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Abstract

We model the dynamic behavior of spot and futures commodity prices with an infinite horizon rational expectations equilibrium model. A new type of proof of existence of the equilibrium is provided. Using simulations with minimal changes between scenarios, we explore the specific effects of market structure, autocorrelation of production, and global risk aversion. The market structure can change a virtually nonstorable commodity into a high-inventory one. A high autocorrelation soften the apparent effects of storage in the short run. Global risk aversion typically decreases when financialization is developed. The effects on the joint price dynamics, risk sharing and physical choices are explored.

Keywords: Commodities; Futures Prices; Risk Premium; Speculation; Rational Expectations; Infinite Horizon.

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1 Introduction

This paper aims at providing a theoretical perspective on the analysis of commodity prices dynamics: since commodity markets are highly volatile, and since volatility is the subject of many analyses and discussion, there is a need to understand why and how their prices vary over time. Such a task is challenging, due to the existence of a large spectrum of commodities: they can, indeed, be expensive to store, or not; they can be produced continuously, or not; they can be perishable, or not, etc. It is all the more important to reach this objective that, since a few years, a least two possible additional sources of volatility have emerged: the financialization of commodity markets and climate change. Yet many other circumstances are at play and a theory-based tool to test the likelihood and consistency of analytical claims could be useful.

To better understand the dynamics of commodity prices, we develop a micro-founded model for spot and futures prices. This infinite-horizon rational-expectations equilibrium model is based on the interaction of heterogeneous risk-averse agents. The relevant economic functions in commodity markets are represented: processing and storage on the physical market; speculation and hedging (with short and long positions) on the futures market. In addition, random factors in primary production and the final demand are included. Finally, commodities being material besides being reserves of value, non-negativity constraints on inventories limit arbitrage and speculation.

On the theoretical point of view, the model is an extension, in a dynamic setting, of the static model proposed by Ekeland et al. (2019). The flexibility of a static model is enhanced: our model allows to describe a wider range of commodities. This dynamic model confirms important findings of the static model: qualitatively, most of the comparative statics on the variability of prices or the effect of the markets structure are kept. Given the 'last-period effect' in finite-horizon storage models, this congruence has to be noted.

Yet, the fact that inventories can be prolonged limitless changes the model output in critical ways. Statistical moments found in time series can be investigated for calibration or comparative statics. The cross-sectional differences between commodity markets as well as longitudinal differences can be explained by parameter changes that are interpretable in economic terms. Among the possible explorations, let's mention the structural aspects of the market (such as the number of operators or the characteristics of the storage, processing, hedging and speculations functions) and the behavioral aspects of the operators (such as risk aversion or expectations).

The proof of the existence of the equilibrium is based on a fixed-point argument and is entirely new. This new equilibrium analysis shows that our model is a generalization of Deaton and Laroque (1992) and Chambers and Bailey (1996). Compared to their framework we add: i) risk-aversion; ii) non linear storage costs; iii) the processing function; iv) the futures market. The Appendix provides a long presentation of the model and the proof of existence of an equilibrium.

Thanks to the simulations, the model emphasizes the existing heterogeneity between different commodity markets and the impact of this heterogeneity on the dynamics of the prices. We run three main sets of simulations.

- First we explore how minimally different market structures, in terms of numbers of player of each categories, can dramatically change the equilibrium behavior. We compare the dynamic behavior of prices for three categories of markets: 'Contango' commodity where inventories are prevalent so that the non-negativity constraint almost never operates (e.g. gold or corn), the 'Backwardation' commodity where the inventories are so low that arbitrage operations are very difficult (e.g. electricity), and the 'Intermediate' commodity which stands in the middle of the two others (e.g. crude oil or copper).

- Second we explore the smoothing effect of storage. Essentially, storage is very useful where production is negatively autocorrelated (rare case) or slightly positively correlated (most of the cases). When production is highly autocorrelated, storage loses its thrust. The experiment consist in showing the intensity of this softening as autocorrelation is reinforced.

- The third experiment shows what happens when the global risk aversion in the economy changes. The effects on risk premia and physical positions are shown.

The article is organized as follows: Section 2 reviews the relevant literature. Section 3 describes the economic framework of the model. The optimality and the market clearing conditions are given in Section 4. The equilibrium is defined and its existence proved in Section 5. Simulations are exposed in Section 6. Section 7 concludes.

2 Literature review

Our article is linked to different strands of the economic literature on commodity markets. In term of modeling, our model is at the cross-road of two literatures: the one regarding traditional competitive storage models (Williams and Wright 1991; Deaton and Laroque 1992; Routledge et al. 2000), and the one based on heterogeneous agents interacting in the physicla and financial markets (Hirshleifer 1989a; Vercammen and Doroudian 2014; Baker 2016). The simulations performed with the model are connected to the empirical literature on the behavior of commodity prices. Pure analysis on the impact of speculative analysis contributes to the literature on the financialization of commodity markets.

The rational expectations competitive storage model described by Williams and Wright (1991) is the reference framework for the study of the dynamic behavior of commodity prices. Like theirs, our model has an infinite horizon. In their case the operators, however, are risk neutral: they are competitive storers who optimize their expected utility by choosing an optimal nonlinear storage behavior. Moreover the only asset is the physical commodity traded on a spot market: there is no futures market. The authors emphasize that in their dynamic rational expectations equilibrium, the presence of the non-negativity constraint on the inventories creates a need for recursive methods to solve the equilibrium. We also work in this direction. Finally the authors indicate that most of the time there are no analytical solutions; numerical solutions are needed. In this article, we propose both of them. Deaton and Laroque (1992) is an econometric prowess applying this framework to real spot data. The article emphasizes storage as critical determinant of the price formation process. Consequently, all models on commodity prices should incorporate this feature, with all its complications. Routledge et al. (2000) provide the first rational-expectations competitive-storage models to include futures markets in the analysis. This is an important improvement, since they can analyze the whole term structure of commodity prices. One limitation, however, is that the model features only homogeneous risk-neutral agents. Consequently, hedging and risk premia cannot be addressed.

Hirshleifer (1989b), in his finite horizon model of storage (dynamic programming principle up to maturity T), is the first to model an active futures market. Compared to the previous models, he proposes an explicit modeling of the behavior of the heterogeneous risk-averse traders on this market. By solving for the joint equilibrium in the spot and the futures markets his model allows to study precisely the determinants of the risk premium. This model is an important reference for the present article. It indeed emphasizes that storage and hedging decisions need to be studied in a dynamic framework. More importantly, we adopt a trading structure on the futures market, between different heterogeneous hedgers having a naturally opposite position, which approaches the one used by the author. The articles by Vercammen and Doroudian (2014) and Baker (2016) are also close to ours, because they gather an infinite horizon storage model, heterogeneous risk-averse agents, and an active futures market. The model proposed by Vercammen and Doroudian (2014) is an extension of Routledge et al. (2000) where they add cross-asset risk-averse speculators. As far as the modeling is concerned, we are also close to Baker (2016), a study of the oil market; besides a preference for a more flexible model and a technical innovation (proof of existence), we are less centered than him on the study of the financialization. We focus instead of experiments about the effect of critical parameters.

Through the simulations performed with the model our work is also connected to the empirical literature on the behavior of commodity prices.

First, the analysis of arbitrage operations between the physical and the paper markets for commodity shows that, in the presence of large inventories, when prices are in contango (i.e. the futures price is higher that the spot price), the basis should be stable and bounded by storage costs. On the contrary, in backwardation (when the spot price is higher than the futures prices), the basis should be volatile and unbounded because there are no inventories to perform arbitrage operations. Nevertheless, as already emphasized by Fama and French (1987) in an empirical study on a wide range of commodities, there exists a more complicated link between the volatility of the basis and the storage cost. Our model allows us discuss these issues.

We also contribute to the empirical literature regarding the classification of commodity markets, which most of the time have been tackled through large scale empirical studies. For instance, the descriptive statistics in the paper of Kang et al. (2017) which is focused on the liquidity provision on futures markets give a good overview of the basis and the risk premia for 26 commodities.

Finally, our article is connected to the growing literature on the financialization of commodity markets.[1] There is an important imbalance between the empirical and the theoretical literature on this subject. On the empirical side, there is a significant and still growing literature on different issues linked to the financialization for a large variety of commodities. Brunetti and Büyükçahin (2009), Buyukşahin and Harris (2011), Singleton (2014), Hamilton and Wu (2015) are some references with mixed results for the direct link between Commodity Index Traders (CITs) and prices. Tang and Xiong (2012) show that the correlations between different commodities have increased after 2004. Büyükçahin and Robe (2014a) and Büyükçahin and Robe (2014b) show that the cross-asset correlation has increased after 2008 and link this to the trading of hedge funds. Hamilton and Wu (2014) have shown that the risk premium in oil futures markets has significantly decreased due to the potential hedging pressure from financial investors. On the other hand, the theoretical side of the literature is still scarce. One paper is the one of Basak and Pavlova (2016) to assess the impact of institutional investors (and a benchmarked investment) on commodity markets. Baker (2016) assesses the impact of the entry of households on the futures markets by calibrating his model to the crude oil market. Finally Boons et al. (2014) study the impact of hedging by investors of their commodity risk, on commodity returns. While those three papers study the impact of the introduction of a new agent on the commodity markets, in our model, we answer a different question, which is: how can we explain the huge heterogeneity encountered in commodity markets?

3 The model

Time and the markets. This is an infinite horizon model. The interest rate r and the associated discounting factor $\delta = (1+r)^{-1}$ are assumed to be positive and constant. At each period $t \geq 0$, two markets are opened. On the spot market there are transactions for the immediate delivery of the physical commodity at price p_t. The clearing of this market implies the equality between the total physical supply (the production and the inventories that are released) and the total physical demand for final consumption and for storage. On the futures market, one trades derivatives contracts based on the commodity. A futures contract at date t implies no cashflow at date t. A random cashflow is realized at date $t+1$. It is equal to $(p_{t+1} - f_t)$, where f_t is the futures price at t.[2]

[1] For a more detailed review see Cheng and Xiong (2014).

[2] f_t corresponds to $F_{t,t+1}$, a notation often used in other studies.

The agents. There are four categories of agents. The first three are different types of price-taking risk-averse agents: storers I, processors P and speculators S. The number of agents of a certain type is n_i $(i = I, P, S)$. These three categories of agents maximize their mean-variance objective. More precisely:

- *The storer* maximizes his profit from carrying over the commodity from one period to the next for a quadratic storage cost. The inventories cannot be negative. He has access to the spot market, where he buys and sells his inventories, and to the futures market, where his natural position is short. At each period, he has to choose his optimal level $x^\star$ of inventories and his optimal position $q_I^\star$ on the futures market.

- *The processor* maximizes his profit from transforming the commodity into a final good, using a costly production process. His margin can not be negative. He has to commit himself, one period in advance, to the level $y^\star$ of input to be processed. This rigidity in the processing activity is due to the commercial contracts he has signed, plus the constraints on the organization of the production. The input acquired on the spot market is processed immediately. The processor must chose his optimal position $q_P^\star$ on the futures contract. His natural hedging position is long.

- *The speculator* maximizes his profit from purely financial operations on the futures market only. At each period, he has to choose the optimal position $q_S^\star$ that he will hold until the the next. He does not operate on the physical market, nor on on other financial markets.

- The fourth type of agent is represented by a short-term excess demand function $D(\cdot)$ which depends on the current price only, and on a shock. Since the value of this excess demand is algebraic, it comprises nonstrategic producers.

The information structure and the uncertainty. At each period t the agents know the quantity z_t freely available on the physical market:

$$z_t = \text{Random new production} + \text{Inventories from period } (t-1) - \text{Inputs pre-ordered at } (t-1)$$

Thus, z_t depends on the random 'harvest' ω_t. The ω_t are assumed to follow a first-order Markov process. In the applications, an AR(1) is taken. We work on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration $(\mathscr{F}_t)_{t\geq 0}$ is the one generated by the process ω and enlarged by the $\mathbb{P}$-null sets.

The z_t also depends on predetermined variables: the storage and processing decisions. Based on this information and relying on the expectation and variance of future spot prices, the agents make new choices on the physical market. We consider only the stochastic processes $(z_t, p_t, f_t)_{t\geq 0}$ that are $\mathscr{F}_t$-measurable for all t.

4 Market clearing

In this section we derive the optimal positions of the agents and the equilibrium market by market.

4.1 Optimal positions

We retain mean-variance objectives. A type-i agent maximises:

$$\mathbb{E}_t[\pi_{i,t+1}] - \frac{\alpha_i}{2}\mathbb{V}_t[\pi_{i,t+1}],$$

where $\pi_{i,t+1}$ is the random profit at date $t+1$ and α_i the risk aversion. $\mathbb{E}_t[\cdot]$ and $\mathbb{V}_t[\cdot]$ are respectively the expectation and the variance at date t of the variable.

The speculator S**.** His random profit $\pi_{S,t+1}$ can be written

$$\pi_{S,t+1}(q_{S,t}) := \delta\ (p_{t+1} - f_t)\ q_{S,t},$$

where $q_{S,t}$ is the number of futures contracts bought ($q_{S,t} > 0$) or sold ($q_{S,t} < 0$) at date t. Maximizing the objective with respect to $q_{S,t}$ gives:

$$q^\star_{S,t} = \frac{1}{\delta}\,\frac{\mathbb{E}_t[p_{t+1}] - f_t}{\alpha_S\,\mathbb{V}_t[p_{t+1}]}. \tag{1}$$

The storer I**.** Storage has quadratic costs $\frac{\gamma}{2}x^2$ (a linear part can be added, as we did in the simulation code), so that the random profit of the storer is written:

$$\pi_{I,t+1}(q_{I,t}, x_t) := \delta\ (p_{t+1} - f_t)\ q_{I,t} + (\delta p_{t+1} - p_t)\ x_t - \frac{\gamma}{2}\,x_t^2.$$

Maximizing the expected utility according to the quantities hold on the futures market $q_{I,t}$ and on the physical market x_t gives:

$$x^\star_t = \frac{1}{\gamma}\ \max\{\delta f_t - p_t, 0\}, \tag{2}$$

$$q^\star_{I,t} = \frac{1}{\delta}\,\frac{\mathbb{E}_t[p_{t+1}] - f_t}{\alpha_I\,\mathbb{V}_t[p_{t+1}]} - x^\star_t. \tag{3}$$

The processor P**.** We denote by Q the fixed price of the normalized output, the processing costs being $\frac{\beta}{2}y^2$. We also assume that the processing activity is instantaneous. The random profit of the processor becomes:

$$\pi_{P,t+1}(q_{P,t}, y_t) := \delta\ (p_{t+1} - f_t)\ q_{P,t} + \delta\ (Q - p_{t+1})\ y_t - \frac{\beta}{2}\,y_t^2.$$

The optimal positions are:

$$y_t^* = \frac{\delta}{\beta} \max\{Q - f_t, 0\}, \tag{4}$$

$$q_{P,t}^\star = \frac{1}{\delta} \frac{\mathbb{E}_t[p_{t+1}] - f_t}{\alpha_P \, \mathbb{V}_t[p_{t+1}]} + y_t^*. \tag{5}$$

As in Anderson and Danthine (1983a), Anderson and Danthine (1983b), Boons et al. (2014) and Ekeland et al. (2019), the storer and the processor have positions on the futures market that can be decomposed into two components: a hedging one (short for the storer, long for the processor) and a speculative one. Moreover, this speculative component has the same structure as the one of the pure speculator's.

Note also that the equations (1) to (5) express the quantities $(q_{S,t}^\star, q_{I,t}^\star, q_{P,t}^\star, x_t^\star, y_t^\star)$ as functions of four numbers p_t, f_t, $\mathbb{E}_t[p_{t+1}]$ and $\mathbb{V}_t[p_{t+1}]$, i.e. two current prices and two beliefs about a yet unrealized price. Thus, prices and beliefs determine the positions of the operators on all markets. For the sake of simplicity, and without loss of generality, we set $\gamma = 1$ and $\beta = 1$. This is similar to what is done in Ekeland et al. (2019). Indeed, modifying the number of agents is equivalent to changing their costs. This observation relies on the well-known fact that quadratic costs can be perfectly aggregated.

4.2 Spot net demand

The final consumers demand is not inter-temporally substitutable. It responds only to the current price. Moreover, we assume that the lowest possible price is $p_{\min}$: the commodity can be destroyed if the price is too low (free disposal). Finally, the price never goes beyond $p_{\max}$: a substitute can replace the commodity when the price is too high. These three assumptions enable us to represent the *net* demand (henceforth simply the demand) as the following continuous correspondence:

$$D(p) : \begin{cases} \in & [D_{\max}, +\infty) & \text{if} \quad p = p_{\min}, \\ = & d(p) & \text{if} \quad 0 < p < p_{\max}, \\ \in & (-\infty, 0] & \text{if} \quad p = p_{\max}. \end{cases}$$

We assume that $d(p)$ is a continuous decreasing function. To simplify matters, we set $p_{\min} = 0$, $d(0) = D_{\max}$ and $d\left(p_{\max}\right) = 0$. See Figure 1.

4.3 The clearing of the futures market

At t, the clearing implies zero net supply. That is:

$$n_S \, q_{S,t}^\star + n_P \, q_{P,t}^\star + n_I \, q_{I,t}^\star = 0,$$

which gives

$$\mathbb{E}_t[p_{t+1}] - f_t = \alpha \, \delta \, \mathbb{V}_t[p_{t+1}] \; h_t, \tag{6}$$

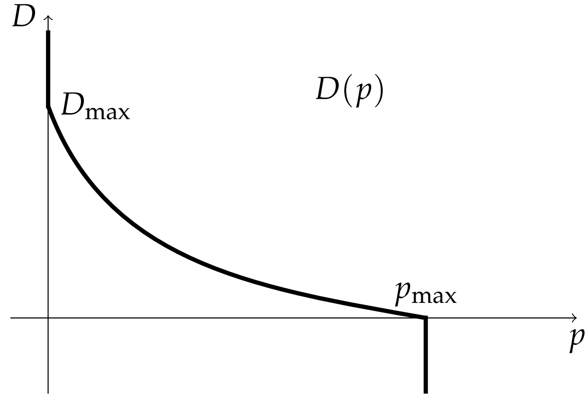


Figure 1: (Net) demand functions

where the constant α represents the risk aversion at the market level:

$$\alpha := \frac{1}{\frac{n_P}{\alpha_P} + \frac{n_I}{\alpha_I} + \frac{n_S}{\alpha_S}},$$

and h_t is the net hedging demand on the futures market, i.e. the hedging pressure:

$$h_t := n_I \max\{\delta f_t - p_t, 0\} - n_P \max\{\delta(Q - f_t), 0\}.$$

Equation (6) is known as the risk premium and provides the relation between the expected spot price $\mathbb{E}_t[p_{t+1}]$ in t for $t+1$ and the futures price f_t in t. As implied by the theory of normal backwardation of Keynes (1930), a risk premium exists only if: (i) agents are risk-averse ($\alpha \neq 0$); (ii) a risk exists ($\mathbb{V}_t[p_{t+1}] \neq 0$); (iii) physical hedgers want to hedge; (iv) and there is an imbalance in the market ($h_t \neq 0$).

4.4 The clearing of the spot market

At t, once the harvest ω_t is observed, the quantity z_t available on the physical market is known:

$$z_t = h_{t-1} + \omega_t. \tag{7}$$

This quantity is fundamental in the model: it serves as the basic state variable with respect to which all the other processes are conditioned. The state variables are z_t only if crops are i.i.d., and (z_t, ω_t) for first-order Markov processes.

Let's turn to the forward view of z_t. The clearing of the spot market at date t implies:

$$z_t \in n_I x_t^\star + D(p_t).$$

Given the definition of the demand, $z_t \leq 0$ is possible. Then, $D(p_t) < 0$, which means that $p = p_{\max}$, and the net demand is in fact a supply, for example of a backstop substitute.

The supply to the physical market is $n_I x_t^\star - z_t$. Taking into account the different aspects of the demand gives:

$$z_t \geq D_{\max} + n_I x_t^\star = D_{\max} + n_I \max\{\delta f_t, 0\} \qquad \text{if } p_t = 0, \tag{8a}$$
$$z_t = D(p_t) + n_I x_t^\star = D(p_t) + n_I \max\{\delta f_t - p_t, 0\} \qquad \text{if } 0 < p_t < p_{\max}, \tag{8b}$$
$$z_t \leq n_I x_t^\star = n_I \max\{\delta f_t - p_{\max}, 0\} \qquad \text{if } p_t = p_{\max}. \tag{8c}$$

In the case of a strict inequality, the gap is bridged either by destruction or substitution, depending on the price.

5 The equilibrium

We define, analyze and characterize the equilibrium before giving a proof of its existence. We search only for a stationary equilibrium. To find a fixed point of a natural operator, we show in two steps how belief-based decisions and rational expectations shape the iterative process.

5.1 Belief-consistent decisions and rational expectations

If two real numbers (e, v) are respectively the price expected in the next period and its variance, then the spot price P and the futures price F verify the following equilibrium equations:

$$\begin{aligned}
&z : \begin{cases} \geq D_{\max} + n_I \max\{\delta F - P, 0\} & \text{if } P = 0, \\ = D(P) + n_I \max\{\delta F - P, 0\} & \text{if } 0 < P < p_{\max}, \\ \leq n_I \max\{\delta F - p_{\max}, 0\} & \text{if } P = p_{\max}, \end{cases} \\
&F = e - \alpha \delta v H, \\
&H = n_I \max\{\delta F - P, 0\} - n_P \max\{\delta(Q - F), 0\}.
\end{aligned}$$

We call *belief-consistent decisions* the functions $P(z, e, v)$ and $F(z, e, v)$ which, to $(z, e, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, associate a solution (P, F) of the equilibrium equations. We also define a consistent hedging pressure:

$$H(z, e, v) := n_I \max\{\delta F(z, e, v) - P(z, e, v), 0\} - n_P \max\{\delta(Q - F(z, e, v)), 0\}.$$

Let us define the mapping Λ :

$$\Lambda : \begin{pmatrix} P \\ F \end{pmatrix} \to \begin{pmatrix} D(P) + n_I \max\{\delta F - P, 0\} \\ F + \delta \alpha v \left[n_I \max\{\delta F - P, 0\} - n_P \max\{\delta(Q - F), 0\} \right] \end{pmatrix}.$$

Clearly, consistent decisions solve the nonlinear system

$$\Lambda \begin{pmatrix} P \\ F \end{pmatrix} = \begin{pmatrix} z \\ e \end{pmatrix}.$$

The equilibrium regimes. In order to analyze the equilibrium, we compute $P(z,e,v)$ and $F(z,e,v)$. As in Ekeland et al. (2019), we start from the space (P,F). In this space, for a given v, and because of the nonlinearity of the equilibrium equations, we consider 6 regions:

Region 1. $0 \le P \le p_{\max}$, $\delta F > P$ and $Q > F$. In this region, all agents are active.

Region 2. $0 \le P \le p_{\max}$, $\delta F > P$ and $Q < F$. In this region the processors are not active.

Region 3. $0 \le P \le p_{\max}$, $\delta F < P$ and $Q < F$. In this region, there is no activity on the physical and the futures markets.

Region 4. $0 \le P \le p_{\max}$, $\delta F < P$ and $Q > F$. In this region the storers do not have an incentive to operate.

Region 5. $P \le p_{\min} = 0$.

Region 6. $P \ge p_{\max}$.

These regions in the space (P,F) are depicted by Figure 2. To have an interesting economy where Region 3 is nonempty, we assume that:

$$p_{\min} = 0 \le \delta\, Q \le p_{\max}.$$

The intersection of the first four regions is the point $M = (\delta Q, Q)$. The other characteristic points have obvious interpretations. The coordinates of these points and of their images by Λ are reported in Table 1. We denote by $\mathscr{R}_i$, with $i = 1$ to 6, the images by Λ of the regions in the space (e,z). For example, the regions $\mathscr{R}_1$ to $\mathscr{R}_4$ are delimited by four half-lines emanating from the point $\Lambda(M)$. The regions in (e,z) are depicted by Figure 3. Note that they are obtained for a given v. The details of the computations are given in the Appendix A.1.

$O = (0,0)$	$\Lambda(O) = (D_{\max}, -n_P\alpha\delta^2 vQ)$
$A = (0,Q)$	$\Lambda(A) = \left(D_{\max} + n_I\delta Q, (1+n_I\alpha\delta^2 v)Q\right)$
$B = \left(p_{\max}, \frac{p_{\max}}{\delta}\right)$	$\Lambda(B) = \left(0, \frac{p_{\max}}{\delta}\right)$
$C = (p_{\max}, Q)$	$\Lambda(C) = (0,Q)$
$M = (\delta Q, Q)$	$\Lambda(M) = (D(\delta Q), Q)$

Table 1: Characteristic points

In each of the regions in the space (e,z), we can compute the values of $P(z,e,v)$ and $F(z,e,v)$. For example, in region 1 there are storage as well as processing activities. Thus, the equilibrium equations can be written:

$$z = D(P) + n_I\,(\delta F - P)\,, \tag{9}$$
$$e = F + \alpha\delta v\left[n_I\,(\delta F - P) - \delta n_P(Q - F)\right]. \tag{10}$$

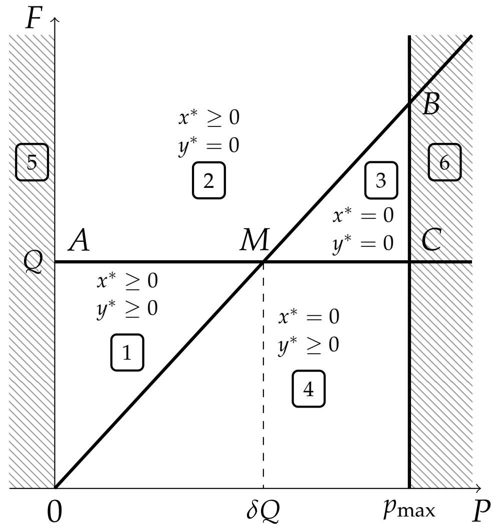


Figure 2: Regions in the space (P, F)

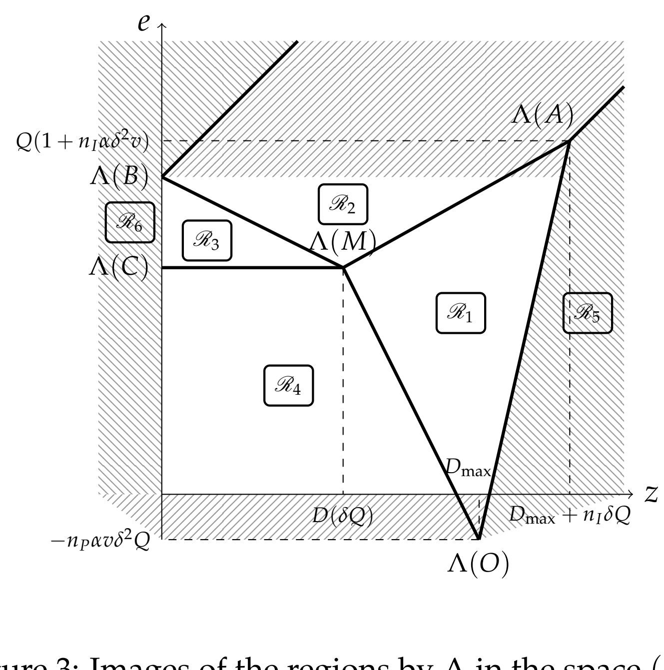


Figure 3: Images of the regions by Λ in the space (z, e)

Solving Equation (10) for F gives:

$$F = \frac{e + \alpha\delta v(n_I P + n_P \delta Q)}{1 + (n_I + n_P)\alpha\delta^2 v}. \tag{11}$$

Plugging this into Equation (9) gives:

$$z = D(P) - \frac{n_I(1 + n_P\alpha\delta^2 v)}{1 + (n_I + n_P)\alpha\delta^2 v}P + \frac{n_I\delta}{1 + (n_I + n_P)\alpha\delta^2 v}e + \frac{n_I n_P \alpha\delta^3 Q v}{1 + (n_I + n_P)\alpha\delta^2 v}. \tag{12}$$

The right-hand side of Equation (12) is strictly decreasing with respect to P, meaning that there is a unique $P(z, e, v)$. Plugging this expression into the Equation (11) gives a unique $F(z, e, v)$. The values of $P(z, e, v)$, $F(z, e, v)$ and $H(z, e, v)$ in the case where $D(\cdot)$ is piecewise linear can be found in the Appendix A.2.

Rational expectations. We say that $E(z, \omega)$ and $V(z, \omega)$, with $V \geq 0$, are rational expectations if we have:

$$E(z,\omega) = \mathbb{E}\left[P(h(z,\omega) + \omega_\bullet, E(h(z,\omega) + \omega_\bullet, \omega_\bullet), V(h(z,\omega) + \omega_\bullet, \omega_\bullet)) \mid z, \omega\right], \tag{13}$$
$$V(z,\omega) = \mathbb{V}\left[P(h(z,\omega) + \omega_\bullet, E(h(z,\omega) + \omega_\bullet, \omega_\bullet), V(h(z,\omega) + \omega_\bullet, \omega_\bullet)) \mid z, \omega\right]. \tag{14}$$

where $\omega_\bullet$ is the successor of ω in the process, and

$$h(z,\omega) := H(z, E(z,\omega), V(z,\omega)). \tag{15}$$

Similarly, we define the functions $p(z, w)$ and $f(z, w)$:

$$p(z,\omega) := P(z, E(z,\omega), V(z,\omega)), \tag{16}$$
$$f(z,\omega) := F(z, E(z,\omega), V(z,\omega)). \tag{17}$$

5.2 Existence of the equilibrium: the fixed point theorem

We say that $(z_t, \omega_t, p_t, f_t)_{t\geq 0}$ is an *equilibrium process* if all markets clear at any time. More precisely, the markets have to verify the equilibrium equations (7)–(6). An equilibrium process is such that for all t, $p_t = p(z_t, \omega_t)$ and $f_t = f(z_t, \omega_t)$ for two real functions p and f, is a *stationary equilibrium process*. We search only for a stationary equilibrium. In that case, the expectations $\mathbb{E}_t[p_{t+1}]$ and $\mathbb{V}_t[p_{t+1}]$ also depend only on (z_t, ω_t), not on time. To characterize a stationary equilibrium, we proceed with two perspectives in order to find a fixed point.

Definition 1. *The functions* (p, f, E, V) *support a* stationary rational expectations equilibrium *if and only if they verify the equations* (13)–(17).

Equipped with (p, f, E, V), we can compute the processes of interest, for example:

$$\begin{aligned} p_t &= p(z_t, \omega_t), \\ f_t &= f(z_t, \omega_t), \\ z_{t+1} &= h(z_t, \omega_t) + \omega_{t+1}. \end{aligned}$$

Assume that the process can be represented by the conditional density $\varphi(\omega_\bullet|\omega)$, where $\omega_\bullet$ is the successor of ω. The fonction $\varphi : \mathbb{R} \times \mathbb{R} \to [0, \infty]$ is assumed to be C^1 jointly with respect to $(\omega, \omega_\bullet)$. In addition, we assume that there is some k_φ such that:

$$\int \partial_\omega \varphi(\omega_\bullet|\omega)\, d\omega_\bullet \leq k_\varphi, \quad \forall \omega. \tag{18}$$

Note that this is satisfied if φ is k_φ-Lipschitz. We define:

$$\mu_\varphi = \sup_\omega \mathbb{E}\left[\left(z_{\min}(\omega), z_{\max}(\omega)\right)|\omega\right]. \tag{19}$$

and we assume:

$$\mu_\varphi(\omega) < 1. \tag{20}$$

These properties are satisfied with many usual distributions, in particular normal and those with finite support, provided they are absolutely continuous (in particular, they should be non-atomic). We shall refer to k_φ and μ_φ as measures of the statistical *concentration* of harvests, and to $(n_I, n_P, n_S, \alpha, \delta, Q, D(p))$ as the *market data*.

The following result shows that if the law is not too concentrated, a stationary equilibrium will exist.

Theorem 1. *Assume the market data are given. Then there are $\bar{\mu} > 0$, $\bar{k} > 0$ such that, if φ satisfies conditions (18), (19), (20) with $\mu_\varphi \leq \bar{\mu}$ and $k_\varphi \leq \bar{k}$, the market has a stationary equilibrium $(\bar{p}, \bar{f}, \bar{E}, \bar{V})$ where $\bar{p}$, $\bar{f}$, $\bar{E}$ and $\bar{V}$ are Lipschitz functions of (z, ω).*

The proof in Appendix B is constructive. We shall compute explicitly $\bar{\mu}$ and $\bar{k}$ (although we shall not attempt to get the best estimates) and shall define an operator $\Gamma \circ \Psi$ such that the iterated sequence $(p_{n+1}, h_{n+1}) = \Gamma \circ \Psi(p_n, h_n)$ converges uniformly to $(\bar{p}, \bar{h})$. The functions $\bar{f}$, $\bar{E}$ and $\bar{V}$ are directly calculated from these limits. This is quite interesting from the algorithmic and economic viewpoints. On the one hand, the iteration is easy to implement and converges geometrically to the solution. On the other, the agents (storers, processors, and speculators) can reach the equilibrium by trial and error.

Note that in the case where $\omega_{t+1} = \rho\omega_t + \varepsilon_t$, with the ε_t i.i.d. and Gaussian, the conditions on φ will be achieved provided the variance of ε_t is large enough.

6 Simulations for contrasted commodities

The simulations presented in this section test the ability of the model at capturing and explaining the huge variability encountered in commodity markets. This heterogeneity has been largely documented by the empirical literature and there are many possible markers for it: the sign and the volatility of the basis; the sign, level and volatility of the hedging pressure; the sign, level and volatility of the risk premium; the quantities stored and the commitments taken on the physical market; the level of the autocorrelation in prices and returns, etc.

This heterogeneity has two dimensions. First, there is huge cross-sectional variation: if we take the sign of the basis as an example, some commodities are most of the time in contango, while others are most often in backwardation. Second, there is a longitudinal instability in price behavior: some markets are sometimes in backwardation, sometimes in contango, with various degrees of persistence of a temporary state. To grasp such a heterogeneity we rely on the equilibrium depicted by Figure 3, where the four regions depict various situations. For example, Region 1 is characterized by contango, and Region 4 by backwardation (among other markers). Our questions become: why is a commodity market situated in one region rather than in another one? Why would a market move from one region to the other?

6.1 Calibration

Our model is very flexible and contains 15 parameters. We thus prioritize the regions to study and the type of behaviors to focus on.

Regions 1 and 4 are the most important. In Region 1, all operators are active; prices are in contango and the premium can be positive or negative. In Region 4, prices are in backwardation. Region 2 is qualitatively a continuation of Region 1; and the industrial operators do not have any incentive to operate in Region 3. Looking at what happens in Regions 2 and 3 is an option we don't expose. It suffices, for example, to decrease the price of the output Q, which reduces the activity of the processors.

Table 2 exhibits the reference values retained across simulations presented in this paper. All the 'external parameters' of the model, situated on the top, are fixed: the discount factor δ, the parameters describing the linear demand (M and m) and the law of the production ω_t. Bellow are presented the values retained for the parameters describing the two industrial activities : storage and processing. Then, the structure of the market, depicted by the number of agents n_i belonging to each category. And finally, risk aversion α_i.

We set the quadratic storage costs (γ) and the quadratic processing costs (β) equal to one: according to the remark made in the end of paragraph 4.1, modifying the number of agents is an alternative and equivalent way to changing their aggregate cost. Finally, for a preliminary set of simulations, we set the autocorrelation coefficient ρ to zero. This parameter demands a focused study that is presented in Subsection 6.3.3.

The parameter α aggregates risk aversion at the market level. It embeds six others representing, for each category of agents, the specific risk aversions α_i, and the number of operators of every type n_i. Therefore changing n_P, n_I and n_S has an impact on α that

Parameters	**Notation**	**Contango** (Region 1)	**Intermediate** (Regions 1/4)	**Backwardation** (Region 4)
Discount factor	δ	1/1.01	1/1.01	1/1.01
Demand: $D(p) = M - mp$				
Maximum demand D_{max}	M	100	100	100
Price elasticity of the demand	m	0.5	0.5	0.5
Maximum price p_{max}	M/m	200	200	200
Production ω:				
Mean of the Normal Law	μ	60	60	60
Variance of the Normal Law	σ^2	10	10	10
Autocorrelation coefficient	ρ	0	0	0
Total variance	$\sigma^2/(1-\rho^2)$	10	10	10
Storage and processing:				
Quadratic storage costs	γ	1	1	1
Quadratic processing costs	β	1	1	1
Price of the output	Q	150	150	150
Number of processors (P)	n_P	0.01	0.01	1
Number of storers (I)	n_I	60	20	10
Number of speculators (S)	n_S	1	1	1
Risk aversion of a processor (P)	α_P	2	2	2
Risk aversion of a storer (I)	α_I	2	2	2
Risk aversion of a speculator (S)	α_S	2	2	2
Risk tolerance, all processors (P)	n_P/α_P	0.005	0.005	0.5
Risk tolerance, all storers (I)	n_I/α_I	30	10	5
Risk tolerance, all speculators (S)	n_S/α_S	0.5	0.5	0.5
Global risk aversion	$\alpha = (\sum_i n_i/\alpha_i)^{-1}$	0.033	0.095	0.166

This table gives the reference values of the model's parameters, for three main categories of commodities: 'Contango' represents markets that are most of the time in contango, 'Backwardation' illustrates situations where backwardation is frequent, and 'Intermediate' is in between. We assume that the demand function is linear.

Table 2: Reference values of the parameters, for three main categories of commodities.

cannot be neutralized without further assumptions. Yet we have limited the range of α to focus on the intensity of activity by storers and processors (and thus their hedging behavior). Risk aversion indeed have second order effects in this experiment.

Focusing on Regions 1 and 4 allows us to target three main categories of commodities. The first category is representative of all markets that are most of the time in contango and situated mainly in Region 1 – possible candidates could be gold or corn. The second category stands for the markets with significant frequencies for both positive and negative carrying charges, and between Regions 1 and 4 – it could be represented by crude oil, or copper. The third category represents the markets where backwardation prevails; it is expected mainly in Region 4 – possible candidates could be electricity or live cattle. In what follows, for the convenience of the exposition, we will label this types of commodities as 'Contango', 'Intermediate', and 'Backwardation'.

6.2 Numerical solution

The algorithm's search for a fixed point follows the same steps as the proof of existence. We choose a finite grid in the space (z, ω) over which all relevant functions are defined and we use linear interpolation when necessary.

1. We start by a guess of the expectation $\hat{E}(z, \omega)$ and the variance $\hat{V}(z, \omega)$.

2. We compute estimations for the spot price $\hat{P}(z, \omega) = P(z, \hat{E}(z, \omega), \hat{V}(z, \omega))$, the futures prices $\hat{F}(z, \omega) = F(z, \hat{E}(z, \omega), \hat{V}(z, \omega))$ and the hedging pressure $\hat{H}(z, \omega) = H(z, \hat{E}(z, \omega), \hat{V}(z, \omega))$, where P, F and H are known exact functions.

3. We obtain Monte-Carlo estimates for the expectation and variance based on

$$\hat{E}(z, \omega) = \mathbb{E}\left[\hat{P}\big(\hat{H}(z, \omega) + \omega_\bullet, \omega_\bullet\big) \mid z, \omega\right], \tag{21}$$
$$\hat{V}(z, \omega) = \mathbb{V}\left[\hat{P}\big(\hat{H}(z, \omega) + \omega_\bullet, \omega_\bullet\big) \mid z, \omega\right]. \tag{22}$$

   where $\omega_\bullet$ is the successor of ω in the process.

4. We iterate and stop when two successive estimates are close enough, according to the uniform norm.

5. We simulate the shock process to obtain trajectories of any length of all variables of interest.

All is implemented with Python 3.

6.3 Numerical experiments and relevance of the model

An important step of the simulations is to check for the relevance of the model. We focus on the storage function and ask whether or not the relative number of industrial hedgers

Indicator	Contango ($R1$) $n_I = 60$	Intermediate ($R1, R4$) $n_I = 20$	Backwardation ($R4$) $n_I = 10$
Bases			
Percentage of contango	84.6	77.2	23.8
Volatility in contango	0.12	0.17	0.36
Percentage of backwardation	15.4	22.8	76.2
Volatility in backwardation	3.16	3.35	3.38
Hedging pressure and risk premium			
Frequency of positive values	71.5%	47.3%	0%
Volatility of the hedging pressure	2.45	1.13	0.97
Volatility of the risk premium	0.32	1.81	0.09
Autocorrelations			
Spot prices, lag 1	0.4	0.27	0.1
Spot returns, lag 1	−0.3	−0.37	−0.45

This table gathers the results obtained for the three main categories of commodities: the first represents markets that are most of the time in contango, the third illustrates situations where backwardation is frequent, and the second represents intermediate situations. The parameters are those depicted in Table 2. The grid for (z, ω) is 100×100. The Monte-Carlo integration of expected values is based on 5,000 draws. The trajectories have 1,000 periods. The tolerance level for the convergence test is $\sigma/4$.

Table 3: Bases, autocorrelations, hedging pressure and risk premia for the three categories of commodities.

is a way to recover the three targeted market categories[3]. We then look at the impact of the speculation, and finally the fundamental structures of the economy, through the autocorrelation coefficient of the production.

6.3.1 The three categories of commodity markets

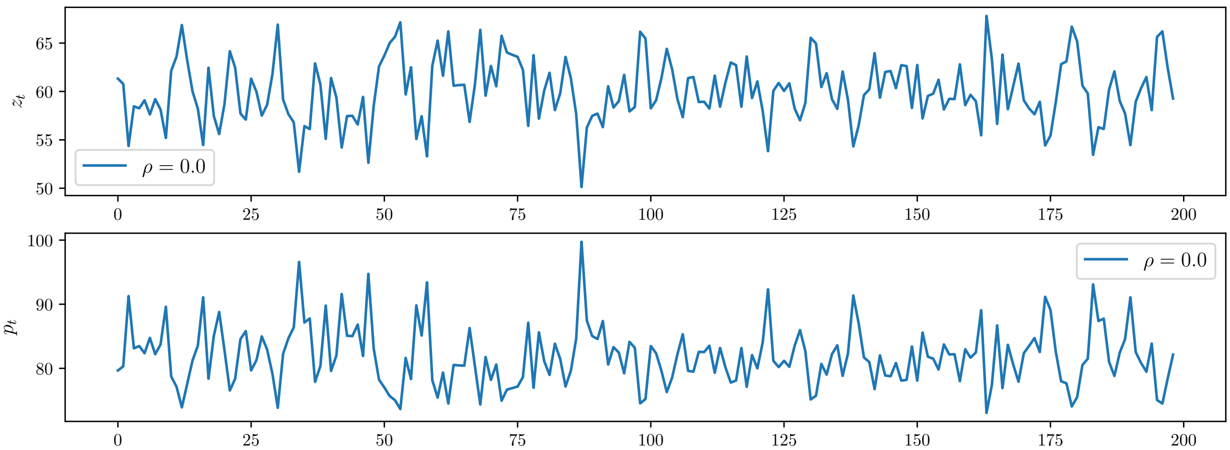
We first look at the ability of the model at representing correctly the three market categories. We retain a large number of indicators: the sign and volatility of the bases, the autocorrelations in spot prices and spot returns, the sign and the volatility of the hedging pressure and the risk premium. An important starting point is that high spot prices come with low available quantities z_t on the physical market, as depicted by Fig 4.

Second point, the simulations give rise to contrasted and relevant results on the dynamic behavior of the bases and prices, and of the quantities recorded on the futures and physical markets. The 'Contango' market is expected to exhibit a majority of positive bases and to be situated mainly in Region 1, which is the case, as depicted by Table 3. The

[3]Another preliminary question is about the possibility of an economy to reach all the regions over time. This has been verified. For the sake of simplicity we do not expose the results on this point.

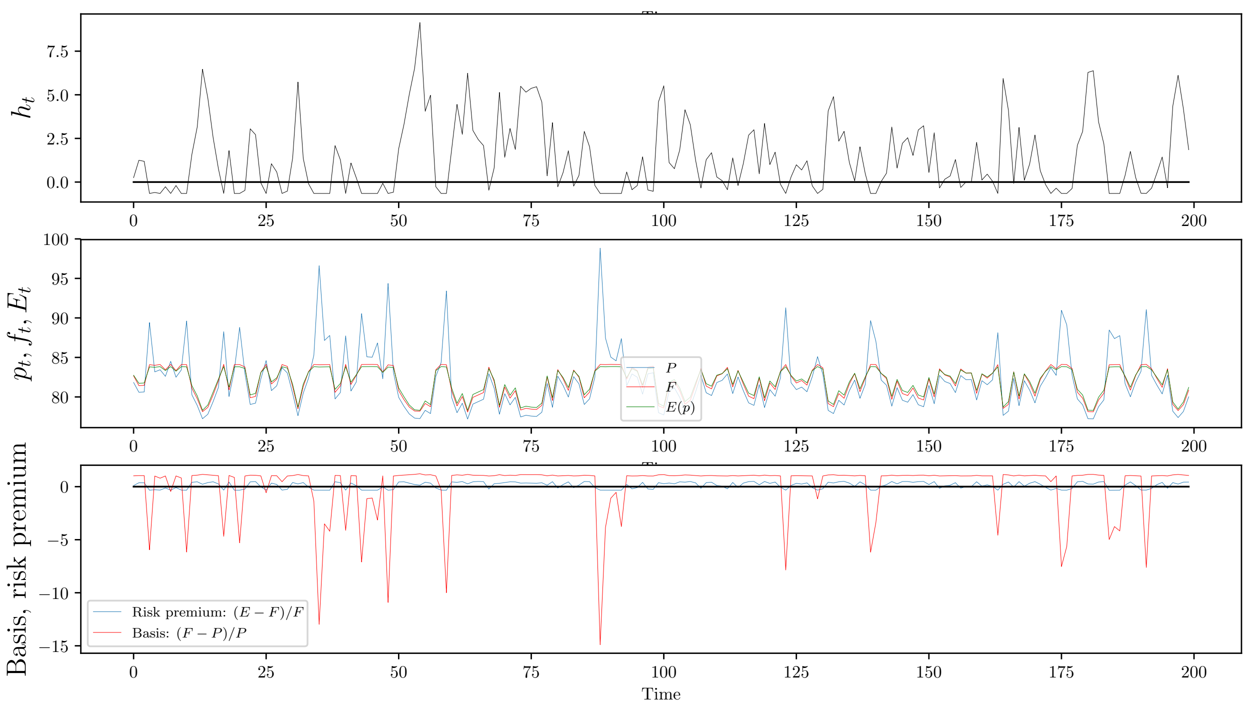
'Intermediate' market is 77% of the time in contango, in Region 1 and 23% in backwardation (Region 4). Finally the 'Backwardation' category is in Region 4 in 76% of the cases. Moreover, in each cases, the volatility of the basis is higher in backwardation than in contango, which reflects the difficulty to undertake arbitrage operations in backwardation.

The Contago scenario is illustrated by Figure 5 and the second column of Table 3, representing the prices, hedging pressure, basis $(f_t - p_t)/p_t$ and risk premium $(E_t - f_t)/f_t$. As the inventories are generally substantial, the hedging pressure is positive: the storers are the dominant hedgers, and they have short hedging positions. Consistently with Equation (6), the risk premium has the same sign. Finally, the futures prices do not exhibit peaks, contrary to the spot prices. In other words, physical arbitrage via storage is visibly active. The basis is stable and limited in contango, when futures prices are higher than the spot price, whereas it can peak when the market is in backwardation. This corresponds to the classical set of predictions of the storage theory and is comforted by the fact that in contango, the basis is less volatile than in backwardation. Finally, the comparison between the three cases depicted by Table 3 shows that the realized autocorrelation of the spot price is bigger when the inventories are higher. This comforts a very common view: inventories and the behavior of the operators on the physical markets are responsible for the autocorrelation in the spot prices (see for example Deaton and Laroque 1992; Bessembinder et al. 1996).



These graphs plot times series for the Intermediate case, synthesized in the third column of Table 3. Only 200 periods over 1,000 are plotted. The quantities available on the physical market, z_t, are on the top. Below are the spot prices p_t.

Figure 4: Physical quantities z and spot prices p in the Intermediate case



These graphs illustrate the different results obtained for the 'Contango' case, synthesized in the second column of Table 3. Only 200 periods over 1,000 are reproduced. The hedging pressure h_t is situated on the top ; in the middle, p_t stands for the spot prices, f_t for the futures prices and E_t for the expected spot prices; the basis $(f_t - p_t)/p_t$ is represented in the bottom, with the risk premium $(E_t - f_t)/f_t$.

Figure 5: Hedging pressure, prices, basis and risk premium in the Contango case

6.3.2 The impact of speculation

The analysis of the impact of speculation further confirms the global relevance of the model. Table 4 shows what happens, in the Intermediate case, when the number of speculators rises in the market, from 1 to 50. A first remark is that the market risk aversion decreases. Further, the volatility of the risk premium decreases; this results from an increased competition among speculators. The level of the risk premium also decreases, as illustrated by Figure 6. Another intuitive result (that we do not reproduce here), is that the speculative activity encourages the building of inventories and has a positive impact on the level of the hedging positions.

Finally, we can also observe a rise in the percentages of contango, which means that the increase in the speculative activity moves the market deeper into Region 1. Besides the relative position of the industrial hedgers, the financial activity influences the position of a market. This is all the more important that, contrary to the number of industrial operators, which describes a fundamental economic structure of the market, the number of speculators can change quickly.

6.3.3 Autocorrelation in, autocorrelation out

In what follows, we explore the 'Intermediate' case when the autocorrelation coefficient of the production, ρ, varies. This parameter indeed describes one of the market's fundamental structures: it represents the rigidity in the production process that characterizes many commodity markets. Economic intuition suggests that if production is highly correlated, low prices tend to follow low prices and storage is globally less profitable.

Table 5 shows what happens when ρ increases from zero (its reference value in Table 2) to 0.95. Other things being equal, the frequency of the contango situation increases, from 77.2 to 98.2%. A market tends to be on the upper zone of Region 1 when the autocorrelation in the production is high. This evolution is very clearly illustrated by Figure 7, that depicts the basis for different values of ρ: backwardation becomes less pronounced when ρ increases.

The behavior of the risk premium depicted by Figure 8 further witnesses that the position of a commodity market on the map is influenced by the level of production's autocorrelation: the risk premium can indeed be positive or negative, according to the level of ρ. Thus the market will be situated in the Upper Region 1 or the Lower Region 1 (or even in Region 4), depending on this correlation parameter.

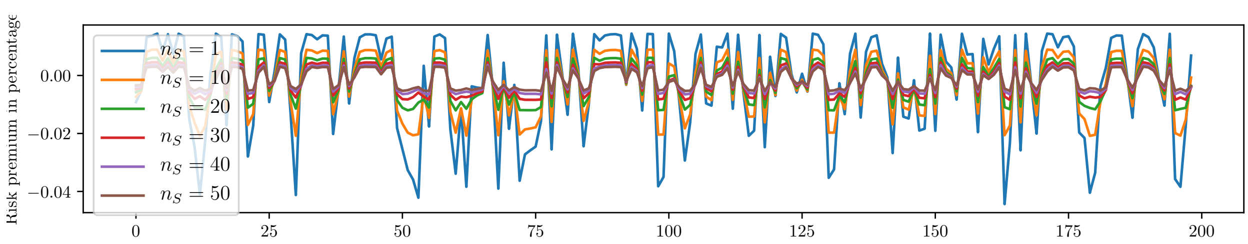
Figure 9 shows that, besides the position on the map represented by Figure 3, the autocorrelation has also an influence on the dynamics of the spot, the futures and the expected prices. While the spot prices seems to be a bit less volatile when the autocorrelation increases, the contrary is true for the futures and expected prices.

The most interesting point of Table 5 is that the autocorrelation in the spot prices, induced by the inventories, is reinforced by the autocorrelation in the production (from 0.27 to 0.95). Even if such result is intuitive, to the best of our knowledge, it has never

Intermediate case $n_I = 20$	$n_S = 1$	$n_S = 10$	$n_S = 20$	$n_S = 30$	$n_S = 40$	$n_S = 50$
Total risk aversion	0.095	0.067	0.05	0.04	0.033	0.028
Bases						
Percentage of bases in contango	77.2	78.9	80.5	82.2	83	83.1
Volatility of the basis in contango	0.17	0.17	0.17	0.19	0.18	0.18
Percentage of bases in backwardation	22.8	21.1	19.5	17.8	17	16.9
Volatility of the basis in backwardation	3.35	3.36	3.34	3.26	3.24	3.24
Hedging pressure and risk premium						
Frequency of positive signs	52.7%	59.1%	64.2%	66.9%	68.9%	69.8%
Volatility of the hedging pressure	1.13	1.51	1.82	1.87	2.15	2.18
Volatility of the risk premium	1.81	1.04	0.65	0.47	0.37	0.32
Autocorrelations						
Spot prices at lag 1	0.3	0.35	0.38	0.4	0.4	0.4
Spot returns at lag 1	-0.37	-0.35	-0.3	-0.3	-0.3	-0.3

This table gathers the results obtained for the intermediate case, when different levels of speculation are taken into account. All prices, for all simulations are either in R1 or in R4. The grid for (z, ω) is 100×100. The Monte-Carlo integration of expected values is based on 5,000 draws. The trajectories have 1,000 periods. The tolerance level for the convergence test is $\sigma/4$.

Table 4: Intermediate case, with different levels of speculation (n_S)



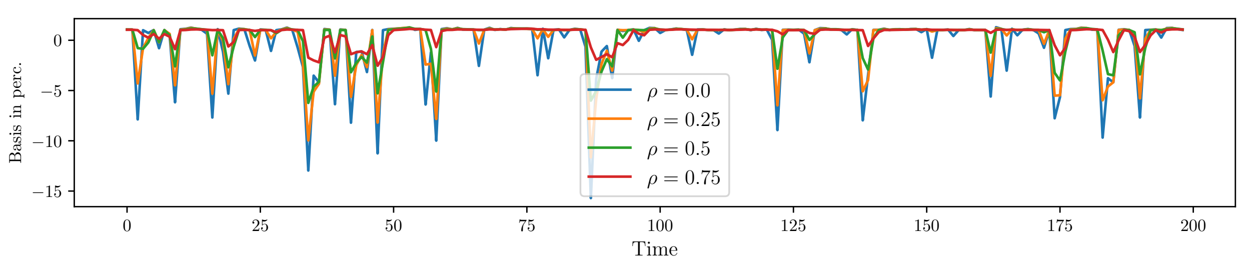
This graphic plots time series of risk premiums in percentage, for the Intermediate case, synthesized in the third column of Table 3, for different levels of speculation n_S. Only 200 periods over 1,000 are plotted.

Figure 6: Risk premium in percentage in the Intermediate case, for different level of speculation.

Intermediate case ($n_I = 20$)	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 0.95$
Variance of the production (ω_t)	10	10	10	10	10
Variance of shock (σ^2)	10	9.375	7.5	4.375	0.975
Bases					
Percentage of contango	77.2	80.6	83.8	89.2	98.2
Volatility in contango	0.17	0.16	0.18	0.2	0.28
Percentage of backwardation	22.8	19.4	16.2	10.8	1.8
Volatility in backwardation	3.35	2.47	1.58	0.65	0.11
Hedging pressure and risk premium					
Frequency of negative values	47.3%	46.6%	48.6%	56.9%	98.8%
Volatility of the hedging pressure	1.13	0.99	0.91	0.75	0.4
Volatility of the risk premium	1.81	1.45	0.87	0.34	0.04
Autocorrelation					
Spot prices at lag 1	0.27	0.42	0.6	0.78	0.95
Spot returns at lag 1	-0.37	-0.25	-0.20	-0.1	-0,05

This table gathers the results obtained for the Intermediate case, synthesized in the second column of Table 3, when different levels of autocorrelation are taken into account, and when the level of the global variance remains the same. All prices, for all simulations are either in R1 or in R4. The second column of this table is identical to the third column of Table 3. The grid for (z, ω) is 100×100. The Monte-Carlo integration of expected values is based on 5,000 draws. The trajectories have 1,000 periods. The tolerance level for the convergence test is $\sigma/4$.

Table 5: Intermediate case, with different levels of autocorrelation ρ in the production, when the global variance of the process remains the same.



This graph represents the bases, in the Intermediate case, when the autocorrelation in the production increases and the global variance is maintained at the same level. Only 200 periods over 1,000 are reproduced.

Figure 7: Basis in the Intermediate case when the autocorrelation increases

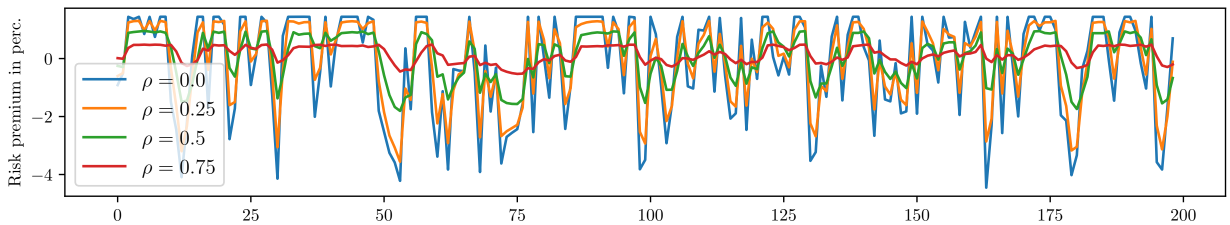
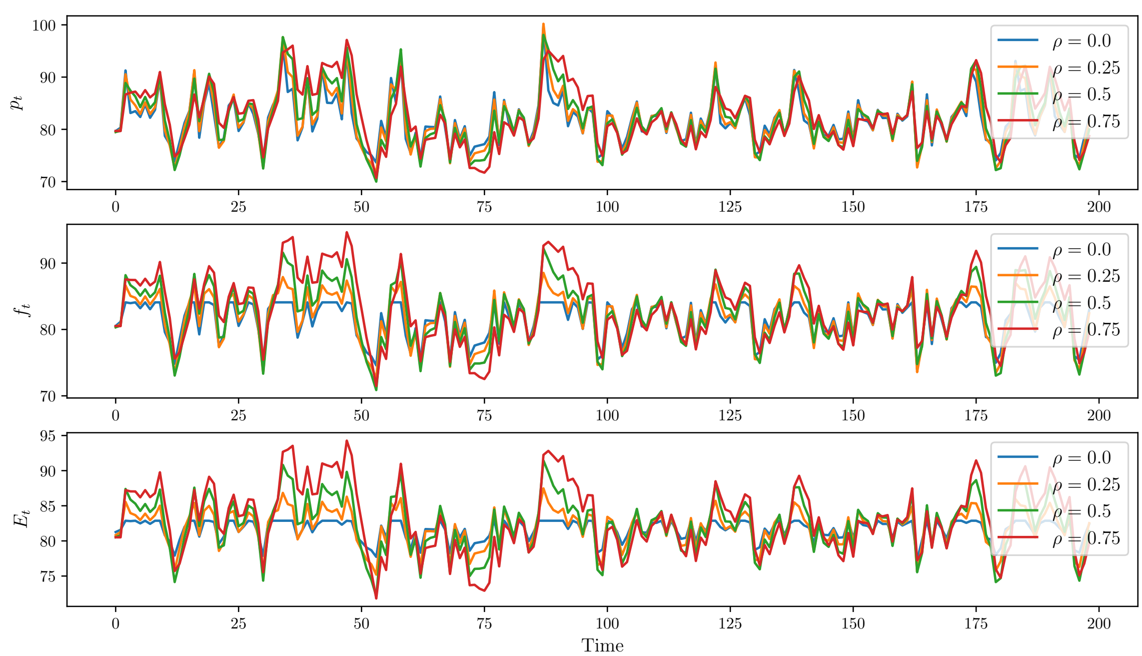


Figure 8: This graphics reproduces the risk premium for different values of the autocorrelation coefficient ρ, in the Intermediate case, when the level of the total variance is maintained at the same level. Only 200 periods over 1,000 are reproduced.



These graphs represent the different results found in the Intermediate case when the autocorrelation in the production increases and the global variance is maintained at the same level. Only 200 periods over 1,000 are reproduced. The spot prices p_t are situated on the top. Below are the futures prices f_t and the expected spot prices E_t.

Figure 9: Spot, futures and expected spot prices when the autocorrelation increases

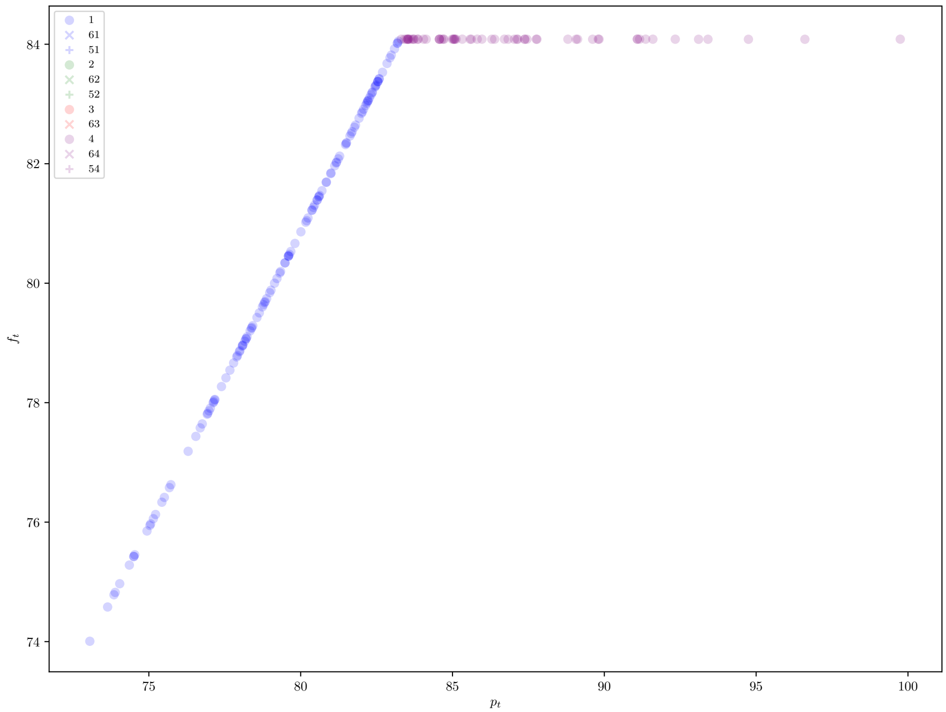


Figure 10: This graphics plots the futures prices against the spot prices for the Intermediate case, synthesized in the second column of Table 3, when ρ is equal to zero. The blue points are those situated in Region 1, the purple ones are situated in Region 4.

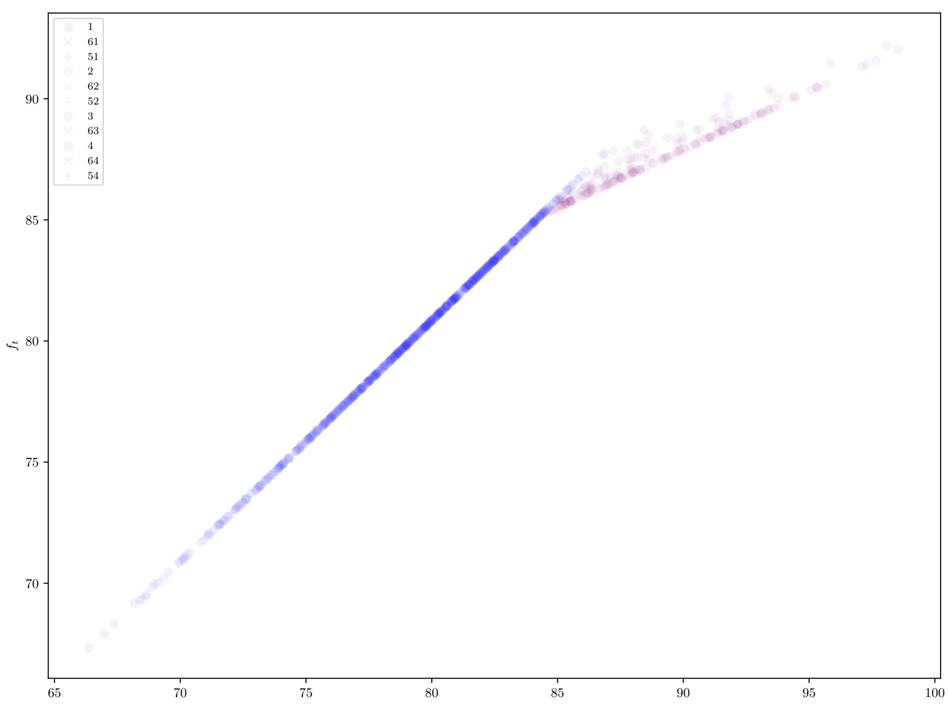


Figure 11: This graphics plots the futures prices against the spot prices for the Intermediate case, when ρ is equal to 0.5. The blue points are those situated in Region 1, the purple ones are situated in Region 4.

been taken into account in a structural model. Nor have its consequences been thoroughly investigated.

It is very common, in commodity markets, to stress that through arbitrage operations, inventories insure the existence of a relation between the physical and the paper markets and consequently, between the present and the future. Figure 10, a scatter plot of the futures prices against the spot prices, illustrates that point: it shows that in Region 1 (blue points), the futures prices rise with the spot prices, when the market is in contango. This parallelism in the behavior of the prices is a well-known phenomenon in commodity markets and is explained by arbitrage operations. On the contrary, in Region 4 (purple points), the futures prices do not depend anymore on the spot price, since the inventories that would be necessary for arbitrage operations are not available.

This result, however, is obtained with a coefficient of production's autocorrelation that is equal to zero. Figure 11, that illustrates the relation between the spot and futures prices when the autocorrelation is equal to 0.5, gives a very different picture. In this case, indeed, a strong positive relation between the two prices remains, even when there are no inventories.[4] In other words, the production replaces the inventories. The usual distinction between storable and non storable commodities is not be so important when there is a high rigidity in the production process.

7 Conclusion

Our infinite horizon rational expectations equilibrium model explains the interaction, in a dynamic setting, between spot and futures markets for commodities. In equilibrium, this model is able to reproduce the dynamic behavior of spot and futures prices for a wide range of commodities including non-storable ones like electricity. We can obtain a contango or a backwardation, the futures prices can be higher or lower than the expected spot prices, inventories can be held or not, the commodity can be processed or not, and adding speculators decreases the risk premiums, while encouraging the building of inventories on the physical market. This variety of situations is found in real commodity markets. Moreover, the analysis of the autocorrelation in the prices shows that the usual distinction between storable and non storable commodities is not so important when one take into account the role of the production process. In other words, when stocks are rare, production replaces inventories.

[4]When the autocorrelation is equal to 0.95, there is no more distinctions between the Regions 1 and 4, as far as the relationship between the spot and futures prices is concerned. This is however quite an extreme value. We thus focus on the intermediate value $\rho = 0.5$.

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A Appendix: Equilibrium

A.1 The images of the regions at the equilibrium

We assume that:

$$\begin{aligned} p_{\min} = 0 \le \delta Q &\le p_{\max}, \\ 0 \le D(\delta Q) &\le D_{\max}. \end{aligned}$$

- $\mathscr{R}_1$ and $\mathscr{R}_2$ are separated by the arc $\mathscr{D}_{12}$, the image of the segment $\overline{AM}$, for which $F = Q$, and $P < \delta Q$:

$$\mathscr{D}_{12} = \left\{ \begin{pmatrix} D(t) + n_I\,(\delta Q - t) \\ Q + n_I \alpha \delta v\,(\delta Q - t) \end{pmatrix} \mid 0 < t < \delta Q \right\}.$$

- $\mathscr{R}_2$ and $\mathscr{R}_3$ are separated by $\mathscr{D}_{23}$, the image of the segment $\overline{MB}$, for which $\delta F = P$, with $F > Q$:

$$\mathscr{D}_{23} = \left\{ \begin{pmatrix} D(t) \\ \frac{t}{\delta} \end{pmatrix} \mid \delta Q < t < p_{\max} \right\}.$$

- $\mathscr{R}_3$ and $\mathscr{R}_4$ are separated by $\mathscr{D}_{34}$, the image of the segment $\overline{MC}$, for which $F = Q$, with $P > \delta Q$:

$$\mathscr{D}_{34} = \left\{ \begin{pmatrix} D(t) \\ Q \end{pmatrix} \mid \delta Q < t < p_{\max} \right\}.$$

- $\mathscr{R}_4$ and $\mathscr{R}_1$ are separated by $\mathscr{D}_{41}$, the image of the segment $\overline{OM}$, for which $P = \delta F$, with $F < Q$:

$$\mathscr{D}_{41} = \left\{ \begin{pmatrix} D(t) \\ \frac{t}{\delta} - n_P \alpha \delta v\,(\delta Q - t) \end{pmatrix} \mid 0 < t < \delta Q \right\}.$$

- $\mathscr{R}_4$ and $\mathscr{R}_5$ are separated by $\mathscr{D}_{45}$, the image of the half-line $P = 0$, $F \leq 0$:
$$\mathscr{D}_{45} = \left\{ \begin{pmatrix} D_{\max} \\ (1 + n_P \alpha \delta^2 v)t - n_P \alpha \delta^2 v Q \end{pmatrix} \mid t \leq 0 \right\}.$$
- $\mathscr{R}_1$ and $\mathscr{R}_5$ are separated by $\mathscr{D}_{15}$, is the image of the segment $\overline{OA}$, for which $P = 0$, $0 \leq F \leq Q$:
$$\mathscr{D}_{15} = \left\{ \begin{pmatrix} D_{\max} + n_I \delta t \\ (1 + (n_I + n_P)\alpha \delta^2 v)t - n_P \alpha \delta^2 v Q \end{pmatrix} \mid 0 \leq t \leq Q \right\}.$$
- $\mathscr{R}_2$ and $\mathscr{R}_5$ are separated by $\mathscr{D}_{25}$, the image of half-line $P = 0$, $F \geq Q$:
$$\mathscr{D}_{25} = \left\{ \begin{pmatrix} D_{\max} + n_I \delta t \\ (1 + n_I \alpha \delta^2 v)t \end{pmatrix} \mid t \geq Q \right\}.$$
- $\mathscr{R}_4$ and $\mathscr{R}_6$ are separated by $\mathscr{D}_{46}$, the image of the half-line $P = p_{\max}$, $F \leq Q$:
$$\mathscr{D}_{46} = \left\{ \begin{pmatrix} 0 \\ (1 + n_P \alpha \delta^2 v)t - n_P \alpha \delta^2 v Q \end{pmatrix} \mid t \leq Q \right\}.$$
- $\mathscr{R}_3$ and $\mathscr{R}_6$ are separated by $\mathscr{D}_{36}$, the image of the segment $\overline{CB}$, for which $P = p_{\max}$, $Q \leq F \leq p_{\max}$:
$$\mathscr{D}_{36} = \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix} \mid Q \leq t \leq \frac{p_{\max}}{\delta} \right\}.$$
- $\mathscr{R}_2$ and $\mathscr{R}_6$ are separated by $\mathscr{D}_{26}$, the image of the half-line $P = p_{\max}$, $F \geq p_{\max}$:
$$\mathscr{D}_{26} = \left\{ \begin{pmatrix} n_I \delta t - n_I p_{\max} \\ (1 + n_I \alpha \delta^2 v)t - n_I \alpha \delta v p_{\max} \end{pmatrix} \mid t \geq \frac{p_{\max}}{\delta} \right\}.$$

A.2 The values of $P(z,e,v)$, $F(z,e,v)$ and $H(z,e,v)$ in the Regions $\mathscr{R}_1$ to $\mathscr{R}_6$.

In the case where $D(P) = M - mP$ in the flexible part of demand, we have:

Region 1.

$$P(z,e,v) = \frac{n_I \delta \left[e + \alpha \delta^2 v n_P Q\right] + (M - z)\left[1 + \alpha \delta^2 v \left(n_I + n_P\right)\right]}{m + n_I + \alpha \delta^2 v \left(n_P m + n_P n_I + n_I m\right)}, \tag{23}$$

$$F(z,e,v) = \frac{\left(m + n_I\right)\left[e + \alpha \delta^2 v n_P Q\right] + \alpha \delta v n_I \left(M - z\right)}{m + n_I + \alpha \delta^2 v \left(n_P m + n_P n_I + n_I m\right)}. \tag{24}$$

Thus we have:

$$H(z,e,v) = \frac{e\delta \left[n_P m + n_P n_I + n_I m\right] - n_P \delta Q \left(m + n_I\right) - n_I \left(M - z\right)}{m + n_I + \alpha \delta^2 v \left(n_P m + n_P n_I + n_I m\right)}. \tag{25}$$

Region 1 being the most complicated one, the values of $P(z,e,v)$, $F(z,e,v)$ and $H(z,e,v)$ in the other regions can be found easily. The details can be found in the Appendix A.2.

Region 2. If $(z, e) \in \mathscr{R}_2$, the processors are inactive in the physical market. The equations are as if $n_P = 0$ in the expression of Region 1:

$$P(z,e,v) = \frac{e\delta n_I + (M-z)\left[1 + \alpha\delta^2 v n_I\right]}{m + n_I + \alpha\delta^2 v n_I m}, \quad F(z,e,v) = \frac{e\,(m+n_I) + \alpha\delta v n_I\,(M-z)}{m + n_I + \alpha\delta^2 v n_I m},$$
$$H(z,e,v) = \frac{e\delta n_I m - n_I\,(M-z)}{m + n_I + \alpha\delta^2 v n_I m}.$$

Region 3. If $(z, e) \in \mathscr{R}_3$, nobody operates. The equilibrium equations can be written:

$$z = D(P), \qquad e = F,$$

which gives:

$$F(z,e,v) = e, \qquad P(z,e,v) = \frac{M-z}{m}, \qquad H(z,e,v) = 0.$$

Region 4. If $(z, e) \in \mathscr{R}_4$, the storers are inactive in the physical market. With $n_I = 0$, the equilibrium equations become:

$$z = D(P), \qquad e = F - \alpha\delta^2 v n_P\,(Q - F)\,.$$

Thus:

$$F(z,e,v) = \frac{e + \alpha\delta^2 v n_P Q}{1 + \alpha\delta^2 v n_P}, \qquad P(z,e,v) = \frac{M-z}{m}, \qquad H(z,e,v) = \frac{\delta n_P\,(e-Q)}{1 + \alpha\delta^2 v n_P}.$$

Region 5. In this case $(z, e) \in \mathscr{R}_5$, and:

$$P(z,e,v) = 0.$$

We will note:

$$H(z,e,v) = H(\pi_5\,(e)\,, e, v)\,,$$

where π_5 parameterize the frontier of $\mathscr{R}_5$.

Region 6. If $(z, e) \in \mathscr{R}_6$ we have:

$$P(z,e,v) = p_{\max}.$$

We will note:

$$H(z,e,v) = H(\pi_6\,(e)\,, e, v)\,,$$

where π_6 parameterize the frontier of $\mathscr{R}_6$.

B Proof of Theorem 1

B.1 Some properties of probability distributions.

Let φ and Φ denote respectively the density and the CDF of a real random variable. In the following, expected values and variances are calculated with respect to that distribution. In the sequel, we shall only use the upper bound in formula (26).

Lemma 1. *Let $g : \mathbb{R} \to [0, +\infty)$ be a measurable function such that*

$$\begin{cases} g(x) = b > 0 & \textit{for} \quad x \leq 0, \\ 0 \leq g(x) \leq b & \textit{for} \quad 0 < x < a, \\ g(x) = 0 & \textit{for} \quad x \geq a > 0. \end{cases}$$

Then:

$$\frac{\mathbb{E}\left[(-\infty, 0)\right] \mathbb{E}\left[(a, +\infty)\right]}{\mathbb{E}\left[(0, a)\right]}\, b^2 \leq \mathrm{Var}[g] \leq \frac{1}{4}\, b^2. \tag{26}$$

Proof. Denote by $\mathscr{C}_{ab}$ the set of all functions g satisfying all the prescribed conditions. It is a convex, closed and bounded subset of $L^2(\mathbb{P})$, and hence weakly compact. Consider the optimization problem:

$$\min \left\{ \mathrm{Var}[g] \mid g \in \mathscr{C}_{ab} \right\}.$$

By standard optimization arguments, it can be shown that the minimum is attained, and that the minimizer $g_{\min}$, which is equal to b on $(-\infty, 0]$ and to 0 on $[a, +\infty)$, must also be constant on $(0,\ a)$, with:

$$g_{\min}(x) = \mathbb{E}[g] = \int_{-\infty}^{+\infty} g\, d\Phi \ \text{ for } 0 < x < a.$$

Let us make the right-hand side explicit:

$$\int_{-\infty}^{+\infty} g\, d\Phi = b\Phi(0) + g_{\min}(x)(\Phi(a) - \Phi(0)).$$

This becomes an equation for $g(x)$, yielding:

$$g_{\min}(x) = \frac{b\Phi(0)}{1 - \Phi(a) + \Phi(0)} \ \text{ for } 0 < x < a,$$

to which we should add $g_{\min}(x) = b$ for $x \leq 0$ and $g(x) = 0$ for $x \geq a$. Computing the variance of $g_{\min}$, we get the lower bound.

Consider now the optimization problem:

$$\max \left\{ \mathrm{Var}[g] \mid g \in \mathscr{C}_{ab} \right\}.$$

The variance also attains its maximum, but this time the constraints $0 \le g \le b$ are binding: the maximum is attained at some step function $g_{\max} \in \mathscr{C}_{ab}$, with $g(x) \in \{0, b\}$ for all $x \in [0, a]$. For any such step function g, set $q = \mathbb{P}[g(x) = b,\ 0 \le x \le a]$. We get:

$$\begin{aligned}
\mathbb{E}[g] &= b(q + \Phi(0)), \\
\mathrm{Var}[g] &= b^2(1 - q - \Phi(0))(q + \Phi(0)).
\end{aligned}$$

If $\Phi(0) \ge 1/2$, the variance is maximal for $q = 0$, so $g_{\max} = b\ 1_{x \le 0}$. If $\Phi(a) \le 1/2$, the variance is maximal for $q = \Phi(a) - \Phi(0)$, so $g_{\max} = b\ 1_{x \le a}$. If $\Phi(0) < \frac{1}{2} < \Phi(a)$, the variance is maximal for $q = \frac{1}{2} - \Phi(0)$. Finally:

$$\mathrm{Var}[g] = \begin{cases} b^2\Phi(0)(1 - \Phi(0)) & \text{if} \quad \Phi(0) \ge 1/2, \\ \frac{1}{4}b^2 & \text{if} \quad \Phi(0) < 1/2 < \Phi(a), \\ b^2\Phi(a)(1 - \Phi(a)) & \text{if} \quad \Phi(a) \le 1/2. \end{cases}$$

Note that, in all cases we have $\mathrm{Var}[g] \le \frac{1}{4}b^2$, which gives the upper bound. □

In our model, since $0 \le p_t \le p_{\max}$, we have:

$$0 \le \mathbb{E}[p_t \mid z_{t-1}, \omega_{t-1}] \le p_{\max}, \tag{27}$$

$$\mathrm{Var}[p_t \mid z_{t-1}, \omega_{t-1}] \le \frac{1}{4}\ p_{\max}^2. \tag{28}$$

B.2 From prices to expectations

Now consider the mapping Ψ which associates to every pair $(p(z, \omega), f(z, \omega))$ of price functions (and thus implicitly $h(z, \omega)$) the corresponding pair of expectations $(E(z, \omega), V(z, \omega))$. More precisely, $\Psi(p, f) = (E, V)$, where:

$$\begin{aligned}
E(z, \omega) &= \int p(h(z, \omega) + \omega_\bullet, \omega_\bullet)\ \varphi(\omega_\bullet | \omega)\ d\omega_\bullet, \\
V(z, \omega) &= \int p(h(z, \omega) + \omega_\bullet, \omega_\bullet)^2\ \varphi(\omega_\bullet | \omega)\ d\omega_\bullet - E(z, \omega)^2.
\end{aligned}$$

Lemma 2. *Suppose p and h are Lipschitz functions with respective constants k_p and k_h:*

$$\begin{aligned}
|p(z_1, \omega_1) - p(z_2, \omega_2)| &\le k_p\, |z_1 - z_2| + k_p\, |\omega_1 - \omega_2|\, , \\
|h(z_1, \omega_1) - h(z_2, \omega_2)| &\le k_h\, |z_1 - z_2| + k_h\, |\omega_1 - \omega_2|\, .
\end{aligned}$$

Then E and V are Lipschitz functions of (z, ω) with respective constants:

$$\begin{aligned}
k_E &= \mu_\varphi k_p k_h + p_{\max} k_\varphi, \\
k_V &= 4\mu_\varphi p_{\max} k_p k_h + 3p_{\max}^2 k_\varphi.
\end{aligned}$$

Proof. By a theorem of Rademacher, since p and h are Lipschitz, they are differentiable almost everywhere. We have $k_h \leq \max |\partial_z h(z,\omega))|$ with similar relations for the other Lipschitz constants. Differentiating under the integral:

$$\partial_z E(z,\omega) = \int \partial_z p(h(z,\omega)+\omega_\bullet,\omega_\bullet)\partial_z h\,(z,\omega)\;\;\varphi(\omega_\bullet|\omega)\;d\omega_\bullet.$$

Note that $p(z,\omega)$ is constant outside the interval $[z_{\min}(\omega), z_{\max}(\omega)]$, so that $\partial_\omega p$ and $\partial_z p$ vanish outside that interval. Using condition (19), we find:

$$|\partial_z E(z,\omega)| \leq \mu_\varphi \max_{z,\omega} |\partial_z p|\;\;|\partial_z h| \leq \mu_\varphi\; k_p\; k_h.$$

Now for the other derivative, which is more complicated:

$$\begin{aligned}\partial_\omega E(z,\omega) = &\int \partial_z p(h(z,\omega)+\omega_\bullet,\omega_\bullet)\;\partial_\omega h\,(z,\omega)\;\;\varphi(\omega_\bullet|\omega)\;d\omega_\bullet \\ &+\int p(h(z,\omega)+\omega_\bullet,\omega_\bullet)\;\partial_\omega\varphi(\omega_\bullet|\omega)\;d\omega_\bullet.\end{aligned}$$

Proceeding as above for the first term, and using condition (18) on the second:

$$|\partial_\omega E(z,\omega)| \leq \mu_\varphi\; k_p\; k_h + p_{\max}\; k_\varphi.$$

We now turn to $V(z,\omega)$. We have:

$$V(z,\omega) = \int p(h\,(z,\omega)+\omega_\bullet,\omega_\bullet)^2\;\varphi(\omega_\bullet|\omega)\;d\omega_\bullet - E(z,\omega)^2.$$

Differentiating w.r.t. z, we get:

$$\begin{aligned}\partial_z V(z,\omega) = &\int 2\partial_z p(h\,(z,\omega)+\omega_\bullet,\omega_\bullet)\;p(h\,(z,\omega)+\omega_\bullet,\omega_\bullet)\;\partial_z h\,(z,\omega)\;\;\varphi(\omega_\bullet|\omega)\;d\omega_\bullet \\ &-2E(z,\omega)\;\partial_z E(z,\omega).\end{aligned}$$

Using the estimate for $\partial_z E(z,\omega)$, this yields:

$$|\partial_z V(z,\omega)| \leq 2\mu_\varphi p_{\max} k_p k_h + 2\max|E(z,\omega)|\,\mu_\varphi\; k_p\; k_h.$$

Differentiating $V(z,\omega)$ w.r.t. ω, we get:

$$\begin{aligned}\partial_\omega V(z,\omega) = 2&\int \partial_z p(h\,(z,\omega)+\omega_\bullet,\omega_\bullet)\;p(h\,(z,\omega)+\omega_\bullet,\omega_\bullet)\;\partial_\omega h\,(z,\omega)\;\;\varphi(\omega_\bullet|\omega)\;d\omega_\bullet \\ &+\int p(h\,(z,\omega)+\omega_\bullet,\omega_\bullet)^2\;\partial_\omega\varphi(\omega_\bullet|\omega)\;d\omega_\bullet - 2E(z,\omega)\;\partial_\omega E(z,\omega).\end{aligned}$$

This gives:

$$|\partial_\omega V(z,\omega)| \leq 2\mu_\varphi p_{\max} k_p k_h + p_{\max}^2 k_\varphi + 2\max|E(z,\omega)|\left(\mu_\varphi\; k_p k_h + p_{\max} k_\varphi\right).$$

From the definition of $E\,(z,\omega)$ it follows that $\max|E(z,\omega)| \leq p_{\max}$. Hence the result. □

B.3 From expectations to prices

Consider the domain $\Delta \subset \mathbb{R}^4$ defined by:

$$\Delta = \left\{ (z, e, v) \mid 0 \le e \le p_{\max},\ 0 \le v \le \frac{1}{4}\, p_{\max}^2 \right\}.$$

It follows from the explicit formulas for $P(z,e,v)$ and $H(z,e,v)$ that they are both Lipchitz functions on Δ (the fact that v is bounded is important here). Note that they are constant on $z \le z_{\min}(\omega)$ and on $z \ge z_{\max}(\omega)$. Denote by c_P and c_H the Lipschitz constants, and take $c = \max(c_P, c_H)$:

$$|P(z_1,e_1,v_1) - P(z_2,e_2,v_2)| \le c\left(|z_1 - z_2| + |e_1 - e_2| + |v_1 - v_2|\right), \tag{29}$$

$$|H(z_1,e_1,v_1) - H(z_2,e_2,v_2)| \le c\left(|z_1 - z_2| + |e_1 - e_2| + |v_1 - v_2|\right). \tag{30}$$

Consider the mapping Γ which associates to every pair of expectations (E, V) the corresponding pair (p, h) of prices and transfers:

$$\begin{aligned} \Gamma(E,V) &= (p,h), \\ p(z,\omega) &= P(z, E(z,\omega), V(z,\omega)), \\ h(z,\omega) &= H(z, E(z,\omega), V(z,\omega)). \end{aligned}$$

Lemma 3. *Suppose E and V are Lipschitz functions of (z,ω) with respective constants k_E and k_V. Suppose:*

$$\begin{aligned} 0 \le E(z,\omega) &\le p_{\max}, \\ 0 \le V(z,\omega) &\le \frac{1}{4}\, p_{\max}^2. \end{aligned}$$

Then p and h are Lipschitz functions of (z,ω) with respective constants:

$$\begin{aligned} k_p &\le c\left(1 + k_E + k_V\right), \\ k_h &\le c\left(1 + k_E + k_V\right). \end{aligned}$$

Proof. Obvious: just substitute. □

B.4 Setting up the fixed point

We are looking for a pair (p,h) such that $\Gamma \circ \Psi(p,h) = (p,h)$, and

$$\begin{aligned} E(z,\omega) &= \int p(h(z,\omega) + \omega_\bullet, \omega_\bullet)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet, \\ V(z,\omega) &= \int p(h(z,\omega) + \omega_\bullet, \omega_\bullet)^2\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet - E(z,\omega)^2, \\ p(z,\omega) &= P(z, E(z,\omega), V(z,\omega)), \\ h(z,\omega) &= H(z, E(z,\omega), V(z,\omega)). \end{aligned}$$

Lemma 4. *Suppose μ_φ and k_φ satisfy the smallness condition:*

$$4\mu_\varphi \left(1 + 4p_{\max}\right) \left(1 + p_{\max} k_\varphi \left(1 + 3p_{\max}\right)\right) \leq \frac{1}{c^2}. \tag{31}$$

Then there exists $\bar{k}$ such that the operator $\Gamma \circ \Psi$ sends the set of functions (p, h) which are both $\bar{k}$-Lipschitz into itself.

Proof. Suppose (p, h) are $\bar{k}$-Lipschitz. Set $(q, g) = \Gamma \circ \Psi(p, h)$. It follows from the above that q and g are Lipschitz, with constants

$$\max\left\{k_q, k_g\right\} \leq c \left(1 + \mu_\varphi \left(1 + 4p_{\max}\right) \bar{k}^2 + p_{\max} k_\varphi \left(1 + 3p_{\max}\right)\right).$$

A sufficient condition for $k_q \leq k_p$ and $k_g \leq k_h$ is:

$$1 + \mu_\varphi \left(1 + 4p_{\max}\right) \bar{k}^2 + p_{\max} k_\varphi \left(1 + 3p_{\max}\right) \leq \frac{\bar{k}}{c}.$$

This is an inequality of the second degree in $\bar{k}$. To have a solution, we must have a non-negative discriminant, that is:

$$\frac{1}{c^2} - 4\mu_\varphi \left(1 + 4p_{\max}\right) \left(1 + p_{\max} k_\varphi \left(1 + 3p_{\max}\right)\right) \geq 0.$$

□

B.5 Existence and uniqueness

Suppose μ_φ and k_φ satisfy condition (31). Apply the preceding lemma, so that $\Gamma \circ \Psi$ sends the set $\mathscr{C}$ of bounded functions $p(z, \omega)$ and $h(z, \omega)$ which are $\bar{k}$-Lipschitz into itself. We endow $\mathscr{C}$ with the metric of uniform convergence:

$$\|f_1 - f_2\| = \sup_{(z,\omega)} \left|f_1(z, \omega) - f_2(z, \omega)\right|$$

It is a closed convex subset of the space of all continuous and bounded functions.

Proposition 1. *There are $\bar{\mu}_\varphi$ and $\bar{k}_\varphi$ so small that if $\mu_\varphi \leq \bar{\mu}_\varphi$ and $k_\varphi \leq \bar{k}_\varphi$ the restriction of $\Gamma \circ \Psi$ to $\mathscr{C}$ is contracting.*

Proof. We know that $\Gamma \circ \Psi$ sends $\mathscr{C}$ into itself. We have to prove that if $(p_i', h_i') = \Gamma \circ \Psi(p_i, h_i)$ for $i = 1, 2$, if p_1, p_2, p_1', p_2',are Lipschitz with constants $\bar{k}$ and h_1, h_2, h_1', h_2' are Lipschitz with constants $\bar{k}$, then there is some $r < 1$ such that:

$$\max_{z,\omega} \left|p_1'(z, \omega) - p_2'(z, \omega)\right| \leq r \left\|p_1 - p_2\right\|,$$

$$\max_{z,\omega} \left|h_1'(z, \omega) - h_2'(z, \omega)\right| \leq r \left\|h_1 - h_2\right\|.$$

Write:

$$\begin{aligned}
E_2(z,\omega) - E_1(z,\omega) &= \int \left(p_2(h_2(z,\omega)+\omega_\bullet,\omega_\bullet) - p_1(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)\right)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet \\
&= \int \left(p_2(h_2(z,\omega)+\omega_\bullet,\omega_\bullet) - p_2(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)\right)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet \\
&+ \int \left(p_2(h_1(z,\omega)+\omega_\bullet,\omega_\bullet) - p_1(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)\right)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet.
\end{aligned}$$

So:

$$\|E_2 - E_1\| \le \mu_\varphi(\bar{k}\,\|h_1 - h_2\| + \|p_1 - p_2\|). \tag{32}$$

Similarly:

$$\begin{aligned}
V_2(z,\omega) - V_1(z,\omega) &= \int \left(p_2(h_2(z,\omega)+\omega_\bullet,\omega_\bullet)^2 - p_1(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)^2\right)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet \\
&- E_2(z,\omega)^2 + E_1(z,\omega)^2 \\
&= \int \left(p_2(h_2(z,\omega)+\omega_\bullet,\omega_\bullet)^2 - p_2(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)^2\right)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet \\
&+ \int \left(p_2(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)^2 - p_1(h_1(z,\omega)+\omega_\bullet,\omega_\bullet)^2\right)\ \varphi(\omega_\bullet|\omega)\ d\omega_\bullet \\
&- \left(E_2(z,\omega) - E_1(z,\omega)\right)\left(E_2(z,\omega) + E_1(z,\omega)\right).
\end{aligned}$$

Hence:

$$\begin{aligned}
\|V_2 - V_1\| &\le 2p_{\max}\,\mu_\varphi\,\bar{k}\,\|h_1 - h_2\| + 2p_{\max}\,\mu_\varphi\ \|p_1 - p_2\| + 2p_{\max}\,\mu_\varphi\ (\bar{k}\,\|h_1 - h_2\| + \|p_1 - p_2\|) \\
&\le 4p_{\max}\,\mu_\varphi\ (\bar{k}\,\|h_1 - h_2\| + \|p_1 - p_2\|).
\end{aligned} \tag{33}$$

Finally, we have, for $i = 1, 2$:

$$\begin{aligned}
p_i'(z,\omega) &= P(z, E_i(z,\omega), V_i(z,\omega)), \\
h_i'(z,\omega) &= H(z, E_i(z,\omega), V_i(z,\omega)).
\end{aligned}$$

Recalling (29) and (30), we get:

$$\begin{aligned}
\left\|p_2' - p_1'\right\| &\le c\left(\|E_2 - E_1\| + \|V_2 - V_1\|\right), \\
\left\|h_2' - h_1'\right\| &\le c\left(\|E_2 - E_1\| + \|V_2 - V_1\|\right).
\end{aligned}$$

Substituting (32) and (33), we see that:

$$\begin{aligned}
\left\|p_2' - p_1'\right\| &\le c\,\mu_\varphi\ (1 + 4p_{\max})\,(\bar{k}\,\|h_1 - h_2\| + \|p_1 - p_2\|), \\
\left\|h_2' - h_1'\right\| &\le c\,\mu_\varphi\ (1 + 4p_{\max})\,(\bar{k}\,\|h_1 - h_2\| + \|p_1 - p_2\|).
\end{aligned}$$

□

The coefficients on the right-hand side can be made smaller than 1 if μ_φ is small enough, namely:

$$c\,\mu_\varphi\ (1 + 4p_{\max})\,\bar{k} < 1.$$

The map $(p, h) \to (p', h')$ is then contracting, the fixed point $(\bar{p}, \bar{h})$ is unique and can be reached by iteration. Once we have the fixed point, then the determination of $\bar{E}$, $\bar{V}$ and $\bar{f}$ is straightforward.