Abstract

We adapt the classical Schaefer model of fisheries management to take into account intergenerational equity. The resulting model exhibits a non-constant discount rate, and preferences then are time inconsistent, so that optimal solutions are not implementable. We define sustainable policies as Markov subgame perfect equilibria of the underlying sequential game. We characterize such sustainable policies and conclude that intergenerational equity should lead us to replace δ, the discount rate of the present generation, with $\delta - n$, where n is the rate of renewal of the population.

JEL: Q01, Q22, C 61

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Sustainable fisheries management

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What is sustainable development? Many people trace this concept to the Brundtland Report: WCED (1987), which states that sustainable development is one that "meets the needs of the present generation without compromising the ability of future generations to meet their own needs". However, this concept can be traced to the Lectures on Jurisprudence of Adam Smith (1766), where he states that "the Earth and the fullness of it belongs to every generation, and the preceding one can have no right to blind it up from posterity". This statement is clearly alluding to what we now called sustainable development, and even though there is no generally accepted definition as yet, it is clear that future generations feature prominently in the concept of sustainable development.

Sustainable development would have to take into account many, mostly conflicting, aspects of benefits and costs. One of these, however, already receives a broad consensus, that is, the time scale involved is so large that benefits and costs are shifted from one generation to another. If we overfish tuna and drive the species to extinction, we reap the benefits of eating sushi, but our descendants will have to bear the ecological cost of an ocean without large predators. In other words, we should bear in mind, not only our own interests, but also those of future generations.

This paper attempts to bring those interests to bear in a classical and well-understood framework, namely the Schaefer model of fisheries management. We show that, if the interests of future generations are taken into account, the equation:

$$f'(\tilde{x}) - \frac{c(\tilde{x})}{p - c(\tilde{x})} f(\tilde{x}) = \delta \tag{1}$$

which defines the optimal stock level $\tilde{x}$ should be replaced by:

$$f'(\bar{x}) - \frac{c(\bar{x})}{p - c(\bar{x})} f(\bar{x}) = \delta - n \tag{2}$$

where $\bar{x}$ is now the equilibrium stock level. In other words, the discount rate δ should be lowered to $\delta - n$, where δ is the individual rate of time preference and n is the rate of renewal of the fishermen population. In other words, concern for future generations will lead the current one to reduce its discount rate from δ to $\delta - n$. This result is remarkably robust. It only requires the present generation to take into account in its welfare the benefits of future generations, by discounting it at a constant rate σ: it does not depend on the value of σ.

The structure of the paper is as follows. We first recall the main features of the Schaefer model. We then introduce intergenerational equity, using an approach pionneered by (Sumaila and Walters 2005) in the case when time is discrete. Classically, if a policy brings a stream of revenue w_t, its net present value is $\sum_{s \geq 0} \delta^s w_s$, where δ_t is the rate of time preference of the current generation. Sumaila and Walters note that the same stream of revenue is worth $\sum_{s \geq t} \delta^s w_s$ for the generation born at time t, and propose to take it into account in the interemporal welfare by setting the net present value at

$$\sum_t \delta^t w_t + \sum_t \sigma^t \left(\sum_{s \geq t} \delta^s w_s \right)$$

where σ is the rate at which the current generation discounts the utility of future ones. This approach has been used in (Liu et al. 2002) for evaluating New Jersey's ecosystem services, and by (Graham, 2010) for evaluating public health policies.

In this paper, we extend the Sumaila-Walters approach to continuous time. This leads to non-constant discount rates, so that the management problem becomes time-inconsistent: denoting by $NPV(h, t)$ the net present value of a policy h computed at time t, it may be the case that $NPV(h_1, 0) > NPV(h_2, 0)$, but that $NPV(h_1, t) < NPV(h_2, t)$ at some later time $t > 0$. In that case, the decision-maker at time t will obviously not carry out the policy planned at time 0, which therefore is not implementable. This phenomenon has been much analysed in the economic literature, since the pioneering paper (Phelps and Pollak, 1968). Various solution concepts have been proposed, by (Krusell and Smith, 2003), and by (Harris and Laibson, 2
We adapt to the present situation the solution developed (Ekeland and Lazrak, 2010) in a dif-

ferent setting: we assume that the decision-makers have a very small window of opportunity: the decision-maker at time t holds power only between t and $t+\varepsilon$, and cannot commit his or her successors. We then seek a *sustainable policy*, that is, a Nash equilibrium of the resulting sequential game, and we compute it explicitly. This leads us to formula (2), which is our main result. Unfortunately, we cannot apply directly the results of (Ekeland and Lazrak, 2010), since the sustainable policy we find is a threshold strategy (apply maximum effort if the stock is below equilibrium level, and zero effort if the stock is above equilibrium level), and hence discontinuous, whereas the results in (Ekeland and Lazrak, 2010) assume that the strategies under consideration are continuous, and even better (continuously differentiable). The computations, however, are elementary, if a bit tedious, and are given in the appendix.

I. Optimal management

A simple model of optimal fisheries management, probably originating with (Schaefer 1957), consists of seeking the optimal harvesting policy $h(t)$ as the solution of the problem:

$$\text{(Opt)} \qquad \begin{gathered} \max_h \int_0^\infty e^{-\delta t}(p - c(x(t)))h(t)dt \\ \frac{dx}{dt} = f(x) - h(t), \quad 0 \le h \le h_{\max} \\ x(0) = x_0 \end{gathered}$$

where $\delta > 0$ is the rate of time preference, p is the price of fish sold on shore, $c(x)$ is the cost of bringing one fish to shore when the total population (consisting of only one species) is x, and $f(x)$ models the natural growth of the fish population. The fishing effort (control variable) is $h(t)$, which is bounded above by $h_{\max}$. We have:

$$f(0) = 0,$$

$$c(0) = 0,\ c(x) > 0 \text{ for } x > 0$$

so that if $x(T) = 0$ for some T, meaning that the fish population has been driven to extinction, there can be no recovery: from then on, we obviously stop the fishing effort: $h(t) = 0$ for $t > T$. Mathematicaly speaking, this is an optimal control problem, where $h(t)$ is the control and $x(t)$ is the state at time t. Schaefer himself took the specification:

$$f(x) = rx\left(1 - \frac{x}{K}\right), \quad c(x) = \frac{c}{qx}$$

but many other choices are possible. We refer to the book (Clark 1990) for a thorough discussion of this problem and its variants. It is shown that the optimal harvesting policy $\tilde{h}(t)$ consists of bringing the population as quickly as possible to a certain size $\tilde{x}$ and maintaining it at that level from then on. Specifically, consider the equation:

$$f'(\tilde{x}) - \frac{c(\tilde{x})}{p - c(\tilde{x})} f(\tilde{x}) = \delta \tag{3}$$

If it has a positive solution $\tilde{x}$, the optimal harvesting policy is given by:

$$\tilde{h}(t) = \begin{cases} 0 & \text{if } 0 \leq x(t) < \tilde{x} \\ f(\tilde{x}) & \text{if } x(t) = \tilde{x} \\ h_{\max} & \text{if } x(t) > \tilde{x} \end{cases} \tag{4}$$

If there are no positive solutions, then the optimal harvesting policy consists of taking $\tilde{x} = 0$, that is, of bringing all the fish population to shore as quickly as possible. In the Schaefer specification, for instance, this will happen if $p > \frac{c}{2q}$ and δ is large enough. In other words, using a high rate of time preference (for instance, taking δ to be equal to the market rate of interest) will result in driving the population to extinction: see for instance (Clark 1960).

In the sequel, we will refer to strategies of type (4), optimal or not, as *threshold strategies*. Note that they are discontinuous.

II. Intergenerational equity.

In the Schaefer model, the crucial parameter is δ: if it is too high, the fish population is driven to extinction. But what exactly is δ ? Problem (Opt), is formulated as if there was a single, immortal, individual, who will reap the future benefits, and who discounts them at the constant rate δ. But this is not true: the benefits accrue to individual fishermen, who have a finite life span. To be sure, their personal interest in the fishing stock extends beyond their own lifetime, because they are interested in leaving a job to their sons, but the same parameter δ cannot be used to describe two different things, the rate at which they discount benefits which accrue to them, and the rate at which they discount benefits which accrue to others, even their own children. One should really use two different parameters, δ and σ.

We make the simplifying assumption that all individuals are identical: both born and unborn apply the same discount rate δ to benefits which accrue to the current generation (people alive at the time when decisions are made), and the same discount rate σ to benefits which accrue to future generations (people yet unborn). More precisely, let us assume that the total population N is constant through time. Let n be its rate of renewal, meaning that $nNdt$ individuals die and $nNdt$ are born between t and $t + dt$. Of the generation born at time s, only a proportion $e^{-n(t-s)}$ is still alive at time $t > s$; so the total population at time t consists of N individuals, e^{-nt} of whom were born at time 0, and $e^{-n(t-s)}ds$ were born between s and $s + ds$, where $0 \leq s \leq t$. These N individuals will share equally the benefits of the harvest at time t, namely $(p - c\,(x\,(t)))\,h\,(t)$.

At time 0, the total discounted benefit accruing to the present generation is:

$$N \int_0^\infty e^{-\delta t}(p - c(x(t)))h(t)dt \tag{5}$$

The present generation also takes into account the benefits which will accrue to individuals born between t and $t+dt$, namely:

$$\int_t^{\infty} e^{-\delta(s-t)}(p-c(x(s)))h(s)ds$$

There are $Nndt$ such people, so the total benefit is:

$$\left(\int_t^{\infty} e^{-\delta(s-t)}(p-c(x(s)))h(s)ds\right) Nndt$$

This occurs at time t, so it has to be discounted to time 0. The present generation will not discount it at the same rate δ it would use for itself, but another, presumably higher, rate σ, yielding:

$$e^{-\sigma t}\left(\int_t^{\infty} e^{-\delta(s-t)}(p-c(x(s)))h(s)ds\right) Nndt$$

Summing up over all future generations, we get the "altruistic" part of the present generation's welfare:

$$\int_0^{\infty} e^{-\sigma t}\left(\int_t^{\infty} e^{-\delta(s-t)}(p-c(x(s)))h(s)ds\right) Nndt$$

which has to be added to its "selfish" part (5). We finally get the intertemporal welfare function:

$$(6)\quad W(h) := N\int_0^{\infty} e^{-\delta t}(p-c(x(t)))h(t)dt \;+nN\int_0^{\infty} e^{-\sigma t}dt\int_t^{\infty} e^{-\delta(s-t)}(p-c(x(s)))h(s)ds$$

Dropping the constant N, we find

$$\begin{aligned} W(h) &= \int_0^{\infty} e^{-\delta t}(p-c(x(t)))h(t)dt \;+\frac{n}{\delta-\sigma}\int_0^{\infty} e^{-\delta t}(p-c(x(t)))h(t)\left[e^{(\delta-\sigma)t}-1\right]dt \\ &= \int_0^{\infty} e^{-\delta t}(p-c(x(t)))h(t)dt \;+\frac{n}{\delta-\sigma}\int_0^{\infty}(p-c(x(t)))h(t)\left[e^{-\sigma t}-e^{-\delta t}\right]dt \\ &= \left(1-\frac{n}{\delta-\sigma}\right)\int_0^{\infty} e^{-\delta t}(p-c(x(t)))h(t)dt+\frac{n}{\delta-\sigma}\int_0^{\infty} e^{-\sigma t}(p-c(x(t)))h(t)dt \end{aligned}$$

Rearranging terms, we find that:

$$W(h) = \int_0^\infty R(t)\,(p - c(x(t)))h(t)dt \tag{7}$$

where the discount factor $R(t)$ is given by:

$$R(t) = \lambda e^{-\delta t} + (1-\lambda)\,e^{-\sigma t} \tag{8}$$

$$\lambda = \left(1 + \frac{n}{\sigma - \delta}\right), \quad \sigma > \delta \tag{9}$$

Note that this corresponds to a non-constant discount rate $r(t)$ (except when $n = 0$)

$$r(t) := -\frac{R'(t)}{R(t)} = \frac{\lambda\delta - (\lambda - 1)\,\sigma e^{(\delta-\sigma)t}}{\lambda - (\lambda - 1)\,e^{(\delta-\sigma)t}}$$

$$r(t) \longrightarrow \delta - n \text{ when } \; t \longrightarrow 0$$

$$r(t) \longmapsto \delta \;\text{ when } t \longrightarrow \infty$$

When $\delta = \sigma$, formula (8) breaks down. It has to be replaced by its limit when $\sigma \longrightarrow \delta$, namely:

$$R(t) = (1 + nt)\,e^{-\delta t} \tag{10}$$

leading to a discount rate which is again non-constant unless $n = 0$:

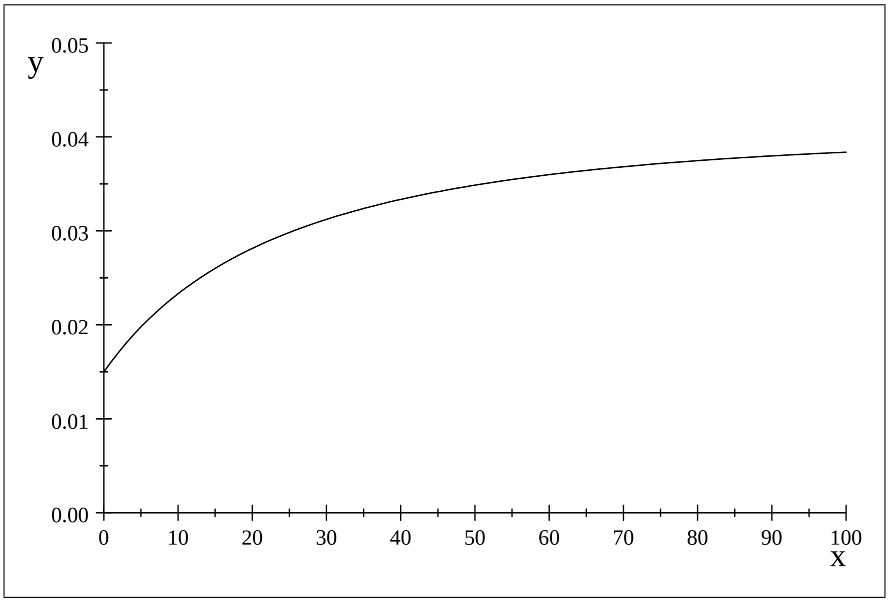
$$r(t) = \delta - \frac{n}{1 + nt}$$

$$r(t) \longrightarrow \delta - n \;\text{ when } t \longrightarrow 0$$

$$r(t) \longrightarrow \delta \;\text{ when } t \longmapsto \infty$$

Note that in both cases, $r(\infty) > r(0)$: the long-term rate is higher than the spot rate. The following picture plots $r(t)$ from $t = 0$ to $t = 100$ with the numerical values $n = 1/40$,

$\delta = 4\%$, $\sigma = 2\%$.



III. Sustainable policies

To sum up, if the present generation discounts the benefits accruing to future generations at a rate σ, then the present value of the welfare arising from a harvesting policy $h(t)$ should be given by formula (7):

$$W(h) = \int_0^\infty R(t)\,(p - c(x(t)))h(t)dt$$

where $R(t)$ is a quasi-exponential discount. Note that even if $\delta = \sigma$, that is, even if the present generation uses the same rate for itself and for future generations, the discount rate $r(t) = -R'(t)/R(t)$ is not constant.

It is well known that the decision-makers then face a problem of *time inconsistency*: see (Harris and Laibson, 2002) for a survey. To see why, consider two harvesting policies $h_1(s)$ and $h_2(s)$, starting at time $T > 0$, and suppose that at a time 0 the first one is preferred:

$$\int_T^\infty R(s)\,(p - c(x(s)))h_1(s)ds \geq \int_T^\infty R(s)\,(p - c(x(s)))h_2(s)dsds \tag{11}$$

Now take a subsequent time t, with $0 < t < T$, and compare the present values at time t. Is it still the case that $h_1(s)$ is preferred to $h_2(s)$? In the exponential case, $R(s) = \exp(-rs)$, we have:

$$\int_T^{\infty} e^{-r(s-t)} R(s) (p - c(x(s)))h(s)ds = e^{rt} \int_T^{\infty} e^{-rs} R(s) (p - c(x(s)))h(s)ds$$

so that both sides of equation (11) are multiplied by a positive constant, and the inequality persists. However, if $R(t)$ is not an exponential (in particular, if it is quasi-exponential), the inequality may well be reversed. In other words, if h_1 seemed better than h_2 at time $t = 0$, it may well be that h_2 will seem better than h_1 at a later time $t > 0$.

The consequences for the planner are considerable. Suppose the current generation is seeking an optimal policy, that is, a harvesting policy $\bar{h}$ that will maximize its discounted intertemporal welfare $W(h)$. Suppose such a policy is found and acted upon at time $t = 0$. As soon as some time has elapsed, it will be found that $\bar{h}$ no longer maximizes $W(h)$ on the remaining interval; in other words, at time $t > 0$, the optimal policy is some $\tilde{h} \neq \bar{h}$. It is then to be expected that the new policy $\tilde{h}$ will be implemented instead of $\bar{h}$ if the decision-maker is free to do so.

As a result, there is no way for the present generation to achieve what is, from her point of view, the first-best solution of the problem, and it must turn to a second-best policy. The best it can do is to guess what future generations are planning to do, and to lay down its own plan accordingly. In other words, we will be looking for a subgame-perfect equilibrium of a certain game in continuous time.

Let us summarize. We replace the exponential discount $e^{-\delta t}$ by a quasi-exponential discount $R(t)$. Problem (Opt) now becomes:

(Eq)

$$\max_h \int_0^\infty R(s)(p - c(x(s)))h(s)ds$$
$$\frac{dx}{ds} = f(x) - h(s), \quad 0 \le h \le h_{\max}$$
$$x(0) = x_0$$

The first-best (optimal) solution for the present generation cannot be implemented, and the decision-maker will have to find a second-best solution. This will be done by considering the situation no longer as an optimization problem, but as a non-cooperative game between successive decision-makers: a *sustainable policy* will be a Nash equilibrium of this leader-follower game. To be able to compute them, we will assume *perfect competition betwen decision-makers*: at every instant t, a new one assumes power, and will hold it for a vanishingly small amount of time ε. To formalize the idea, we need a few notations:

- Given some $x_\infty > 0$, we say that $h(t)$ is a *threshold strategy converging* to x_∞ (shortened to x_∞-strategy) if:

(12)
$$h(t) = \begin{cases} 0 & \text{if } 0 \le x(t) < x_\infty \\ f(x_\infty) & \text{if } x(t) = x_\infty \\ h_{\max} & \text{if } x(t) > x_\infty \end{cases}$$

- Denote by $\xi(t, h, x)$ the population at time t,when the harvesting policy is $h(t)$ and the initial stock is x. The *present value* V of the threshold strategy h, given that the initial stock is x, is given by:

$$V(h, x) = \int_0^\infty R(s)(p - c(\xi(t, h, x)))h(t)dt$$

- Given a fishing effort $h(s)$, a time t, some $a \in R$ and some $\varepsilon > 0$, we define the

(ε, t, a)-*perturbation* of h by:

$$h^{\varepsilon}(s) = \begin{cases} h(t) & \text{if } \ s \notin [t, t+\varepsilon] \\ a & \text{if } \ t \leq s \leq t+\varepsilon \end{cases}$$

Definition 1. *A threshold strategy $h(t)$ is an* sustainable policy *if for every (x, t, a), with $0 \leq a \leq h_{\max}$, we have*

$$\lim_{\varepsilon \longrightarrow 0} \frac{1}{\varepsilon} \left[V\left(h^{\varepsilon}, x\right)\right) - V(h, x) \right] \leq 0 \tag{13}$$

where h^{ε} is the (ε, t, a)-perturbation of h. It is locally sustainable *if (13) holds for all x in some neighbourhood of x_{∞}, all $t \geq 0$ and all a with $0 \leq a \leq h_{\max}$*

The interpretation is straightforward. Suppose a sustainable policy $h(x)$ is public knowledge, and has been implemented until time $t > 0$, leading to a situation $x_t = \xi(t, h, x)$. The generation alive at time t reexamines the situation: its first-best solution cannot be implemented, as we have just seen, so it looks for a second-best. It will be in power between t and $t + \varepsilon$, and during that small interval of time it can exert any fishing effort h_0. After that, decisions will be made by others, and his best guess is that they will revert to the original strategy $h(x)$. This means that, if he exerts effort a, instead of $h(x_t)$, he will be changing the strategy h for the (ε, t, a)-*perturbation* h^{ε} of h, and the present value of doing that is $V(h^{\varepsilon}, x_t)$. On the other hand, the present value of applying $h(x_t)$ like everybody else is $V(h, x_t)$. We want $V(h^{\varepsilon}, x_t) \leq V(h, x_t)$, so there is no incentive to the generation alive at time t for defecting from the agreed strategy $h(x)$. But the difference $V(h^{\varepsilon}, x_t) - V(h, x_t)$ is clearly first-order in ε, hence the condition (13).

In other words, a sustainable policy is a Nash equilibrium: there is no incentive for unilateral deviations; note that the sustainable policy itself is Markovian (it depends only on the current state x), but it is robust against non-Markovian deviations.

If a policy converging to x_{∞} is *locally sustainable*, then (13) holds only in some interval

$]x_\infty - a, \quad x_\infty - b[$. This means that, for any point in that interval, the policy h is proof against unilateral deviations. So, if the starting point $x(0)$ lies in that interval, applying that policy h will keep all the following $x(t)$, $t > 0$, in that interval (because it converges to x_∞), so none of the future decision-makers will have an incentive to deviate.

IV. Sustainable management

Our first result extends Schaefer's relation (3) to the current situation:

Theorem 2. *Suppose a threshold strategy h converging to x_∞ is a sustainable policy. Then the equilibrium population x_∞ solves the equation:*

$$f'(x_\infty) - \frac{c'(x_\infty)}{p - c(x_\infty)} f(x_\infty) = \delta - n \tag{14}$$

Note that $\delta - n$ is the short (spot) discount rate in for quasi-exponential discounts, both for type 1 and type 2. We get the same formula as (3), but with a right-hand side shifted downwards by n, the rate of renewal of the population; the value of σ (the rate at which the current generation discounts benefits which accrue to later generations) plays no role. So we are led to a very simple and robust conclusion: *intergenerational equity should lead us to replace δ, the discount rate of the present generation, by $\delta - n$, where n is the rate of renewal of the population.*

We first prove the existence of a sustainable policy in the simplest case, when the biological growth is linear:

Theorem 3. *Suppose $f(x) = kx$ and there is some population level $x_\infty > 0$ such that:*

$$kx_\infty < h_{\max}, \quad c(x_\infty) < p \tag{15}$$

$$1 - \frac{c'(x_\infty)\, x_\infty}{p - c(x_\infty)} = \frac{\delta - n}{k} > 0 \tag{16}$$

$$(\delta - n)\left(\frac{\delta}{k} - 1\right)\left(\frac{\delta}{k} - 2\right)\frac{p - c(x_\infty)}{x_\infty^2} + c''(x_\infty) > 0 \tag{17}$$

Then there is a locally sustainable policy h, defined on some neighbourhood of x_∞ and converging to x_∞

Note that we are assuming $\delta > n$: the rate of time preference exceeds the rate of renewal. Formula (16) is just (14) in the setting of a linear biological growth. Condition (15) means that fishing can actually decrease the population, and the price should still make it worthwhile to fish at x_∞. Condition (17) is not as transparent, because x_∞ depends on δ, n and k. Note, for instance, that by eliminating δ with the help of (16), we get a relation involving only n, k and x_∞.

Finally, we extend the argument to the general case:

Theorem 4. *Assume $n < \delta$ and there is some population level $x_\infty > 0$ where:*

$$0 < f(x_\infty) < h_{\max}, \quad c(x_\infty) < p \tag{18}$$

$$0 \le (\delta - f'(x_\infty))(\sigma - f'(x_\infty)) \tag{19}$$

$$\delta - n = f'(x_\infty) - \frac{c'(x_\infty)}{p - c(x_\infty)} f(x_\infty) \tag{20}$$

Assume moreover that, in some neighbourhood of x_∞, the function $c(x)$ is decreasing and the function $f(x)(p - c(x))$ is concave, one of them strictly so. Then there is a locally sustainable policy h, defined on some neighbourhood of x_∞ and converging to x_∞.

The proofs of Theorems 2, 3 and 4 will be given in the appendix.

V. Conclusion

We have presented in this paper a new approach to incorporating intergenerational equity concerns in the use of marine fish populations, which can easily be extended and made applicable to a broad range of environmental and natural resources. To do this, we adapted the classical Schaefer model of the fishery and determined sustainable policies as Markov sub-game perfect equilibria. Our new discounting formula achieves many of the results of other alternative approaches to discounting in a way that requires very little additional data, which would make it very useful in terms of its ability to contribute to the ongoing debate on how to achieve sustainable development. A significant contribution is that our approach explicitly links sustainable development to population growth.

The proposed approach recognizes the need for discounting flows of benefits by each generation because we agree that each generation would prefer to have their benefits now rather than later due to various factors, e.g., the fact that capital has an opportunity cost and therefore discounting is necessary. At the same time, our approach builds in the need not to foreclose options to future generations when it comes to their needs from the natural environment. We have produced believe this is a more balanced approach to discounting, which could help policy makers design management solutions for the natural environment that would address the concerns raised by Adam Smith in his 1766 Lecture.

A Mathematical Appendix (not for publication)

A. Threshold strategies

Let $h(t)$ be a threshold strategy (not necessarily a sustainable policy) converging to x_∞. Denoting, as above, by $\xi(t,h,x)$ the population at time t,when the harvesting policy is $h(t)$ and the initial stock is x, we introduce two functions which will play a crucial role:

$$\text{(A1)} \qquad v(x) := \int_0^\infty \lambda e^{-\delta t}(p - c(\xi(t,h,x)))h(\xi(t,h,x))dt$$

$$\text{(A2)} \qquad w(x) := \int_0^\infty (1-\lambda)e^{-\sigma t}(p - c(\xi(t,h,x)))h(\xi(t,h,x))dt$$

so that the present value associated with a harvest function h and the starting population x, is $V(h,x) = v(x) + w(x)$. It is clear from the definition of a threshold strategy that v and w are continously differentiable at every $x \neq x_\infty$. The case $x = x_\infty$ is important and will be handled directly.

We will now give explicit formulas for the derivatives $v'(x)$ and $w'(x)$.

The case $x < x_\infty$**.**—We have $h(x) = 0$ and the fish stock is increasing. Let a small time $\tau > 0$ elapse, so that the population reaches the level $x + \varepsilon < x_\infty$, with $\varepsilon = f(x)\tau$. We have, up to first order in ε:

$$\begin{aligned}
v(x) &= \int_0^\infty \lambda e^{-\delta t}[p - c(\xi(t,h,x))]h(\xi(t,h,x))dt \\
&= \int_0^\tau \lambda e^{-\delta t}[p - c(\xi(t))]h(\xi(t))dt + \int_{\tau+}^\infty \lambda e^{-\delta t}[p - c(\xi(t))]h(\xi(t))dt \\
&= 0 + e^{-\delta\tau}\int_0^\infty \lambda e^{-\delta t}[p - c(\xi(t))]h(\xi(t))dt = \\
&= e^{-\delta\tau}v(x+\varepsilon) = \left(1 - \delta\frac{\varepsilon}{f(x)}\right)(v(x) + \varepsilon v'(x))
\end{aligned}$$

So

(A3) $$v'(x) = \frac{\delta}{f(x)}v(x)$$

and likewise:

(A4) $$w'(x) = \frac{\sigma}{f(x)}w(x)$$

Adding up, we find that

(A5) $$(v'(x) + w'(x))f(x) = \delta v + \sigma w$$

The case $x > x_\infty$**.**—We have $h(x) = h_{\max}$ and the fish stock is decreasing. Let a small time $\tau > 0$ elapse, so that the population reaches the level $x - \varepsilon > x_\infty$, with

$\varepsilon = (h_{\max} - f(x))\,\tau$. We have, up to first order in ε:

$$\begin{aligned}
v(x) &= \int_0^{\infty} \lambda e^{-\delta t}[p - c(\xi(t,h,x))]h(\xi(t,h,x))dt \\
&= \int_0^{\tau} \lambda e^{-\delta t}[p - c(\xi(t))]h_{\max}dt + \int_{\tau}^{\infty} \lambda e^{-\delta t}[p - c(\xi(t))]h(\xi(t))dt \\
&= \lambda[p - c(x)]h_{\max}\tau + e^{-\delta\tau} v\,(x - \varepsilon) \\
&= \lambda[p - c(x)]h_{\max}\tau + (1 - \delta\tau)\,(v\,(x) - \varepsilon v'\,(x)) \\
&= v\,(x) + \tau\,[\lambda\,(p - c(x))\,h_{\max} - \delta v\,(x) - (h_{\max} - f(x))\,v'\,(x)]
\end{aligned}$$

This leads to:

$$v'(x) = \frac{\delta v(x)}{f(x) - h_{\max}} - \lambda \frac{p - c(x)}{f(x) - h_{\max}} h_{\max} \tag{A6}$$

$$w'(x) = \frac{\sigma w(x)}{f(x) - h_{\max}} - (1 - \lambda)\,\frac{p - c(x)}{f(x) - h_{\max}} h_{\max} \tag{A7}$$

Adding up, we find that

$$(v'\,(x) + w'\,(x))\,(f\,(x) - h_{\max}) = \delta v + \sigma w - h_{\max}\,(p - c\,(x)) \tag{A8}$$

The case $x = x_{\infty}$***.***—We have:

$$v(x_{\infty}) = \int_0^{\infty} \lambda e^{-\delta t}(p - c(x_{\infty}))f(x_{\infty})dt = \frac{\lambda}{\delta}(p - c(x_{\infty}))f(x_{\infty})$$

Start from a smaller population $x_{\infty} - \varepsilon$, with $\varepsilon > 0$ small, and apply the strategy h. This means that the fishing effort is $h\,(t) = 0$ until the level x_{∞} is reached again. This will happen after a time $\tau = \varepsilon / f\,(x_{\infty})$, and then the population is stabilized at that level. This leads to:

$$\begin{aligned} v(x_\infty - \varepsilon) &= \int_0^\infty \lambda e^{-\delta t}(p - c(\xi(t,h,x_\infty - \varepsilon)))h(\xi(t,h,x_\infty - \varepsilon)))dt \\ &= \int_\tau^\infty \lambda e^{-\delta t}(p - c(x_\infty))f(x_\infty)dt = \frac{\lambda}{\delta}e^{-\delta\tau}(p - c(x_\infty))f(x_\infty) \end{aligned}$$

where we have taken into account that $h(\xi(t,h,x_\infty - \varepsilon))) = 0$ for $0 \le t \le \tau$. Hence the left derivative:

$$v'_-(x_\infty) = \lambda(p - c(x_\infty))$$

In the same way, we compute the right derivative. This time, we start with a larger population $x_\infty + \varepsilon$, with $\varepsilon > 0$ small, and we apply the fishing level $h(t) = h_{\max}$ until the level x_∞ is reached again. This will happen after at time τ given by $(h_{\max} - f(x_\infty))\tau = \varepsilon$. We have:

$$\begin{aligned} v(x_\infty + \varepsilon) &= \int_0^\tau \lambda e^{-\delta t}(p - c(\xi(t,h,x_\infty + \varepsilon)))h_{\max}dt + \int_\tau^\infty \lambda e^{-\delta t}(p - c(x_\infty))f(x_\infty)dt \\ &= \lambda(p - c(x_\infty))h_{\max}\tau + e^{-\delta\tau}\frac{\lambda}{\delta}(p - c(x_\infty))f(x_\infty) \\ &= v(x_\infty) + [\lambda(p - c(x_\infty))h_{\max} - \lambda(p - c(x_\infty))f(x_\infty)]\tau \\ &= v(x_\infty) + \lambda(p - c(x_\infty))(h_{\max} - f(x_\infty))\tau \end{aligned}$$

and substituting the value for τ, we get $v'_+(x_\infty) = \lambda(p - c(x_\infty))$, which proves that the right and left derivatives are equal, so that v is derivable at x_∞, with $v'(x_\infty) = \lambda(p - c(x_\infty))$.

On the other hand, we also have:

$$v(x_\infty) = \int_0^\infty \lambda e^{-\delta t}(p - c(x_\infty))h(x_\infty))dt = \frac{\lambda}{\delta}h(x_\infty)(p - c(x_\infty))$$

(fishing effort maintains the population at the level x_∞), so that $\lambda(p - c(x_\infty))f(x_\infty) = \delta v(x_\infty)$. A similar argument holds for w. We summarize, bearing in mind that $h(x_\infty) =$

$f\left(x_{\infty}\right)$ in equilibrium:

$$v'\left(x_{\infty}\right)=\lambda(p-c(x_{\infty})) \tag{A9}$$

$$w'\left(x_{\infty}\right)=(1-\lambda)\left(p-c(x_{\infty})\right) \tag{A10}$$

$$v\left(x_{\infty}\right)=\frac{\lambda}{\delta}f\left(x_{\infty}\right)\left(p-c\left(x_{\infty}\right)\right) \tag{A11}$$

$$w\left(x_{\infty}\right)=\frac{1-\lambda}{\sigma}f\left(x_{\infty}\right)\left(p-c\left(x_{\infty}\right)\right) \tag{A12}$$

Note that:

$$v'\left(x_{\infty}\right)=\frac{\delta}{f\left(x_{\infty}\right)}v\left(x_{\infty}\right)=\frac{\delta v(x_{\infty})}{f(x_{\infty})-h_{\max}}-\lambda\frac{p-c(x_{\infty})}{f(x_{\infty})-h_{\max}}h_{\max}$$
$$w'\left(x_{\infty}\right)=\frac{\sigma}{f\left(x_{\infty}\right)}w\left(x_{\infty}\right)=\frac{\sigma w(x_{\infty})}{f(x_{\infty})-h_{\max}}-(1-\lambda)\frac{p-c(x_{\infty})}{f(x_{\infty})-h_{\max}}h_{\max}$$

so that (A3), (A4), (A6) and (A7) all hold at $x=x_{\infty}$. As a consequence, so do (A5) and (A8)

B. Sustainable policy

Characterization.—We now consider the (ε,t,a)-*perturbation* of h. Without loss of generality, we can assume that $t=0$ (that is, we reset our watches if necessary), so that:

$$h^{\varepsilon}(s)=\begin{cases} h(x(t)) & \varepsilon<t \\ a & 0\leq t\leq\varepsilon \end{cases} \tag{A13}$$

Let us write v_{ε} and w_{ε} instead of $v_{h^{\varepsilon}}$ and $w_{h^{\varepsilon}}$, so that $v_0=v$ and $w_0=w$. Keeping only first-order terms in ε, we have, at any point $x \neq x_\infty$:

$$\begin{aligned} v_\varepsilon(x) &= \int_0^\infty \lambda e^{-\delta t}[p - c(\xi(t, h^\varepsilon, x))]h^\varepsilon(\xi(t, h^\varepsilon, x))dt \\ &= \int_0^\varepsilon \lambda e^{-\delta t}[p - c(\xi)]h^\varepsilon(\xi)dt + \int_\varepsilon^\infty \lambda e^{-\delta t}[p - c(\xi)]h^\varepsilon(\xi)dt \\ &= \lambda[p - c(x)]a\varepsilon + \int_0^\infty \lambda e^{-\delta(t+\varepsilon)}[p - c(\xi(t+\varepsilon))]h(\xi(t+\varepsilon))dt \\ &= \lambda[p - c(x)]a\varepsilon + e^{-\delta\varepsilon}\int_0^\infty \lambda e^{-\delta t}[p - c(\xi(t+\varepsilon))]h(\xi(t+\varepsilon))dt \\ &= \lambda[p - c(x)]a\varepsilon + (1 - \delta\varepsilon)\, v_h\left(\xi\left(t+\varepsilon\right)\right) \\ &= v_h\left(x\right) + \varepsilon\left[v_h'\left(x\right)\left(f\left(x\right) - a\right) - \delta v_h\left(c\right) + \lambda\left(p - c(x)\right)a\right] \end{aligned}$$

The term $(f(x) - a)$ comes from the fact that, if the fishing effort is a exerted during a period ε when the stock is x, then the new stock at the end of the period will be $x + (f(x) - a)\,\varepsilon$, up to first order. Similarly, we get:

$$w_\varepsilon(x) = w\left(x\right) + \varepsilon\left[w'\left(x\right)\left(f\left(x\right) - a\right) - \sigma w\left(c\right) + (1 - \lambda)\left(p - c(x)\right)a\right]$$

We then introduce the Hamiltonian $H(x, a)$:

$$H\left(x, a\right) := \left(v'\left(x\right) + w'\left(x\right)\right)\left(f\left(x\right) - a\right) - \delta v - \sigma w + \left(p - c(x)\right)a \tag{A14}$$

$$= a\left[\left(p - c(x)\right) - \left(v'\left(x\right) + w'\left(x\right)\right)\right] + \left(v'\left(x\right) + w'\left(x\right)\right)f\left(x\right) - \delta v - \sigma w \tag{A15}$$

Condition (13) then reduces to the following:

$$\max\left\{H\left(x, a\right)\ \mid 0 \leq a \leq h_{\max}\right\} \leq 0 \tag{A16}$$

By definition, $h(x)$ is a sustainable policy if and only if it satisfies condition (A16). It is reminescent of the classical Hamilton-Jacobi-Bellman equation in optimal control, so once again we emphasize that, in the present situation, with non-constant discounting, it will NOT give an optimal solution, but an equilibrium one.

In (A16) we find ourselves maximizing a linear function of a, so the maximum must be attained at the boundary unless the slope is zero. There are two possible cases for $x \neq x_\infty$, according to the value of the maximand $h(x)$:

- if $h(x) = 0$, so that $x < x_\infty$, the slope must be negative or zero:

$$0 \geq (p - c(x)) - (v'(x) + w'(x)) \tag{A17}$$

- if $h(x) = h_{\max}$, so that $x > x_\infty$, the slope must be positive or zero:

$$0 \leq (p - c(x)) - (v'(x) + w'(x)) \tag{A18}$$

Necessary condition: Theorem 2 ..—We have proved that the function v and w are continuously differentiable everywhere. Conditions (A17) and (A18) mean that function $\varphi(x) := v'(x) + w'(x) - p + c(x)$ goes from ≥ 0 to ≤ 0 when x increases through x_∞. It is continous, and hence much vanish at x_∞:

$$v'(x_\infty) + w'(x_\infty) = p - c(x_\infty)$$

We know that φ is differentiable all $x \neq x_\infty$, but we cannot assume that it is differentiable at x_∞.So we cannot claim that $\varphi'(x_\infty) \leq 0$. However, there is a sequence $x_n \longrightarrow x_\infty$ from the left $(x_n < x_\infty)$ and a sequence $y_n \longrightarrow x_\infty$ from the right $(x_n > x_\infty)$ such that $\varphi'(x_n) \leq 0$ and $\varphi'(y_n) \leq 0$:

$$v''(x_n) + w''(x_n) \leq -c'(x_n) \text{ and } v''(y_n) + w''(y_n) \leq -c'(y_n)$$

Let us write on the first equation. Differentiating (A5) we have:

$$(v''(x_n) + w''(x_n))f(x_n) + (v'(x_n) + w'(x_n))f'(x_n) = \delta v'(x_n) + \sigma w'(x_n)$$

Combining with the preceding inequation, we get:

$$\frac{1}{f\left(x_n\right)}\left[\delta v'(x_n) + \sigma w'(x_n) - (v'(x_n) + w'(x_n))f'(x_n)\right] \leq -c'\left(x_n\right)$$

and taking the limit as $n \longrightarrow \infty$, we get:

$$\text{(A19)} \qquad \frac{1}{f\left(x_\infty\right)}\left[\delta v'(x_\infty) + \sigma w'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty)\right] \leq -c'\left(x_\infty\right)$$

Now let us work on the y_n. Differentiating (A8) we have:

$$(v''(y_n) + w''(y_n))\left(f\left(x\right) - h_{\max}\right) + (v'(y_n) + w'(y_n))f'(y_n) = \delta v'(y_n) + \sigma w'(y_n) + h_{\max}c'\left(y_n\right)$$

Combining with the inequality $\varphi'\left(y_n\right) \leq 0$, and taking the limit as $n \longrightarrow \infty$, we get:

(A20)

$$\frac{1}{f\left(x\right) - h_{\max}}\left[\delta v'(x_\infty) + \sigma w'(x_\infty) + h_{\max}c'\left(x_\infty\right) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty)\right] \leq -c'\left(x_\infty\right)$$

$$\delta v'(x_\infty) + \sigma w'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty) \geq -f\left(x_\infty\right)c'\left(x_\infty\right)$$

Combining (A19) and (A20), we find:

$$\delta v'(x_\infty) + \sigma w'(x_\infty) - (v'(x_\infty) + w'(x_\infty))f'(x_\infty) = -f\left(x_\infty\right)c'\left(x_\infty\right)$$

Plugging in the values for $v'(x_\infty)$ and $w'(x_\infty)$ from (**??**), we find:

$$\text{(A21)} \qquad \left(p - c\left(x_\infty\right)\right)\left(\lambda\delta + (1-\lambda)\,\sigma - f'\left(x_\infty\right)\right) = -f\left(x_\infty\right)c'\left(x_\infty\right)$$

and substituting the value $\lambda = \left(1 - \frac{n}{\delta - \sigma}\right)$, we get formula (14)

Constant growth rate: Theorem 3.—We now assume that $f(x) = kx$. Formulas (A3) and (A4) become:

$$\frac{v'(x)}{v(x)} = \frac{\delta}{k}\frac{1}{x} \text{ and } \frac{w'(x)}{w(x)} = \frac{\sigma}{k}\frac{1}{x}$$

yielding:

$$\text{(A22)} \qquad v(x) = v(x_\infty)\left(\frac{x}{x_\infty}\right)^{\delta/k} = \frac{\lambda k}{\delta}(p - c(x_\infty))\left(\frac{x}{x_\infty}\right)^{\delta/k} x_\infty \text{ for } x < x_\infty$$

$$\text{(A23)} \qquad w(x) = w(x_\infty)\left(\frac{x}{x_\infty}\right)^{\sigma/k} = \frac{(1-\lambda)k}{\sigma}(p - c(x_\infty))\left(\frac{x}{x_\infty}\right)^{\sigma/k} x_\infty \text{ for } x < x_\infty$$

Similarly, formulas (A6) and (A7) become:

$$v'(x) = -\lambda[p - c(x)]h_{\max} + \delta v(x)\frac{1}{kx - h_{\max}}$$
$$w'(x) = -(1-\lambda)[p - c(x)]h_{\max} + \sigma w(x)\frac{1}{kx - h_{\max}}$$

yielding:

$$v(x) = I(x)(kx - h_{\max})^{\delta/k} + \frac{\lambda}{\delta}kx_\infty(p - c_\infty)$$
$$w(x) = J(x)(kx - h_{\max})^{\sigma/k} + \frac{(1-\lambda)}{\sigma}kx_\infty(p - c_\infty)$$

where the auxiliary function $I(x)$ and $J(x)$ are given by:

$$I(x) := -\lambda\int_x^{x_\infty}\frac{p - c(y)}{(ky - h_{\max})^{\delta/k}}dy$$
$$J(x) := -(1-\lambda)\int_x^{x_\infty}\frac{p - c(y)}{(ky - h_{\max})^{\sigma/k}}dy$$

We have $I(x_\infty) = J(x_\infty) = 0$. Introduce the function $\varphi(x) = c(x) + v'(x) + w'(x) - p$. We have to check that $\varphi(x) \geq 0$ for $x < x_\infty$ and $\varphi(x) \leq 0$ for $x > x_\infty$. We know that φ

is continuous at x_∞, with $\varphi(x_\infty) = 0$. Let us compute the left and right derivatives at x_∞. First the left:

$$\begin{aligned}
\varphi'_-(x_\infty) &= v''_-(x_\infty) + w''_-(x_\infty) + c'(x_\infty) \\
&= \left[\lambda\left(\frac{\delta}{k} - 1\right) + (1-\lambda)\left(\frac{\sigma}{k} - 1\right)\right]\left(\frac{p - c(x_\infty)}{x_\infty}\right) + c'(x_\infty) \\
&= [\lambda\delta + (1-\lambda)\sigma - k]\frac{p - c(x_\infty)}{kx_\infty} + c'(x_\infty) = 0
\end{aligned}$$

because of (14). This is not good enough, so we have to take one more derivative:

$$\begin{aligned}
\varphi''_-(x_\infty) &= v'''_-(x_\infty) + w'''_-(x_\infty) + c''(x_\infty) \\
&= (\lambda\delta + (1-\lambda)\sigma)\left(\frac{\delta}{k} - 1\right)\left(\frac{\delta}{k} - 2\right)\frac{p - c(x_\infty)}{x_\infty^2} + c''(x_\infty)
\end{aligned}$$

$$\text{(A24)} \qquad = (\delta - n)\left(\frac{\delta}{k} - 1\right)\left(\frac{\delta}{k} - 2\right)\frac{p - c(x_\infty)}{x_\infty^2} + c''(x_\infty)$$

If condition (17) is satisfied, we have $\varphi''_-(x_\infty) > 0$, with $\varphi'_-(x_\infty) = \varphi(x_\infty) = 0$. So there must be some $a < x_\infty$ such that $\varphi(x) > 0$ for $a < x < x_\infty$. This is (A17), which is the first part of the relation we want.

Now for (A18). Looking at the right derivative, we have:

$$\begin{aligned}
\varphi'_+(x_\infty) &= v''_+(x_\infty) + w''_+(x_\infty) + c'(x_\infty) \\
&= \lambda\left(\frac{p - c(x_\infty)}{kx_\infty - h_{\max}}\delta - c'(x_\infty)\right) + (1-\lambda)\left(\frac{p - c(x_\infty)}{(kx_\infty - h_{\max})}\sigma - c'(x_\infty)\right) + c'(x_\infty) \\
&= \frac{p - c(x_\infty)}{kx_\infty - h_{\max}}(\lambda\delta + (1-\lambda)\sigma)
\end{aligned}$$

$$\text{(A25)} \qquad = (\delta - n)\frac{p - c(x_\infty)}{kx_\infty - h_{\max}}$$

This is negative by (15). So $\varphi'_+(x_\infty) > 0$ and $\varphi(x_\infty) = 0$, which proves that there is some $b > 0$ such that $\varphi(x) < 0$ for $x_\infty < x < b$. The proof is concluded. Note that the function φ is not differentiable at x_∞: we have $\varphi'_-(x_\infty) = 0$ and $\varphi'_+(x_\infty) < 0$.

General growth and cost: Theorem 4.—As in section 3.3, we define v and w by (A1) and (A2). Differentiate equations (A3) and (A4) from the left at x_∞:

$$v''_-(x) = v'(x)\frac{\delta - f'(x)}{f(x)} = \frac{\lambda(p - c(x_\infty))(\delta - f'(x_\infty))}{f(x_\infty)}$$
$$w''_-(x) = w'(x)\frac{\sigma - f'(x)}{f(x)} = \frac{(1-\lambda)(p - c(x_\infty))(\sigma - f'(x_\infty))}{f(x_\infty)}$$

Set $I(x) = f(x)(p - c(x))$ and $\psi(x) = v(x)\delta + w(x)\sigma - I(x)$. Note that, by (A9),(A10).(A11) and.(A12), we have:

$$\psi(x_\infty) = 0 = \psi'(x_\infty)$$

Now consider the (left) second derivative $\psi''_-(x_\infty)$.After some computations, we find:

$$\psi''_-(x_\infty) = \left(\delta\frac{\lambda(p - c(x_\infty))(\delta - f'(x_\infty))}{f(x_\infty)} + \sigma\frac{(1-\lambda)(p - c(x_\infty))(\sigma - f'(x_\infty))}{f(x_\infty)} - I''(x_\infty)\right)$$
$$= (\delta A + \sigma B - I''(x_\infty))$$

with obvious notations. Note that $A + B = -c'(c_\infty)$ by (A21). So $AB \geq 0$ by (18), (19), and $A + B \geq 0$ because $c(x)$ has been assumed to be decreasing. It follows that A and B are positive, and hence:

$$\psi''_-(x_\infty) \geq -\min(\delta, \sigma)c'(x_\infty) - I''(x_\infty) > 0$$

So there exist some $a < x_\infty$ such that $\psi(x) > 0$ for all x in the open interval $]a,\ x_\infty[$.

We redo the preceeding analysis but for $x > x_\infty$. We find:

$$v''_+(x_\infty) = \frac{\lambda c'(x_\infty)h_{\max} + \lambda(p - c(x_\infty)))(\delta - f'(x_\infty))}{f(x_\infty) - h_{\max}}$$
$$w''_+(x_\infty) = \frac{(1-\lambda)c'(x_\infty)h_{\max} + (1-\lambda)(p - c(x_\infty))(\sigma - f'(x_\infty))}{f(x_\infty) - h_{\max}}$$

and hence:

$$\begin{aligned}\psi''_+(x_\infty) &= -(p - c(x_\infty))\frac{\lambda\delta(\delta - f'(x_\infty)) + (1-\lambda)\sigma(\sigma - f'(x_\infty)}{\bar{h} - f(x_\infty)} - \\ &- \left(\frac{(\lambda\delta + (1-\lambda)\sigma)c'(x_\infty)\bar{h}}{\bar{h} - f(x_\infty)} + I''(x)\right) \\ &\geq -(\min(\sigma,\delta)c'(x_\infty) + I''(x_\infty)) > 0\end{aligned}$$

As above, there exists $b > x_\infty$ such that $\psi(x) > 0$ for all x such that $b > x > x_\infty$. So, for all x in the interval $]a,\ b[$ we have $v(x)\delta + w(x)\sigma \geq f(x)(p - c(x)$. Using (A3) and (A4), this yields:

$$v'(x) + w'(x) = \frac{v(x)\delta + w(x)\sigma}{f(x)} \geq p - c(x) \text{ for } a < x < x_\infty$$

while using (A6) and (A7), we get:

$$v'(x) + w'(x) = \frac{-[p - c(x)]h_{\max} + \delta v(x) + \sigma w(x)}{f(x) - h_{\max}} \leq p - c(x) \text{ for } x_\infty < x < b$$

But these two inequalities are precisely (A17) and (A18). So the threshold strategy converging to x_∞ is a sustainable policy, as announced.

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Notes

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