Towards the Schrödinger equation

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- Court's idea is that the price of a car depends on a set of characteristics $z = (z^1, ..., z^d) \in R^d$ (safety, color, upholstery, motorization, and so forth). He then imagines a "standard" car with characteristics, $\bar{z}$, which will serve as a comparison term for the others: only increases in $p(\bar{z})$ qualify as true price increases.

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- His work is now fundamental for constructing *price indices* net of quality

What is a car

A "car" is a generic name for very different objects. Court identified the following characteristics:

- cars come in *discrete* quantities: you buy 0, 1, 2, ...

If the price of cars decrease, you do not buy *more* cars: you sell the old one and buy a *better* one. This is in contrast to classical economic theory, which is concerned with homogeneous (undifferentiated) goods: if the price of bread decreases, you eat more bread. Modern economies are shifting towards hedonic (differentiated) goods.

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- what happens to equilibrium theory ?

A model for hedonic markets

- There are two probability spaces (X, μ) and (Y, ν)

X, Y, Z will be assumed to be bounded subsets of some Euclidean space with smooth boundary, u and v will be smooth. We do not assume that μ and ν are absolutely continuous.

A model for hedonic markets

- There are two probability spaces (X, μ) and (Y, ν)
- There is a third set Z and two maps $u(x, z)$ and $c(y, z)$

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A model for hedonic markets

- There are two probability spaces (X, μ) and (Y, ν)
- There is a third set Z and two maps $u(x, z)$ and $c(y, z)$
- Each $x \in X$ is a *consumer* type, each $y \in Y$ is a *producer* type, and each $z \in Z$ is a quality

X, Y, Z will be assumed to be bounded subsets of some Euclidean space with smooth boundary, u and v will be smooth. We do not assume that μ and ν are absolutely continuous.

Demand and supply

Suppose a (continuous) *price system* $p : Z \to R$ is announced. Then

$$\max_z \left(u\left(x,z\right) - p\left(z\right) \right) \implies \begin{cases} p^{\natural}\left(x\right) = \max_z \\ D_p\left(x\right) = \arg\max_z \end{cases}$$

$$\max_z \left(p\left(z\right) - c\left(y,z\right) \right) \implies \begin{cases} p^{\flat}\left(y\right) = \max_z \\ S_p\left(y\right) = \arg\max_z \end{cases}$$

A *demand distribution* is a measure $\alpha_{X\times Z}$ on $X \times Z$ projecting on μ such that

$$\alpha_{X\times Z} = \int_X \alpha_x d\mu \text{ with Supp } \alpha_x \subset D_p\left(x\right)$$

A *supply distribution* is a measure $\beta_{Y\times Z}$ on $Y \times Z$ projecting on ν such that

$$\beta_{Y\times Z} = \int_Y \beta_y d\nu \text{ with Supp } \beta_y \subset S_p\left(y\right)$$

Equilibrium

Definition

$p : Z \longrightarrow R$ is an equilibrium if

$$pr_Z\left(\alpha_{X\times Z}\right) = pr_Z\left(\beta_{Y\times Z}\right) := \lambda$$

Does it exist ? There is an obvious condition:

$$\begin{aligned} p^{\natural\natural}(z) &:= \max_x \left(u(x,z) - p^{\natural}(x)\right) = \text{ maximum bid price for } z \\ p^{\flat\flat}(z) &:= \min_y \left(p^{\flat}(y) - c(y,z)\right) = \text{ minimum ask price for } z \end{aligned}$$

If $p^{\natural\natural}(z) < p^{\flat\flat}(z)$, then quality z is not traded. Set

$$Z_0 := \left\{ z \mid p^{\natural\natural}(z) < p^{\flat\flat}(z) \right\}$$

Existence

Theorem (Existence)

If $Z_0 \neq \varnothing$. there is an equilibrium price. The set of all equilibrium prices p is convex and non-empty. If $p : Z_0 \longrightarrow R$ is an equilibrium price, then so is every $q : Z \longrightarrow R$ which is admissible, continuous, and satisfies for some constant c:

$$p^{\sharp\sharp}(z) \leq q(z) + c \leq p^{\flat\flat}(z) \quad \forall z \in Z$$

Uniqueness

Theorem (Uniqueness of equilibrium prices)

For λ-almost every quality z which is traded in equilibrium, we have

$$p^{\sharp\sharp}(z) = p(z) = p^{\flat\flat}(z).$$

Theorem (Uniqueness of equilibrium allocations)

Let $\left(p_1, \alpha^1_{X\times Z}, \beta^1_{Y\times Z}\right)$ and $\left(p_2, \alpha^2_{X\times Z}, \beta^2_{Y\times Z}\right)$ be two equilibria. Denote by $D_1(x), D_2(x)$ and $S_1(y), S_2(y)$ the corresponding demand and supply maps. Then:

$$\alpha^2_x\left[D_1(x)\right] = \alpha^1_x\left[D_1(x)\right] = 1 \quad \text{for } \mu\text{-a.e. } x$$
$$\beta^2_y\left[S_1(y)\right] = \beta^1_y\left[S_1(y)\right] = 1 \quad \text{for } \nu\text{-a.e. } y$$

Efficiency and duality

With every pair of demand and supply distributions, $\alpha'_{X\times Z}$ and $\beta'_{Y\times Z}$, we associate the total welfare of society:

$$W\left(\alpha'_{X\times Z},\beta'_{Y\times Z}\right)=\int_{X\times Z}u\left(x,z\right)d\alpha'_{X\times Z}-\int_{Y\times Z}v\left(y,z\right)d\beta'_{Y\times Z}$$

Theorem (Pareto optimality of equilibrium allocations)

Let $\left(p,\alpha_{X\times Z},\beta_{Y\times Z}\right)$ *be an equilibrium. Take any pair of demand and supply distributions* $\alpha'_{X\times Z}$ *and* $\beta'_{Y\times Z}$ *such that* $pr_Z\left(\alpha'_{X\times Z}\right)=pr_Z\left(\beta'_{Y\times Z}\right)$. *Then*

$$\begin{aligned}
W\left(\alpha'_{X\times Z},\beta'_{Y\times Z}\right) &\leq W\left(\alpha_{X\times Z},\beta_{Y\times Z}\right)\\
W\left(\alpha_{X\times Z},\beta_{Y\times Z}\right) &= \int_X p^{\sharp}\left(x\right)d\mu+\int_Y p^{\flat}\left(y\right)d\nu\\
\int_X p^{\sharp}\left(x\right)d\mu+\int_Y p^{\flat}\left(y\right)d\nu &= \min_q\left[\int_X q^{\sharp}\left(x\right)d\mu+\int_Y q^{\flat}\left(y\right)d\nu\right]
\end{aligned}$$

Many-to-one matching

For applications to the job market, it is important to allow employers to hire several workers.

$$\max_{z} \left(p\left(z\right) - c\left(x, z\right)\right)$$

$$\max_{z,n} \left(u\left(y, z, n\right) - np\left(z\right)\right)$$

Let us write the *pure* version of the problem (maps instead of distributions)

$$\max \left\{ \int_Y u\left(y, z_s\left(y\right), n\left(y\right)\right) d\nu - \int_X c\left(x, z_d\left(x\right)\right) d\mu \right\}$$

$$\int_X \varphi\left(z_d\left(x\right)\right) d\mu = \int_Y n\left(y\right) \varphi\left(z_s\left(y\right)\right) d\nu$$

One can then prove existence and quasi-uniqueness in the usual way (IE, unpublished)

An example

$$
\begin{aligned}
u(y,z,n) \quad &: \quad = n\bar{u}(y,z) - \frac{n^2}{2}\bar{c}(n) \\
\max_{z,n}\left(n\bar{u}(y,z) - \frac{n^2}{2}\bar{c}(n) - np(z)\right) &= \max_n\left[\max_z\left\{n\bar{u}(y,z) - np(z)\right\} - \right. \\
&= \max_n\left[n\bar{p}^{\natural}(y) - \frac{n^2}{2}\bar{c}(y)\right] \\
&= \frac{1}{2\bar{c}(y)}\left[\bar{p}^{\natural}(y)\right]^2
\end{aligned}
$$

The dual problem is:

$$
\max_p\left[\int_Y \frac{\bar{p}^{\natural}(y)^2}{2\bar{c}(y)}d\nu - \int_X p^{\flat}(x)\,d\mu\right]
$$

Trade cannot be forced.

Consumer of type x has a *reservation utility* $u_0(x)$ and producer of type y has a *reservation utility* $v_0(y)$

$$\begin{cases} \max_z (u(x,z) - p(z)) > u_0(x) \Longrightarrow x \text{ buys } z \in D_p(x) \\ \max_z (u(x,z) - p(z)) < u_0(x) \Longrightarrow x \text{ does not buy} \end{cases}$$

$$\begin{cases} \max_z (p(z) - c(y,z)) > v_0(y) \Longrightarrow y \text{ produces } z \in S_p(x) \\ \max_z (p(z) - c(y,z)) < v_0(y) \Longrightarrow y \text{ does not produce} \end{cases}$$

We then have a suitable definition of equilibrium and an existence theorem. Note that:

- proofs become quite delicate (Pschenichnyi)
- the absolute level of prices becomes relevant, i.e. the constant c disappears
- we do not need $\mu(X) = \nu(Y)$ any more: prices keep excess people out of the market

What do economists do ?

Economists, like all scientists except mathematicians, are interested in:

- testing theories

In the case of the labor market, one can observe:

One wants to infer the utilities $u(x,z)$ for employers and costs $c(y,z)$ to labourers

There is an added difficulty, namely *unobservable characteristics* ξ and η:

$$\text{utilities are } u(x,\xi,z) \text{ instead of } u(x,z)$$
$$\text{costs are } c(y,\eta,z) \text{ instead of } c(y,z)$$

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What do economists do ?

Economists, like all scientists except mathematicians, are interested in:

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In the case of the labor market, one can observe:

- the distributions of types μ and ν
- the equilibrium prices $p(z)$ and the equilibrium allocations α_x and β_y

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The marriage problem

- For $I = \{1, ..., n\}$, and Σ_n its permutation group, we consider the optimal transportation problem

$$\max_{\sigma} \left\{ \sum_i \Phi_{i,\sigma(i)} \mid \sigma \in \Sigma_n \right\}$$

We cannot infer the $\Phi_{i,j}$ from the optimal matching. Note that there is a fundamental indeterminacy in the problem: $\Phi_{i,j} + a_i + b_j$ and $\Phi_{i,j}$ give the same matching.

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- We consider the relaxed problem:

$$\max \left\{ \sum_{i,j} \pi_{i,j} \Phi_{i,j} \mid \pi_{i,j} \geq 0, \ \sum_j \pi_{i,j} = 1 = \sum_i \pi_{i,j} \right\}$$

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Simulated annealing

- We introduce a parameter $T > 0$ (temperature), and consider the problem:

$$\max \left\{ \sum_{i,j} \pi_{i,j} \left(\Phi_{i,j} + T \ln \pi_{i,j} \right) \mid \pi_{i,j} \geq 0, \ \sum_{j} \pi_{i,j} = 1 = \sum_{i} \pi_{i,j} \right\}$$

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- The solution is given in a quasi-explicit form by

$$\pi_{i,j} = \exp\left(\frac{-\Phi_{i,j} + u_i + v_j}{T} \right)$$

where the u_i and v_j are the Lagrange multipliers

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- Erwin Schrödinger, "Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique", Annales de l'IHP 2 (1932), p. 269-310. If the distribution of the i is p_i and the distribution of j is q_j, the formula becomes:

$$\pi_{i,j} = p_i q_j \exp \left(\frac{-\Phi_{i,j} + u_i + v_j}{T} \right)$$

Identification

If we observe the $\pi_{i,j}$, Schrödinger's equation gives us:

$$\Phi_{i,j} = u_i + v_j + T\left(\ln p_i + \ln q_j\right) + \ln \pi_{i,j}$$

and the surplus function $\Phi_{i,j}$ is identified, up to the fundamental indeterminacy

$$\Phi_{i,j} = \ln \pi_{i,j}$$

Current work (Galichon and Salanié) investigates continuous versions of this problem:

$$\max \int_{X\times Y} \left[\Phi\left(x,y\right) + \ln \pi\left(x,y\right)\right] \pi\left(x,y\right) dxdy$$

$$\int_X \pi\left(x,y\right) dx = q\left(y\right), \quad \int_Y \pi\left(x,y\right) dy = p\left(x\right)$$

References

Symposium on Transportation Methods, "Economic Theory", vol.42, 2, February 2010, Springer