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$\mathbb{Z}_2$-equivariant Ljusternik-Schnirelman theory for non-even functionals

by

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ABSTRACT. – We develop an equivariant Ljusternik-Schnirelman theory for non-even functionals. We show that if one applies a $\mathbb{Z}_2$-equivariant min-max procedure to a non-symmetric functional φ, then one gets either the usual critical points, defined by $\varphi'(x) = 0$ or an interesting new class of points x, defined by $\varphi(x) = \varphi(-x)$ and $\varphi'(x) = \lambda\varphi'(-x)$ for some $\lambda > 0$. We call them "$\mathbb{Z}_2$-resonant points"; by a "virtual critical point" we understand a point which is either critical or $\mathbb{Z}_2$-resonant. We extend the classical existence and multiplicity results of Ljusternik-Schnirelman theory for critical points of even functionals to virtual critical points of non-even functionals. As an application we prove a bifurcation-type result for a class of non-homogenous semi-linear elliptic boundary value problems. © Elsevier, Paris

RÉSUMÉ. – Nous construisons une théorie de Ljusternik-Schnirelman pour des fonctionnelles non symétriques. Plus précisément, nous montrons que si l'on applique une procédure de minimax $\mathbb{Z}_2$-équivariante à une fonctionnelle non paire φ, l'on obtient d'une part les points critiques habituels, définis par $\varphi'(x) = 0$, et d'autre part des points d'un type nouveau, vérifiant, $\varphi(x) = \varphi(-x)$ et $\varphi'(x) = \lambda\varphi'(-x)$ pour certain $\lambda > 0$. Nous les baptisons « $\mathbb{Z}_2$-résonants », et nous appellerons « point critiques virtuels » les points qui sont, soit critiques au sens habituel, soit $\mathbb{Z}_2$-résonants. Nous étendons alors la théorie classique de Ljusternik-Schnirelman pour les points critiques de fonctionnelles paires à la recherche des points critiques virtuels de fonctionnelles impaires. A titre d'application, nous obtenons un résultat de bifurcation pour une classe d'équations semi-linéaires elliptiques avec un second membre.

I. INTRODUCTION

Classical Ljusternik-Schnirelman theory tells us that any smooth and *even* functional φ on an n-dimensional sphere S^n has at least $n+1$ pairs of antipodal critical points. These critical points are found by considering for $1 \leq k \leq n+1$, the numbers

$$c_k := c(\varphi, \mathcal{F}_k) = \inf_{A \in \mathcal{F}_k} \max_{x \in A} \varphi(x)$$

where each $\mathcal{F}_k$ consists of the class of all compact symmetric subsets of S^n whose $\mathbb{Z}_2$-genus is larger than k, and by proving that each of these numbers is in fact a critical level of φ.

The key idea of the proof is that each class $\mathcal{F}_k$ is stable under *odd* (or $\mathbb{Z}_2$- equivariant) homotopies of the identity. Since the functional φ is even, its gradient flow belongs to this class of homotopies, and will not be able to push down the levels c_k, which must therefore contain a critical point.

This procedure, originally due to Ljusternik and Schnirelman, was successfully extended by many authors to the infinite dimensional setting in order to solve nonlinear elliptic partial differential equations of the form

$$(*) \qquad \begin{aligned} -\Delta u &= g(x,u), \quad && x \in \Omega \subset \mathbb{R}^n, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

where $g(x,.)$ is *odd* for each x in the domain Ω. The latter assumption insures that the associated functional – the critical points of which are the solutions of $(**)$– is an even function on the appropriate Sobolev space. Lusternik-Schnirelman theory then enables us to conclude that this functional has infinitely many critical points, that is, that the equation $(*)$ has infinitely many solutions.

Assume now that $g(x,.)$ is no longer odd, so that the associated functional is no longer even. Let us apply to this functional the same $\mathbb{Z}_2$-equivariant min-max procedure. What kind of points do we get? The answer turned out to be quite simple. Let us first state the problem in a general setting:

If D and X are two topological spaces, we shall denote by $C(D;X)$ the set of all continuous maps from D into X. If $\mathbb{Z}_2$ acts on both D and X, we shall say that a map $h : D \to X$ is odd or $\mathbb{Z}_2$-equivariant if $h(-x) = -h(x)$ for every $x \in D$. A deformation $\eta \in C([0,1] \times D; X)$ is said to be $\mathbb{Z}_2$-equivariant if for any t, the function $\eta(t,.)$ is.

A subset A of X is said to be symmetric if $-x \in A$ whenever $x \in A$.

DEFINITION I.1. – Let B be a closed symmetric subset of a Hilbert space H. We shall say that a class $\mathcal{F}$ of subsets of H is a *$\mathbb{Z}_2$-homotopy stable family with boundary* B if:

(a) Every set in $\mathcal{F}$ is compact symmetric and contains B.

(b) For any set A in $\mathcal{F}$ and any $\mathbb{Z}_2$-equivariant $\eta \in C([0,1] \times H; H)$ satisfying $\eta(t,x) = x$ for all (t,x) in $(\{0\} \times H) \cup ([0,1] \times B)$, we have that $\eta(\{1\} \times A) \in \mathcal{F}$. In section II, we shall prove the following

THEOREM (A). – *Let φ be a C^1-functional on a Hilbert space H and consider a $\mathbb{Z}_2$-homotopy-stable family $\mathcal{F}$ in H with a closed symmetric boundary B. Suppose that*

$$\sup \varphi(B) < c := \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x)$$

and that $\varphi(0) \neq c$. Then, there exists a sequence $(x_n)_n$ in H such that $\lim_n \varphi(x_n) = c$ and which also satisfies one of the following (non-mutually exclusive) assertions:

Either

(a) $\lim_n \varphi'(x_n) = 0$,

or

(b) $\lim_n \varphi(-x_n) = c$ *and* $\lim_n \|\varphi'(x_n) - \lambda_n \varphi'(-x_n)\| = 0$ *for some sequence of positive reals* $(\lambda_n)_n$.

To derive an existence result from Theorem (A), we shall need a strengthening of the classical condition of Palais and Smale. Recall that a functional φ is said to satisfy the *Palais-Smale condition at level* c (in short (P-S)$_c$), if any sequence $(x_n)_n$ satisfying $\lim_n \varphi(x_n) = c$ and $\lim_n \varphi'(x_n) = 0$ is relatively compact.

DEFINITION I.2. – Say that a C^1-functional φ on a Hilbert space H satisfies the *symmetrized Palais-Smale condition at level* c ((sP-S)$_c$) if φ satisfies (P-S)$_c$ and if a sequence $(x_n)_n$ in H is relatively compact in H whenever it satisfies the following conditions:

$$\lim_n \varphi(x_n) = \lim_n \varphi(-x_n) = c$$

and

$\lim_n \|\varphi'(x_n) - \lambda_n \varphi'(-x_n)\| = 0$ for some positive sequence of reals $(\lambda_n)_n$.

We can now derive the following result, which tells us which kind of points we obtain by applying a $\mathbb{Z}_2$-equivariant min-max procedure to a non-even functional:

COROLLARY I.3. – *Let φ be a C^1-functional on a Hilbert space H and consider a $\mathbb{Z}_2$-homotopy-stable family $\mathcal{F}$ in H with a closed symmetric boundary B. Suppose that*

$$\sup \varphi(B) < c := \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x),$$

that φ satisfies (sP-S)$_c$ and that $\varphi(0) \neq c$. Then there exists $x \in H$, with $\varphi(x) = c$ such that:
Either

(a) $\varphi'(x) = 0$

or

(b) $\varphi(x) = \varphi(-x) = c$ *and* $\varphi'(x) = \lambda \varphi'(-x)$ *for some* $\lambda \geq 0$.

As usual, we shall denote by K_c the set of critical points at level c, *i.e.*,

$$K_c = \{x \in H; \varphi(x) = c, \varphi'(x) = 0\}.$$

The points satisfying assertion (b) above will be called the $\mathbb{Z}_2$-*resonant points at level* c. We shall denote

$$K_c^f = \{x \in H; \varphi(x) = \varphi(-x) = c, \varphi'(x) = \lambda \varphi(-x), \lambda > 0\}.$$

Note that the set K_c^f of $\mathbb{Z}_2$-resonant points at level c is a symmetric set while the set K_c of true critical points is generally not symmetric. We shall denote

$$E_c := K_c^f \cup K_c$$

whose points will be called the *virtual critical points* at level c. Thus, a virtual critical point is either a true critical point or a Z_2-resonant point.

Sometimes we shall need the symmetrized version of the set E_c, that is

$$E_c^* = E_c \cup -E_c = K_c^f \cup (K_c \cup -K_c)$$

We shall also consider the part of E_c^* that is on level c, that is:

$$\tilde{E}_c = E_c^* \cap \{\varphi = c\} = K_c^f \cup K_c \cup (-K_c \cap \{\varphi = c\}).$$

Theorem (A) and its corollaries will be proved in section II. In section III, we shall show how the virtual critical points found in Theorem (A) can be localized by additional information. We first recall the following concept that was introduced in [A-R] and extensively studied in [G].

DEFINITION I.4. – Let $\mathcal{F}$ be a homotopy-stable family with boundary B. A closed subset F of H is then said to be *dual to* $\mathcal{F}$ if it satisfies

$$F \cap B = \emptyset \quad \text{and} \quad F \cap A \neq \emptyset \quad \text{for every} \quad A \in \mathcal{F}.$$

A typical example of a dual set to a family $\mathcal{F}$ is the set $F = \{\varphi \geq c\}$ where φ is a real-valued function on H and

$$c = c(\varphi; \mathcal{F}) := \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x).$$

Another more geometrical example is given by any sphere S separating two points u and v in a Hilbert space X. It is then dual to the class $\mathcal{F}$ consisting of all continuous paths joining u and v. More elaborate examples will emerge in the sequel. We also refer to [G1,2] where this notion is studied at length.

THEOREM (B). – *Let φ be a C^1-functional on a Hilbert space H and consider a $\mathbb{Z}_2$-homotopy stable family $\mathcal{F}$ in H with a symmetric boundary B and let $c := c(\varphi, \mathcal{F})$. Suppose F is a closed subset of H that is dual to the family $\mathcal{F}$ and such that* $\inf \varphi(F) \geq c$.

If $0 \notin F \cap \varphi^{-1}(c)$, then there exists a sequence $(x_n)_n$ in H such that

(i) $\lim_n \varphi(x_n) = c$ *and* $\overline{\lim}_n \varphi(-x_n) \leq c$,

(ii) $\lim_n \operatorname{dist}(x_n, F \cup -F) = 0$,

and which also satisfies one of the following (non-mutually exclusive) assertions:

Either

(iii.a) $\lim_n \varphi'(x_n) = 0$,*or*

(iii.b) $\lim_n \varphi(-x_n) = c$ *and* $\lim_n(\varphi'(x_n) - \lambda_n \varphi'(-x_n)) = 0$ *for some sequence of positive reals* $(\lambda_n)_n$.

Recall again that a functional φ is said to satisfy the *Palais-Smale condition at level c and around the set D* (in short (P-S)$_{D,c}$), if any sequence $(x_n)_n$ satisfying $\lim_n \varphi(x_n) = c$, $\lim_n \operatorname{dist}(x_n, D) = 0$ and $\lim_n \varphi'(x_n) = 0$ is relatively compact.

We shall need the following strengthening of that compactness condition.

DEFINITION I.5. – Say that a C^1-functional φ on a Hilbert space H satisfies the *symmetrized Palais-Smale condition at level c and around the* *set* F ((sP-S)$_{F,c}$) if φ satisfies (P-S)$_{F\cup -F,c}$ and if a sequence $(x_n)_n$ in H is relatively compact whenever it satisfies the following conditions

$$\lim_n \varphi(x_n) = \lim_n \varphi(-x_n) = c,$$

$$\lim_n (\varphi'(x_n) - \lambda_n \varphi'(-x_n)) = 0 \text{ for some sequence of positive reals } \lambda_n.$$

as well as

$$\lim_n \operatorname{dist}(x_n, F).$$

COROLLARY I.6. – *Under the hypothesis of Theorem (B), if φ satisfies (sP-S)$_{F,c}$, then there exists $x \in H$ such that:*
Either

(i) $x \in F$, $\varphi(x) = c$ *and* $\varphi'(x) = 0$,

or

(ii) $x \in -F$, $\varphi(x) = \varphi(-x) = c$ *and* $\varphi'(x) = 0$,

or

(iii) $x \in F$, $\varphi(x) = \varphi(-x) = c$ *and* $\varphi'(x) = \lambda\varphi'(-x))$ *for some* $\lambda > 0$.
In other words, $\tilde{E}_c \cap F \neq \emptyset$.

In section IV, we will be studying multiplicity. First, let us recall the notion of $\mathbb{Z}_2$-index. Suppose the symmetry group $\mathbb{Z}_2$ is acting on a smooth manifold X and denote by Σ the class of all closed symmetric subsets of X not containing 0. The $\mathbb{Z}_2$-index is defined on Σ in the following way:

$$\gamma(A) = \inf\{k; \text{there exists } f : A \to \mathbb{R}^k \backslash \{0\} \text{ odd and continuous}\}.$$

If no such a finite k exists, we set $\gamma(A) = \infty$. We also let $\gamma(\emptyset) = 0$.

THEOREM (C). – *Let φ be a C^1-functional on a Hilbert space H satisfying (sP-S)$_c$ for any $c \in \mathbb{R}$ and consider a a decreasing sequence of Z_2-homotopy stable families $(\mathcal{F}_j)_{j=1}^N$ (with N possibly infinite) of symmetric compact subsets of H that satisfy the following excision property:*

(E) *For every $1 \leq j \leq j+p \leq N$, any A in $\mathcal{F}_{j+p}$ and any U open and symmetric such that $\gamma(\overline{U}) \leq p$, we have $A \setminus U \in \mathcal{F}_j$.*

Suppose each level $c_j := c(\varphi, \mathcal{F}_j)$ is finite and that $\varphi(0) < c_1$. Then,

(a) *$E_{c_j} \neq \emptyset$ for each $1 \leq j \leq N$.*

(b) *If $c_j = c_{j+p}$ for some $p \geq 0$, we have $\gamma(E^*_{c_j}) \geq p+1$.*

In particular, φ has at least N distinct virtual critical points.

II. EXISTENCE RESULTS

The main idea behind the proofs below is the fact that the equivariant Ljusternik-Schnirelman min-max levels for the original functional φ and the even functional $\psi(x) = \max(\varphi(x), \varphi(-x))$ are the same. Therefore, modulo the obvious smoothness problems, the classical equivariant theory can apply to ψ. One way to deal with this problem (at least when φ is positive) consists of replacing ψ with the smooth and even functional $\varphi_n(x) = \left(\frac{\varphi^{2n}(x)+\varphi^{2n}(-x)}{2}\right)^{1/2n}$ with n large enough. However, we opted to give in this paper a direct proof that may have an independent interest. It consists of constructing an equivariant deformation that "pushes down" parts of–but not all–the non-critical level sets of a non-even functional. This is the object of this section.

We start by stating the following quantitative version of Theorem (A). We use the notation D^η to describe the η-neighborhood of a set D. That is, if D is a subset of H and $\eta > 0$, then $D^\eta = \{x \in H; \operatorname{dist}(x, D) < \eta\}$.

THEOREM II.1. – *Let φ be a C^1-functional on a Hilbert space H and consider a $\mathbb{Z}_2$-homotopy-stable family $\mathcal{F}$ in H with a closed symmetric boundary B. Suppose that* $\sup\varphi(B) < c := c(\varphi, \mathcal{F})$ *and that* $\varphi(0) \neq c$. *Then, for any $\epsilon > 0$ small enough and any $A \in \mathcal{F}$ such that* $\sup\varphi(A) < c + \epsilon^2$, *there exists $x_\epsilon \in H$ such that*

(i) $c - \epsilon^2 \leq \varphi(x_\epsilon) \leq c + \epsilon^2$

(ii) *dist*$(x_\epsilon, A) \leq \epsilon$

while satisfying one of the following (non-mutually exclusive) properties: Either

(iii.a) $\|\varphi'(x_\epsilon)\| \leq 2\sqrt{\epsilon}$

or

(iii.b) $c - \epsilon^2 \leq \varphi(-x_\epsilon) \leq c + \epsilon^2$ *and* $\frac{\langle\varphi'(x),\varphi'(-x)\rangle}{\|\varphi'(x)\|\cdot\|\varphi'(-x)\|} \geq 1 - 2\sqrt{\epsilon}$

First, we establish the basic equivariant deformation lemmas for non-even functionals.

THEOREM II.2. – *Let φ be a C^1-functional on a Hilbert space H and let K be a symmetric compact subset of H that is disjoint from a symmetric closed subset B of H. Suppose K does not contain 0 nor any critical point of φ and that there exists ϵ ($0 < \epsilon < 1/2$) such that for every $x \in K$*

$$\frac{\langle\varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} < 1 - \epsilon. \tag{1}$$

Then, there exists $\eta > 0$, an equivariant and continuous vector field $V : H \to H$ such that

(i) $V(x) = 0$ *for every* $x \in B$.

(ii) $\|V(x)\| \leq 1$ *for every* $x \in H$ *and* $\|V(x)\| = 1$ *on* K^η.

(iii) $\langle \varphi'(x), V(x)\rangle < -\min\{\frac{\epsilon}{2}, 1-2\epsilon\}\|\varphi'(x)\|$ *for every* $x \in K^\eta$.

Proof. – We split K into two symmetric sets:

$$K_1 = \left\{x \in K; -1+\epsilon < \frac{\langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} < 1-\epsilon\right\}$$

and

$$K_2 = \left\{x \in K; \frac{\langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} \leq -1+\epsilon\right\}$$

We first construct an appropriate vector field w_1 on a neighborhood of K_1. For that, choose $\eta_1 > 0$ small enough is such a way that $K_1^{2\eta_1}$ contains no critical points for φ, that

$$\eta_1 < \min\left\{\frac{1}{2}\operatorname{dist}(K,B), \operatorname{dist}(0,K)\right\}, \tag{2}$$

and for all $x \in K_1^{2\eta_1}$

$$-1+\epsilon < \frac{\langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} < 1-\epsilon. \tag{3}$$

Now consider the set

$$D = \{x \in K_1^{\eta_1}; \|\varphi'(x)\| \leq \|\varphi'(-x)\|\}$$

so that $K_1^{\eta_1} = D \cup (-D)$ and define on D the function

$$\mu(x) = \frac{\|\varphi'(x)\|^2 + \langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(-x)\|^2 + \langle \varphi'(x), \varphi'(-x)\rangle}. \tag{4}$$

Clearly, the function μ as well as the vector field

$$w_1(x) = -\varphi'(x) + \mu(x)\varphi'(-x).$$

are continuous on D. Extend w_1 to $K_1^{\eta_1}$ in an equivariant fashion by setting $w_1(-x) = -w_1(x)$ for any $x \in D$. Note that the definition is unambiguous on $D \cap (-D)$ since $\mu(x) = 1$ and $w_1(-x) = -w_1(x)$ for all $x \in D \cap (-D)$. We claim that for every $x \in K_1^{\eta_1}$,

$$\langle \varphi'(x), \frac{w_1(x)}{\|w_1(x)\|} \rangle < -\frac{\epsilon}{2} \|\varphi'(x)\|. \tag{5}$$

To show that, set $u = \varphi'(x)$, $v = \varphi'(-x)$ and $\cos\theta = \frac{\langle u,v \rangle}{\|u\|\|v\|}$.

If $x \in D$, write $\|v\| = \alpha\|u\|$ with $\alpha \geq 1$. Note that

$$\alpha\mu = \frac{1 + \alpha\cos\theta}{\alpha + \cos\theta}$$

which is necessarily between $\cos\theta$ and 1. It follows from (3) that

$$\begin{aligned} \left\langle \varphi'(x), \frac{w_1(x)}{\|w_1(x)\|} \right\rangle &= \frac{-1 + \mu\alpha\cos\theta}{(1 + \alpha^2\mu^2 - 2\mu\alpha\cos\theta)^{1/2}} \|u\| \\ &\leq \frac{-1 + (1-\epsilon)}{2} \|u\| \leq -\frac{\epsilon}{2} \|u\|. \end{aligned} \tag{6}$$

If $\alpha \leq 1$, then assertion (5) follows from the above estimate and the following easily verifiable identity: for all $x \in K_1^{\eta_1}$,

$$\langle \varphi'(x), w(x) \rangle + \langle \varphi'(-x), w(x) \rangle = 0. \tag{7}$$

We now construct an appropriate vector field w_2 on a neighborhood of K_2. For that, choose $\eta_2 > 0$ small enough is such a way that for all $x \in K_2^{2\eta_2}$,

$$\frac{\langle \varphi'(x), \varphi'(-x) \rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} < -1 + 2\epsilon. \tag{8}$$

and such that the set $K_2^{2\eta_2}$ contains no critical points of φ. We now define on $K_2^{\eta_2}$ the equivariant and continuous vector field

$$w_2(x) = -\varphi'(x) + \varphi'(-x).$$

We claim that for any $x \in K_2^{\eta_2}$, we have

$$\langle \varphi'(x), \frac{w_2(x)}{\|w_2(x)\|} \rangle \leq (-1 + 2\epsilon) \|\varphi'(x)\|. \tag{9}$$

Indeed, by using (8) we get that

$$\begin{aligned} \langle \varphi'(x), \frac{w_2(x)}{\|w_2(x)\|} \rangle &= \frac{-\|\varphi'(x)\|^2 + \langle \varphi'(x), \varphi'(-x) \rangle}{\| - \varphi'(x) + \varphi'(-x)\|} \\ &\leq \frac{-\|\varphi'(x)\|^2 + (-1+2\epsilon)\|\varphi'(x)\|.\|\varphi'(-x)\|}{\| - \varphi'(x) + \varphi'(-x)\|} \\ &\leq \left(-1 + \frac{2\alpha\epsilon}{1+\alpha}\right) \|\varphi'(x)\| \end{aligned}$$

where $\alpha = \frac{\|\varphi'(-x)\|}{|\varphi'(x)\|}$. This clearly yields (9).

In order to glue the vector fields w_1 and w_2, take $\eta = \frac{1}{2}\min\{\eta_1, \eta_2\}$ and consider the following partition of unity:

$$\ell_1(x) = \mathrm{dist}(x, K_1^\eta \setminus K_2^\eta) \text{ resp.,} \quad \ell_2(x) = \mathrm{dist}(x, K_2^\eta \setminus K_1^\eta).$$

Since K_1 and K_2 are symmetric, ℓ_1 and ℓ_2 are even functions and the vector field

$$w(x) = \frac{\ell_2(x)}{\ell_1(x)+\ell_2(x)} \frac{w_1(x)}{\|w_1(x)\|} + \frac{\ell_1(x)}{\ell_1(x)+\ell_2(x)} \frac{w_2(x)}{\|w_2(x)\|}$$

is therefore odd and continuous on $K_1^\eta \cup K_2^\eta$. Moreover, for any x in $K_1^\eta \cup K_2^\eta$, we have in view of (5) and (9) that $\|w(x)\| \le 1$ and

$$\langle \varphi'(x), w(x) \rangle < -\min\left\{\frac{\epsilon}{2}, 1-2\epsilon\right\} \|\varphi'(x)\|. \tag{10}$$

Now let $\tilde{w}$ be any continuous extension of w to the whole of H and let $\tilde{v}(x) = \frac{1}{2}\big(\tilde{w}(x) - \tilde{w}(-x)\big)$ be its corresponding equivariant field. Note that $\tilde{v} = w$ on $K_1^\eta \cup K_2^\eta$ and hence is a continuous equivariant extension of w.

We still need to make the vector field uniformly bounded everywhere and zero on B. For that, consider the set $N = \{x \in H; \tilde{v}(x) = 0\}$. It is a closed symmetric set which is disjoint from K^η, since on the latter set $\varphi'(x)$ and $\varphi'(-x)$ are not parallel by assumption (1), and therefore we can clearly assume–modulo taking a smaller η–that $N^\eta \cap K^\eta = \emptyset$.

Let now

$$F_1(x) = \begin{cases} \dfrac{\tilde{v}(x)}{\|\tilde{v}(x)\|} & \text{if } x \notin N \\ 0 & \text{otherwise} \end{cases}$$

and let

$$g(x) = \frac{\mathrm{dist}(x, N)}{\mathrm{dist}(x, H \setminus N^\eta) + \mathrm{dist}(x, N)}$$

and

$$h(x) = \frac{\mathrm{dist}(x, B)}{\mathrm{dist}(x, H \setminus B^\eta) + \mathrm{dist}(x, B)}.$$

It is clear that h and g are even and continuous functions and the vector field defined as $V(x) = h(x)g(x)F(x)$ is continuous, equivariant and satisfies assertions (i) and (ii) while it coincides with $\frac{w(x)}{\|w(x)\|}$ on K^η and hence (iii) is also satisfied.

LEMMA II.3. – *Let φ be a C^1-functional on a Hilbert space H and let B be a given closed symmetric set. Suppose K_1 and K_2 are two compact subsets of H that are disjoint from B and such that for some ϵ $(0 < \epsilon < 1/4)$,*

$$\|\varphi'(x)\| > \epsilon \text{ for every } x \in K_1 \cup K_2. \tag{11}$$

Assume further that $K_2 \cap (-K_2) = \emptyset$ while K_1 is a symmetric set not containing 0 on which we have

$$\frac{\langle \varphi'(x), \varphi'(-x) \rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} < 1 - \epsilon \text{ for every } x \in K_1 \,. \tag{12}$$

Then, there exists $\delta > 0$ and a continuous and equivariant deformation α in $C([0,1] \times H; H)$ such that for some $t_0 > 0$, the following holds for every $t \in [0, t_0)$,

(i) *$\alpha(t,x) = x$ for every $x \in B$.*

(ii) *$\|\alpha(t,x), x)\| \le t$ for every $x \in H$.*

(iii) *$\varphi(\alpha(t,x)) - \varphi(x) \le -\frac{\epsilon^2}{2} t$ for every x in $K_1^\delta \cup K_2^\delta$.*

Proof. – We start by showing that there exists $\eta > 0$ and an equivariant and continuous vector field $V : H \to H$ such that

(a) $V(x) = 0$ for every $x \in B$.

(b) $\|V(x)\| \le 1$ for every $x \in H$ and $\|V(x)\| = 1$ on $K_1^\eta \cup K_2^\eta$.

(c) $\langle \varphi'(x), V(x) \rangle < \frac{-\epsilon}{2} \|\varphi'(x)\|$ for every $x \in K_1^\eta \cup K_2^\eta$.

To do that, we first apply Lemma II.2 to the set K_1 to find $\eta_1 > 0$ and a vector field V_1 that satisfies the conclusion of that lemma on $K_1^{\eta_1}$.

In order to deal with K_2, first note that since $K_2 \cap (-K_2) = \emptyset$, we can take η_2 small enough so that

$$K_2^{\eta_2} \cap -(K_2^{\eta_2}) = \emptyset, \tag{13}$$

and such that for $x \in K^{\eta_2}$, we still have

$$\|\varphi'(x)\| > \epsilon. \tag{14}$$

Define on $K_2^{\eta_2}$ the (standard) vector field $w_2(x) = -\varphi'(x)$ and note that (13) allows us to extend it equivariantly and unambiguously to $K_2^{\eta_2} \cup -K_2^{\eta_2}$ by letting $w_2(-x) = -w_2(x)$ for each $x \in K_2^{\eta_2}$. It is clear that

$$\langle \varphi'(x), \frac{w_2(x)}{\|w_2(x)\|} \rangle < -\|\varphi'(x)\| \tag{15}$$

for every $x \in K_2^\eta$.

In order to glue the two vector fields V_1 and w_2, we proceed as in the proof of the preceeding lemma: that is take $\delta = \min\{\eta_1, \eta_2\}$ and consider an even partition of unity ℓ_1, ℓ_2 associated to the symmetric sets K_1 and $K_2 \cup -K_2$. The vector field

$$w(x) = \frac{\ell_2(x)}{\ell_1(x) + \ell_2(x)} V_1(x) + \frac{\ell_1(x)}{\ell_1(x) + \ell_2(x)} w_2(x)$$

is therefore equivariant and continuous on the δ-neighborhood of $K_1 \cup (K_2 \cup -K_2)$. Moreover, for any x in $K_1^\delta \cup K_2^\delta$, we have in view of (14) and Lemma II.2.(iii) that

$$\langle \varphi'(x), w(x) \rangle < -\min\left\{\frac{\epsilon}{2}, 1\right\} \|\varphi'(x)\|. \tag{16}$$

Now let $\tilde{w}$ be any continuous extension of w to the whole of H and let $\tilde{v}(x) = \frac{1}{2}\big(\tilde{w}(x) - \tilde{w}(-x)\big)$ be its corresponding equivariant field. Note that $\tilde{v} = w$ on the set $K_1^\delta \cup (K_2^\delta \cup -K_2^\delta)$ and hence is a continuous equivariant extension of w.

We still need to make the vector field uniformly bounded everywhere and zero on B. For that, consider again the set $N = \{x \in H; \tilde{v}(x) = 0\}$. It is a closed symmetric set which is disjoint from $K_1^\delta \cup (K_2^\delta \cup -K_2^\delta)$ and we can clearly assume–modulo taking a smaller δ–that the latter set is disjoint from N^δ.

Let now

$$F(x) = \begin{cases} \dfrac{\tilde{v}(x)}{\|\tilde{v}(x)\|} & \text{if } x \notin N \\ 0 & \text{otherwise} \end{cases}$$

and let

$$g(x) = \frac{\operatorname{dist}(x, N)}{\operatorname{dist}(x, H \setminus N^\delta) + \operatorname{dist}(x, N)}$$

and

$$h(x) = \frac{\operatorname{dist}(x, B)}{\operatorname{dist}(x, H \setminus B^\delta) + \operatorname{dist}(x, B)}.$$

Again, h and g are even and continuous functions and the vector field defined as $V(x) = h(x)g(x)F(x)$ is continuous, equivariant and satisfies assertions (a) and (b) while it coincides with $\frac{w(x)}{\|w(x)\|}$ on $K_1^\delta \cup K_2^\delta$ and hence (c) is also satisfied.

For any $(t,x) \in [0,1] \times H$, set $\alpha(t,x) = x + tV(x)$ and let $t_0 = \delta$. Assertions (i) and (ii) are clear. Moreover, for any $x \in K_1^\delta \cup K_2^\delta$ and any $t \leq t_0$, we have for some θ between 0 and 1,

$$\varphi(\alpha(t,x)) = \varphi(x + tv(x)) = \varphi(x) + \langle \varphi'(x + t\theta v(x)), tv(x) \rangle$$

so that by (16), we have $\varphi(\alpha(t,x)) - \varphi(x) \leq -\frac{\epsilon^2}{2}t$, and the proof of the lemma is complete.

Remark. – Note that φ is only decreasing along the deformation lines $\alpha(t,x)$ starting in a neighborhood of $K_1 \cup K_2$ and everywhere in H. Actually, φ is increasing along the flow starting on $-K_2$.

Proof of Theorem I.1. – Let $\epsilon > 0$ be small enough so that

$$\sup_B \varphi < c - \epsilon^2 \tag{17}$$

and

$$\varphi(0) \notin [c - \epsilon^2 \leq c + \epsilon^2] \tag{18}$$

and let A be a set in $\mathcal{F}$ such that

$$c \leq \sup \varphi(A) < c + \varepsilon^2. \tag{19}$$

Consider the subspace $\mathcal{L}$ of $C([0,1] \times H; H)$ consisting of all continuous and equivariant deformations η such that

$$\eta(t,x) = x \quad \text{for all } (t,x) \text{ in } K_0 = (\{0\} \times H) \cup ([0,1] \times B)$$

and $\sup\{\|\eta(t,x) - x\|; t \in [0,1], x \in H\} < +\infty$.

The space $\mathcal{L}$ is a complete metric space once equipped with the following metric

$$\delta(\eta, \eta') = \sup\{\|\eta(t,x) - \eta'(t,x)\|; (t,x) \in [0,1] \times H\}.$$

Define a function $I : \mathcal{L} \to \mathbb{R}$ by $I(\eta) = \sup\{\varphi(\eta(1,x)); x \in A\}$. Let $\bar{\eta}$ be the identity in $\mathcal{L}$, that is $\bar{\eta}(t,x) = x$ for all (t,x) in $[0,1] \times H$ and note that

$$I(\bar{\eta}) = \sup\{\varphi(x); x \in A\} < c + \varepsilon^2 \leq \inf\{I(\eta); \eta \in \mathcal{L}\} + \varepsilon^2. \tag{20}$$

Apply now Ekeland's theorem to get η_0 in $\mathcal{L}$ such that

$$I(\eta_0) \leq I(\bar{\eta}) \tag{21}$$

$$\delta(\eta_0, \bar{\eta}) \leq \varepsilon \tag{22}$$

$$I(\eta) \geq I(\eta_0) - \varepsilon\delta(\eta, \eta_0) \quad \text{for all } \eta \text{ in } \mathcal{L}. \tag{23}$$

Consider the following compact subsets of $\eta_0(\{1\} \times A)$,

$$C = \{x \in \eta_0(\{1\} \times A); \varphi(x) = I(\eta_0)\}$$

$$K_1 = \{x \in \eta_0(\{1\} \times A); \varphi(x) \text{ and } \varphi(-x) \in [c - \epsilon^2, c + \epsilon^2]\}$$

and

$$K_2 = \{x \in \eta_0(\{1\} \times A); \varphi(x) \geq c \text{ while } \varphi(-x) \leq c - \epsilon^2\}.$$

and note that K_1 is symmetric and that $K_2 \cap (-K_2) = \emptyset$. Also,

$$C \subset K_1 \cup K_2 \tag{24}$$

while (17) implies that

$$(K_1 \cup K_2) \cap B = \emptyset. \tag{25}$$

We shall now prove the following:

Claim. – If $\|\varphi'(x)\| \geq 2\sqrt{\epsilon}$ for all $x \in K_1 \cup K_2$, then there exists $x_\varepsilon \in K_1$ such that

$$\frac{\langle \varphi'(x_\epsilon), \varphi'(-x_\epsilon)\rangle}{\|\varphi'(x_\epsilon)\| . \|\varphi'(-x_\epsilon)\|} \geq 1 - 2\sqrt{\epsilon}. \tag{26}$$

This would prove the theorem since first, any point x_ϵ in $K_1 \cup K_2$ necessarily satisfies $c - \epsilon^2 \leq \varphi(x_\epsilon) \leq c + \epsilon^2$ and $\text{dist}(x_\epsilon, A) \leq \epsilon$ in view of (22). Moreover, the above claim says that either $\|\varphi'(x)\| \leq 2\sqrt{\epsilon}$ for some $x \in K_1 \cup K_2$ or there exists x_ϵ such that $\varphi(x_\epsilon)$ and $\varphi(-x_\epsilon)$ belong to $[c - \epsilon^2, c + \epsilon^2]$ and $\frac{\langle \varphi'(x_\epsilon), \varphi'(-x_\epsilon)\rangle}{\|\varphi'(x_\epsilon)\| . \|\varphi'(-x_\epsilon)\|} \geq 1 - 2\sqrt{\epsilon}$.

To establish the above claim, we shall assume it is false and work towards a contradiction. In that event, we have for every $x \in K_1$, $\frac{\langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\| . \|\varphi'(-x)\|} < 1 - 2\sqrt{\epsilon}$, and all the hypothesis of Lemma II.3 are satisfied. Hence, we may construct a continuous and equivariant deformation $\alpha(t, x)$ satisfying the conclusion of that lemma with a suitable $\delta > 0$ and a time $t_0 > 0$. For $0 < \lambda < t_0$, we consider the function $\eta_\lambda(t, x) = \alpha(t\lambda, \eta_0(t, x))$. It belongs to $\mathcal{L}$ since it is clearly continuous on $[0, 1] \times H$ and since for all $(t, x) \in (\{0\} \times H) \cup ([0, 1] \times B)$, we have

$$\eta_\lambda(t, x) = \alpha(t\lambda, \eta_0(t, x)) = \alpha(t\lambda, x) = x$$

Since $\delta(\eta_\lambda, \eta_0) < t\lambda \leq \lambda$, we get from (23) that $I(\eta_\lambda) \geq I(\eta_0) - \epsilon\lambda$.

Since A is compact, let $x_\lambda \in A$ be such that $\varphi(\eta_\lambda(1, x_\lambda)) = I(\eta_\lambda)$. We then have

$$\varphi(\eta_\lambda(1, x_\lambda)) - \varphi(\eta_0(1, x)) \geq -\varepsilon\lambda \quad \text{for every } x \in A. \tag{27}$$

If x_0 is any cluster point of (x_λ) when $\lambda \to 0$, we have from (27) that φ attains its maximum on $\eta_0(1, A)$ at the point $\eta_0(1, x_0)$ which means that $\eta_0(1, x_0) \in C$ and hence it belongs to $K_1 \cup K_2$.

It follows that for λ small enough, $\eta_0(1, x_\lambda)$ belongs to $K_1^\delta \cup K_2^\delta$. But for such $\lambda's$ we have from (iii) of Lemma II.3, that

$$\begin{aligned} \varphi(\eta_\lambda(1, x_\lambda)) - \varphi(\eta_0(1, x_\lambda)) &= \varphi(\alpha(\lambda, \eta_0(1, x_\lambda)) - \varphi(\eta_0(1, x_\lambda)) \\ &\leq -2\epsilon\lambda. \end{aligned} \tag{28}$$

Combining (27) and (28) we get the obvious contradiction $-\epsilon \leq -2\epsilon$ and the proof of the theorem is complete.

III. LOCATION OF THE VIRTUAL CRITICAL POINTS

We shall now establish the following quantitative version of Theorem (B).

THEOREM III.1. – *Let φ be a C^1-functional on a Hilbert space H and consider a $\mathbb{Z}_2$-homotopy stable family $\mathcal{F}$ in H with a symmetric boundary B and let $c := c(\varphi, \mathcal{F})$. Suppose F is a closed subset oh H that is dual to the family $\mathcal{F}$ such that* $0 \notin F \cap \varphi^{-1}(c)$ *and* $\inf \varphi(F) \geq c - \delta$ *for some* $\delta > 0$.

If δ is small enough, then for any A in $\mathcal{F}$ satisfying $\sup \varphi(A) \leq c + \delta$, *there exists $x_\delta \in H$ such that*

(i) $c - 4\delta \leq \varphi(x_\delta) \leq c + 11\delta$ *and* $\varphi(-x_\delta) \leq c + 11\delta$,

(ii) $\operatorname{dist}(x_\delta, A) \leq 3\sqrt{\delta}$,

(iii) $\operatorname{dist}(x_\delta, F \cup -F) \leq 5\sqrt{\delta}$,

while satisfying one of the following (non-mutually exclusive) properties: Either

(iv.a) $\|\varphi'(x_\delta)\| < 2\delta^{1/4}$,

or

(iv.b) $c - 4\delta \leq \varphi(-x_\delta)$ *and* $\frac{\langle \varphi'(x_\delta), \varphi'(-x_\delta)\rangle}{\|\varphi'(x_\delta)\| \cdot \|\varphi'(-x_\delta)\|} \geq 1 - 2\delta^{1/4}$.

Besides Corollary I.6, it is worth noting the following consequences of the above theorem.

COROLLARY III.2. – *Let φ be a C^1-functional on a Hilbert space H and consider a Z_2-homotopy-stable family $\mathcal{F}$ in H with a symmetric boundary B and let $c := c(\varphi, \mathcal{F})$. Let F be a symmetric closed subset of H that is dual to the family $\mathcal{F}$ and such that* $0 \notin F \cap \varphi^{-1}(c)$ *while* $\inf \varphi(F) \geq c$.

If φ satisfies $(sP\text{-}S)_{F,c}$, then there exists a virtual critical point at the level c on the set F.

Moreover, for any $\delta > 0$ and $\epsilon > 0$, there exists $A \in \mathcal{F}$ such that

$$\sup \varphi(A) < c + \epsilon \quad \textit{and} \quad A \subset (H \backslash F) \cup (F \cap \tilde{E}_c)^{\delta}.$$

Proof. – The first part is immediate from Theorem III.1. Now, if the second claim were not true, then for some $\epsilon_0 > 0$ and some $\delta_0 > 0$, the set $F' = F \backslash (F \cap \tilde{E}_c)^{\delta_0}$ would be dual to the class$\tilde{\mathcal{F}}_{\epsilon_0} = \{A \in \mathcal{F}; \sup \varphi(A) \leq \epsilon_0\}$. By the first part, this would imply that $F' \cap \tilde{E}_c \neq \emptyset$ which is absurd.

COROLLARY III.3. – *Under the hypothesis of Corollary III.2 on φ and $\mathcal{F}$, let F be a closed subset of H that is dual to the family $\mathcal{F}$ and which now satisfies* $0 \notin \varphi^{-1}(c)$ *and* $\sup \varphi(B) \leq \inf \varphi(F)$.

If φ verifies $(sP\text{-}S)_c$, then c is a virtual critical value for φ.

Proof. – Indeed, the fact that F is dual to $\mathcal{F}$ implies that $\inf \varphi(F) \leq c$. So we distinguish the two cases:

(a) Either $\sup \varphi(B) < c$, which means that Theorem (A) applies,

(b) or $\sup \varphi(B) = \inf \varphi(F) = c$, which means that the conditions of Corollary III.2 are satisfied and therefore we get a virtual critical point on $F \cup -F$.

For the sequel we denote

$$G_c = \{x \in H;\ \varphi(x) < c\} \quad \text{and} \quad L_c = \{x \in H; \varphi(x) \geq c\}$$

COROLLARY III.4. – *Under the hypothesis of Corollary* I.3, *for any $\delta > 0$ and any $\epsilon > 0$, there exists $A \in \mathcal{F}$ such that*

$$\sup \varphi(A) < c + \epsilon \quad \textit{and} \quad A \subset G_c \cup (\tilde{E}_c)^{\delta}.$$

Proof. – If not, then for some $\epsilon_0 > 0$ and some $\delta_0 > 0$, the set $F = L_c \setminus (\tilde{E}_c)^{\delta_0}$ will be dual to the class $\tilde{\mathcal{F}}_{\epsilon_0} = \{A \in \mathcal{F}; \sup \varphi(A) \leq \epsilon_0\}$. By Corollary I.6, this would imply that $F \cap \tilde{E}_c \neq \emptyset$ which is absurd.

COROLLARY III.5. – *Let φ be a C^1-functional on a Hilbert space H and consider a $\mathbb{Z}_2$-homotopy-stable family $\mathcal{F}$ in H with a closed symmetric boundary B. Suppose that* $\sup \varphi(B) < c := c(\varphi, \mathcal{F})$ *and that* $\varphi(0) \neq c$. *If φ' is uniformly continuous around the level c, then for every $\epsilon > 0$, there exist $\delta > 0$ and $A \in \mathcal{F}$ with* $\sup \varphi(A) < c + \delta$ *such that:*

For any $x \in A$ satisfying $\varphi(x) \geq c - \delta$, we have:
Either

(i) $\|d\varphi(x)\| < \epsilon$,

or

(ii) $|\varphi(x) - \varphi(-x)| < \epsilon$ and $\frac{\langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\| \cdot \|\varphi'(-x)\|} > 1 - \epsilon$.

Proof. – If the claim does not hold, there exists then an $\epsilon_0 > 0$ such that for every $\delta > 0$, the set

$$F = L_{c-\delta} \cap \{x; \|d\varphi(x)\| \geq \epsilon_0\} \cap \left(\{x; |\varphi(x) - \varphi(-x)| \geq \epsilon_0\} \cup \left\{\frac{\langle \varphi'(x), \varphi'(-x)\rangle}{\|\varphi'(x)\| \cdot \|\varphi'(-x)\|} \leq 1 - \epsilon_0\right\}\right)$$

satisfy the conditions of Theorem III.2: that is F is dual to $\mathcal{F}$ and $\inf_F \varphi \geq c - \delta$. Hence, there exists x_δ that satisfies the conclusions of that Theorem. By letting $\delta \to 0$, we construct a sequence (x_δ) whose distance to F goes to zero while satisfying the assertions (i), (ii) and (iii) of Theorem III.1. This contradicts the uniform continuity of φ'.

Here is another immediate application of Theorem III.1.

COROLLARY III.6. – *Let φ be a C^1-functional on a Hilbert space H and consider a Z_2-homotopy-stable family $\mathcal{F}$ in H with a symmetric boundary B. Let ψ be a continuous even functional such that $\psi \leq \varphi$ on H while $c = c(\varphi, \mathcal{F}) = c(\psi, \mathcal{F})$. Suppose* $\sup \psi(B) < c$ *and that φ verifies (sP-S)$_c$, then there exists $x \in H$, such that*

(i) $\varphi(x) = \psi(x) = c$

and one of the following conditions hold:
Either

(ii.a) $\varphi'(x) = \psi'(x) = 0$,

or

(ii.b) $\varphi'(x) = \psi'(x) = \lambda\varphi'(-x) = -\psi'(-x)$ *for some* $\lambda > 0$.

Proof. – It is enough to realize that the closed symmetric set $F = \{\psi \geq c\}$ is then dual to $\mathcal{F}$ while $\inf \varphi(F) = c$. Corollary III.2 then applies to give the claim.

Proof of Theorem III.1. – Suppose δ is small enough so that

$$0 < \delta < \max\left\{\frac{1}{32}\mathrm{dist}^2(B, F); \frac{1}{8}[\inf \varphi(F) - \sup \varphi(B)]\right\}$$

and

$$\delta < \max\left\{\frac{1}{32}|\varphi(0) - c|, \frac{16}{9}\mathrm{dist}^2(0, F)\right\}$$

Let $\delta = \varepsilon^2/8$ which implies that

$$0 < \varepsilon < \max\left\{\frac{1}{2}\,\mathrm{dist}(B, F); \sqrt{[\inf\varphi(F) - \sup\varphi(B)]^+}\right\} \tag{1}$$

where $\alpha^+ = \alpha \vee 0$, that

$$\inf\varphi(F) \geq c - \varepsilon^2/8 \tag{2}$$

and that

$$\text{either } \varphi(0) \notin [c - \epsilon^2/2, c + 2\epsilon^2] \text{ or } \mathrm{dist}(0, F) < 3\epsilon/2. \tag{3}$$

We shall prove the existence of $x_\varepsilon \in H$ that satisfies

(i) $c - \varepsilon^2/2 \leq \varphi(x_\epsilon) \leq c + 11\varepsilon^2/8$ and $\varphi(-x_\epsilon) \leq c + 11\varepsilon^2/8$,

(ii) $\mathrm{dist}(x_\varepsilon, F \cup -F) \leq 3\varepsilon/2$,

(iii) $\mathrm{dist}(x_\varepsilon, A) \leq \varepsilon/2$,

and such that one of the following assertions holds:

(iv.a) Either $\|\varphi'(x_\epsilon)\| \leq 2\sqrt{\epsilon}$,

(iv.b) or $\varphi(-x_\epsilon) \geq c - \varepsilon^2/2$ and $\frac{\langle\varphi'(x_\epsilon), \varphi'(-x_\epsilon)\rangle}{\|\varphi'(x_\epsilon)\|\cdot\|\varphi'(-x_\epsilon)\|} \geq 1 - 2\sqrt{\epsilon}$.

This will clearly imply the claims of Theorem III.1.

Let $F_\varepsilon = \{x \in H; \mathrm{dist}(x, F) < \varepsilon\}$ and consider the subspace $\mathcal{L}$ of $C([0,1] \times H; H)$ consisting of all equivariant deformations η such that

$$\eta(t, x) = x \text{ for all } (t, x) \in K_0 = \{0\} \times H \cup [0,1] \times (B \cup (A \setminus (F_\epsilon \cup -F_\epsilon)))$$

and $\sup\{\|\eta(t, x) - x\|; t \in [0,1], x \in H\} < +\infty$.

Since $(\{0\} \times H) \cup ([0,1] \times B) \subset K_0$, we get that $\eta(\{1\} \times A) \in \mathcal{F}$ for all η in $\mathcal{L}$. The space $\mathcal{L}$ equipped with the uniform metric δ is a complete metric space.

Set now $\psi(x) = \max\{0, \varepsilon^2 - \varepsilon\,\mathrm{dist}(x, F)\}$ and define a lower semi-continuous function $I : \mathcal{L} \to \mathbb{R}$ by

$$I(\eta) = \sup\{(\varphi + \psi)(\eta(1, x)); x \in A\})$$

Let $d = \inf\{I(\eta); \eta \in \mathcal{L}\}$. Since $\eta(\{1\} \times A) \in \mathcal{F}$ for all $\eta \in \mathcal{L}$ and since $\psi = \varepsilon^2$ on F we get from the fact that F is dual to $\mathcal{F}$ and estimate (2) that

$$I(\eta) \geq \sup\{(\varphi + \psi)(x); x \in \eta(\{1\} \times A) \cap F\} \geq c - \varepsilon^2/8 + \varepsilon^2.$$

Hence

$$d \geq c + 7\varepsilon^2/8. \tag{4}$$

Consider again the identity element $\bar{\eta}$ in $\mathcal{L}$ and note that

$$d \leq I(\bar{\eta}) = \sup\{(\varphi+\psi)(x); x \in A\} < c + \varepsilon^2/8 + \varepsilon^2 = c + 9\varepsilon^2/8. \tag{5}$$

Combine (4) and (5) to get that $\bar{\eta}$ satisfies

$$I(\bar{\eta}) < c + 9\varepsilon^2/8 \leq d + \varepsilon^2/4 = \inf\{I(\eta); \eta \in \mathcal{L}\} + \varepsilon^2/4. \tag{6}$$

Apply Ekeland's theorem to get η_0 in $\mathcal{L}$ such that

$$I(\eta_0) \leq I(\bar{\eta}) \tag{7}$$

$$\delta(\eta_0, \bar{\eta}) \leq \varepsilon/2 \tag{8}$$

and

$$I(\eta) \geq I(\eta_0) - \frac{\epsilon}{2}\delta(\eta, \eta_0) \quad \text{for all } \eta \text{ in } \mathcal{L}. \tag{9}$$

We now show that

$$\sup(\varphi+\psi)((A \setminus F_\varepsilon) \cup B) \leq d - 3\varepsilon^2/4. \tag{10}$$

Indeed, since $\psi = 0$ outside F_ε we get from (7) that

$$\sup(\varphi+\psi)(A \setminus F_\varepsilon) \leq \sup\varphi(A) < c + \varepsilon^2/8 \leq d - 3\varepsilon^2/4.$$

We now distinguish two cases:

- either $0 < \varepsilon < \frac{1}{2}\operatorname{dist}(B, F)$ which means that $B \subset A \setminus F_\varepsilon$ and we are done;
- or $0 < \varepsilon < \sqrt{[\inf\varphi(F) - \sup\varphi(B)]^+}$ which means that the latter is strictly positive and $\sup\varphi(B) \leq \inf\varphi(F) - \varepsilon^2 \leq c - \varepsilon^2$ since F is dual to $\mathcal{F}$. Hence $\sup(\varphi+\psi)(B) \leq c \leq d - 7\varepsilon^2/8$ by (4).

In both cases, (10) is verified.

Consider now the following compact subsets of $\eta_0(\{1\} \times A)$,

$$C = \{x \in \eta_0(\{1\} \times A); (\varphi+\psi)(x) = I(\eta_0)\}$$

$$K_1 = \left\{x \in \eta_0(\{1\} \times A); (\varphi+\psi)(x) \text{ and } (\varphi+\psi)(-x) \in \left[d - \frac{3\epsilon^2}{8}, d + \frac{\epsilon^2}{4}\right]\right\},$$

$$K_2 = \left\{x \in \eta_0(\{1\} \times A); (\varphi+\psi)(x) \geq d \text{ while } (\varphi+\psi)(-x) \leq d - \tfrac{3\epsilon^2}{8}\right\}$$

and let $B' = B \cup (A \setminus (F_\epsilon \cup -F_\epsilon))$ be the new (symmetric!) boundary.

Note that K_1 is also symmetric, while $K_2 \cap (-K_2) = \emptyset$. Moreover,

$$C \subset K_1 \cup K_2 \tag{11}$$

while (10) implies that

$$(K_1 \cup K_2) \cap B' = \emptyset. \tag{12}$$

Also note that $0 \notin K_1$. Indeed, otherwise we would have

$$c - \varepsilon^2/2 \leq \varphi(0) \leq c + 11\varepsilon^2/8$$

and $0 = \eta_0(1, x)$ for some x which, in view of (12), must be in F_ϵ or -F_ϵ. We may assume without loss that $\text{dist}(x, F) \leq \epsilon$. On the other hand, by (8) we have $\|x\| = \|\eta_0(1, x) - x\| \leq \delta(\eta_0, \bar{\eta}) \leq \epsilon/2$. Hence $\text{dist}(0, F) \leq 3\epsilon/2$ which in view of the above estimate on $\varphi(0)$, contradicts (3).

We shall now prove the following:

Claim. – If $\|\varphi'(x)\| \geq 2\sqrt{\epsilon}$ for all $x \in K_1 \cup K_2$, then there exists $x_\varepsilon \in K_1$ such that

$$\frac{\langle \varphi'(x_\epsilon), \varphi'(-x_\epsilon) \rangle}{\|\varphi'(x_\epsilon)\|.\|\varphi'(-x_\epsilon)\|} > 1 - 2\sqrt{\epsilon}.$$

Before proving it, let us show how it implies Theorem III.1. Indeed, any point x_ϵ in $K_1 \cup K_2$ satisfies $d - \frac{3\epsilon^2}{8} \leq (\varphi + \psi)(x_\varepsilon) \leq d + \frac{\epsilon^2}{4}$. Since $0 \leq \psi \leq \varepsilon^2$, we get from (4) that $c - \varepsilon^2/2 \leq \varphi(x_\varepsilon) \leq c + 11\varepsilon^2/8$ and since $-x_\epsilon \in \eta_0(\{1\} \times A)$, we have that $(\varphi + \psi)(-x_\epsilon) \leq I(\eta_0) \leq I(\bar{\eta}) \leq d + \epsilon^2/4$ and hence $\varphi(-x_\epsilon) \leq c + 11\epsilon^2/8$. Assertion (i) is therefore established.

For (ii) write $x_\varepsilon = \eta_0(1, x)$ where, in view of (12), x is necessarily in the set $F_\epsilon \cup -F_\epsilon$. Hence $\text{dist}(x, F \cup -F) \leq \epsilon$. On the other hand, by (8) we have $\|x_\varepsilon - x\| = \|\eta_0(1, x) - x\| \leq \delta(\eta_0, \bar{\eta}) \leq \epsilon/2$. Hence $\text{dist}(x_\epsilon, F \cup -F) \leq 3\epsilon/2$.

Note that (iv) is also satisfied since $x \in A$ and $\text{dist}(x_\epsilon, A) \leq \epsilon/2$ in view of (8). Finally, if x_ϵ satisfies the claim, then since it belongs to K_1, we have

$$\varphi(x_\epsilon) \text{ and } \varphi(-x_\epsilon) \in c - \frac{\epsilon^2}{2}, c + 11\frac{\epsilon^2}{8} \text{ and } \frac{\langle \varphi'(x), \varphi'(-x) \rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} \geq 1 - 2\sqrt{\epsilon}.$$

Proof of the claim. – Suppose it is false. This means that for every $x \in K_1$, $\frac{\langle \varphi'(x), \varphi'(-x) \rangle}{\|\varphi'(x)\|.\|\varphi'(-x)\|} < 1 - 2\sqrt{\epsilon}$, and since $0 \notin K_1$, all the hypothesis of Lemma II.3 on K_1, K_2 and B' are therefore satisfied. Hence, we may construct a continuous and equivariant deformation $\alpha(t, x)$ satisfying the conclusion of that lemma with a suitable $\delta > 0$ and a time $t_0 > 0$. For $0 < \lambda < t_0$, we consider the function $\eta_\lambda(t,x) = \alpha(t\lambda, \eta_0(t,x))$. It belongs to $\mathcal{L}$ since it is clearly continuous on $[0,1] \times H$ and since for all $(t,x) \in (\{0\} \times H) \cup ([0,1] \times B')$, we have

$$\eta_\lambda(t,x) = \alpha(t\lambda, \eta_0(t,x)) = \alpha(t\lambda, x) = x$$

Since $\delta(\eta_\lambda, \eta_0) < t\lambda \leq \lambda$, we get from (6) that $I(\eta_\lambda) \geq I(\eta_0) - \epsilon\lambda/2$.

A being compact, we let $x_\lambda \in A$ be such that $(\varphi+\psi)(\eta_\lambda(1,x_\lambda)) = I(\eta_\lambda)$. That is

$$(\varphi+\psi)(\eta_\lambda(1,x_\lambda)) - (\varphi+\psi)(\eta_0(1,x)) \geq -\varepsilon\lambda/2 \quad \text{for every } x \in A. \tag{13}$$

Since the Lipschitz constant of ψ is less than ϵ we get

$$\varphi(\eta_\lambda(1,x_\lambda)) - \varphi(\eta_0(1,x_\lambda)) \geq -3\varepsilon\lambda/2. \tag{14}$$

If x_0 is any cluster point of (x_λ) when $\lambda \to 0$, we have from (13) that $\varphi + \psi$ attains its maximum on $\eta_0(1,A)$ at the point $\eta_0(1,x_0)$ which means that $\eta_0(1,x_0) \in C$ and hence in $K_1 \cup K_2$.

It follows that for λ small enough, $\eta(1,x_\lambda)$ belongs to $K_1^\delta \cup K_2^\delta$. For such $\lambda's$ we have from (iii) of Lemma II.3, that

$$\begin{aligned}\varphi(\eta_\lambda(1,x_\lambda)) - \varphi(\eta_0(1,x_\lambda)) &= \varphi(\alpha(\lambda, \eta_0(1,x_\lambda)) - \varphi(\eta_0(1,x_\lambda)) \\ &\leq -3\epsilon\lambda. \end{aligned}\tag{15}$$

Combining (14) and (15) we get the obvious contradiction $-3\epsilon/2 \leq -2\epsilon$ and the proof of the theorem is complete.

IV. MULTIPLICITY RESULTS FOR VIRTUAL CRITICAL POINTS

We first recall the well known properties of the $\mathbb{Z}_2$-index denoted by γ. For more details, we refer to [R] or [S]. It satisfies the following:

(I1) $\gamma(A) = 0$ if and only if $A = \emptyset$.

(I2) $\gamma(A_2) \geq \gamma(A_1)$ if there is an odd continuous map from A_1 to A_2.

(I3) If K is compact symmetric, there exists a closed symmetric neighborhood $K^\delta = \{x; \mathrm{dist}(x,K) \leq \delta\}$ of K so that $\gamma(K^\delta) = \gamma(K)$.

(I4) $\gamma(A_1 \cup A_2) \leq \gamma(A_1) + \gamma(A_2)$ for all closed symmetric sets A_1, A_2.

(I5) If K is compact symmetric, $0 \notin K$ and $\gamma(K) \geq 2$ then K is infinite.

(I6) If K is compact symmetric and $0 \notin K$, then $\gamma(K) < +\infty$.

Here is the first multiplicity result

THEOREM IV.1. – *Let φ be a C^1-functional on a Hilbert space H and consider a Z_2-homotopy-stable family $\mathcal{F}$ in H with a symmetric boundary B and let $c := c(\varphi, \mathcal{F})$. Suppose F is a symmetric closed subset of H that is dual to the family $\mathcal{F}$ and such that $0 \notin F \cap \varphi^{-1}(c)$ while $\inf \varphi(F) \geq c$. If φ satisfies (sP-S)$_{F,c}$, then we have*

$$\gamma(E_c^* \cap F) \geq \inf\{\gamma(A \cap F); A \in \mathcal{F}\}.$$

Proof. – Let $n = \inf\{\gamma(A \cap F); A \in \mathcal{F}\}$. By property (I3) of the index, there exists a symmetric neighborhood U of $E_c^* \cap F$ such that $\gamma(\overline{U}) = \gamma(E_c^* \cap F)$. Let $F' = H \setminus U$. It is clearly closed and symmetric. By property (I4) of the index, we have for A in $\mathcal{F}$,

$$\begin{aligned}\gamma(A \cap F) &\leq \gamma(A \cap F \setminus U) + \gamma(\overline{U}) \\ &= \gamma(A \cap F \cap F') + \gamma(E_c^* \cap F).\end{aligned}$$

It follows that if $\gamma(E_c^* \cap F) \leq n-1$, then $\gamma(A \cap F \cap F') \geq 1$ and in particular $A \cap F \cap F' \neq \emptyset$ for all A in $\mathcal{F}$ by property (I1). In other words, the set $F \cap F'$ is dual to the class $\mathcal{F}$. On the other hand $\inf \varphi(F \cap F') \geq \inf \varphi(F) \geq c$. Hence Corollary III.2 applies to $F \cap F'$ and we get that $E_c \cap F \cap F' \neq \emptyset$ which is clearly a contradiction.

The following multiplicity result is more reminescent of the L-S theory.

THEOREM IV.2. – *Let φ be a C^1-functional on a Hilbert space H satisfying (sP-S)$_c$ for any $c \in \mathbb{R}$ and consider a decreasing sequence of Z_2-homotopy stable families $(\mathcal{F}_j)_{j=1}^N$ in H (with N possibly infinite) with boundaries $(B_j)_{j=1}^N$, that satisfy the following excision property:*

(E) *For every $1 \leq j \leq j+p < N$, any A in $\mathcal{F}_{j+p}$ and any U open and symmetric such that $\overline{U} \cap B_j = \emptyset$ and $\gamma(\overline{U}) \leq p$, we have $A \setminus U \in \mathcal{F}_j$.*

Let F be a closed symmetric set that is dual to $\mathcal{F}_j$ while $\sup \varphi(B_j) \leq d := \inf \varphi(F)$ for each $1 \leq j \leq N$.

Set $c_j = c(\varphi, \mathcal{F}_j)$, let $M = \sup\{k \geq 0; c_k = d\}$ and suppose $\varphi(0) < c_1$. Then,

(a) $\gamma(E_{c_M}^* \cap F) \geq M$.

(b) *For very $M < j \leq j+p \leq N$ such that $c_j = c_{j+p}$, we have $\gamma(E_{c_j}^*) \geq p+1$.*

In particular, if $0 \in (X \setminus F) \cap \{\varphi \leq d\}$ then

(c) *φ has at least N distinct virtual critical points.*

(d) *φ has an unbounded sequence of virtual critical values whenever the sequence $(\mathcal{F}_j)_{j=1}^N$ is infinite ($N = +\infty$).*

Proof. – (a) Assume $1 \leq M$ since otherwise there is nothing to prove. In view of Theorem IV.1, it is enough to show that $\gamma(A \cap F) \geq M$ for every A in $\mathcal{F}_M$. Suppose this was false for some A in $\mathcal{F}_M$. By property (I3) of the index and since $F \cap B_1 = \emptyset$ we can find U open and symmetric such that $A \cap F \subset U$, $\overline{U} \cap B_1 = \emptyset$ and $\gamma(\overline{U}) = \gamma(A \cap F) \leq M - 1$. The excision property implies that $A \setminus U \in \mathcal{F}_1$ and hence that $(A \setminus U) \cap F \neq \emptyset$ which is clearly a contradiction.

(b) Suppose now $M < j < j + p \leq N$ such that $c_j = c_{j+p}$. If we suppose $\gamma(E^*_{c_j}) \leq p$ and since $\sup \varphi(B_j) \leq \inf \varphi(F) < c_j$, we can find, as above, an open symmetric neighborhood U of $E^*_{c_j}$ such that $\overline{U} \cap B_j = \emptyset$ and $\gamma(\overline{U}) \leq p$. By the excision property, we have $A \setminus U \in \mathcal{F}_j$ for every A in $\mathcal{F}_{j+p}$, hence $(A \setminus U) \cap \{\varphi \geq c_j\}$ is non-empty. It follows that $F' = \{\varphi \geq c_{j+p}\} \setminus U = \{\varphi \geq c_j\} \setminus U$ is dual to the class $\mathcal{F}_{j+p}$. By Corollary I.6, $E^*_{c_j} \setminus U = E^*_{c_{j+p}} \cap F' \neq \emptyset$ which is clearly a contradiction.

(c) Follows immediately from the above estimates and property (I5) of the index since $0 \notin K_{c_i} \cap F$ when $1 \leq i \leq M$ and $0 \notin K_{c_i}$ if $M + 1 \leq i \leq N$ since then $d < c_i$.

(d) Suppose now that $(\mathcal{F}_j)_j$ is an infinite sequence of $\mathbb{Z}_2$-homotopy stable classes. We note first that M must be finite. Indeed, we have by (a) that $\gamma(E^*_d \cap F) \geq M$. Since φ satisfies $(sP - S)_d$, $E^*_d \cap F$ is compact and since the latter does not contain 0, we cannot have $\gamma(E^*_d \cap F) = +\infty$ in view of property (I6) of the index.

The same reasoning shows that the sequence $(c_j)_{j>M}$ cannot become stationary. It remains to show that $c_\infty = \lim_j c_j$ must be $+\infty$. Suppose not and consider

$$E = \bigcup_{c_{M+1} \leq c \leq c_\infty} E^*_c.$$

Since it is symmetric, compact and does not contain 0, we have $\gamma(E) = q < +\infty$. Let U be an open symmetric set containing K, disjoint from $\cup_j B_j \subset \{\varphi \leq d\}$ and such that $\gamma(\overline{U}) = q$. We shall show the existence of a sequence $(x_n)_n$ with $\lim_n \varphi(x_n) = c$, $\lim_n \operatorname{dist}(x_n, H \setminus U) = 0$ and such that either $\lim_n \varphi'(x_n) = 0$, or $\lim_n \varphi(-x_n) = c$ and $\lim_n(\varphi'(x_n) - \lambda_n \varphi'(-x_n)) = 0$ for some positive sequence of reals λ_n.

Since φ has (sP-S)c for any c, we will then get that $E_c \setminus U \neq \emptyset$ which is clearly a contradiction.

To get the sequence of approximate virtual critical points, fix $\delta > 0$ such that $c_{M+1} < c - \delta$ and find j large enough so that $c_{M+1} < c - \delta \leq c_j \leq c_{j+q} \leq c$. By the excision property, we have for every A in $\mathcal{F}_{j+q}$ that

$A \setminus U \in \mathcal{F}_j$. Hence if we set $F' = \{\varphi \geq c_j\} \setminus U$ we get that $F' \cap A \neq \emptyset$ for all A in $\mathcal{F}_{j+q}$ and $F' \cap B_{j+q} = \emptyset$, since $\sup \varphi(B_{j+q}) \leq d < c_{M+1} < c_{j+q}$. Moreover, $\inf \varphi(F') \geq c_j \geq c - \delta \geq c_{j+q} - \delta$.

It follows from Theorem III.1 that if we choose δ initially to be less than $\frac{1}{8}(c_{M+1} - d) \leq \frac{1}{8}[\inf \varphi(F') - \sup \varphi(B_{j+q})]$, then there is $x_\delta \in H$ so that

(i) $c_{j+q} - 4\delta \leq \varphi(x_\delta) \leq c_{j+q} + 11\delta$, hence $c - 5\delta \leq \varphi(x_\delta) \leq c + 12\delta$

(ii) $\operatorname{dist}(x_\delta, F' \cup -F') \leq 5\sqrt{\delta}$,

and

(iii.a) either $\|\varphi'(x_\delta)\| \leq 2\delta^{1/4}$

(iii.b) or $c_{j+q} - 4\delta \leq \varphi(-x_\delta) \leq c_{j+q} + 9\delta$ (hence $c - 5\delta \leq \varphi(-x_\delta) \leq c + 10\delta$) and $\frac{\langle \varphi'(x_\delta), \varphi'(-x_\delta)\rangle}{\|\varphi'(x_\delta)\| \cdot \|\varphi'(-x_\delta)\|} > 1 - 2\delta^{1/4}$.

We get the required contradiction by letting δ go to 0 .

We can now establish the following *symmetric mountain pass theorem for non-even functionals*. We shall use the notation $B_R(E)$ (resp., $S_R(E)$) to denote the ball (resp., the sphere) of radius $R > 0$ in the Banach space E.

COROLLARY IV.3 (the finite case). – *Let φ be a C^1-functional on a Hilbert space $H = Y \oplus Z$ where Y is a finite dimensional subspace of dimension k. Assume the following conditions:*

(a) *There is $\rho > 0$ and $\alpha \geq 0$ such that* $\inf \varphi(S_\rho(Z)) \geq \alpha$.

(b) *There exist a subspace E_n of H containing Y with* $\dim(E_n) = n > k$ *and $R > \rho$ such that* $\sup \varphi(S_R(E)) \leq 0$.

(c) $\varphi(0) = 0$ *and φ verifies (sP-S)$_c$ for every $c \in \mathbb{R}^+$.*

Then, there exist levels $c_n \geq \ldots \geq c_{k+1} \geq \alpha \geq 0$ such that:

(i) *If $0 < c_i$ for some i ($k < i \leq n$), there exists a virtual critical point at level c_i.*

(ii) *If $0 < c_i = c_{i+p}$ for some $k < i \leq i + p \leq n$, then $\gamma(E^*_{c_i}) \geq p + 1$.*

(iii) *If $c_{k+p} = \alpha \geq 0$ for some $1 \leq p \leq n - k$, then $\gamma(E^*_\alpha \cap S_\rho(Z)) \geq p$.*

In particular, φ has at least $n - k$ distinct pairs of non-trivial virtual critical points.

Proof. – As in the standard symmetric mountain pass theorem, consider the classes of functions

$$\mathcal{L}_n = \{h \in C(\bar{B}_R(E_n), X); h \text{ odd and equal the identity on } S_R(E_n)\}.$$

Write D_n for $B_R(E_n)$ and S_n for $S_R(E_n)$ and for $k < j \leq n$, define

$$\mathcal{F}_j = \mathcal{F}_j(E_n, R) = \{h(\overline{D_n \setminus K}); h \in \mathcal{L}_n, \gamma_{Z_2}(K) \leq n - j\}.$$

It is well known (*See* Rabinowitz [R]) that the classes $\mathcal{F}_j$ $(k < j \leq n)$ are non-empty and they satisfy the following properties:

(a) (Monotonicity) $\mathcal{F}_{j+1} \subset \mathcal{F}_j$.

(b) (Stability) $\mathcal{F}_j$ is a $\mathbb{Z}_2$-homotopic class of dimension n with boundary S_n.

(c) (Excision) For any A in $\mathcal{F}_{j+p}$, any open symmetric set U such that $\bar{U} \cap S_n = \emptyset$ and $\gamma(\bar{U}) \leq p$, we have $A \setminus U \in \mathcal{F}_j$.

(d) (Linking) For all A in $\mathcal{F}_j$, $\gamma_{Z_2}(A \cap S_\rho(Z)) \geq j - k$ provided $\rho < R$.

It is now enough to set $c_j = c(\varphi, \mathcal{F}_j)$ and to apply Theorem IV.2, the dual set F being $S_\rho(Z)$.

Now we can prove the following

COROLLARY IV.4 (the infinite case). – *Let φ be a C^1-functional satisfying $sP - S)_c$ on a Hilbert space $X = Y \oplus Z$ with $\dim(Y) < \infty$. Assume $\varphi(0) = 0$ as well as the following conditions:*

(1) *There is $\rho > 0$ and $\alpha \geq 0$ such that $\inf \varphi(S_\rho(Z)) \geq \alpha$.*

(2) *There exists an increasing sequence $(E_n)_n$ of finite dimensional subspaces of X, all containing Y such that $\lim_n \dim(E_n) = \infty$ and for each n, $\sup \varphi(S_{R_n}(E_n)) \leq 0$ for some $R_n > \rho$.*

Then φ has an unbounded sequence of virtual critical values.

Proof. – For each $k < j$, let $\widetilde{\mathcal{F}}_j = \bigcup_n \{\mathcal{F}_j(E_n, R_n); \dim(E_n) \geq j\}$ where $\mathcal{F}_j(E_n, R_n)$ are the families defined in the preceeding corollary. Each $\widetilde{\mathcal{F}}_j$ is a union of homotopy-stable classes with boundaries $(S_{R_n}(E_n))_n$ that satisfy properties (a), –, (d) stated above.

By the monotonicity and linking properties, we have:

$$\alpha \leq c_{k+1} \leq c_{k+2} \leq \cdots \leq c_n \leq \cdots$$

where $c_j = c(\varphi, \widetilde{\mathcal{F}}_j)$.

Property (c) above implies that $(\widetilde{\mathcal{F}}_j)_j$ verifies the excision axiom. Moreover, 0 does not belong to $(H \setminus S_\rho(Z)) \cap \{\varphi \leq \alpha\}$. Hence Corollary IV.3 apply (with $F = S_\rho(Z)$) to get the result.

V. AN APPLICATION

We now consider the following non-homogeneous Dirichlet boundary value problem

$$(*)_f \qquad \begin{cases} -\Delta u = |u|^{p-2}u + f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain ($N \geq 3$), $H = H_0^1(\Omega)$, $f \in H^{-1}$. We shall deal with the subcritical case, that is when $p < 2^* = \frac{2N}{N-2}$, the limiting exponent in the Sobolev embedding.

Weak solutions for this problem are the critical points of the functional

$$\varphi_f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx - \int_\Omega fu\,dx$$

defined on H. It is clear that the difficulty comes from the fact that the non-homogeneous term adds a linear term to an otherwise $\mathbb{Z}_2$-invariant functional.

If $f = 0$, then $(*)_f$ is known to have an infinite number of solutions ([R],[St]). The same is known to hold for the non-homogeneous case provided $p < \frac{N}{N-2}$ (See [Ba-Be], [Ba-L]). It is still an open problem whether this remains true for all p up to 2^*. On the other hand, Bahri [Ba] had shown the following generic result:

If $2 < p < 2^*$, then the set of $f \in H^{-1}$ such that $(*)_f$ has an infinite number of solutions is a dense residual in H^{-1}.

The relevance of the results of this paper on this problem comes from the easily verifiable fact that $\mathbb{Z}_2$-*resonant points* of φ_f correspond to (weak) solutions of the following equation

$$(**)_f \qquad \begin{cases} -\Delta u = |u|^{p-2}u + \mu f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \\ -1 \leq \mu \leq 1 & \text{and } \displaystyle\int_\Omega fu dx = 0. \end{cases}$$

The theory developed in the preceding sections yields the following:

THEOREM V.1. – *Assume that* $2 < p < 2^*$, *then for every* $f \in H^{-1}$, *either* $(*)_f$ *or* $(**)_f$ *has an unbounded sequence of solutions.*

Proof. – We shall show that φ_f satisfies the hypothesis of Corollary IV.4.

Denote $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ the eigenvalues of $-\Delta$ on Ω with homogeneous Dirichlet data and let v_j be the corresponding eigenfunctions.

Claim (i). – For k_0 sufficiently large, there exist $\rho > 0$ such that $\varphi_f(u) \geq 1$ for all $u \in Z := \text{span}\{v_k; k \geq k_0\}$ with $\|u\|_{H_0^1} = \rho$.

Indeed, by the Sobolev's embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and Hölder's inequality, we get by setting $C = \|f\|_{H^{-1}}$, that for $u \in Z$,

$$\begin{aligned}\varphi_f(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx - \int_\Omega fu\,dx\\ &\geq \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{1}{p}\|u\|_2^r\|u\|_{2^*}^{p-r} - C\|u\|_{H_0^1}\\ &\geq \left(\frac{1}{2} - \frac{1}{p}\lambda_{k_0}^{-r/2}\|u\|_{H_0^1}^{p-2}\right)\|u\|_{H_0^1}^2 - C\|u\|_{H_0^1}\end{aligned}$$

where $\frac{r}{2}+\frac{p-r}{2^*} = 1$. In particular, $r = N(1-\frac{p}{2^*}) > 0$. Let now $\rho > 0$ be such that $\rho^2 - 4(C\rho + 1) = 0$ and choose $k_0 \in \mathbb{N}$ such that $\frac{1}{p}\lambda_{k_0}^{-r/2}\rho^{p-2} \leq \frac{1}{4}$ and therefore the claim.

Let now $Y = \text{span}\{v_j; j < k_0\}$ be its orthogonal complement. We now show the following:

Claim (ii). – On any finite dimensional subspace $E \subset H_0^1$, there exist constants C_1, C_2, C_3 (depending on E) such that

$$\sup_{u\in\partial B_R(E)} \varphi(u) \leq C_1R^2 - C_2R^q + C_3R.$$

Indeed, for any fixed $u \in H_0^1$ and any $R > 0$, we have

$$\varphi(Ru) \leq \frac{R^2}{2}\|u\|_{H_0^1}^2 - \frac{R^p}{p}\|u\|_p^p + CR\|u\|_{H_0^1}.$$

This easily implies Claim (ii).

Claim (iii). – The functional φ_f satisfies the symmetrized (sP-S)$_c$ condition at any level $c > 0$.

First, it is well known that φ_f satisfies $(P-S)_c$ for any c. Now assume that $(u_n)_n$ is a sequence in $H_0^1(\Omega)$ that satisfies the following conditions:

$$\lim_n \varphi_f(u_n) = \lim_n \varphi_f(-u_n) = c \tag{1}$$

and

$$\lim_n \|\varphi_f'(u_n) - \lambda_n\varphi_f'(-u_n)\|_{H^{-1}} = 0 \text{ for some positive sequence of reals } (\lambda_n)_n. \tag{2}$$

This easily imply that

$$\int_\Omega f u_n \, dx \to 0 \tag{3}$$

and that for any $v \in H_0^1(\Omega)$,

$$\langle \varphi_0'(u_n), v\rangle - \frac{1-\lambda_n}{1+\lambda_n} \int_\Omega f v \, dx \to 0. \tag{4}$$

It follows that $(\|\varphi_0'(u_n)\|)_n$ is bounded, that $\langle \varphi_0'(u_n), u_n\rangle \to 0$ and $\varphi_0(u_n) \to c$. Therefore, there is $K > 0$ such that for n large enough,

$$c + 1 + K\|u_n\|_{H_0^1} \geq \varphi_0(u_n) - \frac{1}{p}\langle \varphi_0'(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_{H_0^1}^2,$$

which clearly implies that $(u_n)_n$ is bounded in $H_0^1(\Omega)$.

Let now u be a weak cluster point for $(u_n)_n$ in $H_0^1(\Omega)$ and use the compactness of the Sobolev embedding to deduce that $\|u_n - u\|_p \to 0$. This combined with the fact that $\langle \varphi_0'(u_n), u_n\rangle \to 0$, yields that $\|u_n\|_{H_0^1}^2 \to \|u\|_p^p$. On the other hand (4) and the weak to norm continuity of φ_0' yields that $\|u\|_p^p = \|u\|_{H_0^1}^2$, which means that $\|u_n\|_{H_0^1}^2 \to \|u\|_{H_0^1}^2$ and therefore we have strong convergence of $(u_n)_n$ in H_0^1.

Let now μ be a cluster point for the sequence $\mu_n = \frac{1-\lambda_n}{1+\lambda_n}$. It is clear that the limit u will then satisfy $-\Delta u = |u|^{p-2}u + \mu f$ on Ω, $u = 0$ on $\partial\Omega$, $0 \leq \mu \leq 1$ and $\int_\Omega f u \, dx = 0$.

Remark. – One is tempted to understand the parameter μ as a Lagrange multiplier that appears by way of finding solutions to equation $(**)_f$ by seeking the critical points of φ_f on the hyperplane:

$$H_f = \{u \in H_0^1(\Omega); \int_\Omega f u \, dx = 0\}.$$

However, we cannot see how to get in this way the additional information $-1 \leq \mu \leq 1$. Moreover, there may be some difficulties in constructing the decreasing and infinite sequence of min-max families $(\mathcal{F}_k)_k$ inside the hyperplane H_f. These difficulties are more apparent when one considers the more general problem

$$(*) \qquad \begin{aligned} -\Delta u &= g(x, u), \quad && x \in \Omega \subset \mathbb{R}^n, \\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

where $g(x,.)$ is not necessarily odd. The results of this paper yield –modulo the standard superquadratic conditions on g– that there exists an infinite number of solutions to the problem

$$(**)\qquad \begin{aligned} &-\Delta u = g_1(x,u) + \mu g_2(x,u), && x \in \Omega \subset \mathbb{R}^n, \\ &\qquad u = 0, && x \in \partial\Omega \\ &-1 \leq \mu \leq 1 \qquad \text{and} && \int_\Omega G_2(x,u)dx = 0. \end{aligned}$$

Where g_1 (resp., g_2) are the odd (resp., even) part of g and where $G_2(x,t) = \int_0^t g_2(x,s)\,ds$.

We can also prove the following bifurcation type result, which is originally due to [A]:

THEOREM V.2. – *Assume that $2 < p < 2^*$ and that $(*)_0$ has at most a countable number of solutions. Then, there exists a dense G_δ subset G of H^{-1} such that for every $f \in G$ and any integer $n \geq 1$, there exists $\delta_n > 0$ such that for any $0 \leq a \leq \delta_n$, the problem $(*)_{af}$ has at least n solutions.*

Proof. – Suppose $(*)_0$ has a countable number of non-trivial solutions $(v_n)_n$ and consider the following dense G_δ subset of $H^{-1}(\Omega)$.

$$G = \cap_n \{f \in H^{-1}(\Omega);\ \int_\Omega v_n f \neq 0\}.$$

Fix $f \in G$ and any integer $N \geq 1$ and suppose the above statement does not hold. This means that for any $\delta > 0$, there exists a, $0 < a < \delta$ such that φ_{af} has less than $N - 1$ solutions.

Let Y and Z be the subspaces associated to φ_f in the proof of Theorem V.1 and let $\{\mathcal{F}_j; k_0 \leq j \leq k_0 + N\}$ be the $\mathbb{Z}_2$-homotopy stable families associated to them in Corollary IV.4. Set $c_j(a) = c(\varphi_{af}, \mathcal{F}_j)$ and note that, since φ_{af} has less than $N - 1$ true critical points, then by Corollary IV.3 there exists at least one virtual critical point at one of the levels $c_1(a) \leq ... \leq c_N(a)$. That is there exists b, $0 \leq b < a \leq \delta$ such that $(*)_{bf}$ has a solution u_a satisfying $\int_\Omega u_a f\ dx = 0$, and $\varphi_{af}(u_a) = c_j$ for some j, $1 \leq j \leq N$, which means that $1 \leq \varphi_{af}(u_a) \leq c_N(a)$.

Letting $\delta \to 0$, it is then easy to see that the sequence $(u_a)_a$, is such that $\varphi_0'(u_a) \to 0$ and $\varphi_0(u_a)$ is bounded above. Since φ_0 satisfies the $(P-S)$ condition, we get then a solution u of $(*)_0$ such that $\int_\Omega uf\ dx = 0$. This is a contradiction since $f \in G$ and since the fact that $1 \leq \varphi_{u_a}(u_a)$ yields that u is a non-trivial solution for $(*)_0$.

This is essentially a bifurcation result. It means that, for any given function f in G, the problem

$$(**)_{af} \qquad \begin{cases} -\Delta u = |u|^{p-2}u + af & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has more and more solutions as $a \to 0$, and that these solutions converge to the (countably many) solutions of the homogeneous problem

$$(**)_0 \qquad \begin{cases} -\Delta u = |u|^{p-2}u & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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