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SYMPOSIUM

Existence, uniqueness and efficiency of equilibrium in hedonic markets with multidimensional types

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Abstract We study equilibrium in hedonic markets, when consumers and suppliers have reservation utilities, and the utility functions are separable with respect to price. There is one indivisible good, which comes in different qualities; each consumer buys 0 or 1 unit, and each supplier sells 0 or 1 unit. Consumer types, supplier types and qualities can be either discrete of continuous, in which case they are allowed to be multidimensional. Prices play a double role: they keep some agents out of the market, and they match the remaining ones pairwise. We define equilibrium prices and equilibrium distributions, and we prove that equilibria exist, we investigate to what extend equilibrium prices and distributions are unique, and we prove that equilibria are efficient. In the particular case when there is a continuum of types, and a generalized Spence–Mirrlees condition is satisfied, we prove the existence of a pure equilibrium, where demand distributions are in fact demand functions, and we show to what extent it is unique. The proofs rely on convex analysis, and care has been given to illustrate the theory with examples.

Keywords Hedonic goods · Competitive markets · Equilibrium · Optimal transportation

JEL Classification C62 · C78 · D41 · D50

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1 Introduction

1.1 Main results

In this paper, we show the existence and uniqueness of equilibrium in a hedonic market, and we give uniqueness results. The main features of our model are as follows:

- There is a single, indivisible, good in the market, and it comes in different qualities z.
- Consumers and producers are price-takers and utility-maximizers. They are characterized by the values of some variables; each set of values is called a (multi-dimensional) *type*.
- Consumers buy at most one unit of the good, and they buy none if their reservation utility is not met; producers supply at most one unit of the good, and they supply none if their reservation utility is not met. In other words, agents always have the option of staying out of the market.
- The utilities of consumers and of producers are quasi-linear with respect to price: the utility consumers with type x derive from buying one unit of quality z at price $p(z)$ is $u(x, z) - p(z)$, and the utility producers with type y derive from selling one unit of quality z at price $p(z)$ is $p(z) - v(y, z)$.

Our results are valid in the discrete case and in the continuous case. We show that there is a (nonlinear) price system $p(z)$ such that, for every quality z, the number (or the aggregate mass) of consumers who demand z is equal to the number (or the aggregate mass) of suppliers who produce z. In addition, agents who are staying out of the market are doing so because by entering they would lower their utility. In other words this price system exactly matches a subset of consumers with a subset of producers, and the remaining consumers or producers are priced out of the market. This is called an *equilibrium price*, and the resulting allocation of qualities is called an *equilibrium allocation*. An example is given in Sect. 4.4, and the reader may proceed there directly. We should stress, however, that we prove existence in full generality, beyond the one-dimensional situation described in that example.

Every price system $p(z)$ creates a matching between consumers and producers: for every unit traded, there is a pair consisting of a consumer who buys it and a producer who sells it. When summing their utilities, the price of the traded item cancels out, so that the resulting utility of the pair is independent of the price system. Unmatched consumers and producers get their reservation utility. It is then meaningful to take the social planner's point of view, and to ask for a matching between consumers and producers which will maximize aggregate utility, where the utility of matched pairs is the maximum utility they can get by trading, and the utility of unmatched agents is their reservation utility. We will show that the solution of this problem coincides with the equilibrium matching. This implies that *every equilibrium is efficient.*

An interesting feature of equilibrium pricing is that, even tough all technologically feasible qualities are priced, not all of them will be traded in equilibrium. For each non-traded quality, there is a non-empty bid-ask range: all prices which fall within that range are equilibrium prices, that is, they will not lure customers or suppliers away from traded qualities. This means that equilibrium prices cannot be uniquely defined on non-traded qualities. On the other hand, they are *uniquely defined on traded qualities*. There is a corresponding degree of uniqueness for the equilibrium allocation.

The main drawback of our model is the assumption that utilities are quasi-linear. It is quite a restriction, from the economic point of view, since it means that the marginal utility of money is constant, but our proof seems to require it in an essential way. On the other hand, it also enables us to prove some uniqueness results, which are probably not to be expected in the more general case.

1.2 The literature

This paper inherits from two traditions in economics. On the one hand, it can be seen as a contribution to the research program on hedonic pricing that was outlined by Shervin Rosen in his seminal paper (Rosen 1974). The idea of defining a good as a bundle of attributes (originating perhaps with Houthakker (1952), and developed by Lancaster (1966); Becker (1965) and Muth (1966)), provides a systematic framework for the economic analysis of the supply and demand for quality. The main direction of investigation, however, has been towards econometric issues, such as the construction of price indices net of changes in quality; see for instance the seminal work of Court (1941) and the book Griliches (1971). The identification of hedonic models raises specific questions which have been first discussed by Rosen (1974), and most recently by Ekeland et al. (2004). Theoretical question, such as the existence and characterization of equilibria, have attracted less attention. The papers by Rosen (1974) and later Mussa and Rosen (1978) study the one-dimensional situation, that is, the case when agents are fully characterized by the value of a single parameter. The multidimensional situation has been investigated by Rochet and Choné (1998), but it deals with monopoly pricing. The issue of equilibrium pricing in the multidimensional situation, had to my knowledge not been addressed up to now (nor, for that matter, has the issue of oligopoly pricing).

One of Rosen's main achievement has been to recognize hedonic pricing as nonlinear, against the prevailing tradition in econometric usage. As noted in Rosen (1974), a buyer can force prices to be linear with respect to quality if certain types of arbitrage are allowed. In the present paper, buyers and sellers are restricted to trading one unit of a single quality, and there is no second-hand market, so this kind of arbitrage is unavailable, and prices will be inherently nonlinear. This would not be the case if consumers and producers were allowed to buy and sell several qualities simultaneously.

On the other hand, this paper also belongs to the tradition of assignment problems. This tradition has several strands, one of which originates with Koopmans and Beckmann (1957), and the other with Shapley and Shubik (1972). We refer to the papers by Gretzki et al. (1992) and Gretzki et al. (1999), and to Ramachandran and Ruschendorf (2002) for more recent work. In this literature, producers are not free to choose the quality they sell: each quality is associated with a single producer, who can produce that one and not any other one. The Shapley–Shubik model, for instance, describes a market for houses. There are a certain number of sellers, each one is endowed with a house, and a certain number of buyers. No seller can sell a house other than his own, but a buyer can buy any house. This is in contrast with the situation in the present paper, where both buyers and sellers are free to choose the quality they buy or sell.

1.3 Structure of the paper

Section 2 describes the mathematical model and the basic assumptions. As we mentioned earlier, we do not require that the distribution of types be continuous, nor that the number of consumers equals the number of producers. Mathematically speaking, there is a positive measure μ on the set of consumer types X, and a measure ν on the set of producer types Y, both μ and ν can have atoms, and typically $\mu(X) \neq \nu(Y)$. These features, although very appealing from the point of view of economic modelling, introduce great complications in the mathematical treatment. In earlier work (Ekeland 2005), the author has given a streamlined proof in the particular case when μ and ν are non-atomic, $\mu(X) = \nu(Y)$ and an additional sorting assumption on utilities is satisfied (extending to multidimensional types the classical Spence–Mirrlees single-crossing assumption), so that all agents with the same type do the same thing. Beside the fact that it does not apply when X or Y are finite, such a model does not capture one of the essential role of prices, which serve not only to match consumers and producers which enter the market (there must necessarily be an equal number of both) but also to keep out of the market enough agents so that matching becomes possible. The latter function is an essential focus of the present paper.

In our model, there is a single indivisible good, consumers are restricted to buying one or zero unit, and producers are restricted to supply one or zero unit. The price is a nonlinear function $p(z)$ of the quality z. It is an equilibrium price if the market for every quality clears. This implies that the number of consumers who trade is equal to the number of suppliers who trade. The remaining, non-trading, agents, are kept out of the market by the price system, which is either too high (for consumers) or too low (for producer) to allow them to make more than their reservation utility.

It is important to note that in equilibrium consumers (or producers) which have the same type may not be doing the same thing. This will typically occur when utility maximisation does not result in a single quality being selected. To be precise, given an equilibrium price $p(z)$, consumers of type x maximize $u(x, z) - p(z)$ with respect to z. But there is no reason why there should be a unique optimal quality: even if we assumed $u(x, z)$ to be strictly concave with respect to z, the price $p(z)$ typically is nonlinear with respect to z, and no conclusion can be derived about uniqueness.

If $p(z)$ is an equilibrium price, and if there is a non-trivial subset $D(x) \subset Z$ such that any $z \in D(x)$ is a utility maximizer for x, there will be a certain equilibrium probability P_x^α on $D(x)$. This means that, given $A \subset D(x)$, the number $P_x^\alpha[A] \in [0, 1]$ is the proportion of agents of type x whose demands lie in A. Similarly, there will be an equilibrium probability P_y^β for every producer y, and the resulting demand and supply for every quality z will balance out. A formal definition is given in Sect. 3. In other words, in equilibrium, we cannot tell which agent of a given type does what, but we can tell how many of them do this or that.

The main results of the paper, together with the definition of equilibrium, are stated in Sect. 3: equilibria exist, equilibrium prices are not unique, there is a unique equilibrium allocation, and it is efficient (Pareto optimal). Proofs are deferred to Appendices C and D. These proofs combine two mathematical ingredients, the Hahn–Banach separation theorem on the one hand, and duality techniques which extend the classical Fenchel duality for convex functions, and which have been developed in the context of optimal transportation (see Villani (2003) for a recent survey). Everything relies in studying a certain optimization problem (33), which is novel.

Section 4 gives additional assumptions which ensure that all agents of the same type do the same thing in equilibrium: μ and ν should be non-atomic, and conditions (9) and (10) should be satisfied. These conditions extend to multidimensional types the classical single-crossing assumption of Spence and Mirrlees. The resulting equilibria are called *pure*, in reference to pure and mixed equilibria in game theory. Note however that, even in this case, one cannot fully determine the behaviour of agents in equilibrium: if consumers of type x are indifferent between entering the market or not (either decision giving them their reservation utility), then, even with these additional assumptions, we cannot say which ones will stay out and which ones will come in. The equilibrium relations will only determine the proportion of each.

Section 4.4 describes an explicit example. It is strictly one-dimensional (types and qualities are real numbers), which makes calculations possible, and a complete description of the equilibrium is provided. Unfortunately, the method uses does not extend to multidimensional types.

Appendix A gives the mathematical results on u-convex and v-concave analysis which will be in constant use in the text. Appendix B gives general mathematical notations, and references about Radon measures. Appendices C and D contain proofs.

2 The model

2.1 Standing assumptions

Let $X \subset R^{d_1}$, $Y \subset R^{d_2}$, and $Z_0 \subset R^{d_3}$ be compact subsets. We are given non-negative finite measures μ on X and ν on Y. They are allowed to have point masses.

Typically, we will have $\mu(X) \neq \nu(Y)$.

Let Ω_1 be a neighbourhood of $X \times Z_0$ in $R^{d_1+d_3}$, and Ω_2 be a neighbourhood of $Y \times Z_0$ in $R^{d_2+d_3}$. We are given continuous functions $u : \Omega_1 \to R$ and $v : \Omega_2 \to R$. It is assumed that u is differentiable with respect to x, and that the derivative:

$$D_x u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{d_1}} \right)$$

is continuous with respect to (x, z). Similarly it is assumed that v is differentiable with respect to y, and that the derivative $D_y v$ is continuous with respect to (y, z).

Note that X, Y and/or Z_0 are allowed to be finite. If X is finite, the assumption on u is satisfied. If Y is finite, the assumption on v is satisfied.

2.2 Bid and ask prices

We are describing the market for a quality good: it is indivisible, and units differ by their characteristics $\left(z_1, \ldots, z_{d_3}\right) \in Z_0$. The bundle $z = \left(z_1, \ldots, z_{d_3}\right)$ will be referred to as a (multidimensional) *quality*. So Z_0 is the set of all technologically feasible qualities; it is to be expected that they will not all be traded in equilibrium.

Points in X represent consumer types, points in Y represent producer types. If X is finite, then $\mu(x)$ is the number of consumers of type x. If Y is finite, then $\nu(y)$ is the number of producers of type y. If X is infinite, then μ is the distribution of types in the consumer population, and the same interpretation holds for (Y, ν).

Each consumer buys zero or one unit, and each supplier sells zero or one unit. There is no second-hand trade.

For the time being, we define a price system to be a continuous map $p : Z_0 \to R$. This definition will be modified in a moment, as the set Z_0 will be extended to a larger set Z. Typically, pricing is nonlinear with respect to the characteristics. Once the price system is announced, agents make their decisions according to the following rules:

- Consumers of type x maximize $u(x, z) - p(z)$ over Z_0. If the value of that maximum is strictly positive, the consumer enters the market and buys one unit of the maximizing quality z. If there are several maximizing qualities, he is indifferent between them, and the way he chooses which one to buy is not specified at this stage. If the value of the maximum is 0, he is indifferent between staying out of the market, and entering it to buy one unit of the maximizing quality. Again, the way he chooses is not specified at this stage.
- Producers of type y maximize $p(z) - v(y, z)$ over Z_0. If the value of that maximum is strictly positive, the producer enters the market and sells one unit of the maximizing quality z. If there are several maximizing qualities, he is indifferent between them. If the value of the maximum is 0, he is indifferent between staying out of the market, and entering it to sell one unit of the maximizing quality.

To model this procedure by a straightforward maximization, we introduce two extra points $\varnothing_d \notin Z_0$ and $\varnothing_s \notin Z_0$, with $\varnothing_d \neq \varnothing_s$, and we extend utilities and prices as follows:

$$p(\varnothing_d) = u(x, \varnothing_d) = 0 \quad \forall x \in X \tag{1}$$

$$p(\varnothing_s) = v(y, \varnothing_s) = 0 \quad \forall y \in Y \tag{2}$$

$$u(x, \varnothing_s) = -1, \quad v(y, \varnothing_d) = 1 \tag{3}$$

The set of possible decisions for agents is now

$$Z = Z_0 \cup \{\varnothing_d\} \cup \{\varnothing_s\}$$

so that

$$\max\{u(x, z) - p(z) \mid z \in Z\} \geq u(x, \varnothing_d) - p(\varnothing_d) = 0$$

$$\max\{p(z) - v(y, z) \mid z \in Z\} \geq p(\varnothing_s) - v(y, \varnothing_s) = 0$$

and the procedure we just described amounts to maximizing over Z instead of Z_0. The relations (1) to (3) imply that consumers will never choose $\varnothing_s$ (it is always better to choose $\varnothing_d$), and producers will never choose $\varnothing_d$ (it is always better to choose $\varnothing_s$). So our model does capture the intended behaviour.

Note that we have normalized reservation utilities to 0. This does not cause any loss of generality. The behaviour of consumers, for instance, is fully specified by $u(x, z)$ and $\bar{u}(x)$, the latter being the reservation utility, and we get the same behaviour by replacing $u(x, z)$ by $u(x, z) - \bar{u}(x)$ and $\bar{u}(x)$ by 0, the only restriction being that we would require $\bar{u}$ to be C^1, to preserve the regularity properties of u.

Normalizing reservation utilities to 0, we find that $u(x, z)$ is the bid price for quality z by consumers of type x, that is, the highest price that they are willing to pay for that quality. Similarly, $v(y, z)$ is the asking price for quality z by producers of type y, that is, the lowest price they are willing to accept for supplying that quality. For a given quality $z \in Z$, it is natural to consider the highest bid price from consumers and the lowest ask price from producers:

Definition 1 The *highest bid price* $b : Z \to R$ is given by:

$$b(z) = \max_x u(x, z)$$

and the *lowest ask price* $a : Z \to R$ is given by:

$$a(z) = \min_y v(y, z)$$

Note that $b(\varnothing_d) = a(\varnothing_s) = 0$ and that $a(\varnothing_d) = -b(\varnothing_s) = 1$.

It follows from their definitions that b is u-convex and a is v-concave. More precisely, we have $b(z) = 0_x^\sharp$ and $a(z) = 0_y^\flat$ where 0_x and 0_y denote the maps $x \to 0$ and $y \to 0$ on X and Y. Conversely, we have $0 = \max_z \{u(x, z) - b(z)\}$ and $0 = \min_z \{v(y, z) - a(z)\}$, so that $b^\sharp(x) = 0$ and $a^\flat(y) = 0$.

Note that if the price system is such that $p(z) > b(z)$ for some quality z, then there will be no buyers for this quality, and so it cannot be traded at that price. Similarly, if $p(z) < a(z)$, then there will be no sellers for this quality, and it cannot be traded at that price. The following is obvious:

Proposition 2 (No-trade equilibrium) *If* $a(z) > b(z)$ *everywhere, then all consumers and all producers stay out of the market.*

2.3 Demand and supply

From now on, a *price system* will be a continuous map $p : Z \to R$ such that $p(\varnothing_d) = p(\varnothing_s) = 0$. We will use the notations and results of Appendix A.

Given a price system p, the map $p : Z \to R$ is continuous and the set Z is compact, so that the functions $u(x, z) - p(z)$ and $p(z) - v(y, z)$ attain their maximum on Z.

Definition 3 Given a price system p, we define:

$$D(x) = \arg\max\{u(x,z) - p(z) \mid z \in Z\}$$
$$S(y) = \arg\min\{v(y,z) - p(z) \mid z \in Z\}$$

Both are compact and non-empty subsets of Z. We shall refer to $D(x)$ as the *demand* of type x consumers, and to $S(y)$ as the *supply* of type y producers.

It follows from the definitions that if a consumer of type x is out of the market, then we must have $\varnothing_d \in D(x)$. If there is no other point in $D(x)$, then all consumers of the same type stay out of the market. If, on the other hand, $D(x)$ contains some point $z \in Z_0$, then all consumers of type x are indifferent between staying out or buying quality z, and we may expect that some of them actually buy quality z instead of staying out. This remark will be at the core of our equilibrium analysis. Of course, the same observation is valid for producers.

The following result clarifies the relation between $D(x)$ and $S(y)$ on the one hand, and the sub- and supergradients $\partial p^\sharp(x)$ and $\partial p^\flat(y)$ on the other. Recall that:

$$p^\sharp(x) = \max\{u(x,z) - p(z) \mid z \in Z\}$$
$$p^\flat(y) = \min\{v(y,z) - p(z) \mid z \in Z\}$$

Proposition 4 *We have* $D(x) \subset \partial p^\sharp(x)$ *and* $S(y) \subset \partial p^\flat(y)$. *More precisely:*

$$D(x) = \left\{z \in \partial p^\sharp(x) \mid p(z) = p^{\sharp\sharp}(z)\right\}$$
$$S(y) = \left\{z \in \partial p^\flat(y) \mid p(z) = p^{\flat\flat}(z)\right\}$$

Proof The point $x \in X$ being fixed, consider the functions $\varphi : Z \to R$ and $\psi : Z \to R$ defined by $\varphi(z) = u(x,z) - p(z)$ and $\psi(z) = u(x,z) - p^{\sharp\sharp}(z)$. The subgradient $\partial p^\sharp(x)$ is the set of points z where ψ attains its maximum (see Appendix A), while $D(x)$ is the set of points z where φ attains its maximum. But $\psi \geq \varphi$ and $\max\psi = \max\varphi$. The result follows. □

Definition 5 Given a price system $p(z)$, consumers of type x are *inactive* if $p^\sharp(x) < 0$, so that $D(x) = \{\varnothing_d\}$, and they are *active* if $p^\sharp(x) > 0$, so that $\{\varnothing_d\} \notin D(x)$. They are *indifferent* if $p^\sharp(x) = 0$, so that $D(x) \supset \{\varnothing_d\} \cup \{z\}$ for some $z \in Z_0$. Similarly, producers of type y are inactive, active or indifferent according to whether $p^\flat(y)$ is positive, negative or zero.

2.4 Admissible price systems

We have seen that, if $a(z) > b(z)$ everywhere, there is a no-trade equilibrium. We are concerned with the more interesting case when $a(z) \leq b(z)$ for some z.

Definition 6 Quality $z \in Z$ is *marketable* if $a(z) \leq b(z)$. The set of marketable qualities will be denoted by Z_1 :

$$\begin{aligned} Z_1 &= \{z \in Z \mid a(z) \leq b(z)\} \\ &= \{z \in Z \mid \exists x\ , \exists y : v(y, z) \leq\ u(x, z)\} \end{aligned}$$

Note that staying out is not a marketable option: $a(\varnothing_d) > b(\varnothing_d)$ and $a(\varnothing_s) > b(\varnothing_s)$. As mentioned earlier, this means that consumers will never choose $\varnothing_s$ and that suppliers will never choose $\varnothing_d$. We have therefore the inclusions:

$$Z_1 \subset Z_0 \subsetneq Z$$

If a quality z is not marketable, one will never be able to find a buyer/seller pair that trade z. If a quality z is marketable, there is no sense in setting its price to be higher than $b(z)$ (there would be no buyers), or lower than $a(z)$ (there would be no sellers). Hence:

Definition 7 A price system $p : Z \to R$ will be called *admissible* if:

$$\forall z \in Z_1, \quad a(z) \leq p(z) \leq b(z)$$

Let p be an admissible price system, so that $a(z) \leq p(z) \leq b(z)$. Recall that $p^\sharp(x)$ is the indirect utility of type x consumers, and that $-p^\flat(y)$ is the indirect utility of type y producers. Taking conjugates, we get:

$$\begin{aligned} \forall x \in X, &\quad 0 \leq p^\sharp(x) \\ \forall y \in Y, &\quad 0 \geq p^\flat(y) \end{aligned}$$

which means that all consumers and producers achieve at least their reservation utility.

3 Equilibrium

3.1 Demand distribution and supply distribution

Assume a price system $p : Z \to R$ is given. Let $D(x)$ and $S(y)$ be the associated demand and supply. Recall that their graphs are compact sets.

We refer to Appendix B for notations and definitions concerning Radon measures and probabilities.

Definition 8 A *demand distribution* associated with p is a positive measure $\alpha_{X \times Z}$ on $X \times Z$ such that:

- $\alpha_{X \times Z}$ is carried by the graph of D
- its marginal α_X is equal to μ

Similarly, a *supply distribution* associated with p is a positive measure $\beta_{Y \times Z}$ on $Y \times Z$ such that:

- $\beta_{Y\times Z}$ is carried by the graph of S
- its marginal β_Y is equal to ν

The conditional probabilities P_x^α and P_y^β then are carried by $D(x)$ and $S(y)$ respectively. Given $A \subset Z$, the numbers $P_x^\alpha[A]$ and $P_y^\beta[A]$ are readily interpreted as the probability that consumers of type x demand some $z \in A$ and the probability that producers of type y supply some $z \in A$.

If $S(y)$ is a singleton, so that the supply of type y producers is uniquely defined, then P_y^β reduces to a Dirac mass:

$$S(y) = \{s(y)\} \Longrightarrow P_y^\beta = \delta_{s(y)}$$

and similarly for consumers.

3.2 Definition of equilibrium

Definition 9 An *equilibrium* is a triplet $(p, \alpha_{X\times Z}, \beta_{Y\times Z})$, where p is an admissible price system and $\alpha_{X\times Z}$ and $\beta_{Y\times Z}$ are demand and supply distributions associated with p, such that:

$$\alpha_{Z_0} = \beta_{Z_0}$$

By α_{Z_0} and β_{Z_0} we denote the marginals of $\alpha_{X\times Z}$ and $\beta_{Y\times Z}$ on Z_0. Let us write down explicitly all the conditions on (p, α, β) implied by this definition:

1. $p : Z \to R$ is continuous, and $p(z) \in [a(z),\ b(z)]$ whenever $a(z) \leq b(z)$
2. the marginal α_X is equal to μ
3. the conditional probability P_x^α is carried by $D(x)$
4. the marginal β_Y is equal to ν
5. the conditional probability P_y^β is carried by $S(y)$
6. the marginals α_Z and β_Z coincide on Z_0:

$$\alpha_Z[A] = \beta_Z[A] \quad \forall A \subset Z_0$$

The interpretation is as follows. Given p, consumers of type x maximize their utility, thereby defining their individual demand set $D(x)$. If that set is a singleton, $D(x) = \{d(x)\}$, the probability P_x^α must be the Dirac mass carried by $d(x)$, and all consumers of type x do the same thing: they stay out of the market if $d(x) = \emptyset_d$, and they buy $z \in Z_0$ if $d(x) = z$. If $D(x)$ contains several points, then consumers of type x are indifferent among these alternatives, and they all do different things. For any Borel subset $A \subset D(x)$, the probability $P_x^\alpha[A]$ gives us the proportion of consumers of type x who choose some $z \in A$ in equilibrium.

Similar considerations hold for suppliers. Condition 6 just states that markets clear in equilibrium: for every quality $z \in Z_0$, the number (or the aggregate mass) of buyers equals the number (or the aggregate mass) of suppliers. Note that this number (or this mass) might be zero, meaning that this particular quality is not traded. This will happen, for instance, if $a(z) > b(z)$, so that quality z is not marketable. It follows that, in equilibrium, demand and supply are carried by Z_1, the set of marketable qualities:

$$\alpha_Z[Z_1] = \alpha_Z[Z_0] = \beta_Z[Z_0] = \beta_Z[Z_1]$$

The number (or the aggregate mass) of consumers who stay out of the market is $\alpha_Z(\{\varnothing_d\})$, and the number (or the aggregate mass) of producers who stay out of the market is $\beta_Z(\{\varnothing_s\})$. As we mentioned several times before, we must have $\alpha_Z(\{\varnothing_s\}) = 0$ and $\beta_Z(\{\varnothing_d\}) = 0$.

3.3 Main results

We begin by an existence result:

Theorem 10 (Existence) *Under the standing assumptions, there is an equilibrium.*

As noted above, if the set Z_1 of marketable qualities is empty, there is an equilibrium, namely the no-trade equilibrium, and it is unique. From now on we assume $Z_1 \neq \varnothing$. The Existence Theorem will be proved in Appendix C.

There is no uniqueness of equilibrium prices. For instance, if a quality $z \in Z_0$ is non-marketable, its price $p(z)$ can be specified arbitrarily. More generally, in Appendix C we will prove the following (see Proposition 37):

Theorem 11 (Non-uniqueness of equilibrium prices) *The set of all equilibrium prices p is convex and non-empty. If $p : Z \to R$ is an equilibrium price, then so is every $q : Z \to R$ which is admissible, continuous, and satisfies:*

$$p^{\sharp\sharp}(z) \leq q(z) \leq p^{\flat\flat}(z) \quad \forall z \in Z \tag{4}$$

For α- and β-almost every quality z which is traded in equilibrium, we have $p^{\sharp\sharp}(z) = p(z) = p^{\flat\flat}(z)$.

Note that q is also required to be admissible, so that in addition to (4) it has to satisfy the inequality:

$$a \leq q \leq b \tag{5}$$

The economic interpretation is as follows. If $(p, \alpha_{X\times Z}, \beta_{Y\times Z})$ is an equilibrium, there will be qualities z which are marketable, but which are not traded in equilibrium, because every supplier type y and every consumer type x prefers some other quality, which means that the price $p(z)$ is too low to interest suppliers, and too high to interest consumers. Formulas (4) and (5) give the range of prices for which this situation will persist. As long as the price $p(z)$ stays in the open interval

$$\left]\max\left\{a(z),\ p^{\sharp\sharp}(z)\right\}, \min\left\{b(z),\ p^{\flat\flat}(z)\right\}\right[$$

the quality z will not be traded. In other words, the price of non-traded qualities can be changed, within a certain range, without affecting $\alpha_{X\times Z}$ or $\beta_{Y\times Z}$, that is, the equilibrium distribution of consumers and suppliers. This is the major source of non-uniqueness in equilibrium prices. On the other hand, if a quality z is traded in equilibrium, one cannot change the price $p(z)$ without affecting $\alpha_{X\times Z}$ and $\beta_{Y\times Z}$, that is, without destroying the given equilibrium.

The equilibrium price p is not unique, but the following result shows that the demand and supply maps $D(x)$ and $S(y)$ almost are:

Theorem 12 (Quasi-uniqueness of equilibrium allocations) *Let* $\left(p_1, \alpha^1_{X\times Z}, \beta^1_{Y\times Z}\right)$ *and* $\left(p_2, \alpha^2_{X\times Z}, \beta^2_{Y\times Z}\right)$ *be two equilibria. Denote by* $D_1(x)$, $D_2(x)$ *and* $S_1(y)$, $S_2(y)$ *the corresponding demand and supply maps. Denote by* P^1_x, P^1_y *and* P^2_x, P^2_y *the corresponding conditional probabilities of demand and supply. Then:*

$$\begin{aligned} P^2_x\left[D_1(x)\right] &= P^1_x\left[D_1(x)\right] = 1 \quad \text{for } \mu\text{-}a.e.\ x \\ P^2_y\left[S_1(y)\right] &= P^1_y\left[S_1(y)\right] = 1 \quad \text{for } \nu\text{-}a.e.\ y \end{aligned}$$

In other words, any z which types x demands in the second equilibrium, when prices are p_2, must belong to the demand set of x when prices are p_1 (even though x might not demand it in the second equilibrium)

Corollary 13 *If the demand of consumers of type x is single-valued in the first equilibrium,* $D_1(x) = \{d_1(x)\}$, *then* $d_1(x) \in D_2(x)$. *If their demand is single-valued in the second equilibrium as well, then* $d_1(x) = d_2(x)$.

Proof We have $P^2_x\left[d_1(x)\right] = 1 = P^2_x\left[D_2(x)\right]$. So $d_1(x)$ must belong to $D_2(x)$, and the remainder must have zero probability:

$$P^2_x\left[D_2(x) \diagdown \{d_1(x)\}\right] = 0$$

□

Corollary 14 *Let* $\left(p_1, \alpha^1_{X\times Z}, \beta^1_{Y\times Z}\right)$ *and* $\left(p_2, \alpha^2_{X\times Z}, \beta^2_{Y\times Z}\right)$ *be two equilibria. If consumers of type x are inactive in the first equilibrium, they cannot be active in the second.*

Proof Since $D_1(x) = \{\varnothing_d\}$, we must have $\varnothing_d \in D_2(x)$. Assume consumers of type x are active in the second equilibrium. We must have $u(x, z) - p(z) > 0$ for all $z \in D_2(x)$, including $z = \varnothing_d$. Since $u(x, \varnothing_d) = p(\varnothing_d) = 0$, this is a contradiction. □

Finally, we will show that we can find equilibrium demand and supply as solutions of the planner's problem. With every pair of demand and supply distributions, $\alpha'_{X\times Z}$ and $\beta'_{Y\times Z}$, we associate the number:

$$\begin{aligned} J\left(\alpha'_{X\times Z},\beta'_{Y\times Z}\right) &= \int\limits_{X\times Z} u\left(x,z\right)\mathrm{d}\alpha'_{X\times Z} - \int\limits_{Y\times Z} v\left(y,z\right)\mathrm{d}\beta'_{Y\times Z} \\ &= \int\limits_{X} P_x^{\alpha'}\left[u\left(x,z\right)\right]\mathrm{d}\mu\left(x\right) - \int\limits_{Y} P_y^{\beta'}\left[v\left(y,z\right)\right]\mathrm{d}\nu\left(y\right) \end{aligned}$$

Note that all expectations are taken over $Z = Z_0 \cup \{\varnothing_d\} \cup \{\varnothing_s\}$. For a given x, the first one $E_x^{\alpha'}\left[u\left(x,z\right)\right]$ represents the average utility of consumers of type x. If they are all out of the market, this average utility is zero, if some of them are out and others in, the contribution of those who are out is zero. Similarly, the second one $E_y^{\beta'}\left[v\left(y,z\right)\right]$ represents the average cost of producers of type y. The sum J therefore is the aggregate utility of society resulting from $\alpha'_{X\times Z}$ and $\beta'_{Y\times Z}$ consumers and suppliers being equally weighted.

In the following, we restrict attention to demand and supply distributions $\alpha'_{X\times Z}$ and $\beta'_{Y\times Z}$ such that the marginals α'_{Z_0} and β'_{Z_0} are equal. These are the only ones that are relevant to the planner's problem, which consists of matching producers and consumers so as to maximize social surplus. The solution to that problem turns out to be precisely the equilibrium allocation.

Theorem 15 (Pareto optimality of equilibrium allocations) *Let $(p, \alpha_{X\times Z}, \beta_{Y\times Z})$ be an equilibrium. Take any pair of demand and supply distributions $\alpha'_{X\times Z}$ and $\beta'_{Y\times Z}$ such that $\alpha'_{Z_0} = \beta'_{Z_0}$. Then*

$$J\left(\alpha'_{X\times Z},\beta'_{Y\times Z}\right) \leq J\left(\alpha_{X\times Z},\beta_{Y\times Z}\right) = \int\limits_{X} p^{\sharp}\left(x\right)\mathrm{d}\mu - \int\limits_{Y} p^{\flat}\left(y\right)\mathrm{d}\nu \tag{6}$$

The proof of the two last theorems will be given in Appendix D.

3.4 Example 1: the case of a single quality

Let $Z_0 = \{z\}$. In other words, there is a single technologically feasible quality. While this example does not have great economic interest, it is quite illuminating to see what the various assumptions mean and how the preceding results apply to this case.

We introduce $Z = \{z\} \cup \{\varnothing_d\} \cup \{\varnothing_s\}$. For the sake of simplicity, consider the case when X and Y are finite. Set $u\left(x,z\right) = u\left(x\right)$ and $v\left(y,z\right) = v\left(y\right)$ and $p\left(z\right) = p$. Indirect utilities are given by:

$$\begin{aligned} \max\left\{u\left(x\right) - p, 0\right\} &= p_x^{\sharp} \qquad \text{for } x \\ \max\left\{p - v\left(y\right), 0\right\} &= -p_y^{\flat} \quad \text{for } y \end{aligned}$$

The highest bid price for z is $b = \max_x u\left(x\right)$, and the lowest ask price is $a = \min_y v\left(y\right)$.

If $b < a$, then the quality z is not marketable, and the no-trade equilibrium prevails.

Suppose $b \geq a$. A price p is admissible if $a \leq p \leq b$. Set:

$$I_1(p) = \{x \in X \mid u(x) < p\}$$
$$I_2(p) = \{x \in X \mid u(x) = p\}$$
$$I_3(p) = \{x \in X \mid u(x) > p\}$$

and define $J_1(p)$, $J_2(p)$, $J_3(p)$ in a similar way for producers. An equilibrium is a set (p, α, β) such that

- $\alpha = (\alpha_x)$, $x \in X$, where each α_x is a probability on $\{z\} \cup \{\varnothing_d\}$
- $\beta = (\beta_y)$, $y \in Y$, where each β_y is a probability on $\{z\} \cup \{\varnothing_s\}$
- $\sum_x \alpha_x(z) = \sum_y \beta_y(z)$

Let us translate this. If $x \in I_1(p)$, then consumers of type x stay out of the market, so that $\alpha_x(z) = 0$. If $x \in I_3(p)$, then consumers of type x buy z, so that $\alpha_x(z) = 1$. If $i \in I_2(p)$, then $\alpha_x(z)$ is the proportion of consumers of type x who buy z in equilibrium. Denote by $\#[A]$ the number of elements in a finite set A. The equilibrium condition implies that:

$$\#[I_3(p)] \leq \#[J_2(p) \cup J_3(p)] \tag{7}$$

$$\#[J_3(p)] \leq \#[I_2(p) \cup I_3(p)] \tag{8}$$

Conversely, if these two inequalities are satisfied, we will always be able to find numbers α_x and β_y such that $0 \leq \alpha_x \leq 1$, $\alpha_x = 0$ if $x \in I_1(p)$ and $\alpha_x = 1$ if $x \in I_3(p)$, with corresponding constraints for the β_y. So, in that particular case, the equilibrium conditions boil down to the inequalities (7) and (8).

Note that there is no uniqueness of the equilibrium price p. If for instance $u_{\bar{x}} > v_{\bar{y}}$, with $u_x < v_{\bar{y}}$ for all $x \neq \bar{x}$ and $v_y > u_{\bar{x}}$ for all $y \neq \bar{y}$, then any price $p \in [u_{\bar{x}}, v_{\bar{y}}]$ is an equilibrium price. There is no uniqueness of the equilibrium allocation either. If for instance $u_x = v_y = p$ for all x, y, then the unique equilibrium price is p, so that all consumers and producers are indifferent in equilibrium. For any choice of coefficients $\alpha_x(z)$ and $\beta_y(z)$ such that:

$$0 \leq \alpha_x(z) \leq 1, \quad 0 \leq \beta_y(z) \leq 1, \quad \sum_x \alpha_x(z) = \sum_y \beta_y(z)$$

(p, α, β) is an equilibrium allocation.

3.5 Example 2: more on uniqueness

We give an example to clarify the uniqueness statement in Theorem 12. There are three goods, z_1, z_2, z_3, two consumers $x_1, x_2,$ three producers y_1, y_2, y_3. The utility functions are:

$$u(x_1, z_1) = 2, \quad u(x_1, z_2) = 1, \quad u(x_1, z_3) = 0.1$$
$$u(x_2, z_1) = 3, \quad u(x_2, z_2) = 2, \quad u(x2, z_3) = 0.1$$

and the cost functions are:

$$v(y_1, z_1) = 0, \quad v(y_1, z_2) = 5, \quad v(y_1, z_3) = 5$$
$$v(y_2, z_1) = 5, \quad v(y_2, z_2) = 0, \quad v(y_2, z_3) = 5$$
$$v(y_3, z_1) = 5, \quad v(y_1, z_2) = 5, \quad v(y_1, z_3) = 0$$

It is easy to check that there are two equilibria:

1 y_1 produces z_1, y_2 produces z_2, y_3 produces nothing; x_1 chooses z_1, x_2 chooses z_2; prices are $p(z_1) = 1$, $p(z_2) = 0$, $p(z_3) = 0$
2 y_1 produces z_1, y_2 produces z_2, y_3 produces nothing; x_1 chooses z_2, x_2 chooses z_1; prices are $p(z_1) = 1.9$, $p(z_2) = 0.9$, $p(z_3) = 0$

The demand set of x_1 is $\{z_1, z_2\} := D_1(x_1)$ in the first equilibrium and $\{z_1, z_2, z_3\} := D_2(x_1)$ in the second. The demand *distribution*, on the other hand, is $P_{x_1}^1(z) = \delta_{z_1}$ (Dirac mass at z_1) in the first equilibrium (simply expressing the fact that x_1 chooses z_1 and nothing else in her demand set) and $P_{x_1}^2(z) = \delta_{z_2}$ in the second. Theorem 12 then states that $\delta_{z_2}[D_1(x_1)] = \delta_{z_1}[D_1(x_1)] = 1$, which simply expresses the fact that both z_1 and z_2 belong to $D_1(x_1)$.

Note for instance that the social utility is the same for both equilibria, namely 4:

1. In the first one:

$$u(x_1, z_1) - v(y_1, z_1) + u(x_2, z_2) - v(y_2, z_2) = 2 - 0 + 2 - 0 = 4$$

2. In the second one:

$$u(x_1, z_2) - v(y_2, z_2) + u(x_2, z_1) - v(y_1, z_1) = 1 - 0 + 3 - 0 = 4$$

This is a general fact: the social utility is *the same at all equilibria*. Indeed, equilibrium prices are found by maximizing the right-hand side of (6): it may be achieved at different p_1 and p_2, but the value of the maximum is the same.

4 Pure equilibrium

4.1 Definition

In equilibrium, consumers of type x demand quality z with probability $P_x^\alpha(z)$, and suppliers of type y supply quality z with probability $P_y^\beta(z)$. The equilibrium is *pure* if all agents of the same type who are in the market at the same time are doing the same thing (buying or selling the same quality), so that these probabilities are Dirac masses. Formally:

Definition 16 An equilibrium $(p, \alpha_{X\times Z}, \beta_{Y\times Z})$ is *pure* if:

- for μ-almost every x, the set $D(x) \cap Z_0$ contains at most one point
- for ν-almost every y, the set $S(y) \cap Z_0$ contains at most one point

Denote by X_p the set of active or indifferent consumers. If $(p, \alpha_{X\times Z}, \beta_{Y\times Z})$ is a pure equilibrium, there is a Borel map $d : X_p \to Z_0$ with $d(x) \in D(x)$ such that, for μ-almost every x, one and only one of the following holds:

- either consumers of type x are inactive, so that $D(x) = \varnothing_d$
- or consumers of type x are indifferent; then $D(x) = \varnothing_d \cup \{d(x)\}$
- or consumers of type x are active; then $D(x) = \{d(x)\}$

We can then rewrite the definition of equilibrium directly in terms of s and d.

Definition 17 A *pure equilibrium* is a triplet (p, d, s) where:

1. d is a Borel map from the set $X_p = \left\{x \mid p^\sharp(x) \geq 0\right\}$ into Z_0
2. s is a Borel map from the set $Y_p = \left\{y \mid p^\flat(y) \leq 0\right\}$ into Z_0
3. For μ- almost every x with $p^\sharp(x) > 0$, the function $z \to u(x,z) - p(z)$ attains its maximum at a single point $z = d(x) \in Z_0$
4. For ν-almost every y with $p^\flat(y) < 0$, the function $z \to p(z) - v(y,z)$ attains its maximum at a single point $z = s(y) \in Z_0$
5. For μ- almost every x with $p^\sharp(x) = 0$, the function $z \to u(x,z) - p(z)$ attains its maximum at two points, $\varnothing_d$ and $z = d(x) \in Z_0$
6. For ν-almost every y with $p^\flat(y) = 0$, the function $z \to p(z) - v(y,z)$ attains its maximum at two points, $\varnothing_s$ and $z = s(y) \in Z_0$
7. The demand and supply distributions α and β associated with d and s have the same marginals on Z_0:

$$\forall A \subset Z_0, \quad \mu[x \mid d(x) \in A] = \nu[y \mid s(y) \in A]$$

For the sake of simplicity, we shall now assume that $a(z) < b(z)$ for every $z \in Z$. As a consequence, $Z_1 = Z$.

4.2 Uniqueness

Theorem 18 *Let* (p_1, d_1, s_1) *and* (p_2, d_2, s_2) *be two pure equilibria. Every consumer* x *who is active in one equilibrium is active or indifferent in the other, and we have* $d_1(x) = d_2(x)$. *Similarly, every producer* y *who is active in one equilibrium is active or indifferent in the other, and* $s_1(y) = s_2(y)$.

Proof It is an immediate consequence of the quasi-uniqueness theorem for equilibrium allocations. □

4.3 Existence

Theorem 19 *Assume that the standard assumptions hold. Assume moreover that* μ *and* ν *are absolutely continuous with respect to the Lebesgue measure, and that the* *partial derivatives $D_x u$ and $D_y v$ with respect to z are injective:*

$$\forall x \in X, \quad D_x u\,(x, z_1) = D_x u\,(x, z_2) \Longrightarrow z_1 = z_2 \tag{9}$$

$$\forall y \in Y, \quad D_y v\,(y, z_1) = D_y v\,(y, z_2) \Longrightarrow z_1 = z_2 \tag{10}$$

Then any equilibrium is pure.

Corollary 20 *In the above situation, there is a pure equilibrium.*

Proof We know that there is an equilibrium, by the Existence Theorem, and we know that it has to be pure. □

If X and Z are one-dimensional intervals, condition (9) is satisfied if

$$\frac{\partial^2 u}{\partial x \partial z} \neq 0$$

so that condition (9), or (10) for that matter, is a multi-dimensional generalization of the classical Spence–Mirrlees condition in the economics of assymmetric information (Carlier 2003). It is satisfied, for instance, by $u\,(x, z) = \|x - z\|^{\alpha}$, provided $\alpha \neq 0$ and $\alpha \neq 1$; if $\alpha < 1$, one should add the requirement that $X \cap Z \neq \varnothing$, so that u is differentiable on $X \times Z$.

4.4 Example

4.4.1 A case when $Z_a = \varnothing = Z_b$

Set $X = [1, 2]$ and $Y = [2, 3]$. Both are endowed with the Lebesgue measure. Set $Z_0 = [0, 1]$ and

$$u\,(x, z) = -\frac{1}{2}z^2 + xz, \quad \bar{u}\,(x) = 0$$
$$v\,(y, z) = \frac{1}{2}yz^2, \quad \bar{v}\,(y) = 0$$

so that suppliers are ordered on the line according to efficiency, the most efficient ones (those with the lowest cost, near $y = 2$) being on the left, and consumers are ordered according to taste, the most avid ones (those with the highest utility, near $x = 2$) being on the right (note the order reversal).

We compute the lowest ask $a\,(z)$ and the highest bid $b\,(z)$:

$$b\,(z) = \bar{u}^{\sharp}\,(z) = \max_{1 \leq x \leq 2} \left\{-\frac{1}{2}z^2 + xz - 0\right\} = -\frac{1}{2}z^2 + 2z$$
$$a\,(z) = \bar{v}^{\flat}\,(z) = \min_{2 \leq y \leq 3} \left\{\frac{1}{2}yz^2 - 0\right\} = z^2$$

Note that $b(z)$ is the bid price for consumer $x = 2$ (the most avid one), and $a(z)$ is the ask price for supplier $y = 2$ (the least efficient one). We have $a \leq b$ as expected.

Note that the generalized Spence–Mirrlees assumptions (9) and (10) are satisfied:

$$D_x u(x, z) = z$$
$$D_y v(y, z) = \frac{1}{2} z^2$$

and both are injective with respect to z. So Theorem 19 applies, and there is a pure equilibrium, with some degree of uniqueness.We shall now compute it.

Assume for the moment that every agent is active. This is possible here since $\mu(X)$ happens to be equal to $\nu(Y)$ (in other words, there are as many consumers as suppliers). This means that $Z_a = \varnothing = Z^b$, and $Z_1 = Z_0$, so that we can try the reduction method we described in the preceding section.

We start with finding the optimal matching between X and Y. Given x and y, the quality $z(x, y)$ which maximizes the utility of the pair (x, y) is obtained by maximizing the expression $-z^2/2 + xz - yz^2/2$ with respect to z, which yields:

$$z(x, y) = \frac{x}{1+y}$$
$$w(x, y) = \frac{1}{2} \frac{x^2}{1+y}$$

where $w(x, y)$ is the resulting utility for the pair. We then seek the measure-preserving map $\sigma : [1, 2] \to [2, 3]$ which maximizes the integral:

$$\int_1^2 w(x, \sigma(x))\, dx = \int_1^2 \frac{x^2}{1+\sigma(x)} dx$$

We have:

$$\frac{\partial^2 w}{\partial x \partial y} = -\frac{x}{(1+y)^2} < 0$$

so w satisfies the Spence–Mirrlees assumption. By the general theory of optimal transportation (Villani 2003), the map σ is uniquely defined. We find that:

$$\sigma(x) = y = 4 - x$$

either by deciding that σ must be continous and comparing directly the two candidates $y = 4 - x$ (decreasing) and $y = x + 1$ (increasing), or, more rigorously, by checking directly that σ is the subgradient of a w-convex function, which, by the general theory again, implies that it is the minimizer. Hence the supply and demand maps $s(y)$ and $d(x)$:

$$d(x) = \frac{x}{5-x} \tag{11}$$

$$s(y) = \frac{4-y}{1+y} \tag{12}$$

and the set of traded qualities is $Z_t = \left[\frac{1}{4}, \frac{2}{3}\right]$, which is a strict subset of Z_0: again, not all technologically feasible qualiities are traded in equilibrium. On Z_t, the price is uniquely defined, and is found by writing the first-order condition for optimality, $p'(z) = \frac{\partial u}{\partial z}(x, z)$ where $z = d(x)$. Inverting this map, we get a differential equation for p, namely $p'(z) = z + 5z(1+z)^{-1}$, yielding:

$$p(z) = -\frac{1}{2}z^2 + 5z - 5\ln(z+1) + c \quad \text{for } \frac{1}{4} \le z \le \frac{2}{3} \tag{13}$$

We can now try to validate our assumption that every agent is active. Compute the indirect utilities:

$$p^\sharp(x) = u(x, d(x)) - p(d(x)) = x + 5(\ln 5 - \ln(5-x)) - c$$
$$-p^\flat(y) = p(s(y)) - v(y, s(y)) = \frac{(4-y)(6+y)}{2(1+y)} - 5(\ln 5 - \ln(1+y)) + c$$

Every agent is active if and only if $p^\sharp(x) > \bar{u}(x)$ for every x and $p^\flat(y) < -\bar{v}(y)$ for every y. This leads us to explicit bounds for c:

$$-0.00928 = -\frac{9}{8} + 5\ln\frac{5}{4} \le c \le 1 + 5\ln\frac{5}{4} = 2.1157 \tag{14}$$

For any c in that interval, the function $p(z)$ given by formula (13) is the restriction to $Z_t = \left[\frac{1}{4}, \frac{2}{3}\right]$ of an equilibrium price, the equilibrium supply and demand being given by (12) and (11).

We now have to extend p_t to $Z_0 = [0, 1]$ in such a way that the qualities $z \in \left[0, \frac{1}{4}\right] \cup \left[\frac{2}{3}, 1\right]$ are not traded. For $z = \frac{1}{4}$, the least efficient supplier $y = 3$ provides the least avid consumer $x = 1$, and the price of qualities $z \le \frac{1}{4}$ must be such that each of them prefers staying at $\frac{1}{4}$. This yields the inequalities:

$$p(z) - v(3, z) \le p\left(\frac{1}{4}\right) - v\left(3, \frac{1}{4}\right)$$
$$u(1, z) - p(z) \le u\left(1, \frac{1}{4}\right) - p\left(\frac{1}{4}\right)$$

and hence:

$$-\frac{1}{2}z^2 + z + 1 - 5\ln\frac{5}{4} + c \le p(z) \le \frac{3}{2}z^2 + \frac{9}{8} - 5\ln\frac{5}{4} + c \quad \text{for } 0 \le z \le \frac{1}{4} \tag{15}$$

Similarly, for $z \geq \frac{2}{3}$, we get the inequalities:

$$-\frac{1}{2}z^2 + 2z + 2 - 5\ln\frac{5}{3} + c \leq p(z) \leq z^2 + \frac{8}{3} - 5\ln\frac{5}{3} + c \quad \text{for } \frac{2}{3} \leq z \leq 1 \tag{16}$$

In summary, given any c satisfying (14), any function $p(z)$ satisfying (13), (15), and (16) is an equilibrium price. By Theorem 18, s and d are uniquely determined, in the sense that any pure equilibrium such that all agents are active will have the same supply and demand. This implies that the pure equilibria we have just found are the only ones for which $Z_0 = \bar{Z}$.

4.4.2 A case when Z_a is non-empty

Let us now increase the number of consumers: say $Y = [2,\ 3]$ is unchanged, while $X = [h,\ 2]$ with $0 < h < 1$. Both intervals are endowed with the Lebesgue measure. In equilibrium, if all suppliers are active, then consumers in the range $[h,\ 1]$ must be priced out of the market. This is done by fixing c in formula (14) to its highest possible value, namely $1 + 5\ln\frac{5}{4}$:

$$p(z) = -\frac{1}{2}z^2 + 5z - 5\ln\frac{4(z+1)}{5} + 1 \quad \text{for } \frac{1}{4} \leq z \leq \frac{2}{3} \tag{17}$$

Then consumer $x = 1$ makes precisely his/her reservation utility, which means that he/she is indifferent.

Recall that $d(1) = \frac{1}{4} = s(3)$. For $0 < z < \frac{1}{4}$, consider the bid price for quality z by consumer $x = 1$:

$$b(1, z) = -\frac{1}{2}z^2 + z$$

Consumers of type $x < 1$ will have a lower bid price. Choose a continuous function p such that:

$$-\frac{1}{2}z^2 + z < p(z) < p\left(\frac{1}{4}\right) - v\left(3, \frac{1}{4}\right) + v(3, z) = \frac{3}{2}z^2 + \frac{17}{8} \quad \text{for } 0 \leq z \leq \frac{1}{4} \tag{18}$$

The left inequality ensures that consumers or type $x < 1$ are not bidders for quality z, so they just buy quality 0 at price 0, that is, they revert to their reservation utility. The right inequality ensures that the least efficient producer will not become interested in producing quality z, so that the more efficient ones will not either.

Any function $p(z)$ satisfying (17), (18) and (16) (with $c = 1 + 5\ln\frac{5}{4}$) is an equilibrium price. Note that for all consumers $x \in [h,\ 1[$ demand is uniquely defined: $d(x) = 0$.

5 Open problems

In this paper, we have assumed that the good is indivisible, and that consumers and producers are limited to buying and selling one unit. That assumption can be relaxed. Indeed, our results carry through if we assume that suppliers, for instance, are restricted to producing one quality, but have the choice of the quantity they produce, their profit then being $np - v(y, z, n)$, where z is the quality produced, n the quantity, p the price, and y the type of the supplier.

As we mentioned in the beginning, the main limitation of our model is the assumption that utilities are separable. A truly general model would introduce a quantity good beside the quality good, and consumers of type x would solve the problem:

$$\max \{u(x, z, t) \mid p(z) + \pi t \leq w\}$$

where t is the quantity of the second good, and π its (linear) price. Our methods do not readily apply to this situation, and we plan to investigate it further.

Finally, we wish to stress that although we have what appears as a complete equilibrium theory for multidimensional hedonic models, the numerical aspects are far from being as well understood. The method we used in the example is strictly one-dimensional, and there is no easy way to extend it to the multidimensional case. The obvious way to proceed is to follow the theoretical argument, and try to minimize the integral $I(p)$ in (33), but we have made no progress in that direction. It certainly is a good topic for future research. So will all the econometric aspects (characterization and identification). This investigation has been started in Ekeland et al. (2004), but is far from being complete.

Appendix A: Fundamentals of u-convex analysis

In this section, we basically follow Carlier (2003).

A.1 u-convex functions

We will be dealing with function taking values in $\mathbb{R} \cup \{+\infty\}$.

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ will be called *u-convex* iff there exists a non-empty subset $A \subset Z \times \mathbb{R}$ such that:

$$\forall x \in X, \quad f(x) = \sup_{(z,\alpha)\in A} \{u(x, z) + a\} \tag{19}$$

A function $p : Z \to \mathbb{R} \cup \{+\infty\}$ will be called *u-convex* iff there exists a non-empty subset $B \subset X \times \mathbb{R}$ such that:

$$p(z) = \sup_{(x,b)\in B} \{u(x, z) + b\} \tag{20}$$

A.2 Subconjugates

Let $f : X \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$, be given. We define its *subconjugate* $f^\sharp : Z \to \mathbb{R} \cup \{+\infty\}$ by:

$$f^\sharp(z) = \sup_x \{u(x,z) - f(x)\} \tag{21}$$

It follows from the definitions that $f^\sharp$ is a u-convex function on Z (it might be identically $\{+\infty\}$).

Let $p : Z \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$, be given. We define its *subconjugate* $p^\sharp : X \to \mathbb{R} \cup \{+\infty\}$ by:

$$p^\sharp(x) = \sup_z \{u(x,z) - p(z)\} \tag{22}$$

It follows from the definitions that $p^\sharp$ is a u-convex function on X (it might be identically $\{+\infty\}$).

Example 21 Set $f(x) = u(x,\bar{z}) + a$. Then

$$f^\sharp(\bar{z}) = \sup_x \{u(x,\bar{z}) - u(x,\bar{z}) - a\} = -a$$

Conjugation reverses ordering: if $f_1 \le f_2$, then $f_1^\sharp \ge f_2^\sharp$, and if $p_1 \le p_2$, then $p_1^\sharp \ge p_2^\sharp$. As a consequence, if f is u-convex, not identically $\{+\infty\}$, then $f^\sharp$ is u-convex, not identically $\{+\infty\}$. Indeed, since f is u-convex, we have $f(x) \ge u(x,z) + a$ for some (z,a), and then $f^\sharp(z) \le -a < \infty$.

Proposition 22 (the Fenchel inequality) *For any functions* $f : X \to \mathbb{R} \cup \{+\infty\}$ *and* $p : Z \to \mathbb{R} \cup \{+\infty\}$*, not identically* $\{+\infty\}$*, we have:*

$$\forall (x,z), \quad f(x) + f^\sharp(z) \ge u(x,z)$$
$$\forall (x,z), \quad p(z) + p^\sharp(x) \ge u(x,z)$$

A.3 Subgradients

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be given, not identically $\{+\infty\}$. Take some point $x \in X$. We shall say that a point $z \in Z$ is a *subgradient* of f at x if the points x and z achieve equality in the Fenchel inequality:

$$f(x) + f^\sharp(z) = u(x,z) \tag{23}$$

The set of subgradients of f at x will be called the *subdifferential* of f at x and denoted by $\partial f(x)$. Specifically:

Definition 23 $\partial f(x) = \arg\max_z \left\{ u(x, z) - f^\sharp(z) \right\}$.

Similarly, let $p : Z \to \mathbb{R} \cup \{+\infty\}$ be given, not identically $\{+\infty\}$. Take some point $z \in Z$. We shall say that a point $x \in X$ is a *subgradient* of p at z if:

$$p^\sharp(x) + p(z) = u(x, z) \tag{24}$$

The set of subgradients of p at z will be called the *subdifferential* of p at z and denoted by $\partial p(z)$.

Definition 24 $\partial p(z) = \arg\max_x \left\{ u(x, z) - p^\sharp(x) \right\}$.

Proposition 25 *The following are equivalent:*

1. $z \in \partial f(x)$
2. $\forall x', \quad f\left(x'\right) \geq f(x) + u\left(x', z\right) - u(x, z)$

If equality holds for some x', then $z \in \partial f\left(x'\right)$ as well.

Proof We begin with proving that the first condition implies the second one. Assume $z \in \partial f(x)$. Then, by (23) and the Fenchel inequality, we have:

$$f\left(x'\right) \geq u\left(x', z\right) - f^\sharp(z) = u\left(x', z\right) - [u(x, z) - f(x)]$$

We then prove that the second condition implies the first one. Using the inequality, we have:

$$\begin{aligned}
f^\sharp(z) &= \sup_{x'} \left\{ u\left(x', z\right) - f\left(x'\right) \right\} \\
&\leq \sup_{x'} \left\{ u\left(x', z\right) - f(x) - u\left(x', z\right) + u(x, z) \right\} \\
&= u(x, z) - f(x)
\end{aligned}$$

so $f(x) + f^\sharp(z) \leq u(x, z)$. We have the converse by the Fenchel inequality, so equality holds.

Finally, if equality holds for some x' in condition (2), then $f\left(x'\right) - u\left(x', z\right) = f(x) - u(x, z)$, so that:

$$\begin{aligned}
\forall x'', \quad f\left(x''\right) &\geq f(x) - u(x, z) + u\left(x'', z\right) \\
&= f\left(x'\right) - u\left(x', z\right) + u\left(x'', z\right)
\end{aligned}$$

which implies that $z \in \partial f\left(x'\right)$. $\square$

There is a similar result for functions $p : Z \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$: we have $x \in \partial p(z)$ if and only if

$$\forall \left(x', \bar{Z}\right), \quad p\left(\bar{Z}\right) \geq p(z) + u\left(x, \bar{Z}\right) - u(x, z) \tag{25}$$

A.4 Biconjugates

It follows from the Fenchel inequality that, if $p : Z \to \mathbb{R} \cup \{+\infty\}$ is not identically $\{+\infty\}$:

$$p^{\sharp\sharp}(z) = \sup_x \left\{ u(x,z) - p^{\sharp}(x) \right\} \leq p(z) \tag{26}$$

Example 26 Set $p(z) = u(\bar{x}, z) + b$. Then

$$\begin{aligned} p^{\sharp\sharp}(z) &= \sup_x \left\{ u(x,z) - p^{\sharp}(x) \right\} \\ &\geq u(\bar{x}, z) - p^{\sharp}(\bar{x}) \\ &= u(\bar{x}, z) + b = p(z) \end{aligned}$$

This example generalizes to all u-convex functions. Denote by $\mathbb{C}_u(Z)$ the set of all u-convex functions on Z.

Proposition 27 *For every function $p : Z \to \mathbb{R} \cup \{+\infty\}$, not identically $\{+\infty\}$, we have*

$$p^{\sharp\sharp}(z) = \sup_{\varphi} \{ \varphi(z) \mid \varphi \leq p, \ \varphi \in \mathbb{C}_u(Z) \}$$

Proof Denote by $\bar{p}(z)$ the right-hand side of the above formula. We want to show that $p^{\sharp\sharp}(z) = \bar{p}(z)$.

Since $p^{\sharp\sharp} \leq p$ and $p^{\sharp\sharp}$ is u-convex, we must have $p^{\sharp\sharp} \leq \bar{p}$.

On the other hand, $\bar{p}$ is u-convex because it is a supremum of u-convex functions. So there must be some $B \subset X \times \mathbb{R}$ such that:

$$\bar{p}(z) = \sup_{(x,b) \in B} \{ u(x,z) + b \}$$

Let $(x, b) \in B$. Since $\bar{p} \leq p$, we have $u(x,z) + b \leq \bar{p}(z) \leq p(z)$. Taking biconjugates, as in the preceding example, we get $u(x,z) + b \leq p^{\sharp\sharp}(z)$. Taking the supremums over $(x, b) \in B$, we get the desired result. □

Corollary 28 *Let $p : Z \to \mathbb{R} \cup \{+\infty\}$ be a u-convex function, not identically $\{+\infty\}$. Then $p = p^{\sharp\sharp}$, and the following are equivalent:*

1. $x \in \partial p(z)$
2. $p(z) + p^{\sharp}(x) = u(x,z)$
3. $z \in \partial p^{\sharp}(x)$

Proof We have $p^{\sharp\sharp} \leq p$ always by relation (26). Since p is u-convex, we have:

$$p(z) = \sup_{(x,b) \in B} \{ u(x,z) + b \}$$

for some $B \subset X \times \mathbb{R}$. By proposition 27, we have:

$$\sup_{(x,b)\in B} \{u(x,z)+b\} \le p^{\sharp\sharp}(z)$$

and so we must have $p = p^{\sharp\sharp}$. Taking this relation into account, as well as the definition of the subgradient, we see that condition (2) is equivalent both to (1) and to (3) □

Definition 29 We shall say that a function $p : Z \to \mathbb{R}\cup\{+\infty\}$ is *u-adapted* if it is not identically $\{+\infty\}$ and there is some $(x,b) \in X \times R$ such that:

$$\forall z \in Z, \quad p(z) \ge u(x,z) + b$$

It follows from the above that if p is u-adapted, then so are $p^\sharp$, $p^{\sharp\sharp}$ and all further subconjugates. Note that a u-convex function which is not identically $\{+\infty\}$ is u-adapted.

Corollary 30 *Let $p : Z \to \mathbb{R}\cup\{+\infty\}$ be u-adapted. Then :*

$$p^{\sharp\sharp\sharp} = p^\sharp$$

Proof If p is u-adapted, then $p^\sharp$ is u-convex and not identically $\{+\infty\}$. The result then follows from Corollary 28. □

A.5 Smoothness

Since u is continuous and $X \times Z$ is compact, the family

$$\{u(x,\cdot) \mid x \in X\}$$

is uniformly equicontinuous on Z. It follows from Definition 19 that all u-convex functions on Z are continuous (in particular, they are finite everywhere).

Denote by k the upper bound of $\|D_x u(x,z)\|$ for $(x,z) \in X \times Z$. Since $D_x u$ is continuous and $X \times Z$ is compact, we have $k < \infty$, and the functions $x \to u(x,z)$ are all k-Lipschitzian on X. Again, it follows from the Definition 19 that all u-convex functions on X are k-Lipschitz (in particular, they are finite everywhere). By Rademacher' theorem, they are differentiable almost everywhere with respect to the Lebesgue measure.

Let $f : X \to R$ be convex. Since $f = f^{\sharp\sharp}$, we have:

$$f(x) = \sup_z \left\{u(x,z) - f^\sharp(z)\right\}$$

Since $f^\sharp$ is u-convex, it is continuous, and the supremum is achieved on the right-hand side, at some point $z \in \partial f(x)$. This means that all u-convex functions on X are subdifferentiable everywhere on X.

The following result will also be useful:

Proposition 31 *Let $p : Z \to R$ be u-adapted, and let $x \in X$ be given. Then there is some point $z \in \partial p^\sharp(x)$ such that $p(z) = p^{\sharp\sharp}(z)$.*

Proof Assume otherwise, so that for every $z \in \partial p^\sharp(x)$ we have $p^{\sharp\sharp}(z) < p(z)$. For every $z \in \partial p^\sharp(x)$, we have $x \in \partial p^{\sharp\sharp}(z)$, so that, by Proposition 25, we have

$$p^{\sharp\sharp}\left(z'\right) \geq u\left(x, z'\right) - u(x, z) + p^{\sharp\sharp}(z)$$

for all $z' \in Z$, the inequality being strict if $z' \notin \partial p^\sharp(x)$. Set $\varphi_z\left(z'\right) = u\left(x, z'\right) - u(x, z) + p^{\sharp\sharp}(z)$. We have:

$$\begin{aligned} z' \notin \partial p^\sharp(x) &\Longrightarrow \varphi_z\left(z'\right) < p^{\sharp\sharp}\left(z'\right) \leq p\left(z'\right) \\ z' \in \partial p^\sharp(x) &\Longrightarrow \varphi_z\left(z'\right) \leq p^{\sharp\sharp}\left(z'\right) < p\left(z'\right) \end{aligned}$$

so that $\varphi_z\left(z'\right) < p\left(z'\right)$ for all $\left(z, z'\right)$. Since Z is compact, there is some $\varepsilon > 0$ such that $\varphi_z\left(z'\right) + \varepsilon \leq p\left(z'\right)$ for all $\left(z, z'\right)$. Taking the subconjugate with respect to z', we get:

$$\begin{aligned} p^\sharp(x) &\leq \sup_{z'}\left\{u\left(x, z'\right) - \varphi_z\left(z'\right)\right\} - \varepsilon \\ &= \sup_{z'}\left\{u\left(x, z'\right) - u\left(x, z'\right) + u(x, z) - p^{\sharp\sharp}(z)\right\} - \varepsilon \\ &= u(x, z) - p^{\sharp\sharp}(z) - \varepsilon = p^\sharp(x) - \varepsilon \end{aligned}$$

which is a contradiction. The result follows □

Corollary 32 *If $\partial p^\sharp(x) = \{z\}$ is a singleton, then:*

$$p(z) = p^{\sharp\sharp}(z) \tag{27}$$

and:

$$p^\sharp(x) = u(x, z) - p(z) \tag{28}$$

Proof Just apply the preceding proposition, bearing in mind that $\partial p^\sharp(x)$ contains only one point, namely $\nabla_u p^\sharp(x)$. This yields Eq. (27). Equation (28) follows from the definition of the subgradient and Eq. (27). □

A.6 v-concave functions

Let us now consider the duality between Y and Z. Given $v : Y \times Z \to R$, we say that a map $g : Y \to \mathbb{R} \cup \{-\infty\}$ is *v-concave* iff there exists a non-empty subset $A \subset Z \times \mathbb{R}$ such that:

$$\forall y \in Y, \quad g(y) = \inf_{(z,a) \in A}\{v(y, z) + a\} \tag{29}$$

and a function $p : Z \to \mathbb{R} \cup \{-\infty\}$ will be called *v-concave* iff there exists a non-empty subset $B \subset X \times \mathbb{R}$ such that:

$$p(z) = \inf_{(x,b)\in B} \{v(y, z) + b\} \tag{30}$$

All the results on u-convex functions carry over to v-concave functions, with obvious modifications. The *superconjugate* of a function $g : Y \to \mathbb{R} \cup \{-\infty\}$, not identically $\{-\infty\}$, is defined by:

$$g^{\flat}(z) = \inf_{y} \{v(y, z) - g(y)\} \tag{31}$$

and the *superconjugate* of a function $p : Z \to \mathbb{R} \cup \{-\infty\}$, not identically $\{-\infty\}$, is given by:

$$p^{\flat}(y) = \inf_{z} \{v(y, z) - p(z)\} \tag{32}$$

The superdifferential $\partial p^{\flat}$ is defined by:

$$\partial p^{\flat}(y) = \arg\min_{z} \{v(y, z) - p(z)\}$$

and we have the Fenchel inequality:

$$p(z) + p^{\flat}(y) \leq v(y, z) \quad \forall (y, z)$$

with equality iff $z \in \partial p^{\flat}(y)$. Note finally that $p^{\flat\flat} \geq p$, with equality if p is v-concave.

Appendix B: Some notations and definitions

B.1 Radon measures and probabilities

With a locally compact set Ω (such as an open subset of the compact set Z) we will associate the following sets of functions and measures on Ω:

- $\mathcal{K}(\Omega)$, the space of continous functions on Ω with compact support
- $\mathcal{C}^b(\Omega)$, the space of bounded continous functions on Ω
- $\mathcal{C}_+(\Omega)$, the cone of non-negative functions
- $\mathcal{M}(\Omega)$, the space of measures on Ω
- $\mathcal{M}_+(\Omega) \subset \mathcal{M}(\Omega)$, the cone of positive measures
- $\mathcal{M}_b(\Omega) \subset \mathcal{M}(\Omega)$, the cone of finite measures
- $\mathcal{M}_+^b(\Omega) = \mathcal{M}_b(\Omega) \cap \mathcal{M}_+(\Omega)$, the cone of positive finite measures
- $\mathcal{P}(\Omega) \subset \mathcal{M}_+^b(\Omega)$ the set of probabilities on Ω

The space $\mathcal{K}(\Omega)$ will be endowed with the topology of uniform convergence on compact subsets of Ω, and the space $\mathcal{C}^b(\Omega)$ with the uniform norm. Then $\mathcal{C}^b(\Omega)$ is a Banach space, but $\mathcal{K}(\Omega)$ is not, unless Ω is compact, in which case all continuous functions on Ω are bounded, and we have $\mathcal{C}(\Omega) = K(\Omega) = \mathcal{C}^b(\Omega)$. When Ω is finite and has d elements, all these spaces coincide with R^d.

We take measures in the sense of Radon, that is, $\mathcal{M}(\Omega)$ is defined to be the dual of $\mathcal{K}(\Omega)$ and $\mathcal{M}_b(\Omega)$ is defined to be the dual of $\mathcal{C}^b(\Omega)$. So $\mathcal{M}_b(\Omega)$ is a Banach space, but $\mathcal{M}(\Omega)$ is not, unless Ω is compact, in which case $\mathcal{M}(\Omega) = \mathcal{M}_b(\Omega)$, that is, all Radon measures on Ω are finite. For $\gamma \in \mathcal{M}(\Omega)$ and $\varphi \in \mathcal{K}(\Omega)$,we write indifferently $\langle \gamma, \varphi \rangle$ or $\int_Z \varphi d\gamma$.

A *probability* $\gamma \in \mathcal{P}(Z)$ is defined as a non-negative bounded measure such that $\langle \gamma, 1 \rangle = 1$. The set $\mathcal{P}(Z)$ is convex, and is compact in the weak* topology: $\gamma_n \to \gamma$ if $\langle \gamma_n, \varphi \rangle \to \langle \gamma, \varphi \rangle$ for every $\varphi \in \mathcal{C}^b(\Omega)$.

We say that a measure γ is *carried by* K if $\langle \gamma, \varphi \rangle = 0$ for all $\varphi \in \mathcal{K}(\Omega)$ which vanish on K. If γ is carried by a subset K, it is also carried by its closure. The *support* of a measure γ, denoted by Supp (γ), is the smallest closed set K such that γ is carried by K.

B.2 Conditional probabilities and marginals

Given a positive measure $\alpha_{X\times Z} \in \mathcal{M}_+(X \times Z)$ (which has to be finite, since $X \times Z$ is compact) we define its marginals $\alpha_X \in \mathcal{M}_+(X)$ and $\alpha_Z \in \mathcal{M}_+(Z)$as follows:

$$\int_X \varphi(x)\, \mathrm{d}\alpha_X = \int_{X\times Z} \varphi(x)\, \mathrm{d}\alpha_{X\times Z} \quad \forall \varphi \in \mathcal{K}(X)$$
$$\int_Z \psi(z)\, \mathrm{d}\alpha_Z = \int_{X\times Z} \psi(z)\, \mathrm{d}\alpha_{X\times Z} \quad \forall \psi \in \mathcal{K}(Z)$$

and we denote the probability of the second coordinate being z conditional on the first coordinate being x by $P_x^\alpha(z)$. The mathematical expectation with respect to this probability will be denoted by E_x^α:

$$E_x^\alpha[\psi] = \int_Z \psi(z)\, dP_x^\alpha(z)$$

This conditional probability is related to the first marginal by the formula:

$$\int_{X\times Z} f(x,z)\, \mathrm{d}\alpha_{X\times Y} = \int_X E_x^\alpha[f(x,z)]\, \mathrm{d}\alpha_X \quad \forall f \in \mathcal{K}(X \times Z)$$

Similar considerations hold for positive measures $\beta_{Y\times Z} \in \mathcal{M}_+(Y \times Z)$, We have:

$$\int_Y \varphi(y)\, \mathrm{d}\beta_Y = \int_{Y\times Z} \varphi(y)\, \mathrm{d}\beta_{Y\times Z} \quad \forall \varphi \in \mathcal{K}(Y)$$

$$\int_{Z} \psi(z)\,\mathrm{d}\beta_Z = \int_{X\times Z} \psi(z)\,\mathrm{d}\beta_{Y\times Z} \quad \forall\psi \in \mathcal{K}(Z)$$

$$\int_{Y\times Z} g(y,z)\,\mathrm{d}\beta_{Y\times Z} = \int_{Y} E_y^{\beta}[g(y,z)]\,\mathrm{d}\beta_Y \quad \forall g \in \mathcal{K}(Y\times Z)$$

Appendix C: Proof of the existence theorem

C.1 The dual problem: existence

Recall that $Z = \{\varnothing_d\} \cup Z_0 \cup \{\varnothing_s\}$,with $Z_1 = \{z \mid a(z) \leq b(z)\}$ a compact non-empty subset of Z_0. Denote by $\mathcal{A}$ the set of all admissible price systems on Z, that is, the set of all continuous maps $p : Z \to R$ which satisfy:

$$\forall z \in Z_1, \quad a(z) \leq p(z) \leq b(z)$$

$\mathcal{A}$ is a non-empty, convex and closed subset of $\mathcal{K}(Z)$, the space of all continuous functions on Z. Now define a map $I : \mathcal{K}(Z) \to R$ by:

$$I(p) = \int_X p^{\sharp}(x)\,\mathrm{d}\mu - \int_Y p^{\flat}(y)\,\mathrm{d}\nu \tag{33}$$

Proposition 33 *The map I is convex.*

Proof Take p_1 and p_2 in $\mathcal{A}$. Take s and t in [0, 1] with $s + t = 1$. Then:

$$\begin{aligned}
(sp_1 + tp_2)^{\sharp}(x) &= \sup_z \{u(x,z) - sp_1(z) - tp_2(z)\} \\
&= \sup_z \{s[u(x,z) - p_1(z)] + t[u(x,z) - p_2(z)]\} \\
&\leq s \sup_z \{u(x,z) - p_1(z)\} + t \sup_z \{u(x,z) - p_2(z)\} \\
&= sp_1^{\sharp}(x) + tp_2^{\sharp}(x)
\end{aligned}$$

Similarly, we find that:

$$(sp_1 + tp_2)^{\flat}(y) \geq sp_1^{\flat}(x) + tp_2^{\flat}(x)$$

Integrating, we find that I is convex, as announced. □

Now consider the convex optimization problem:

$$\inf_{p\in\mathcal{A}} I(p) \tag{34}$$

Proposition 34 *The set of solutions of problem (P) is convex.*

This follows from the fact that we are minimizing a convex function on a convex set. We have to show that this set is non-empty. The following lemma will be useful.

Lemma 35 *Assume p is admissible. Set:*

$$p_a^{\sharp\sharp}(z) = \max\left\{p^{\sharp\sharp}(z), a(z)\right\}$$

Then $\left(p_a^{\sharp\sharp}\right)^{\sharp} = p^{\sharp}$.

Proof We have $p^{\sharp\sharp} \leq p_a^{\sharp\sharp} \leq p$. Taking conjugates, we get $p^{\sharp} \leq \left(p_a^{\sharp\sharp}\right)^{\sharp} \leq \left(p^{\sharp\sharp}\right)^{\sharp} = p^{\sharp}$. □

Similarly, we find that $\left(p_b^{\flat\flat}\right)^{\flat} = p^{\flat}$, with $p_b^{\flat\flat} = \min\left\{p^{\flat\flat}, b\right\}$.

Proposition 36 *Problem (P) has a solution.*

Proof Take a minimizing sequence p_n. Since the functions $p_n^{\sharp}$ (resp. $p_n^{\flat}$), $n \in N$, are u-convex (resp. v-concave), they are uniformly Lipschitzian (see Appendix A), and hence equicontinuous. By Ascoli's theorem we can extract uniformly convergent subsequences (still denoted by $p_n^{\sharp}$ and $p_n^{\flat}$) :

$$p_n^{\sharp} \to f$$
$$p_n^{\flat} \to g$$

so that:

$$\int_X f(x)\,\mathrm{d}\mu - \int_Y g(y)\,\mathrm{d}\nu = \inf_{a \leq p \leq b}\left[\int_X p^{\sharp}(x)\,\mathrm{d}\mu - \int_Y p^{\flat}(y)\,\mathrm{d}\nu\right] \tag{35}$$

It is easy to see that f is u-convex and g is v-concave. In addition, $p_n^{\sharp\sharp} \to f^{\sharp}$ and $p_n^{\flat\flat} \to g^{\flat}$ everywhere (and uniformly as well, since the functions are equicontinuous). Since $p_n^{\sharp\sharp} \leq p_n^{\flat\flat}$, we get $f^{\sharp} \leq g^{\flat}$ in the limit. Since $p_n \leq b$, we have $p_n^{\sharp\sharp} \leq b^{\sharp\sharp} = b$, and letting $n \to \infty$, we find that $f^{\sharp} \leq b$. Since $f^{\sharp}$ is u-convex, it is continuous (and even Lipschitzian, see Appendix A). Similarly, $g^{\flat}$ is v-concave, hence continuous, and satisfies $g^{\flat} \geq a$.

Now take any continuous price schedule $\bar{p}$ such that

$$\left(f^{\sharp}\right)_a^{\sharp\sharp} = \max\left\{f^{\sharp}, a\right\} \leq \bar{p} \leq \min\left\{g^{\flat}, b\right\} = \left(g^{\flat}\right)_b^{\flat\flat} \tag{36}$$

for instance $\bar{p} = \frac{1}{2}\left(\max\left\{f^{\sharp}, a\right\} + \min\left\{g^{\flat}, b\right\}\right)$. By Lemma 35, we have

$$\left(\left(f^{\sharp}\right)_a^{\sharp\sharp}\right)^{\sharp} = f^{\sharp\sharp} = f$$
$$\left(\left(g^{\flat}\right)_b^{\flat\flat}\right)^{\flat} = g^{\flat\flat} = g$$

the last equalities occuring because f is u-convex and g is v-concave. Taking conjugates in formula (36), we get $g \leq \bar{p}^\flat$ and $f \geq \bar{p}^\sharp$.Substituting in the integral, we get:

$$\int\limits_X \bar{p}^\sharp (x)\, \mathrm{d}\mu - \int\limits_Y \bar{p}^\flat (y)\, \mathrm{d}\nu \leq \int\limits_X f(x)\, \mathrm{d}\mu - \int\limits_Y g(y)\, \mathrm{d}\nu$$

and hence, by formula (35):

$$\int\limits_X \bar{p}^\sharp (x)\, \mathrm{d}\mu - \int\limits_Y \bar{p}^\flat (y)\, \mathrm{d}\nu \leq \inf_{a \leq p \leq b} \left[\int\limits_X p^\sharp (x)\, \mathrm{d}\mu - \int\limits_Y p^\flat (y)\, \mathrm{d}\nu \right]$$

Since $\bar{p}$ is admissible, $\bar{p}$ must be a minimizer, and the result follows. □

The proof indicates that uniqueness is not to be expected. The following result is the Non-Uniqueness Theorem for prices:

Proposition 37 *Let p be a solution of problem (P). Then $p_a^{\sharp\sharp}$ and $p_b^{\flat\flat}$ are also solutions. More generally, if q is an admissible price schedule such that:*

$$p_a^{\sharp\sharp}(z) \leq q(z) \leq p_b^{\flat\flat}(z) \quad \forall z \in Z_1$$

then q is a solution of problem (P).

Proof From $p_a^{\sharp\sharp} \leq q \leq p_b^{\flat\flat}$, we deduce that $p^\flat = \left(p_b^{\flat\flat}\right)^\flat \leq q^\flat$ and that $q^\sharp \leq \left(p_a^{\sharp\sharp}\right)^\sharp = p^\sharp$. Substituting into the integral, we get:

$$\int\limits_X q^\sharp (x)\, \mathrm{d}\mu - \int\limits_Y q^\flat (y)\, \mathrm{d}\nu \leq \int\limits_X p^\sharp (x)\, \mathrm{d}\mu - \int\limits_Y p^\flat (y)\, \mathrm{d}\nu = \inf (P)$$

and since q is admissible, it must be a minimizer. □

Corollary 38 *Let p be a solution of problem (P). Then $p^\sharp = \left(p_b^{\flat\flat}\right)^\sharp$, μ-almost everywhere, and $p^\flat = \left(p_a^{\sharp\sharp}\right)^\flat$, ν-almost everywhere.*

Proof By the preceding Proposition, $p_b^{\flat\flat}$ is a solution of problem (P), so that $I\left(p_b^{\flat\flat}\right) = I(p)$ .Substituting in the integrals, we get:

$$\int\limits_X \left(p_b^{\flat\flat}\right)^\sharp \mathrm{d}\mu - \int\limits_Y \left(p_b^{\flat\flat}\right)^\flat \mathrm{d}\nu = \int\limits_X p^\sharp (x)\, \mathrm{d}\mu - \int\limits_Y p^\flat (y)\, \mathrm{d}\nu$$

and since $\left(p_b^{\flat\flat}\right)^{\flat} = p^{\flat}$, this reduces to:

$$\int\limits_X \left(p_b^{\flat\flat}\right)^{\sharp} \mathrm{d}\mu = \int\limits_X p^{\sharp}(x)\,\mathrm{d}\mu$$

Since $p_b^{\flat\flat} \leq p$, we have $\left(p_b^{\flat\flat}\right)^{\sharp} \geq p^{\sharp}$, and since the integrals are equal, it follows that $p^{\sharp} = \left(p_b^{\flat\flat}\right)^{\sharp}$, μ-a.e. The same argument shows that $p^{\flat} = \left(p_a^{\sharp\sharp}\right)^{\flat}$, ν-a.e. □

Corollary 39 *Let p be a solution of problem (P). Then, for μ-almost every x in X, there is a point $z \in D(x)$ such that $p(z) = p_b^{\flat\flat}(z)$, and for ν-almost every y in Y, there is a point $z \in S(y)$ such that $p(z) = p_a^{\sharp\sharp}(z)$.*

Proof Fix an x such that $p^{\sharp}(x) = \left(p_b^{\flat\flat}\right)^{\sharp}(x)$ and consider the functions φ and ψ defined by $\varphi(z) = u(x, z) - p(z)$ and $\psi(z) = u(x, z) - p_b^{\flat\flat}(z)$. We have $\varphi \geq \psi$, and $\max \varphi = \max \psi$. So there must be a point $\bar{z}$ such that $\max \varphi = \max \psi = \varphi(\bar{z}) = \psi(\bar{z})$ .The result follows. □

Note that we already have $p(z) = p^{\sharp\sharp}(z)$ for every $z \in D(x)$, and $p(z) = p^{\flat\flat}(z)$ for every $z \in S(y)$.

C.2 The dual problem: optimality conditions

Recall that we have defined a map $I : \mathcal{K}(Z) \to R$ by:

$$I(p) = \int\limits_X p^{\sharp}(x)\,\mathrm{d}\mu - \int\limits_Y p^{\flat}(y)\,\mathrm{d}\nu$$

We have checked that the function I is convex. It is easily seen to be continuous: if $p_n \to p$ uniformly on Z, then $p_n^{\sharp} \to p^{\sharp}$ uniformly on X and $p_n^{\flat} \to p^{\flat}$ uniformly on Y. On the other hand, the set $\mathcal{A}$ is non-empty, convex and closed in $\mathcal{K}(Z)$. This means that the constraint qualification conditions hold in problem (P): a necessary and sufficient condition for $\bar{p}$ to be optimal is that:

$$0 \in \partial I(\bar{p}) + N_{\mathcal{A}}(\bar{p}) \tag{37}$$

where $\partial I(\bar{p})$ is the subgradient of I at $\bar{p}$ in the sense of convex analysis, and $N_{\mathcal{A}}(\bar{p})$ is the normal cone to $\mathcal{A}$ at $\bar{p}$. All we have to do now is to compute both of them.

C.2.1 Computing $\partial I(p)$

Lemma 40 *Let $p \in \mathcal{K}(Z)$ and $\varphi \in \mathcal{K}(Z)$. Then, for every $x \in X$ and every $y \in Y$, we have:*

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{1}{h} \left[(p + h\varphi)^{\sharp}(x) - p^{\sharp}(x) \right] = -\min \{ \varphi(z) \mid z \in D(x) \}$$
$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{1}{h} \left[(p + h\varphi)^{\flat}(y) - p^{\flat}(y) \right] = -\max \{ \varphi(z) \mid z \in S(y) \}$$

Proof Let us prove the second equality; the first one is derived in a similar way. Take $z \in S_p(y)$ and $z_h \in S_{p+h\varphi}(y)$. From the definition of $S_p(y)$ and $S_{p+h\varphi}(y)$, we have:

$$v(y, z_h) - p(z_h) \geq p^{\flat}(y) = v(y, z) - p(z)$$
$$v(y, z) - p(z) - h\varphi(z) \geq (p + h\varphi)^{\flat}(y) = v(y, z_h) - p(z_h) - h\varphi(z_h)$$

Substracting, we find that:

$$-h\varphi(z) \geq (p + h\varphi)^{\flat}(y) - p^{\flat}(y) \geq -h\varphi(z_h) \tag{38}$$

Since z is an arbitrary point in $S_p(y)$, we can take it to be the minimizer on the left-hand side, and this inequality becomes:

$$-h \max \left\{ \varphi(z) \mid z \in S_p(y) \right\} \geq (p + h\varphi)^{\flat}(y) - p^{\flat}(y) \geq -h\varphi(z_h)$$

Now let $h \to 0$. The family $z_h \in S_{p+h\varphi}(y)$ must have cluster points, because Z is compact, and any cluster point $\bar{z}$ must belong to $S_p(y)$. Taking limits in inequality (38), we find that, for some $\bar{z} \in S_p(y)$:

$$-\max \left\{ \varphi(z) \mid z \in S_p(y) \right\} \geq \lim_{\substack{h \to 0 \\ h > 0}} \frac{1}{h} \left[(p + h\varphi)^{\flat}(y) - p^{\flat}(y) \right] \geq -\varphi(\bar{z}) \tag{39}$$

and the result follows. □

Because of inequality (39), we can apply the Lebesgue convergence theorem, and we get:

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{1}{h} \left[I(p + h\varphi) - I(p) \right] = \int_Y \max \{ \varphi(z) \mid z \in S(y) \} \, \mathrm{d}\nu - \int_X \min \{ \varphi(z) \mid z \in D(x) \} \, \mathrm{d}\mu \tag{40}$$

We now work on the right-hand side of formula (40). Define $B(X, D)$ to be the set of all Borel maps $d : X \to Z$ such that $d(x) \in D(x)$ for every x. Similarly, $B(Y, S)$ is the set of all Borel maps $s : Y \to Z$ such that $s(y) \in S(y)$ for every y.

Lemma 41 *For every $\varphi \in \mathcal{C}(Z)$, we have:*

$$\int_X \min\{\varphi(z) \mid z \in D(x)\} \, \mathrm{d}\mu = \min\left\{\int_X \varphi(d(x)) \, \mathrm{d}\mu \mid d \in B(X, D)\right\} \tag{41}$$

$$\int_Y \max\{\varphi(z) \mid z \in S(y)\} \, \mathrm{d}\nu = \max\left\{\int_Y \varphi(s(y)) \, \mathrm{d}\mu \mid s \in B(Y, S)\right\} \tag{42}$$

Proof Given $\varphi \in \mathcal{C}(Z)$, the multivalued maps Γ_1 and Γ_2 defined by:

$$\Gamma_1(x) = \arg\min\{\varphi(z) \mid z \in D(x)\}$$
$$\Gamma_2(y) = \arg\max\{\varphi(z) \mid z \in S(y)\}$$

have compact graph. Formulas (41) and (42) then follow from a standard measurable selection theorem. □

Define $\mathcal{M}_+(X, D)$ to be the set of all demand distributions, that is, the set of all positive measures $\alpha_{X\times Z}$ on $X \times Z$ which are carried by the graph of D and which have μ as marginal:

$$\alpha_X = \mu$$

Recall that $\alpha_Z \in \mathcal{M}_+(Z)$ denotes the second marginal of $\alpha_{X\times Z}$.

Lemma 42 *For every $\varphi \in \mathcal{K}(Z)$, we have:*

$$\int_X \min\{\varphi(z) \mid z \in D(x)\} \, \mathrm{d}\mu = \min\left\{\int_Z \varphi \mathrm{d}\alpha_Z \mid \alpha_{X\times Z} \in \mathcal{M}_+(X, D)\right\} \tag{43}$$

Proof Let us investigate the right-hand side of formula (41). Let $f \in B(X, D)$ be such that $\varphi(f(x)) = \min\{\varphi(z) \mid z \in D(x)\}$ for μ-almost every x, and define $\gamma_{X\times Z} \in \mathcal{M}_+(X, D)$ by:

$$\forall \psi \in \mathcal{K}(X \times Z), \quad \int_{X\times Z} \psi(x, z) \, d\gamma_{X\times Z} = \int_X \psi(x, f(x)) \, \mathrm{d}\mu$$

Clearly:

$$\min\left\{\int_X \varphi(d(x))\,\mathrm{d}\mu \mid d\in B(X,D)\right\} = \int_X \varphi(f(x))\,\mathrm{d}\mu$$
$$= \int_{X\times Z} \varphi d\gamma_{X\times Z} = \int_Z \varphi d\gamma_Z$$
$$\geq \min\left\{\int_Z \varphi \mathrm{d}\alpha_Z \mid \alpha_{X\times Z}\in \mathcal{M}_+(X,D)\right\}$$

For the reverse inequality, we take any $\alpha_{X\times Z}\in\mathcal{M}_+(X,D)$. Taking conditional expectations, we have:

$$E_x^\alpha[\varphi] \geq \min\{\varphi(z) \mid z\in D(x)\}$$

and by integrating with respect to μ, we get the desired result:

$$\int_Z \varphi \mathrm{d}\alpha_Z \geq \int_X \min\{\varphi(z) \mid z\in D(x)\}\,\mathrm{d}\mu$$
$$= \min\left\{\int_X \varphi(z)\,\mathrm{d}\mu \mid z\in D(x)\right\}$$
$$= \min\left\{\int_X \varphi(d(x))\,\mathrm{d}\mu \mid d\in B(X,D)\right\}$$

□

Considering the set $\mathcal{M}_+(Y,S)$ of supply distributions, we get similar results:

$$\int_Y \max\{\varphi(z) \mid z\in S(y)\}\,\mathrm{d}\nu = \max\left\{\int_Z \varphi \mathrm{d}\beta_Z \mid \beta_{Y\times Z}\in\mathcal{M}_+(Y,S)\right\} \tag{44}$$

Writing formulas (43) and (44) in formula (40), we get:

$$\lim_{\substack{h\to 0\\ h>0}} \frac{1}{h}\left[I(p+h\varphi)-I(p)\right]$$
$$= \max\left\{\int_Z \varphi \mathrm{d}\beta_Z \mid \beta_{Y\times Z}\in\mathcal{M}_+(Y,S)\right\} - \min\left\{\int_Z \varphi \mathrm{d}\alpha_Z \mid \alpha_{X\times Z}\in\mathcal{M}_+(X,D)\right\}$$

$$= \max \left\{ \int_Z \varphi \mathrm{d}\beta_Z - \int_Z \varphi \mathrm{d}\alpha_Z \mid \beta_{Y\times Z} \in \mathcal{M}_+(Y, S),\ \alpha_{X\times Z} \in \mathcal{M}_+(X, D) \right\}$$

Proposition 43 *The subdifferential of I at p is given by:*

$$\partial I\,(p) = \{\beta_Z - \alpha_Z \mid \beta_{Y\times Z} \in \mathcal{M}_+(Y, S),\ \alpha_{X\times Z} \in \mathcal{M}_+(X, D)\}$$

Proof Take $\lambda \in \mathcal{M}\,(Z) = \mathcal{M}_b\,(Z)$. By definition of the subgradient, $\lambda \in \partial I\,(p)$ if and only if, for every $\varphi \in \mathcal{K}\,(Z)$ and $h > 0$, we have:

$$I\,(p + h\varphi) \geq I\,(p) + h \int_Z \varphi \mathrm{d}\lambda$$

Since I is convex, this is equivalent to:

$$\lim_{\substack{h\to 0\\ h>0}} \frac{1}{h}\left[I\,(p + h\varphi) - I\,(p)\right] \geq \int_Z \varphi \mathrm{d}\lambda$$

Because of formula (40), this is equivalent to:

$$\max \left\{ \int_Z \varphi \mathrm{d}\beta_Z - \int_Z \varphi \mathrm{d}\alpha_Z \mid \beta_{Y\times Z} \in \mathcal{M}_+(Y, S),\ \alpha_{X\times Z} \in \mathcal{M}_+(X, D) \right\} \geq \int_Z \varphi \mathrm{d}\lambda$$

This means that λ belongs to the closed convex set:

$$\{\beta_Z - \alpha_Z \mid \beta_{Y\times Z} \in \mathcal{M}_+(Y, S),\ \alpha_{X\times Z} \in \mathcal{M}_+(X, D)\}$$

□

C.2.2 Computing $N_{\mathcal{A}}\,(p)$

Take $\lambda \in \mathcal{M}\,(Z) = \mathcal{M}_b\,(Z)$. By definition, $\lambda \in N_{\mathcal{A}}\,(p)$ if and only if, for every $q \in \mathcal{A}$, we have:

$$\int_Z (q - p)\,\mathrm{d}\lambda \ \leq 0$$

Since $q\,(\varnothing_d) = p\,(\varnothing_d) = 0$ and $q\,(\varnothing_s) = p\,(\varnothing_s) = 0$ for every $q \in \mathcal{A}$, this condition is equivalent to:

$$\int_{Z_0} (q - p)\,\mathrm{d}\lambda \ \leq 0 \tag{45}$$

To interpret this condition, we need some notation. Set:

$$\begin{aligned}
Z^b &= \{z \in Z_0 \mid a(z) < p(z) = b(z)\} \\
Z_a^b &= \{z \in Z_0 \mid a(z) < p(z) < b(z)\} \\
Z_a &= \{z \in Z_0 \mid a(z) = p(z) < b(z)\} \\
M &= \{z \in Z_0 \mid a(z) = p(z) = b(z)\} \\
N &= \{z \in Z_0 \mid a(z) > b(z)\}
\end{aligned}$$

so that we have a partition of Z_0 into subsets $Z_0 = Z_a \cup Z_a^b \cup Z^b \cup M \cup N$, where $Z_a \cup Z_a^b \cup Z^b \cup M = Z_1$, the set of marketable qualities.

Denote by $\lambda^b,\ \lambda_a^b,\ \lambda_a,\ \lambda_M,\ \lambda_N$ the restrictions of λ to $Z^b,\ Z_a^b,\ Z_a,\ Z_M,\ Z_N$ respectively. Note that since λ was a bounded measure, so are $\lambda^b,\ \lambda_a^b,\ \lambda_a$, λ_M and λ_N. Condition (45) is equivalent to the following:

$$\lambda^b \geq 0,\ \lambda_a^b = 0, \quad \lambda_a \leq 0, \lambda_N = 0 \tag{46}$$

C.2.3 Concluding the proof

Let $\bar{p}$ be a solution of problem (P). By condition (37), we have $0 \in \partial I(\bar{p}) + N_{\mathcal{A}}(\bar{p})$. By Proposition 43, this means that there exists $\beta_{Y\times Z} \in \mathcal{M}_+(Y, S)$, $\alpha_{X\times Z} \in \mathcal{M}_+(X, D)$ and $\lambda \in \mathcal{M}(Z)$ satisfying (46) such that $\alpha_Z - \beta_Z = \lambda$.

In other words, the restriction of $\alpha_Z - \beta_Z$ to $Z^b,\ Z_a^b,\ Z_a$ respectively are positive, zero and negative:

$$\alpha_Z \geq \beta_Z \quad \text{on } Z^b \tag{47}$$

$$\alpha_Z = \beta_Z \quad \text{on } Z_a^b \tag{48}$$

$$\alpha_Z \leq \beta_Z \quad \text{on } Z_a \tag{49}$$

$$\alpha_Z = \beta_Z \quad \text{on } N \tag{50}$$

There is no condition on the restriction of α_Z or β_Z to $\{\varnothing_d\}$, $\{\varnothing_s\}$ or M. Since P_x^α is carried by $D(x)$, we must have $P_x^\alpha(z) = 0$ whenever $z \notin D(x)$, which certainly is the case when $p(z) > b(z)$. Similarly, $P_y^\beta(z) = 0$ when $p(z) < a(z)$. If $z \in N$, either $p(z) > b(z)$ or $p(z) < a(z)$, so either $P_x^\alpha(z) = 0$ or $P_y^\beta(z) = 0$. The condition $\alpha_Z = \beta_Z$ on N then implies that:

$$\alpha_Z = \beta_Z = 0 \quad \text{on } N$$

We will now show that there exists $\alpha'_{X\times Z} \in \mathcal{M}_+(X, D)$ and $\beta'_{Y\times Z} \in \mathcal{M}_+(Y, S)$ such that $\alpha'_{Z_0} = \beta'_{Z_0}$. This will be done by suitably modifying $\alpha_{X\times Z}$ and $\beta_{Y\times Z}$ on the subsets Z^b and Z_a (note that they are both subsets of Z_0). In the sequel, we will denote by $\alpha_{X\times A}$ (resp. $\beta_{Y\times B}$) the restriction of $\alpha_{X\times Z}$ (resp. $\beta_{Y\times Z}$) to $X \times A$ (resp. $Y \times B$), for $A \subset X$ (resp. $B \subset Y$), and by α_A (resp. β_B) the marginal on A (resp. B).

On $X \times Z^b$, we have, by:

$$\alpha_{X\times Z^b} = \int_Z P_z^\alpha \mathrm{d}\alpha_{Z^b} \quad \text{and} \quad \beta_{X\times Z^b} = \int_Z P_z^\beta \mathrm{d}\beta_{Z^b}$$

with $\alpha_{Z^b} \geq \beta_{Z^b}$ by (47). Define $\alpha'_{X\times Z}$ by:

$$\alpha'_{X\times Z^b} = \int_Z P_z^\alpha \mathrm{d}\beta_{Z^b}$$

$$\alpha'\left(X \times \{\varnothing_d\}\right) = \alpha\left(X \times \{\varnothing_d\}\right) + \alpha\left(Z^b\right) - \beta\left(Z^b\right)$$

$$\alpha'_{X\times(Z-Z^b\cup\{\varnothing_d\})} = \alpha_{X\times(Z-Z^b\cup\{\varnothing_d\})}$$

Clearly $\alpha'_{X\times Z}$ is a positive measure. It follows from the first equation that $\alpha'_{Z^b} = \beta_{Z^b}$, and from the second that $\alpha'_x = \alpha_X = \mu$. It remains to check that $\alpha'_{X\times Z} \in \mathcal{M}_+(X, D)$. We already know that $\alpha_{X\times Z} \in \mathcal{M}_+(X, D)$, meaning that for $P_x^\alpha\left[D(x)\right] = 1$ for μ-a.e. x, and it differs from $\alpha'_{X\times Z}$ only in the region where $z \in Z^b$ or $z = \varnothing_d$. If $D(x) \cap Z^b = \varnothing$ then $P_x^\alpha\left[D(x)\right] = 1$ as well. If $D(x)$ intersects Z^b, so that $z \in Z^b \cap D(x)$, then consumer x is paying the highest bid price for z, and so he must be indifferent between z and $\varnothing_d$; this shows that $\varnothing_d$ also belongs to $D(x)$. In the new allocation $\alpha'_{X\times Z}$, some of the demand may be transferred from $Z^b \cap D(x)$ to $\varnothing_d$ with positive probability, but this redistribution occurs within $D(x)$ and does not affect the total probability, so that $P_x^{\alpha'}\left[D(x)\right] = 1$.

In words, for every quality z where the highest bid price is paid, we clear the market by letting some of the demand go unsatisfied: all producers y have sold, but there is total quantity $\alpha\left(Z^b\right) - \beta\left(Z^b\right)$ of potential buyers which are thrown out of the market. However, they don't care, because the price asked is the highest bid price, and they are indifferent between buying or nor.

We then shift some of the supply to $\varnothing_s$, as we did for the demand. We end up with $\alpha'_{X\times Z} \in \mathcal{M}_+(X, D)$ and $\beta'_{Y\times Z} \in \mathcal{M}_+(Y, S)$ which satisfy the conclusions of the Existence Theorem.

Appendix D: Remaining proofs

D.1 Pareto optimality of equilibrium allocations

With every pair of demand and supply distributions, $\alpha'_{X\times Z} \in \mathcal{M}_+(X, D)$ and $\beta'_{Y\times Z} \in \mathcal{M}_+(Y, S)$, we associate the number:

$$\begin{aligned} J\left(\alpha'_{X\times Z}, \beta'_{Y\times Z}\right) &= \int_{X\times Z} u(x, z)\, \mathrm{d}\alpha'_{X\times Z} - \int_{Y\times Z} v(y, z)\, \mathrm{d}\beta'_{Y\times Z} \\ &= \int_X E_x^{\alpha'}\left[u(x, z)\right] \mathrm{d}\mu(x) - \int_Y E_y^{\beta'}\left[v(y, z)\right] \mathrm{d}\nu(y) \end{aligned}$$

Assume that $\alpha'_{Z_0} = \beta'_{Z_0}$.We claim that:

$$\int_X E_x^{\alpha'} [p(z)] \, d\mu(x) - \int_Y E_y^{\beta'} [p(z)] \, d\nu(y) = 0 \tag{51}$$

Indeed, the left-hand side can be written as:

$$\left(\int_{Z_0} p(z) \, d\alpha'_Z - \int_{Z_0} p(z) \, d\beta'_Z \right) + p(\varnothing_d) \left(\alpha'_Z [\varnothing_d] - \beta'_Z [\varnothing_d] \right)$$
$$+ p(\varnothing_s) \left(\alpha'_Z [\varnothing_s] - \beta'_Z [\varnothing_s] \right)$$

The first term vanishes because $\alpha'_{Z_0} = \beta'_{Z_0}$, and the two next terms vanish because $p(\varnothing_d) = p(\varnothing_s) = 0$.

Subtracting (51) from J, we get:

$$J\left(\alpha'_{X\times Z}, \beta'_{Y\times Z}\right) = \int_X E_x^{\alpha'} [u(x,z) - p(z)] \, d\mu(x)$$
$$- \int_Y E_y^{\beta'} [v(y,z) - p(z)] \, d\nu(y) \tag{52}$$

By Fenchel's inequality, $(u(x,z) - p(z)) \leq p^\sharp(x)$ for all $z \in Z$. Taking expectations with respect to the probability $P_x^{\alpha'}$, we get:

$$E_x^{\alpha'} [u(x,z) - p(z)] \leq p^\sharp(x) \tag{53}$$

with equality if and only if $u(x,z) - p(z) = p^\sharp(x)$ (in other words, $z \in D(x)$) for $P_x^{\alpha'}$-almost every $z \in Z$. Similarly, we have:

$$E_y^{\beta'} [v(y,z) - p(z)] \geq p^\flat(y) \tag{54}$$

with equality if and only if $v(y,z) - p(z) = p^\flat(y)$ (in other words, $z \in S(y)$) for $P_y^{\beta'}$-almost every $z \in Z$. Writing this in (52), and treating the second term in the same way, we get:

$$J\left(\alpha'_{X\times Z}, \beta'_{Y\times Z}\right) \leq \int_X p^\sharp(x) \, d\mu - \int_Y p^\flat(y) \, d\nu \tag{55}$$

The right-hand side is equal to $J(\alpha_{X\times Z}, \beta_{Y\times Z})$, for any equilibrium allocation (α, β). This proves that equilibrium allocations solve the planner's problem, and as such they are Pareto optimal.

D.2 Uniqueness of equilibrium allocations

Observe that equality holds in (55) if and only if equality holds in (53) for μ-almost every x, and equality holds in (54) for ν-almost every y. This means that $P_x^{\alpha'}[D(x)] = 1$ for μ-almost every x and $P_y^{\beta'}[S(y)] = 1$ for ν-almost every y.

D.3 Proof of Theorem 19

Let $(p, \alpha_{X\times Z}, \beta_{Y\times Z})$ be an equilibrium. By Rademacher's theorem, since $p^\sharp : X \to R$ is Lipschitz, and μ is absolutely continuous with respect to the Lebesgue measure, $p^\sharp$ is differentiable μ-almost everywhere.

Consider the set $A = \{x \mid p^\sharp(x) \geq 0\}$. Let $x \in A$ be a point where $p^\sharp$ is differentiable, with derivative $D_x p^\sharp(x)$. Since x is active or indifferent, the set $D(x) \cap Z_0$ is non-empty, and we may take some $z \in D(x) \cap Z_0$. Consider the function $\varphi(x') = u(x', z) - p(z)$. By Proposition 25, since $D(x) \subset \partial p^\sharp(x)$, we have $\varphi \leq f$ and $\varphi(x) = f(x)$, so that φ and f must have the same derivative at x :

$$D_x f(x) = D_x u(x, z) \tag{56}$$

By condition (9), this equation defines z uniquely. In other words, for μ-almost every point $x \in A$, the set $D(x) \cap Z_0$ consists of one point only. Similarly, for ν-almost every point $y \in B = \{y \mid p^\flat(y) \leq 0\}$, the set $S(y) \cap Z_0$ consists of one point only. This is the desired result.

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