions implies any of the following assertions:

4. *there exists an individually rational weakly efficient allocation,*
5. *there exists a weak equilibrium with inertia.*
Furthermore,
6. *if u has no half-line for every $u \in \mathcal{U}^i$ and i , then assertions 1-5 are equivalent and any weak equilibrium price is a no-arbitrage price.*
7. *If u is strictly concave for every $u \in \mathcal{U}^i$ and i , then 1-5 are equivalent to the existence of an individually rational efficient allocation and equivalent to the existence of an equilibrium .*

Proof: As the set of useful vectors of \succeq^i and V_{E^i} coincide and the set of individually rational attainable allocations for the economy with utilities $(V_{E^i})_i$ coincides with the set of individually rational attainable allocations for the economy with preferences $(\succeq^i)_i$, the equivalence between 1, 2 and 3 follows from standard results on arbitrage with complete preferences. It is also standard that any of these assertions implies the existence of an individually rational efficient allocation $(\bar{X}^i)_{i=1}^m$ or of an equilibrium (X^*, p^*) for the economy with utilities $(V_{E^i})_i$. By the same proofs as in theorem 1, $(\bar{X}^i)_{i=1}^m$ is an individually rational weakly efficient allocation for the economy with preferences $(\succeq^i)_i$ and (X^*, p^*) is a weak equilibrium with inertia.

Let u have no half-line for every $u \in \mathcal{U}^i$, and let (X^*, p^*) be a weak equilibrium with inertia. Then for any useful vector $W^i \neq 0$, $u(X^{*i} + tW^i) > u(X^{*i})$, $\forall u \in \mathcal{U}^i$, for any $t > 0$. Hence $p^* \cdot W^i > 0$ which proves that p^* is a no-arbitrage price for the economy and that 5 implies 1. To show that 4 implies 2, let $(\bar{X}^i)_{i=1}^m$

be an individually rational weakly efficient allocation and suppose that there exists a feasible trade W^1, \dots, W^m with $W^i \in R^i$ for all i and $W^i \neq 0$ for some i . We have $u(\bar{X}^i + tW^i) \geq u(\bar{X}^i)$, $\forall u \in \mathcal{U}^i$, for all i , and $u(\bar{X}^i + tW^i) > u(\bar{X}^i)$, $\forall u \in \mathcal{U}^i$, for any i such that $W^i \neq 0$. The allocation $(\bar{X}^i + tW^i)_{i \in I}$ being feasible, this contradicts the weak efficiency of $(\bar{X}^i)_{i=1}^m$. If u is strictly concave, then u has no half-line. Hence assertion 6 is fulfilled. Furthermore, from lemma 4, any weak equilibrium with inertia is an equilibrium with inertia and any weakly efficient allocation is efficient. ■

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