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SECOND-ORDER EVOLUTION EQUATIONS ASSOCIATED WITH CONVEX HAMILTONIANS[1]

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§0. **Introduction.** Many problems in mathematical physics can be formulated as differential equations of second order in time:

$$(\mathscr{E}) \qquad \ddot{x} = -\text{grad } V(x)$$

with V a convex functional. This is the Euler equation for the Lagrangian

$$\tfrac{1}{2}|\dot{x}|^2 - V(x) = \wedge(x, \dot{x})$$

which is convex with respect to $\dot{x}$, and concave with respect to x. On the other hand, the associated Hamiltonian, by the Legendre transform, is seen to be:

$$\tfrac{1}{2}|p|^2 + V(x) = \Gamma(x, p)$$

It is convex in both variables. It is the purpose of this paper to show how the convexity of the Hamiltonian can be systematically used in the study of equations ($\mathscr{E}$). In the first part, we shall show that the solutions of ($\mathscr{E}$), although they are only extremal for the original Lagrangian $\wedge$, are actually minimizing for another, more complex , Lagrangian K. In the second part, we shall show how this characterization can be used to prove the existence of solutions to ($\mathscr{E}$) satisfying various initial or boundary conditions.

§I. **Characterization.** Let H be some Hilbert space; for the sake of convenience, it will be assumed to contain a countable dense subset. Denote by $H^1(0, T; H)$ the Sobolev space of all functions x in $L^2(0, T; H)$ with derivative $\dot{x} = dx/dt$ also in $L^2(0, T; H)$. When the simpler notations H^1 and L^2 are used, they will always refer to these particular spaces.

Let Γ be a lower semi-continuous (l.s.c.) function on $L^2 \times L^2$, with values in $R \cup \{+\infty\}$. It will be assumed to be jointly convex in the variables (x, p), and to be proper (i.e. there exists (x_0, p_0) such that $\Gamma(x_0, p_0) < \infty$). We shall refer to Γ as the Hamiltonian.

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We define the Lagrangian $\wedge$ from the Hamiltonian by taking the Fenchel conjugate with respect to the second variable:

$$\wedge(x, v) = \sup_{p\in L^2}\{p \cdot v - \Gamma(x, p)\} \tag{1}$$

It is a function on $L^2 \times L^2$ with values in $R \cup \{\pm\infty\}$(note that the value $-\infty$ is now allowed). For any $x \in L^2$, it is a convex l.s.c. function of v in L^2; indeed, formula (1) shows that it is the pointwise supremum of a family of l.s.c. functions. For any $v \in L^2$, it is a concave function of x in L^2, as the following lemma shows (note that it need not be u.s.c.).

LEMMA. *Take v in L^2, x_1 and x_2 in L^2 with $\wedge(x_1, v) < +\infty$ and $\wedge(x_2, v) < +\infty$, α_1 and α_2 non-negative with $\alpha_1 + \alpha_2 = 1$. Then:*

$$\wedge(\alpha_1 x_1 + \alpha_2 x_2, v) \geq \alpha_1 \wedge(x_1, v) + \alpha_2 \wedge(x_2, v).$$

Proof. Clear if either $\wedge(x_1, v)$ or $\wedge(x_2, v)$ are equal to $-\infty$. If both are finite, take $\varepsilon > 0$, and pick p_1 and p_2 in L^2 such that:

$$p_1 \cdot v - H(x_1, p_1) \geq \wedge(x_1, v) - \varepsilon$$

$$p_2 \cdot v - H(x_2 . p_2) \geq \wedge(x_2, v) - \varepsilon$$

Multiplying these inequalities by α_1 and α_2, and adding them, we get (set $\alpha_1 p_1 + \alpha_2 p_2 = p$):

$$p \cdot v - [\alpha_1 H(x_1, p_1) + \alpha_2 H(x_2, p_2)] \geq \alpha_1 \wedge(x_1, v) + \alpha_2 \wedge(x_2, v) - \varepsilon$$

It follows from the definition of $\wedge$ that:

$$\wedge(\alpha_1 x_1 + \alpha_2 x_2, v) \geq p \cdot v - H(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 p_1 + \alpha_2 p_2).$$

Since H is convex, we have:

$$-H(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 p_1 + \alpha_2 p_2) \geq -[\alpha_1 H(x_1, p_1) + \alpha_2 H(x_2, p_2)].$$

Comparing the last three inequalities, and using the fact that ε is arbitrarily small, we get the desired result./

As usual, we denote by $\partial\Gamma(x, p)$ the subgradient of Γ at (x, p), i.e. the (closed convex) set of all $(x', p') \in L^2 \times L^2$ such that:

$$(y - x) \cdot x' + (q - p) \cdot p' \leq \Gamma(y, q) - \Gamma(x, p), \quad \forall (y, q) \in L^2 \times L^2.$$

We shall also denote by $\partial_v \wedge(x, w)$ the subgradient at w of the convex l.s.c. function $v \to \wedge(x, v)$, and by $\partial_x(-\wedge)(y, v)$ the subgradient at y of the convex function $x \to -\wedge(x, v)$. Note that all these subgradients can very well be empty. Note also that we can get Γ from $\wedge$:

$$\Gamma(x, p) = \sup_{v\in L^2}\{p \cdot v - \wedge(x, v)\}. \tag{2}$$

Our interest lies in studying the Hamiltonian equation $(-p, x) \in \partial\Gamma(x, p)$. We must first define some boundary conditions; as is usual in convex analysis, we shall do so by using two l.s.c. convex proper functions ϕ_0 and ϕ_1 from H to $R \cup \{+\infty\}$. We can now state in two equivalent forms the equations we wish to study:

PROPOSITION 1. *The pair* $(x, p) \in H^1 \times H^1$ *satisfies the Euler–Lagrange equations:*

$$(\mathscr{E}) \qquad \begin{gathered} p \in \partial_v \wedge (x, \dot{x}) \\ \frac{dp}{dt} + \partial_x(-\wedge)(x, \dot{x}) \\ p(0) + \partial\phi_0(x(0)) \ni 0 \\ p(T) - \partial\phi_1(x(T)) \ni 0 \end{gathered}$$

if and only if it satisfies Hamilton's equations:

$$(\mathscr{H}) \qquad \begin{gathered} \left(-\frac{dp}{dt}, \frac{dx}{dt}\right) \in \partial\Gamma(x, p) \\ p(0) + \partial\phi_0(x(0)) \ni 0 \\ p(T) - \partial\phi_1(x(T)) \ni 0 \end{gathered}$$

Proof. The boundary conditions are the same. From the definition of $\wedge$, it follows that:

$$p \in \partial_v \wedge (x, \dot{x}) \Leftrightarrow \dot{x} \in \partial_p \Gamma(x, p) \tag{3}$$

There only remains to show the equivalence of relations $-\dot{p} \in \partial_x(-\wedge)(x, \dot{x})$ and $-\dot{p} \in \partial_x \Gamma(x, p)$. The first one means that:

$$(-\dot{p}) \cdot (y - x) \geq \wedge(x, \dot{x}) - \wedge(y, \dot{x}) \qquad \forall y \in L^2$$

Since relations (3) hold, we have:

$$\wedge(x, \dot{x}) = p \cdot \dot{x} - \Gamma(x, p).$$

Writing that into the preceding inequality, we get:

$$(-\dot{p}) \cdot (y - x) \geq p \cdot \dot{x} - \Gamma(x, p) - \wedge(y, \dot{x})$$

Using formula (1) yields:

$$(-\dot{p}) \cdot (y - x) \geq -\Gamma(x, p) + \Gamma(y, p)$$

which means precisely that $-\dot{p} \in \partial_x \Gamma(x, p)$. We can retrace our steps, and get the first relation from the second one. The equivalence is thus proved./

For instance, if the Lagrangian happens to be differentiable, the Euler–Lagrange equations can be written in classical form:

$$\frac{d}{dt}\frac{\partial \wedge}{\partial \dot{x}}(x, \dot{x}) - \frac{\partial \wedge}{\partial x}(x, \dot{x}) = 0.$$

As another example, take $\Gamma(x, p) = \frac{1}{2}\|p\|^2_{L^2} + V(x)$. Then $\wedge(x, v) = \frac{1}{2}\|v\|^2_{L^2} - V(x)$; the Lagrangian and Hamiltonian systems are:

$$(\mathscr{E}) \qquad \ddot{x} \in -\partial V(x), \qquad x(0) \in -\partial\phi_0(x(0)), \qquad x(T) \in \partial\phi_1(x(T))$$

$$(\mathscr{H}) \qquad -\dot{p} \in \partial V(x), \qquad \dot{x} = p, \qquad p(0) \in -\partial\phi_0(x(0)), \quad p(T) \in \partial\phi_1(x(T))$$

We now shall formulate equations $(\mathscr{E})$ and $(\mathscr{H})$ as variational problems. Briefly, there is a non-negative functional $K(x, p)$ on $H^1 \times H^1$ which the solutions of $(\mathscr{E})$ and $(\mathscr{H})$ minimize; moreover, this minimum has to be zero.

PROPOSITION 3. *Solutions of problems* $(\mathscr{E})$ *and/or* $(\mathscr{H})$ *are exactly the pairs* (x, p) *which satisfy:*

$$(\mathscr{P}) \qquad 0 = K(x, p) \le K(y, q) \qquad \forall y \in H^1 \qquad \forall q \in H^1$$

with $K(y, q) = \Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) - 2q \cdot \dot{y} + \phi_0(y(0)) + \phi_0^*(-q(0)) + \phi_1(y(T)) + \phi_1^*)(q(T))$.

Proof. By Fenchel's inequality:

$$\Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) \ge q \cdot \dot{y} - y \cdot \dot{q}$$
$$\phi_0(y(0)) + \phi_0^*(-q(0)) \ge -y(0)q(0)$$
$$\phi_1(y(T)) + \phi_1^*(q(T)) \ge y(T)q(T)$$

equality holding if and only if $(-\dot{q}, \dot{y}) \in \partial\Gamma(y, q)$, $-q(0) \in \partial\phi_0(y(0))$, $q(T) \in \partial\phi_1(y(T))$, i.e. if (y, q) solves problem $(\mathscr{H})$. Adding up these inequalities:

$$K(y, q) \ge -\int_0^T (y(t)\dot{q}(t) + q(t)\dot{y}(t))\, dt - y(0)q(0) + y(T)q(T)$$

Integrating by part, we get $K(y, q) \ge 0$. Equality will hold for solutions of problem $(\mathscr{H})$ only, which is the desired result./

We can reformulate problem $(\mathscr{P})$ as a variational inequality:

PROPOSITION 4. *Solutions of problem* $(\mathscr{E})$ *and/or* $(\mathscr{H})$ *are exactly the pairs* $(x, p) \in H^1 \times H^1$ *which satisfy:*

$$(\mathscr{Q}) \qquad A(x, p; y, q) \le 0 \qquad y \in H^1, \qquad q \in H^1$$

with $A(x, p; y, q) = \Gamma^*(-\dot{p}, \dot{x}) - \Gamma^*(-\dot{q}, \dot{y}) + \phi_0(x(0)) - \phi_0(y(0)) + \phi_1^*(p(T)) - \phi_1^*(q(T)) - p \cdot (\dot{x} - \dot{y}) + x \cdot (\dot{p} - \dot{q}) + p(0)(x(0) - y(0)) - x(T)(p(T) - q(T))$.

Proof. By Proposition 3, the solutions of problems ($\mathscr{E}$) and ($\mathscr{H}$) are the pairs $(x, p) \in H^1 \times H^1$ which satisfy:

$$K(x, p) \leq 0$$

By the definition of Fenchel conjugates:

$$\Gamma(x, p) = \sup_{\dot q, \dot y}\{-\dot q \cdot x + \dot y \cdot p - \Gamma^*(-\dot q, \dot y)\}$$

$$\phi_0^*(-p(0)) = \sup_{y(0)}\{-p(0)y(0) - \phi_0(y(0))\}$$

$$\phi_1(x(T)) = \sup_{q(T)}\{q(T)x(T) - \phi_1^*(q(T))\}$$

Writing these formulas into $K(x, p)$, and noting that $\dot q$, $\dot y$, $y(0)$ and $q(T)$ can be specified independently, we get:

$$K(x, p) = \sup_{y, q} A(x, p; y, q) - [p \cdot \dot x + x \cdot \dot p + p(0)x(0) - p(T)x(T)]$$

Integrating by parts, we see that the bracket is identically zero. Clearly, $K(x, p) \leq 0$ if and only if the pair (x, p) solves the variational inequality ($\mathscr{Q}$)./

We will now give some examples.

EXAMPLE 1. Newton's equation, Neumann boundary conditions.

Let f be a convex l.s.c. proper function on H. Let p_0 and p_1 be two points in H. Consider the problem:

$$\ddot x(t) \in -\partial f(x, (t)) \quad \text{a.e.}$$
$$\dot x(0) = p_0, \qquad \dot x(T) = p_1 \tag{5}$$

These are equations ($\mathscr{E}$), for:

$$\Gamma(x, p) = \tfrac{1}{2}\int_0^T |p(t)|^2\, dt + \int_0^T f(x(t))\, dt$$

$$\phi_0(\xi) = -p_0\xi, \qquad \phi_1(\xi) = p_1\xi$$

Indeed, it is a standard result in convex analysis (see [2], [5]) that the subdifferential at $x \in L^2$ of the map $y \to \int_0^T f(y(t))\, dt$ is just the set of all $z \in L^2$ such that $z(t) \in \partial f(x(t))$ almost everywhere. Proposition 3 now tells us that x solves equations (5) if and only if it belongs to $H^1(0, T; H)$, and there exists $p \in H^1$ such that:

$$0 = K(x, p) \leq K(y, q) \qquad (y, q) \in H^1 \times H^1 \tag{6}$$

with $K(y,q)=\int_0^T[f(y(t))+f^*(-\dot{q}(T))+\frac{1}{2}|q(t)|^2+\frac{1}{2}|\dot{y}(t)|^2-2q(t)\dot{y}(t)]\,dt-p_0y(0)+p_1y(T)$ if $q(0)=p_0$ and $q(T)=p_1$, $+\infty$ otherwise. It then follows that $p=\dot{x}$.

EXAMPLE 2. Newton's equation, Dirichlet boundary conditions

Consider the problem:

$$\begin{aligned}&\ddot{p}(t)\in-\partial f(p(t)) \quad \text{a.e.}\\ &p(0)=p_0, \qquad p(T)=p_1\end{aligned} \tag{7}$$

These are equations ($\mathcal{H}$) for:

$$\Gamma(x,p)=\tfrac{1}{2}\int_0^T|x(t)|^2+\int_0^T f(p(t))\,dt$$

$$\phi_0(\xi)=-p_0\xi, \qquad \phi_1(\xi)=p_1\xi.$$

Indeed, the relation $(-\dot{p},\dot{x})\in\partial\Gamma(x,p)$ decomposes as $-\dot{p}=x$ and $\dot{x}(t)\in\partial f(p(t))$ a.e., which is equivalent to the differential equation (7). It follows that p solves problem (7) if and only if it belongs to $H^1(0,T;H)$, satisfies the boundary conditions, and there exists $x\in H^1$ such that:

$$0=K(x,p)\leq K(y,q) \quad \forall(y,q)\in H^1\times H^1 \tag{8}$$

with $K(y,q)=\int_0^T[\frac{1}{2}|y(t)|^2+\frac{1}{2}|\dot{q}(t)|+f(q(t))+f^*(\dot{y}(t))-2q(t)\dot{y}(t)]\,dt-p_0y(0)+p_1y(T)$ if $q(0)=p_0$ and $q(T)=p_1$, $+\infty$ otherwise. It then follows that $x=-\dot{p}$.

EXAMPLE 3. The wave equation, prescribed initial and final state.

Let Ω be an open bounded domain in R^n. We set $H=L^2(\Omega)$, and we define a convex l.s.c. proper function f on H by:

$$\begin{aligned}&f(\xi)=\tfrac{1}{2}\int_\Omega|\text{grad}\,\xi(w)|^2\,dw \quad \text{if} \quad \xi\in H_0^1(\Omega)\\ &f(\xi)=+\infty \quad \text{otherwise.}\end{aligned}$$

We then consider the Hamiltonian:

$$\Gamma(x,p)=\tfrac{1}{2}\int_0^T|x(t)|^2+\int_0^T f(p(t))\,dt$$

and we prescribe boundary conditions in time by $\phi_0(\xi)=-p_0\xi$, and $\phi_1(\xi)=p_1\xi$, with p_0 and p_1 given in $L^2(\Omega)$. Equations ($\mathcal{H}$) then become: (see [2])

$$\begin{aligned}&p(t)\in H_0^1(\Omega)\cap H^2(\Omega) \quad \text{a.e.}\\ &\frac{d^2p}{dt^2}(t)=\Delta p(t) \quad \text{a.e.}\\ &p(0)=p_0, \qquad p(T)=p_1\end{aligned} \tag{9}$$

which is the wave equation, with homogeneous Dirichlet conditions in the space variables ($p(t)|_{\partial\Omega}=0$). By Proposition 3, p will solve problem (9) if and only if it belongs to $H^1(0, T; L^2(\Omega))$, and there exists $x \in H^1(0, T; L^2(\Omega))$ such that:

$$0 = K(x, p) \leq K(y, q) \tag{10}$$

with

$$K(y, q) = \int_0^T \int_\Omega [\tfrac{1}{2} y(t, w)^2 + \frac{1}{2} \frac{\partial q}{\partial t}(t, w)^2 + \frac{1}{2} \sum_{i=1}^n \frac{\partial q}{\partial w_i}(t, w)^2 + \frac{1}{2} \sum_{i=1}^n \frac{\partial z}{\partial w_i}(t, w)^2 - 2q(t, w)\dot{y}(t, w)]\, dt\, dw - p_0 y(0) + p_1 y(T)$$

if $q(0) = p_0$ $q(T) = p_1$ and $q(t) \in H_0^1(\Omega)$ a.e. Here the function $z(t)$ is defined as the solution of the homogeneous Laplace equation:

$$\forall t,\ \dot{y}(t) = -\Delta z(t), \qquad z(t) \in H_0^1(\Omega)./$$

Finally, we will show how to treat initial-value problems, such as:

$$(\mathcal{H}') \qquad \begin{gathered} (-\dot{p}, \dot{x}) \in \partial\Gamma(x, p) \\ x(0) = x_{0'} \qquad p(0) = p_0 \end{gathered}$$

PROPOSITION 5. *Solutions of problem* ($\mathcal{H}'$) *on the time interval* $[0, T]$ *are exactly the pairs* (x, p) *which satisfy:*

$$(\mathcal{P}') \qquad 0 = K'(x, p) \leq K'(y, q)$$

for $(y, q) \in H^1 \times H^1$, $y(0) = x_0$, $q(0) = p_0$. *Here:*

$$K'(y, q) = \Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) - 2q \cdot \dot{y} + y(T)q(T) - x_0 p_0.$$

Proof. By Fenchel's inequality:

$$\Gamma(y, q) + \Gamma^*(-\dot{q}, \dot{y}) \geq q \cdot \dot{y} - y \cdot \dot{q}$$

equality holding if and only if $(-\dot{q}, \dot{y}) \in \partial\Gamma(y, q)$. It follows that:

$$K'(y, q) \geq -\int_0^T (y(t)\dot{q}(t) + q(t)\dot{y}(t))\, dt - x_0 p_0 + y(T)q(T).$$

Integrating by parts, we get $K'(y, q) \geq 0$. Equality holds for solutions of problem ($\mathcal{H}'$) only, which is the desired result./

PROPOSITION 6. *Solutions of problem* ($\mathcal{H}'$) *on the time interval* $[0, T]$ *are exactly the pairs* (x, p) *such that* $x(0) = x_0$, $p(0) = p_0$, *and*

$$(\mathcal{Q}') \qquad A'(x, p; y, q) \leq 0$$

for $(y, q) \in H^1 \times H^1$, $y(0) = x_0$, $q(0) = p_0$. *Here:*

$$A'(x, p; y, q) = \Gamma^*(-\dot{p}, \dot{x}) - \Gamma^*(-\dot{q}, \dot{y}) - p \cdot (\dot{x} - \dot{y}) + x \cdot (\dot{p} - \dot{q})$$

Proof. By definition of the Fenchel conjugate:

$$\Gamma(x, p) = \sup_{\dot{q}, \dot{y}} \{-\dot{q} \cdot x + \dot{y} \cdot p - \Gamma^*(-\dot{q}, \dot{y})\}$$

Writing this into $K'(x, p)$, and noting that $\dot{q}$ and $\dot{y}$ can be specified arbitrarily, we get:

$$K'(x, p) = \sup A'(x, p; y, q) - [p \cdot \dot{x} + x \cdot \dot{p} + p_0 x_0 - p(T)x(T)]$$

the supremum being taken over all pairs $(y, q) \in H^1 \times H^1$ and that $y(0) = x_0$ and $q(0) = p_0$. The bracket vanishes, so that $K'(x, p) \leq 0$ if and only if $(\mathcal{Q}')$ is satisfied./

Of course, we can apply the same trick to $\Gamma^*(-\dot{p}, \dot{x})$ instead of $\Gamma(x, p)$. In this way, we get statements equivalent to propositions 6 and 4 (we leave the proof to the reader):

PROPOSITION 6 BIS. *Solutions of problem* $(\mathcal{H}')$ *on the time interval* $[0, T]$ *are exactly the pairs* (x, p) *such that* $x(0) = x_0$, $p(0) = p_0$ *and:*

$(\mathcal{R}')$ $\quad B'(x, p; y, q) \leq 0 \quad \forall (y, q) \in H^1 \times H^1, \quad y(0) = y_0, \quad x(0) = x_0.$

where $B'(x, p; y, q) = \Gamma(x, p) - \Gamma(y, q) + \dot{p}(x - y) - \dot{x}(q - y)$.

PROPOSITION 4 BIS. *Solutions of problem* $(\mathcal{H})$ *are exactly the pairs* $(x, p) \in H^1 \times H^1$ *which satisfy:*

$(\mathcal{R})$ $$B(x, p; y, q) \leq 0 \qquad \forall (y, q) \in H^1 \times H^1$$

with $B(x, p; y, q) = \Gamma(x, p) - \Gamma(y, q) + \dot{p}(x - y) - \dot{x}(q - y) + \phi_0(x(0)) - \phi_0(y(0)) + p(0)(x(0) - y(0)) + \phi_1(y(T)) - \phi_1(x(T)) + p(T)(x(T) - y(T))$.

We can now give some more examples:

EXAMPLE 4. Newton's equation, Cauchy problem.

Let f be a convex l.s.c. proper function on H, and $x_0 p_0$ two points of H. Consider the problem:

(11) $$\ddot{x}(t) \in -\partial f(x(t)) \quad \text{a.e.}$$
$$x(0) = x_0, \qquad x(0) = p_0$$

A function $x \in H^1(0, T; H$ solves problem (11) on the time interval $[0, T]$ if and only if $x(0) = x_0$, and there exists $p \in H^1$ such that $p(0) = p_0$ and:

(12) $\quad A'(x, p; y, q) \leq 0 \;\; \forall (y, q) \in H^1 \times H^1, \quad y(0) = x_0, \quad p(0) = p_0.$

Here $A'(x, p; y, q) = \int_0^T [\frac{1}{2}|\dot p(t)|^2 + f^*(x(t)) - \frac{1}{2}|\dot q(t)|^2 - f^*(\dot y(t)) - p(t)\dot x(t) - y(t)) + x(t)(p(t) - \dot q(t))]\, dt$. It then follows that $p = \dot x$.

We can also state this variational inequality as:

$$(13) \qquad B'(x, p; y, q) \le 0 \quad \forall (y, q) \in H^1 \times H^1, \qquad y(0) = x_0, \qquad p(0) = p_0$$

with $B'(x, p; y, q) = \int_0^T [\frac{1}{2}|p(t)|^2 + f(x(t))^2 - \frac{1}{2}|q(t)|^2 - f(y(t))^2 + \dot p(t)(x(t) - y(t)) - \dot x(t)(q(t) - y(t))]\, dt$. It also follows that $p = \dot x$

EXAMPLE 5. The wave equation, Cauchy problem.
Consider, as before, the problem:

$$(14) \qquad \begin{gathered} p(t) \in H_0^1(\Omega) \cap H^2(\Omega) \quad \text{a.e.} \\ \frac{d^2 p}{dt^2}(t) = \Delta p(t) \quad \text{a.e.} \\ p(0) = p_0, \qquad \dot p(0) = -x_0 \end{gathered}$$

The function $p \in H^1(0, T; L^2(\Omega))$ will solve problem (14) on the time interval $[0, T]$ if and only if $p(0) = p_0$, and there exists $x \in H^1(0, T; L^2(\Omega))$ such that $x(0) = x_0$ and:

$$\begin{gathered} A'(x, p; y, q) \le 0 \qquad \forall (y, q) \in H^1 \times H^1, \qquad y(0) = x_0, \\ p(0) = p_0, \qquad q(t) \in H_0^1(\Omega) \quad \text{a.e.} \end{gathered}$$

with

$$\begin{aligned} A'(x, p; y, q) = \int_0^T \int_\Omega \Big[&\frac{1}{2}\frac{\partial p}{\partial t}(t, w)^2 - \frac{1}{2}\frac{\partial q}{\partial t}(t, w)^2 + \frac{1}{2}\sum_{i=1}^n \frac{\partial u}{\partial w_i}(t, w)^2 \\ &- \frac{1}{2}\sum_{i=1}^n \frac{\partial z}{\partial w_i}(t, w)^2 - p(t, w)\Big(\frac{\partial x}{\partial t}(t, w) - \frac{\partial y}{\partial t}(t, w)\Big) \\ &+ x(t, w)\Big(\frac{\partial p}{\partial t}(t, w) - \frac{\partial q}{\partial t}(t, w)\Big)\Big]\, dt\, dw \end{aligned}$$

Here $u(t)$ and $z(t)$ are the solutions of the Laplace equations:

$$\begin{aligned} &\forall t,\ \dot x(t) = -\Delta u(t), \qquad u(t) \in H_0^1(\Omega) \\ &\forall t,\ \dot y(t) = -\Delta z(t), \qquad z(t) \in H_0^1(\Omega) \end{aligned}$$

This will also be written as:

$$\begin{aligned} A'(x, p; y, q) = \int_0^T &[\tfrac{1}{2}\|\dot p(t)\|^2 - \tfrac{1}{2}\|\dot q(t)\|^2 + \tfrac{1}{2}\|\operatorname{grad}(-\Delta)^{-1}\dot x(t)\|^2 \\ &- \tfrac{1}{2}\|\operatorname{grad}(-\Delta)^{-1}\dot y(t)\|^2 - p(t)(\dot x(t) - \dot y(t)) \\ &+ x(t)(\dot p(t) - \dot q(t))]\, dt, \end{aligned}$$

all norms to be taken in $H = L^2(\Omega)$.

§2. **Existence.** The question now is whether these characterizations can actually be used to solve problems ($\mathscr{E}$) and/or ($\mathscr{H}$). We shall show that, in some cases, they can. Our main tool will be a refined version of Ky Fan's inequality (see [$\mathscr{Q}$]):

PROPOSITION 1. *Let $\mathscr{H}$ be a closed subspace of $H^1(0, T; H)^2$, and $\mathscr{B}$ its unit ball. Let Φ be a real function on $\mathscr{H} \times \mathscr{H}$. Assume that, for any $(x, p) \in \mathscr{H}$:*

(1) *the function $(y, q) \to \Phi(x, p; y, q)$ is concave*
(2) $\Phi(x, p; x, p) = 0$
and that, for any $(y, q) \in \mathscr{H}$, and any $n \in N$:
(3) *the function $(x, p) \to \Phi(x, p; y, q)$ is weakly l.s.c. on $n\mathscr{B}$*
Assume moreover that:
(4) $\exists m \in N : \{(x, p) \mid \Phi(x, p; y, q) \leq 0 \quad \forall (y, q) \in \mathscr{B}\} \subset m\mathscr{B}$.
Then there exists $(\bar{x}, \bar{p}) \in \mathscr{H}$ such that:
(5) $\Phi(\bar{x}, \bar{p}; y, q) \leq 0 \qquad \forall (y, q) \in H^1 \times H^1$.

Proof. Let $n \in N$ be given. By the usual Ky Fan inequality ([6]), applied to Φ on the set $n\mathscr{B} \times n\mathscr{B}$, there exists (x_n, p_n) in $n\mathscr{B}$ such that:

$$\Phi(x_n, p_n; y, q) \leq 0 \qquad \forall (y, q) \in n\mathscr{B}.$$

Let $n \to \infty$. By assumption (4), the sequence (x_n, p_n) is bounded, and therefore we can extract a subsequence converging weakly to some $(\bar{x}, \bar{p})$. Take any $(y, q) \in H^1 \times H^1$; it belongs to $n\mathscr{B}$ for n large enough, and by assumption (3):

$$\Phi(\bar{x}, \bar{p}; y, q) \leq \lim_{n \to \infty} \Phi(x_n, p_n; y, q) \leq 0/$$

As a particular case, conclusion (5) will still hold if (4) is replaced by the stronger assumption (see [4]):

$$(6) \qquad \exists m \in N, \qquad (y_0, q_0) \in H^1 \times H^1 : \{(x, p) \mid \Phi(x, p; y_0, q_0) \leq 0\} \subset m\mathscr{B}.$$

We shall apply these results to the variational inequalities in Propositions 4 and 6; the subspace $\mathscr{H}$ being defined by appropriate boundary conditions. Example 2 and 4 will be taken up again.

EXAMPLE 2. Newton's equation, Dirichlet boundary condition.
Consider, as before, the problem (with prescribed $T > 0$):

$$(7) \qquad \begin{aligned} &p(t) \in -\partial f(p(t)) \quad \text{a.e.} \\ &p(0) = p_0, \qquad p(T) = p_1 \end{aligned}$$

By Proposition 4, p solves that problem if and only if it belongs to $H^1(0, T; H)$, satisfies the boundary conditions and there exists $x \in H^1$ such that:

$$A(Hx, p; y, q) \le 0 \quad \text{whenever} \quad q(0) = p_0, \qquad q(T) = p_1$$

with $A(x, p; y, q) = \int_0^T [\frac{1}{2}|\dot p(t)|^2 + f^*(\dot x(t)) - \frac{1}{2}|\dot q(t)|^2 - f^*(\dot y(t)) - p(t)(\dot x(t) - \dot y(t)) + x(t)(\dot p(t) - \dot q(t))]\, dt$

Note that if a constant is added to x, and another one to y, the value of A is unchanged. Indeed:

$$A(x + x_0, p; y + y_0, q) - A(x, p; y, q) = \int_0^T x_0(\dot p(t) - \dot q(t))\, dt$$

$$= 0 \quad \text{since} \quad p(0) = q(0) \quad \text{and} \quad p(T) = q(T).$$

So we can always assume that $x(0) = y(0) = 0$.

PROPOSITION 2. *Let the function $f: H \to R$ be convex, continuous, and satisfy the following growth condition:*

(8) $$\exists K > 0, \qquad \exists k > 0: \qquad f(\xi) \le K|\xi|^2 + k, \quad \text{all} \quad \xi \in H.$$

Then there exists $T_K > 0$ such that problem (7) has at least one solution whenever $T \in]0, T_K[$. Moreover, $T_K \to \infty$ when $K \to 0$.

COROLLARY. *Assume the growth of f is less than quadratic:*

(9) $$f(\xi)/|\xi|^2 \to 0 \quad \textit{uniformly as} \quad |x| \to \infty$$

Then problem (7) *has a solution for all T.*

Proof. The function $A(x, p; y, q)$ satisfies all the assumptions of Proposition 1, where $\mathcal{H}$ is taken to be the subspace of $H^1(0, T; H)$ defined by the boundary conditions $x(0) = 0$, $p(0) = p_0$ and $p(T) = p_1$. Indeed, (1) and (2) are obvious. As for (3), consider a bounded sequence (x_n, p_n) in $\mathcal{H}$ converging weakly to (x, p). Then (x_n, p_n) converges weakly in $L^2(0, T; H)$, which implies that:

$$x_n(t) = 0 + \int_0^t \dot x_n(x)\, ds$$

$$p_n(t) = p_0 + \int_0^t \dot p_n(s)\, ds$$

converge to $x(t)$ and $p(t)$ for all t. Moreover, x_n and p_n are bounded, and hence they converge strongly in $L^2(0, T; H)$ by Lebesgue's theorem. It follows that $\dot x_n \cdot p_n$ and $x_n \cdot \dot p_n$ converge to $\dot x \cdot p$ and $x \cdot \dot p$. By Fatou's lemma, the function $x \to \int_0^T f^*(\dot x(t))\, dt$ is strongly l.s.c. on $H^1(0, T; H)$; since it is convex, it is also weakly l.s.c. Taking everything into account, we see that the function $(x, p) \to A(x, p; y, q)$ is weakly l.s.c. on bounded sets.

There only remains to prove estimate (4), or (6). We shall use the well-known inequality $ab \leq \frac{1}{2}ca^2 + \frac{1}{2}(b^2/c)$ for all non-negative a, b, c. We have:

$$A(x, p; y, q) = \int_0^T [\tfrac{1}{2} |\dot{p}(t)|^2 + f^*(\dot{x}(t)) - p(t)\dot{x}(t) + x(t)\dot{p}(t)]\, dt$$
$$+ \int_0^T [p(t)\dot{y}(t) - x(t)\dot{q}(t)]\, dt - \int_0^T [\tfrac{1}{2} |\dot{q}(t)^2 + f^*(\dot{y}(t))]\, dt$$

Once (y, q) is fixed, the last term on the right-hand side is a constant, the second one is a linear function of (p, q), and the first one, by the inequality just mentioned, is greater than or equal to:

$$\int_0^T \left[\tfrac{1}{2} |\dot{p}(t)|^2 + f^*(\dot{x}(t)) - \frac{c}{2} |\dot{x}(t)|^2 - \frac{1}{2c} |\dot{p}(t)|^2 - \frac{d}{2} |(t)|^2 - \frac{1}{2d} |x(t)|^2\right] dt$$

the constants $c > 0$ and $d > 0$ to be chosen later. Taking into account the initial conditions $x(0) = 0$ and $p(0) = p_0$, we easily get:

$$\|x\|_{L^2} \leq T \|\dot{x}\|_{L^2} \quad \text{and} \quad \|p - p_0\|_{L^2} \leq T \|\dot{p}\|_{L^2}$$

It follows that expression (10) is greater than or equal to:

$$\int_0^T \frac{1}{2}\left(1 - d - \frac{T^2}{c}\right) |\dot{p}(t)|^2\, dt + \int_0^T \left[f^*(\dot{x}(t)) - \frac{1}{2}\left(c + \frac{T^2}{d}\right) |\dot{x}(t)|\right] dt$$
$$- \frac{T^{3/2}}{c} |p_0|\, \|\dot{p}\|_{L^2} - T\, |p_0|^2$$

Now hypothesis (8) comes into play. Taking the Fenchel conjugate of both sides, we get $f^*(\xi) \geq (1/4K)|\xi|^2 - k$ for all $\xi \in H$. Taking that of into account, as well as the preceding inequalities, we get:

$$A(x, p; y, q) \geq \frac{1}{2}\left(1 - d - \frac{T}{c}\right) \|\dot{p}\|^2 + \frac{1}{2}\left(\frac{1}{2K} - c - \frac{T}{d}\right) \|\dot{x}\|^2 - kT$$
$$- \frac{T^{3/2}}{c} |p_0|\, \|\dot{p}\| - T\, |p_0|^2 - \|\dot{y}\|\, (|p_0| + T\, \|\dot{p}\|)$$
$$- T\, \|\dot{q}\|\, \|\dot{x}\| - \tfrac{1}{2} \|\dot{q}\|^2 - \int_0^T f^*(\dot{y}(t))\, dt$$

Take for instance $d = \frac{1}{2}$ and $c = 1/4K$. Then

$$\frac{1}{2}\left(1 - d - \frac{T}{c}\right) = \alpha \quad \text{and} \quad \frac{1}{2}\left(\frac{1}{2K} - c - \frac{T}{d}\right) = \beta$$

both are strictly positive whenever $T < 1/8K$. If y, q, and T are fixed, we have the inequality:

$$A(x, p; y, q) \geq \alpha \|\dot{p}\|^2 + \beta \|\dot{x}\|^2 - \gamma \|\dot{p}\| - \delta \|\dot{x}\| - \zeta \tag{10}$$

with $\alpha, \beta, \gamma, \delta, \zeta$ denoting various constants (depending on y, q, and T). If $T < 1/8K$, it is clear that assumption (6) is satisfied, so Proposition 2 is proved with $T_K = 1/8K$. The corollary immediately follows, since inequality (8) is seen to hold for any K./

The growth condition (8) is natural in this context. For instance, the one-dimensional problem $\ddot{p} = -p$, $p(0) = p_0$, $p(T) = p_1$, can be solved for all $(p_0, p_1) \in R^2$ if and only if $T < 1$, since the solutions have to be 1–periodic.

EXAMPLE 4. Newton's equation, Cauchy problem.
Consider the problem described in the preceding section:

$$\begin{aligned} &\ddot{x}(t) \in -\partial f(x(t)) \quad \text{a.e.} \\ &x(0) = x_0, \qquad \dot{x}(0) = p_0 \end{aligned} \tag{11}$$

The variational inequality (12) characterizing (x, p) with $p = \dot{x}$, is exactly the same as the one in the preceding example; only the boundary conditions have changed ($y(0) = x_0$, $p(0) = p_0$ instead of $y(0) = 0$, $p(0) = p_0$, $p(T) = p_1$). The same arguments leads us to an analogous result:

PROPOSITION 3 (GLOBAL EXISTENCE). *Let the function $f: H \to R$ be convex, continuous, and satisfy the growth condition* (8). *Then problem* (11) *has a solution on the time interval* $[-1/8K, 1/8K]$. *If growth condition* (9) *is satisfied, there is a solution for all times $t \in R$.*/

This can easily be transformed into a local existence result:

PROPOSITION 4 (LOCAL EXISTENCE). *Let the function $f: H \to R \cup \{+\infty\}$ be l.s.c. convex. Let $x_0 \in H$ be a point of continuity for f. Then, for any $p_0 \in H$, problem* (11) *has a solution on some time interval* $[-T, T]$, *with* $T > 0$.

Proof. Since f is l.s.c. convex and continuous at x_0, it is finite and continuous in some neighbourhood $\mathcal{U}$ of x_0. Moreover:

$$\exists M: (\eta \in \partial f(\xi), \eta \in \mathcal{U}) \Rightarrow |\eta| \leq M.$$

We then define a function $g: H \to R$ by the formula:

$$\forall \zeta \in H, \qquad g(\zeta) = \sup\{(\zeta - \xi)\eta + f(\xi) \mid \xi \in \mathcal{U}, \eta \in \partial f(\xi)\}$$

The function g is easily seen to be convex, finite, and to coincide with f on

$\mathcal{U}$, Moreover, it is lipschitzian with constant K:

$$\forall(\xi, \eta) \in H, \qquad |g(\xi)-g(\eta)| \leq K\,|\xi-\eta|$$

so that it certainly satisfies condition (9).

The initial-value problem:

$$\ddot{x}(t) \in -\partial g(x(t)) \quad \text{a.e.}$$
$$x(0)=x_0, \qquad \dot{x}(0)=p_0$$

has a global solution, by Proposition 3. This solution x is also a solution of (11) as long as $x(t) \in \mathcal{U}$. Hence the result./

For sharper results on the Cauchy problem for Newton's equation, we refer to [7]. As for the wave equation, we did not succeed in proving existence by our method.

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