ARTICLE IN PRESS
ANIHPC:2451

Available online at www.sciencedirect.com
ScienceDirect

Ann. I. H. Poincaré – AN ••• (••••) •••–•••

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

An inverse function theorem in Fréchet spaces

Ivar Ekeland [1]

Canada Research Chair in Mathematical Economics, University of British Columbia, Canada

Received 23 September 2010; received in revised form 23 October 2010; accepted 23 October 2010

Abstract

I present an inverse function theorem for differentiable maps between Fréchet spaces which contains the classical theorem of Nash and Moser as a particular case. In contrast to the latter, the proof does not rely on the Newton iteration procedure, but on Lebesgue's dominated convergence theorem and Ekeland's variational principle. As a consequence, the assumptions are substantially weakened: the map F to be inverted is not required to be C^2, or even C^1, or even Fréchet-differentiable.

Résumé

Je présente un théorème d'inversion pour des applications différentiables entre espaces de Fréchet, qui contient le théorème classique de Nash et Moser. Contrairement à ce dernier, la démonstration donnée ici ne repose pas sur l'algorithme itératif de Newton, mais sur le théorème de convergence dominée de Lebesgue et le principe variationnel d'Ekeland. Comme conséquence, les hypothèses sont substantiellement affaiblies : on ne demande pas que l'application F à inverser soit de classe C^2, ni même C^1, ni même différentiable au sens de Fréchet.

Keywords: Inverse function theorem; Implicit function theorem; Fréchet space; Nash–Moser theorem

1. Introduction

Recall that a Fréchet space X is *graded* if its topology is defined by an increasing sequence of norms $\| \|_k$, $k \geqslant 0$:

$$\forall x \in X, \quad \|x\|_k \leqslant \|x\|_{k+1}.$$

Denote by X_k the completion of X for the norm $\| \|_k$. It is a Banach space, and we have the following scheme:

$$\to X_{k+1} \to_{i_k} X_k \to_{i_{k-1}} X_{k-1} \to \cdots \to X_0$$

E-mail address: ekeland@math.ubc.ca.
URL: http://www.pims.math.ca/~ekeland.

[1] The author thanks Eric Séré, Louis Nirenberg, Massimiliano Berti and Philippe Bolle, who were the first to see this proof. Particular thanks go to Philippe Bolle for spotting a significant mistake in an earlier version, and to Eric Séré, whose careful reading led to several improvements and clarifications. Their contribution and friendship is gratefully acknowledged.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2010.11.001

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001

where each identity map i_k is injective and continuous, in fact $\|i_k\| \leqslant 1$. By definition, X is a dense subspace of X_k, we have $X = \bigcap_{k=0}^{\infty} X_k$, and $x^j \to \bar{x}$ in X if and only if $\|x^j - \bar{x}\|_k \to 0$ for every $k \geqslant 0$.

Our main example will be $X = C^\infty(\bar{\Omega}, \mathbb{R}^d)$, where $\bar{\Omega} \subset \mathbb{R}^n$ is compact, is the closure of its interior Ω, and has smooth boundary. It is well known that the topology of $C^\infty(\bar{\Omega}, \mathbb{R}^d)$ can be defined in two equivalent ways. On the one hand, we can write $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \bigcap C^k(\bar{\Omega}, \mathbb{R}^d)$, where $C^k(\bar{\Omega}, \mathbb{R}^d)$ is the Banach space of all functions continuously differentiable up to order k, endowed with the sup norm:

$$\|x\|_k := \max_{p_1+\cdots+p_n \leqslant p} \max_{\omega \in \bar{\Omega}} \left| \frac{\partial^{p_1+\cdots+p_n} x}{\partial^{p_1}\omega_1 \cdots \partial^{p_n}\omega_n}(\omega) \right|. \tag{1.1}$$

On the other, we can also write $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \bigcap H^k(\Omega, \mathbb{R}^d)$, where $H^k(\Omega, \mathbb{R}^d)$ is the Sobolev space consisting of all functions with square-integrable derivatives, up to order k, endowed with the Hilbert space structure:

$$\|x\|_k^2 := \sum_{p_1+\cdots+p_n \leqslant p} \int_{\Omega} \left| \frac{\partial^{p_1+\cdots+p_n} x}{\partial^{p_1}\omega_1 \cdots \partial^{p_n}\omega_n} \right|^2 d\omega. \tag{1.2}$$

We will prove an inverse function theorem between graded Fréchet spaces. Let us give a simple version in C^∞; in the following statement, either definition of the k-norms, (1.1) or (1.2), may be used:

Theorem 1. *Let F be a map from a graded Fréchet space $X = \bigcap_{k \geqslant 0} X_k$ into C^∞. Assume that there are integers d_1 and d_2, and sequences $m_k > 0$, $m'_k > 0$, $k \in \mathbb{N}$, such that, for all x is some neighborhood of* 0 *in X, we have*:

(1) $F(0) = 0$.
(2) *F is continuous and Gâteaux-differentiable, with derivative $DF(x)$.*
(3) *For every $u \in X$, we have*

$$\forall k \geqslant 0, \quad \|DF(x)u\|_k \leqslant m_k \|u\|_{k+d_1}. \tag{1.3}$$

(4) *$DF(x)$ has a right-inverse $L(x)$*:

$$\forall v \in C^\infty, \quad DF(x)L(x)v = v.$$

(5) *For every $v \in C^\infty$, we have*:

$$\forall k \geqslant 0, \quad \|L(x)v\|_k \leqslant m'_k \|v\|_{k+d_2}. \tag{1.4}$$

Then for every $y \in C^\infty$ such that

$$\|y\|_{k_0+d_2} < \frac{R}{m'_{k_0}}$$

and every $m > m'_{k_0}$ there is some $x \in X$ such that:

$$\|x\|_{k_0} < R,$$
$$\|x\|_{k_0} \leqslant m\|y\|_{k_0+d_2},$$

and:

$$F(x) = y.$$

The full statement of our inverse function theorem and of its corollaries, such as the implicit function theorem, will be given in the text (Theorems 4 and 5). Note the main feature of our result: there is a *loss of derivatives* both for $DF(x)$, by condition (1.3), and for $L(x)$, by condition (1.4).

Since the pioneering work of Andrei Kolmogorov and John Nash [10,2–4,13], this loss of derivatives has been overcome by using the Newton procedure: the equation $F(x) = y$ is solved by the iteration scheme $x_{n+1} = x_n - L(x_n)F(x_n)$ ([15,11,12]; see [9,1,5] for more recent expositions). This method has two drawbacks. The first one is

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001

that it requires the function F to be C^2, which is quite difficult to satisfy in infinite-dimensional situations. The second is that it gives a set of admissible right-hand sides y which is unrealistically small: in practical situations, the equation $F(x) = y$ will continue to have a solution long after the Newton iteration starting from y ceases to converge.

Our proof is entirely different. It gives the solution of $F(x) = y$ directly by using Ekeland's variational principle ([7,8]; see [6] for later developments). Since the latter is constructive, so is our proof, even if it does not rely on an iteration scheme to solve the equation. To convey the idea of the method in a simple case, let us now state and prove an inverse function theorem in Banach spaces:

Theorem 2. *Let X and Y be Banach spaces. Let $F : X \to Y$ be continuous and Gâteaux-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded in a neighborhood of* 0:

$$\forall v \in Y, \quad DF(x)L(x)v = v,$$
$$\|x\| \leqslant R \quad \Longrightarrow \quad \big\|L(x)\big\| \leqslant m.$$

Then, for every $\bar{y}$ such that:

$$\|\bar{y}\| < \frac{R}{m}$$

and every $\mu > m$ there is some $\bar{x}$ such that:

$$\|\bar{x}\| < R,$$
$$\|\bar{x}\| \leqslant \mu\|\bar{y}\|,$$

and:

$$F(\bar{x}) = \bar{y}.$$

Proof. Consider the function $f : X \to R$ defined by

$$f(x) = \big\|F(x) - \bar{y}\big\|.$$

It is continuous and bounded from below, so that we can apply Ekeland's variational principle. For every $r > 0$, we can find a point $\bar{x}$ such that:

$$f(\bar{x}) \leqslant f(0),$$
$$\|\bar{x}\| \leqslant r,$$
$$\forall x, \quad f(x) \geqslant f(\bar{x}) - \frac{f(0)}{r}\|x - \bar{x}\|.$$

Note that $f(0) = \|\bar{y}\|$. Take $r = \mu\|\bar{y}\|$, so that:

$$f(\bar{x}) \leqslant f(0),$$
$$\|\bar{x}\| \leqslant \mu\|\bar{y}\|,$$
$$\forall x, \quad f(x) \geqslant f(\bar{x}) - \frac{1}{\mu}\|x - \bar{x}\|. \tag{1.5}$$

The point $\bar{x}$ satisfies the inequality $\|\bar{x}\| \leqslant \mu\|\bar{y}\|$. Since $m < R\|\bar{y}\|^{-1}$, we can assume without loss of generality that $\mu < R\|\bar{y}\|^{-1}$, so $\|\bar{x}\| < R$. It remains to prove that $F(\bar{x}) = \bar{y}$.

We argue by contradiction. Assume not, so $F(\bar{x}) \neq \bar{y}$. Then, write the inequality (1.5) with $x = \bar{x} + tu$. We get:

$$\forall t > 0,\ \forall u \in X, \quad \frac{\|F(\bar{x} + tu) - \bar{y}\| - \|F(\bar{x}) - \bar{y}\|}{t} \geqslant -\frac{1}{\mu}\|u\|. \tag{1.6}$$

As we shall see later on (Lemma 1), the function $t \to \|F(\bar{x} + tu) - \bar{y}\|$ is right-differentiable at $t = 0$, and its derivative is given by

$$\lim_{\substack{t\to 0\\ t>0}} \frac{\|F(\bar{x}+tu)-\bar{y}\| - \|F(\bar{x})-\bar{y}\|}{t} = \langle y^*, DF(\bar{x})u\rangle$$

for some $y^* \in Y^*$ with $\|y^*\|^* = 1$ and $\langle y^*, F(\bar{x}) - \bar{y}\rangle = \|F(\bar{x}) - \bar{y}\|$. In the particular case when Y is a Hilbert space, we have

$$y^* = \frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}\|}.$$

Letting $t \to 0$ in (1.6), we get:

$$\forall u, \quad \langle y^*, DF(\bar{x})u\rangle \geqslant -\frac{1}{\mu}\|u\|.$$

We now take $u = -L(\bar{x})(F(\bar{x}) - \bar{y})$, so that $DF(\bar{x})u = -(F(\bar{x}) - \bar{y})$. The preceding inequality yields:

$$\|F(\bar{x}) - \bar{y}\| \leqslant \frac{1}{\mu}\|u\| \leqslant \frac{m}{\mu}\|F(\bar{x}) - \bar{y}\|$$

which is a contradiction since $\mu > m$. $\square$

The reader will have noted that this is much stronger than the usual inverse function theorem in Banach spaces: we do not require that F be Fréchet-differentiable, nor that the derivative $DF(x)$ or its inverse $L(x)$ depend continuously on x. All that is required is an upper bound on $L(x)$. Note that it is very doubtful that, with such weak assumptions, the usual Euler or Newton iteration schemes would converge.

Our inverse function theorem will extend this idea to Fréchet spaces. Ekeland's variational principle holds for any complete metric space, hence on Fréchet spaces. Our first result, Theorem 3, depends on the choice of a non-negative sequence β_k which converges rapidly to zero. An appropriate choice of β_k will lead to the facts that F is a local surjection (Corollary 1), that the (multivalued) local inverse F^{-1} satisfies a Lipschitz condition (Corollary 2), and that the result holds also with finite regularity (if $\bar{y}$ does not belong to Y_k for every k, but only to $Y_{k_0+d_2}$, then we can still solve $F(\bar{x}) = \bar{y}$ with $\bar{x} \in X_{k_0}$). These results are gathered together in Theorem 4, which is our final inverse function theorem. As usual, an implicit function theorem can be derived; it is given in Theorem 5.

The structure of the paper is as follows. Section 2 introduces the basic definitions. There will be no requirement on X, while Y will be asked to belong to a special class of graded Fréchet spaces. This class is much larger than the one used in the Nash–Moser literature: it is not required that X or Y be tame in the sense of Hamilton [9] or admit smoothing operators. Section 3 states Theorem 3 and derives the other results. The proof of Theorem 3 is given in Section 4.

Before we proceed, it will be convenient to recall some well-known facts about differentiability in Banach spaces. Let X be a Banach space. Let $\mathcal{U}$ be some open subset of X, and let F be a map from $\mathcal{U}$ into some Banach space Y.

Definition 1. F is *Gâteaux-differentiable* at $x \in \mathcal{U}$ if there exists some linear continuous map from X to Y, denoted by $DF(x)$, such that:

$$\forall u \in X, \quad \lim_{t\to 0}\left\| \frac{F(x+tu) - F(x)}{t} - DF(x)u\right\| = 0.$$

Definition 2. F is *Fréchet-differentiable* at $x \in \mathcal{U}$ if it is Gâteaux-differentiable and:

$$\lim_{u\to 0} \frac{F(x+u) - F(x) - DF(x)u}{\|u\|} = 0.$$

Fréchet-differentiability implies Gâteaux-differentiability and continuity, but Gâteaux-differentiability does not even imply continuity.

If Y is a Banach space, then the norm $y \to \|y\|$ is convex and continuous, so that it has a non-empty subdifferential $N(y) \subset Y^*$ at every point y:

$$y^* \in N(y) \quad \Longleftrightarrow \quad \forall z \in Y, \quad \|y+z\| \geqslant \|y\| + \langle y^*, z\rangle.$$

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001

When $y \neq 0$ we have the alternative characterization:

$$y^* \in N(y) \iff \|y^*\|^* = 1 \quad \text{and} \quad \langle y^*, y\rangle = \|y\|.$$

$N(y)$ is convex and compact in the $\sigma(X^*, X)$-topology. It is a singleton if and only if the norm is Gâteaux-differentiable at y. Its unique element then is the Gâteaux-derivative of F at y:

$$N(y) = \{y^*\} \iff DF(x) = y^*.$$

If $N(y)$ contains several elements, the norm is not Gâteaux-differentiable at y, and the preceding relation is then replaced by the following classical result (see for instance Theorem 11, p. 34 in [14]):

Proposition 1. *Take y and z in Y. Then there is some $y^* \in N(y)$ (depending possibly on z) such that*:

$$\lim_{\substack{t\to 0\\ t>0}} \frac{\|y+tz\| - \|y\|}{t} = \langle y^*, z\rangle.$$

The following result will be used repeatedly:

Lemma 1. *Assume $F : X \to Y$ is Gâteaux-differentiable. Take x and ξ in X, and define a function $f : [0, 1] \to R$ by*

$$f(t) := \|F(x+t\xi)\|, \quad 0 \leqslant t \leqslant 1.$$

Then f has a right-derivative everywhere, and there is some $y^* \in N(F(x+t\xi))$ such that:

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(t+h) - f(t)}{h} = \langle y^*, DF(x+t\xi)\xi\rangle.$$

Proof. We have:

$$\begin{aligned} h^{-1}\big[f(t+h) - f(t)\big] &= h^{-1}\big[\big\|F\big(x+(t+h)\xi\big)\big\| - \big\|F(x+t\xi)\big\|\big] \\ &= h^{-1}\big[\big\|F(x+t\xi) + hz(h)\big\| - \big\|F(x+t\xi)\big\|\big] \end{aligned} \tag{1.7}$$

with:

$$z(h) = \frac{F(x+(t+h)\xi) - F(x+t\xi)}{h}.$$

Since F is Gâteaux-differentiable, we have:

$$\lim_{h\to 0} z(h) = DF(x+t\xi)\xi := z(0).$$

By the triangle inequality, we have:

$$\big|\big\|F(x+t\xi) + hz(h)\big\| - \big\|F(x+t\xi) + hz(0)\big\|\big| \leqslant h\big\|z(h) - z(0)\big\|.$$

Writing this into (1.7) and using Proposition 1, we find:

$$\begin{aligned} \lim_{\substack{h\to 0\\ h>0}} \frac{[f(t+h) - f(t)]}{h} &= \lim_{\substack{h\to 0\\ h>0}} \frac{\|F(x+t\xi) + hz(0)\| - \|F(x+t\xi)\|}{h} \\ &= \langle y^*, DF(x+t\xi)\xi\rangle \end{aligned}$$

for some $y^* \in N(F(x+t\xi))$. This is the desired result. □

One last word. Throughout the paper, we shall use the following:

Definition 3. A sequence α_k has *unbounded support* if $\sup\{k \mid \alpha_k \neq 0\} = \infty$.

2. The inverse function theorem

Let $X = \bigcap_{k \geqslant 0} X_k$ be a graded Fréchet space. The following result is classical:

Proposition 2. *Let $\alpha_k \geqslant 0$ be a sequence with unbounded support such that $\sum \alpha_k < \infty$. Let $r > 0$ be a positive number. Then the topology of X is induced by the distance*:

$$d(x, y) := \sum_k \alpha_k \min\{r, \|x - y\|_k\} \tag{2.1}$$

and x_n is a Cauchy sequence for d if and only if it is a Cauchy sequence for all the k-norms. It follows that (X, d) is a complete metric space.

The main analytical difficulty with Fréchet spaces is that, given $x \in X$, there is no information on the sequence $k \to \|x\|_k$, except that it is positive and increasing. For instance, it can grow arbitrarily fast. So we single out some elements x such that $\|x\|_k$ has at most exponential growth in k.

Definition 4. A point $x \in X$ is *controlled* if there is a constant $c_0(x)$ such that:

$$\|x\|_k \leqslant c_0(x)^k. \tag{2.2}$$

Definition 5. A graded Fréchet space is *standard* if, for every $x \in X$, there is a constant $c_3(x)$ and a sequence x_n such that:

$$\forall k, \quad \lim_{n\to\infty} \|x_n - x\|_k = 0, \tag{2.3}$$

$$\forall n, \quad \|x_n\|_k \leqslant c_3(x)\|x\|_k \tag{2.4}$$

and each x_n is controlled:

$$\|x_n\|_k \leqslant c_0(x_n)^k. \tag{2.5}$$

Proposition 3. *The graded Fréchet spaces $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \bigcap C^k(\bar{\Omega}, \mathbb{R}^d)$ and $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \bigcap H^k(\Omega, \mathbb{R}^d)$ are both standard.*

Proof. In fact, much more is true. It can be proved (see [9], [1, Proposition II.A.1.6]) that both admit a family of smoothing operators $S_n : X \to X$ satisfying:

(1) $\|S_n x - x\|_k \to 0$ when $n \to \infty$,
(2) $\|S_n x\|_{k+d} \leqslant c_1 n^d \|x\|_k$, $\forall d \geqslant 0$, $\forall k \geqslant 0$, $\forall n \geqslant 0$,
(3) $\|(I - S_n)x\|_k \leqslant c_2 n^{-d} \|x\|_{k+d}$, $\forall d \geqslant 0$, $\forall k \geqslant 0$, $\forall n \geqslant 0$,

where c_1 and c_2 are positive constants. For every $x \in X$, condition (2) with $k = 0$ implies that:

$$\|S_n x\|_d \leqslant c_1 n^d \|x\|_0.$$

So (2.2) is satisfied and the point $x_n = S_n x$ is controlled. Condition (2.3) follows from (1) and condition (2.4) follows from (2) with $d = 0$. □

Now consider two graded Fréchet spaces $X = \bigcap_{k \geqslant 0} X_k$ and $Y = \bigcap_{k \geqslant 0} Y_k$. We are given a map $F : X \to Y$, a number $R \in (0, \infty]$ and an integer $k_0 \geqslant 0$. We consider the ball:

$$B_X(k_0, R) := \{x \in X \mid \|x\|_{k_0} < R\}.$$

Note that $R = +\infty$ is allowed; in that case, $B_X(k_0, R) = X$.

Theorem 3. *Assume Y is standard. Let there be given two integers d_1, d_2 and two non-decreasing sequences $m_k > 0$, $m'_k > 0$. Assume that, on $B_X(k_0, R)$, the map F satisfies the following conditions*:

(1) $F(0) = 0$.
(2) *F is continuous, and Gâteaux-differentiable.*
(3) *For every $u \in X$ there is a number $c_1(u) > 0$ for which*:

$$\forall k \in \mathbb{N}, \quad \left\|DF(x)u\right\|_k \leqslant c_1(u)\left(m_k\|u\|_{k+d_1} + \left\|F(x)\right\|_k\right).$$

(4) *There exists a linear map $L(x) : Y \to X$ such that $DF(x)L(x) = I_Y$*:

$$\forall v \in Y, \quad DF(x)L(x)v = v.$$

(5) *For every $v \in Y$*:

$$\forall k \in \mathbb{N}, \quad \left\|L(x)v\right\|_k \leqslant m'_k\|v\|_{k+d_2}.$$

Let $\beta_k \geqslant 0$ be a sequence with unbounded support satisfying:

$$\forall n \in \mathbb{N}, \quad \sum_{k=0}^{\infty} \beta_k m_k m'_{k+d_1} n^k < \infty. \tag{2.6}$$

Then, for every $\bar{y} \in Y$ such that:

$$\sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \frac{\beta_{k_0+d_2}}{m'_{k_0}} R \tag{2.7}$$

there exists a point $\bar{x}$ such that:

$$F(\bar{x}) = \bar{y}, \tag{2.8}$$

$$\|\bar{x}\|_{k_0} < R. \tag{2.9}$$

The proof is given in the next section. We will now derive the consequences. Before we do that, note that condition (3) is equivalent to the following, seemingly more general, one:

$$\forall k, \quad \left\|DF(x)u\right\|_k \leqslant c'_1(u)\left(m_k\|u\|_{k+d_1} + \left\|F(x)\right\|_k + 1\right). \tag{2.10}$$

Indeed, condition (3) clearly implies (2.10) with $c'_1(u) = c_1(u)$. Conversely, if (2.10) holds, for $u \neq 0$ we set:

$$c_1(u) = c'_1(u)\left(1 + \frac{1}{m_0\|u\|_0}\right)$$

and then:

$$c_1(u)\left(m_k\|u\|_{k+d_1} + \left\|F(x)\right\|_k\right) \geqslant c'_1(u)\left(m_k\|u\|_{k+d_1} + \left\|F(x)\right\|_k + 1\right),$$

so condition (3) holds as well.

Corollary 1 *(Local surjection). Let $X = \bigcap_{k\geqslant 0} X_k$ and $Y = \bigcap_{k\geqslant 0} Y_k$ be graded Fréchet spaces, with Y standard, and let $F : X \to Y$ satisfy conditions* (1) *to* (5). *Suppose*:

$$\|y\|_{k_0+d_2} < \frac{R}{m'_{k_0}}. \tag{2.11}$$

Then, for every $\mu > m'_{k_0}$, there is some $x \in B_X(k_0, R)$ such that:

$$\|x\|_{k_0} \leqslant \mu\|y\|_{k_0+d_2}, \tag{2.12}$$

and:

$$F(x) = y.$$

Proof. Given $\mu > m'_{k_0}$, take $R' < R$ such that $m'_{k_0}\|y\|_{k_0+d_2} < R' < \mu\|y\|_{k_0+d_2}$ and choose $h > 0$ so small that:

$$\|y\|_{k_0+d_2} + h\sum_{k=0}^{\infty} k^{-k} < \frac{R'}{m'_{k_0}}. \tag{2.13}$$

Define a sequence β_k by

$$\beta_k = \begin{cases} 0 & \text{for } k < k_0 + d_2, \\ 1 & \text{for } k = k_0 + d_2, \\ hk^{-k}[\max\{\|y\|_k, m_k m'_{k+d_1}\}]^{-1} & \text{for } k > k_0 + d_2. \end{cases} \tag{2.14}$$

Then (2.6) is satisfied. On the other hand:

$$\sum_{k=0}^{\infty} \beta_k \|y\|_k \leqslant \|y\|_{k_0+d_2} + h\sum k^{-k}$$

and using (2.13) we find that:

$$\sum_{k=0}^{\infty} \beta_k \|y\|_k < \frac{1}{m'_{k_0}} R' < \infty. \tag{2.15}$$

We now apply Theorem 3 with R' instead of R, and we find some $x \in X$ with $F(x) = y$ and $\|x\|_{k_0} \leqslant R'$. The result follows. □

Consider the two balls:

$$B_X(k_0, R) = \{x \mid \|x\|_{k_0} < R\},$$
$$B_Y\left(k_0 + d_2, \frac{R}{m'_{k_0}}\right) = \left\{y \,\middle|\, \|y\|_{k_0+d_2} < \frac{R}{m'_{k_0}}\right\}.$$

The preceding corollary tells us that F maps the first ball onto the second. The inverse map

$$F^{-1} : B_Y\left(k_0 + d_2, \frac{R}{m'_{k_0}}\right) \to B_X(k_0, R)$$

is multivalued:

$$F^{-1}(y) = \{x \in B_X(k_0, R) \mid F(x) = y\} \tag{2.16}$$

and has non-empty values: $F^{-1}(y) \neq \emptyset$ for every y. The following result shows that it satisfies a Lipschitz condition.

Corollary 2 *(Lipschitz inverse). For every y_0 and y_1 in $B_Y(k_0 + d_2, R(m'_{k_0})^{-1})$, every $x_0 \in F^{-1}(y_0)$, and every $\mu > m'_{k_0}$, we have*:

$$\begin{aligned} \inf\{\|x_0 - x_1\|_{k_0} \mid x_1 \in F^{-1}(y_1)\} &= \inf\{\|x_0 - x_1\|_{k_0} \mid F(x_1) = y_1\} \\ &\leqslant \mu\|y_0 - y_1\|_{k_0+d_2}. \end{aligned}$$

Proof. Take some $R' < R$ with $\max\{\|y_0\|_{k_0+d_2}, \|y_1\|_{k_0+d_2}\} < R'(m'_{k_0})^{-1}$ and consider the line segment $y_t = y_0 + t(y_1 - y_0)$, $0 \leqslant t \leqslant 1$, joining y_0 to y_1. We have $\|y_t\| < R'(m'_{k_0})^{-1}$ for every t, so that, by Corollary 1, there exists some $x_t \in F^{-1}(y_t)$ with $\|x_t\| < R'$. The function $F_t(x) = F(x + x_t) - y_t$ then satisfies conditions (1) to (5) with R replaced by $\rho = R - R'$.

Pick some $x_0 \in F^{-1}(y_0)$. By Corollary 1 applied to F_0 we find that, for every y such that $\|y - y_0\|_{k_0+d_2} \leqslant \rho(m'_{k_0})^{-1}$ we have some $x \in F^{-1}(y)$ with:

$$\|x_0 - x\|_{k_0} \leqslant \mu\|y_0 - y\|_{k_0}.$$

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001

We can connect y_0 and y_1 by a finite chain $y'_0 = y_0, y'_2, \dots y'_N = y_1$ of aligned points, such that the distance between y'_n and y'_{n+1} is always less than $\rho(m'_{k_0})^{-1}$, and for each y'_n choose some $x_n \in F^{-1}(y'_n)$ such that:

$$\|x_n - x_{n+1}\|_{k_0} \leqslant \mu \|y'_n - y'_{n+1}\|_{k_0+d_2}.$$

Summing up:

$$\|x_0 - x_1\|_{k_0} \leqslant \mu \sum_{n=0}^{N} \|y'_n - y'_{n+1}\|_{k_0+d_2} = \mu \|y_1 - y_0\|_{k_0+d_2}. \qquad \square$$

Note that we are not claiming that the multivalued map F^{-1} has a Lipschitz section over $B_Y(k_0 + d_2, R(m'_{k_0})^{-1})$, or even a continuous one.

As a consequence of Corollary 2, we can solve the equation $F(x) = y$ when the right-hand side no longer is in Y, but in some of the Y_k, with $k \geqslant k_0 + d_2$.

Corollary 3 *(Finite regularity). Suppose F extends to a continuous map $\bar{F} : X_{k_0} \to Y_{k_0-d_1}$. Take some $y \in Y_{k_0+d_2}$ with $\|y\|_{k_0+d_2} < R(m'_{k_0})^{-1}$. Then there is some $x \in X_{k_0}$ such that $\|x\|_{k_0} < R$ and $\bar{F}(x) = y$.*

Proof. Let $y_n \in Y$ be such that $\|y_n - y\|_{k_0+d} \leqslant 2^{-n}$. By Corollary 2, we can find a sequence $x_n \in X$ such that $F(x_n) = y_n$ and

$$\|x_n - x_{n+1}\|_{k_0} \leqslant \mu \|y_n - y_{n+1}\|_{k_0+d} \leqslant \mu 2^{-n}.$$

So $\|x_n - x_p\|_{k_0} \leqslant \mu 2^{-n+1}$ for $p > n$, and the sequence x_n is Cauchy in X_{k_0}. It follows that x_n converges to some $x \in X_{k_0}$, with $\|x\|_{k_0} < R$, and we get $F(x) = y$ by continuity. $\square$

Let us sum up our results in a single statement:

Theorem 4 *(Inverse function theorem). Let $X = \bigcap_{k\geqslant 0} X_k$ and $Y = \bigcap_{k\geqslant 0} Y_k$ be graded Fréchet spaces, with Y standard, and let F be a map from X to Y. Assume there exist some integer k_0, some $R > 0$ (possibly equal to $+\infty$), integers d_1, d_2, and non-decreasing sequences $m_k > 0$, $m'_k > 0$ such that, for $\|x\|_{k_0} < R$, we have:*

(1) $F(0) = 0$.
(2) *F is continuous, and Gâteaux-differentiable with derivative $DF(x)$.*
(3) *For every $u \in X$ there is a number $c_1(u)$ such that:*

$$\forall k, \quad \|DF(x)u\|_k \leqslant c_1(u)\big(m_k \|u\|_{k+d_1} + \|F(x)\|_k\big).$$

(4) *There exists a linear map $L(x) : Y \to X$ such that $DF(x)L(x) = I_Y$*

$$\forall u \in X, \quad DF(x)L(x)v = v.$$

(5) *For every $v \in Y$, we have:*

$$\forall k \in \mathbb{N}, \quad \|L(x)v\|_k \leqslant m'_k \|v\|_{k+d_2}.$$

Then F maps the ball $\{\|x\|_{k_0} < R\}$ in X onto the ball $\{\|y\|_{k_0+d_2} < R(m'_{k_0})^{-1}\}$ in Y, and for every $\mu > m'_{k_0}$ the inverse F^{-1} satisfies a Lipschitz condition:

$$\forall x_1 \in F^{-1}(y_1), \quad \inf\big\{\|x_1 - x_2\|_{k_0} \;\big|\; x_2 \in F^{-1}(y_2)\big\} \leqslant \mu \|y_1 - y_2\|_{k_0+d_2}.$$

If F extends to a continuous map $\bar{F} : X_{k_0} \to Y_{k_0-d_1}$, then $\bar{F}$ maps the ball $\{\|x\|_{k_0} < R\}$ in X_{k_0} onto the ball $\{\|y\|_{k_0+d_2} < R(m'_{k_0})^{-1}\}$ in $Y_{k_0+d_2}$, and the inverse $\bar{F}^{-1}$ satisfies the same Lipschitz condition.

We conclude by rephrasing Theorem 4 as an implicit function theorem.

Theorem 5. *Let $X = \bigcap_{k\geqslant 0} X_k$ and $Y = \bigcap_{k\geqslant 0} Y_k$ be graded Fréchet spaces, with Y standard, and let $F(\varepsilon, x) = F_0(x) + \varepsilon F_1(x)$ be a map from $X \times \mathbb{R}$ to Y. Assume there exist integers k_0, d_1, d_2, sequences $m_k > 0$, $m'_k > 0$, and numbers $R > 0$, $\varepsilon_0 > 0$ such that, for every (x, ε) such that $\|x\|_{k_0} \leqslant R$ and $|\varepsilon| < \varepsilon_0$, we have*:

(1) *$F_0(0) = 0$ and $F_1(0) \neq 0$.*
(2) *F_0 and F_1 are continuous, and Gâteaux-differentiable.*
(3) *For every $u \in X$ there is a number $c_1(u)$ such that*:

$$\forall k \geqslant 0, \quad \big\|DF(\varepsilon, x)u\big\|_k \leqslant c_1(u)\big(m_k\|u\|_{k+d_1} + \big\|F(\varepsilon, x)\big\|_k\big).$$

(4) *There exists a linear map $L(\varepsilon, x) : Y \to X$ such that*:

$$DF(\varepsilon, x)L(\varepsilon, x) = I_Y.$$

(5) *For every $v \in Y$, we have*:

$$\forall k \geqslant 0, \quad \big\|L(\varepsilon, x)v\big\|_k \leqslant m'_k\|v\|_{k+d_2}.$$

Then, for every ε such that:

$$|\varepsilon| < \min\left\{\frac{R}{m'_{k_0}}\big\|F_1(0)\big\|^{-1}_{k_0+d_2}, \varepsilon_0\right\}$$

and every $\mu > m'_{k_0}$, there is an x_ε such that:

$$F(\varepsilon, x_\varepsilon) = 0$$

and:

$$\|x_\varepsilon\|_{k_0} \leqslant \mu|\varepsilon|\big\|F_1(0)\big\|_{k_0+d_2}.$$

Proof. Fix ε with $|\varepsilon| < \varepsilon_0$. Consider the function:

$$G_\varepsilon(x) := F_0(x) + \varepsilon\big(F_1(x) - F_1(0)\big).$$

It satisfies conditions (1) to (5) of Theorem 4. The equation $F(\varepsilon, x_\varepsilon) = 0$ can be rewritten $G_\varepsilon(x) = -\varepsilon F_1(0) := y$. By Theorem 4, we will be able to solve it provided:

$$\|y\|_{k_0+d_2} = |\varepsilon|\,\big\|F_1(0)\big\|_{k_0+d_2} < R\frac{1}{m'_{k_0}}$$

and the solution x_ε then satisfies

$$\|x_\varepsilon\|_{k_0} \leqslant \mu\|y\|_{k_0+d_2} = \mu|\varepsilon|\big\|F_1(0)\big\|_{k_0+d_2}. \qquad \square$$

3. Proof of Theorem 3

The proof proceeds in three steps. Using Ekeland's variational principle, we first associate with $\bar{y} \in Y$ a particular point $\bar{x} \in X$. We then argue by contradiction, assuming that $F(\bar{x}) \neq \bar{y}$. We then identify for every n a particular direction $u_n \in X$, and we investigate the derivative of $x \to \sum \beta_k\|F(x) - \bar{y}\|_k$ in the direction u_n. We finally let $n \to \infty$ and derive a contradiction.

3.1. Step 1

We define a new sequence α_k by

$$\alpha_k = \frac{\beta_{k+d_2}}{m'_k}$$

and we endow X with the distance d defined by

$$d(x_1, x_2) := \sum_k \alpha_k \min\{R, \|x_1 - x_2\|_k\}. \tag{3.1}$$

By Proposition 2, (X, d) is a complete metric space. Now consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$ (the value $+\infty$ is allowed) defined by

$$f(x) = \sum_{k=0}^{\infty} \beta_k \big\| F(x) - \bar{y} \big\|_k. \tag{3.2}$$

It is obviously bounded from below, and

$$0 \leqslant \inf f \leqslant f(0) = \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \infty. \tag{3.3}$$

It is also lower semi-continuous. Indeed, let $x_n \to x$ in (X, d). Then $x_n \to x$ in every X_k. By Fatou's lemma, we have:

$$\liminf_n f(x_n) = \liminf_n \sum_k \beta_k \big\| F(x_n) - \bar{y} \big\|_k \geqslant \sum_k \beta_k \lim \big\| F(x_n) - \bar{y} \big\|_k$$

and since $F : X \to Y$ is continuous, we get:

$$\sum_k \beta_k \lim \big\| F(x_n) - \bar{y} \big\|_k = \sum_k \beta_k \big\| F(x) - \bar{y} \big\|_k = f(x)$$

so that $\liminf_n f(x_n) \geqslant f(x)$, as desired.

By assumption (2.7), we can take some $R' < R$ with:

$$\sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \frac{\beta_{k_0+d_2}}{m'_{k_0}} R'. \tag{3.4}$$

We now apply Ekeland's variational principle to f (see [7,8]). We find a point $\bar{x} \in X$ such that:

$$f(\bar{x}) \leqslant f(0), \tag{3.5}$$

$$d(\bar{x}, 0) \leqslant R' \alpha_{k_0}, \tag{3.6}$$

$$\forall x \in X, \quad f(x) \geqslant f(\bar{x}) - \frac{f(0)}{R' \alpha_{k_0}} d(x, \bar{x}). \tag{3.7}$$

Replace $f(x)$ by its definition (3.2) in inequality (3.5). We get:

$$\sum \beta_k \big\| F(\bar{x}) - \bar{y} \big\|_k \leqslant \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \infty \tag{3.8}$$

and from the triangle inequality it follows that:

$$\sum_{k=0}^{\infty} \beta_k \big\| F(\bar{x}) \big\|_k \leqslant 2 \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \infty. \tag{3.9}$$

If $\|\bar{x}\|_{k_0} > R'$, then $d(\bar{x}, 0) > R' \alpha_{k_0}$, contradicting formula (3.6). So we must have $\|\bar{x}\|_{k_0} \leqslant R' < R$, and (2.9) is proved.

We now work on (3.7). To simplify notations, we set:

$$A = \frac{\sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k}{R' \alpha_{k_0}} = \frac{f(0)}{R' \alpha_{k_0}}. \tag{3.10}$$

It follows from (3.7) that, for every $u \in X$ and $t > 0$, we have:

$$-\big(f(\bar{x} + tu) - f(\bar{x})\big) \leqslant A d(\bar{x} + tu, \bar{x})$$

and hence, dividing by t:

$$-\frac{1}{t}\left[\sum_{k=0}^{\infty}\beta_k\left\|\bar{y}-F(\bar{x}+tu)\right\|_k-\sum_{k=0}^{\infty}\beta_k\left\|\bar{y}-F(\bar{x})\right\|_k\right]\leqslant\frac{A}{t}\sum_{k\geqslant 0}\alpha_k\min\left\{R,t\|u\|_k\right\}. \tag{3.11}$$

3.2. Step 2

If $F(\bar{x})=\bar{y}$, the proof is over. If not, we set:

$$v=F(\bar{x})-\bar{y},$$
$$u=-L(\bar{x})v$$

so that:

$$DF(\bar{x})u=-\left(F(\bar{x})-\bar{y}\right).$$

Since Y is standard, there is a sequence v_n such that:

$$\forall k,\quad \|v_n-v\|_k\to 0, \tag{3.12}$$
$$\forall n,\quad \|v_n\|_k\leqslant c_3(v)\|v\|_k, \tag{3.13}$$
$$\|v_n\|_k\leqslant c_0(v_n)^k. \tag{3.14}$$

Set $u_n=-L(\bar{x})v_n$. Clearly $\|u_n-u\|_k\to 0$ for every k. We have, using condition (5):

$$\begin{aligned}\|u_n\|_k&\leqslant m_k'\|v_n\|_{k+d_2}\\&\leqslant m_k'c_0(v_n)^{k+d_2}.\end{aligned} \tag{3.15}$$

We now substitute u_n into formula (3.11), always under the assumption that $F(\bar{x})-\bar{y}\neq 0$, and we let $t\to 0$. If convergence holds, we get:

$$-\lim_{\substack{t\to 0\\ t>0}}\frac{1}{t}\left[\sum_{k=0}^{\infty}\beta_k\left\|\bar{y}-F(\bar{x}+tu_n)\right\|_k-\sum_{k=0}^{\infty}\beta_k\left\|\bar{y}-F(\bar{x})\right\|_k\right]\leqslant A\lim_{\substack{t\to 0\\ t>0}}\sum_{k\geqslant 0}\frac{\alpha_k}{t}\min\left\{R,t\|u_n\|_k\right\}. \tag{3.16}$$

We shall treat the right- and the left-hand side separately, leaving n fixed throughout.

We begin with the right-hand side. We have:

$$\frac{\alpha_k}{t}\min\left\{R,t\|u_n\|_k\right\}=\alpha_k\min\left\{\frac{R}{t},\|u_n\|_k\right\}=:\gamma_k(t).$$

We have $\gamma_k(0)\geqslant 0$, $\gamma_k(t')\geqslant\gamma_k(t)$ for $t'\leqslant t$, and $\gamma_k(t)\to\alpha_k\|u_n\|_k$ when $t\to 0$. By the monotone convergence theorem:

$$\lim_{\substack{t\to 0\\ t>0}}\sum_{k=0}^{\infty}\frac{1}{t}\alpha_k\min\left\{R,t\|u_n\|_k\right\}=\sum_{k=0}^{\infty}\alpha_k\|u_n\|_k. \tag{3.17}$$

Now for the left-hand side of (3.16). Rewrite it as

$$-\sum_{k=0}^{\infty}\beta_k\frac{g_k(t)-g_k(0)}{t} \tag{3.18}$$

where:

$$g_k(t):=\left\|\bar{y}-F(\bar{x}+tu_n)\right\|_k.$$

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001

We have $\|\bar{x}+tu_n\|_k \leqslant \|\bar{x}\|_k + t\|u_n\|_k$. We have seen that $\|\bar{x}\|_{k_0} \leqslant R' < R$, so there is some $\bar{t}>0$ so small that, for $0<t<\bar{t}$, we have $\|\bar{x}+tu_n\|_{k_0} < R$. Without loss of generality, we can assume $\bar{t}\leqslant 1$. Since F is Gâteaux-differentiable, by Lemma 1 g_k has a right derivative everywhere, and:

$$\left|(g_k)'_+(t)\right| = \lim_{\substack{h\to 0\\ h>0}} \left|\frac{g_k(t+h)-g_k(t)}{h}\right| \leqslant \left\|DF(\bar{x}+tu_n)u_n\right\|_k. \tag{3.19}$$

Introduce the function $f_k(t) = \|F(\bar{x}+tu_n)\|_k$. It has a right derivative everywhere, and $(f_k)'_+(t) \leqslant \|DF(\bar{x}+tu_n)u_n\|_k$, still by Lemma 1. We shall henceforth write the right derivatives g'_k and f'_k instead of $(g_k)'_+$ and $(f_k)'_+$. By condition (3), we have:

$$\begin{aligned}
&f'_k(t) \leqslant c_1(u_n)\big(m_k\|u_n\|_{k+d_1} + f_k(t)\big),\\
&f'_k(t) - c_1(u_n)f_k(t) \leqslant c_1(u_n)m_k\|u_n\|_{k+d_1}.
\end{aligned}$$

Integrating, we get:

$$\begin{aligned}
&e^{-tc_1(u_n)}f_k(t) - f_k(0) \leqslant \big(1-e^{-tc_1(u_n)}\big)m_k\|u_n\|_{k+d_1},\\
&m_k\|u_n\|_{k+d_1} + f_k(t) \leqslant e^{tc_1(u_n)}\big[m_k\|u_n\|_{k+d_1} + \big\|F(\bar{x})\big\|_k\big].
\end{aligned}$$

Substituting this into condition (3) and using (3.15), we get:

$$\begin{aligned}
\left\|DF(\bar{x}+tu_n)u_n\right\|_k &\leqslant c_1(u_n)\big(m_k\|u_n\|_{k+d_1} + f_k(t)\big)\\
&\leqslant c_1(u_n)e^{tc_1(u_n)}\big[m_k\|u_n\|_{k+d_1} + \big\|F(\bar{x})\big\|_k\big]\\
&\leqslant C_1(u_n)m_k m'_{k+d_1}c_0(v_n)^{k+d_1+d_2} + C_1(u_n)\big\|F(\bar{x})\big\|_k =: \ell_k
\end{aligned}$$

where the term $C_1(u_n) := c_1(u_n)e^{c_1(u_n)}$ depends on u_n, but not on k. We have used the fact that $0<t<1$.

It follows from (3.19) that the function g_k is ℓ_k-Lipschitzian. So we get, for every k and $0<t<\bar{t}$:

$$\beta_k\left|\frac{g_k(t)-g_k(0)}{t}\right| \leqslant C_1(u_n)\beta_k m_k m'_{k+d_1}c_0(v_n)^{k+d_1+d_2} + C_1(u_n)\beta_k\big\|F(\bar{x})\big\|_k.$$

By assumption (2.6), the first term on the right-hand side belongs to a convergent series. By inequality (3.9), the second term is also summable. So we can apply Lebesgue's dominated convergence theorem to the series (3.18), yielding:

$$\sum_{k=0}^{\infty} -\beta_k g'_k(0) = \lim_{\substack{t\to 0\\ t>0}} \sum_{k=0}^{\infty} -\beta_k \frac{g_k(t)-g_k(0)}{t}. \tag{3.20}$$

Writing (3.17) and (3.20) into formula (3.11) yields:

$$\sum_{k=0}^{\infty} -\beta_k g'_k(0) \leqslant A\sum_{k\geqslant 0}\alpha_k\|u_n\|_k. \tag{3.21}$$

We now apply Lemma 1. Denote by N_k the subdifferential of the norm in Y_k:

$$\forall y\neq 0,\quad y^*\in N_k(y) \iff \big\|y^*\big\|_k^* = 1 \quad\text{and}\quad \langle y^*, y\rangle_k = \|y\|_k.$$

There is some $y_k^*(n)\in N_k(F(\bar{x})-\bar{y})$ such that:

$$g'_k(0) = \big\langle y_k^*(n), DF(\bar{x})u_n\big\rangle_k. \tag{3.22}$$

Substituting into (3.21) we get:

$$\sum_{k=0}^{\infty} -\beta_k\big\langle y_k^*(n), DF(\bar{x})u_n\big\rangle_k \leqslant A\sum_{k\geqslant 0}\alpha_k\|u_n\|_k. \tag{3.23}$$

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001

Since $F(\bar{x}) - \bar{y} \neq 0$, the characterization of $N_k(F(\bar{x}) - \bar{y})$ gives:

$$\langle y_k^*(n), F(\bar{x}) - \bar{y}\rangle_k = \|F(\bar{x}) - \bar{y}\|_k, \tag{3.24}$$

$$\|y_k^*(n)\|_k^* = 1. \tag{3.25}$$

3.3. Step 3

We now remember that $u_n = -L(\bar{x})v_n$, so that $DF(\bar{x})u_n = -v_n$. Formula (3.23) becomes:

$$\sum_{k=0}^{\infty} \beta_k \langle y_k^*(n), v_n\rangle_k \leqslant A \sum_{k\geqslant 0} \alpha_k \|L(\bar{x})v_n\|_k. \tag{3.26}$$

Set $\varphi_k(n) = \beta_k \langle y_k^*(n), v_n\rangle_k$ and $\psi_k(n) = \alpha_k \|L(\bar{x})v_n\|_k$.

On the one hand, by formula (3.13), since $\|y_k^*(n)\|_k^* = 1$, we have:

$$\begin{aligned} |\varphi_k(n)| &= |\beta_k \langle y_k^*(n), v_n\rangle_k| \leqslant \beta_k \|y_k^*(n)\|_k^* \|v_n\|_k \\ &\leqslant c_3(v)\beta_k \|v\|_k = c_3(v)\beta_k \|F(\bar{x}) - \bar{y}\|_k \end{aligned} \tag{3.27}$$

and on the other, still by formula (3.13), we have:

$$\begin{aligned} |\psi_k(n)| &= \alpha_k \|L(\bar{x})v_n\|_k = \frac{\beta_{k+d_2}}{m_k'} \|L(\bar{x})v_n\|_k \\ &\leqslant \frac{\beta_{k+d_2}}{m_k'} m_k' \|v_n\|_{k+d_2} = \beta_{k+d_2} \|v_n\|_{k+d_2} \\ &\leqslant c_3(v)\beta_{k+d_2} \|v\|_{k+d_2} = c_3(v)\beta_{k+d_2} \|F(\bar{x}) - \bar{y}\|_{k+d_2}. \end{aligned} \tag{3.28}$$

By inequality (3.8), the series $\sum \beta_k \|F(\bar{x}) - \bar{y}\|_k$ is convergent, so the last terms in (3.27) and (3.28), which are independent of n, form convergent series.

From (3.12) and condition (5), for each k, $\|L(\bar{x})(v_n - v)\|_k \to 0$ as $n \to \infty$. We thus get the pointwise convergence of $\psi_k(n) = \alpha_k \|L(\bar{x})v_n\|_k$:

$$\lim_{n\to\infty} \psi_k(n) = \alpha_k \|L(\bar{x})v\|_k.$$

Remembering that $v = F(\bar{x}) - \bar{y}$, we have, by formulas (3.24) and (3.25):

$$\left|\langle y_k^*(n), v_n\rangle_k - \|v\|_k\right| = \left|\langle y_k^*(n), v_n - v\rangle_k\right| \leqslant \|v_n - v\|_k,$$

hence, from (3.12), the pointwise convergence of $\varphi_k(n) = \beta_k \langle y_k^*(n), v_n\rangle_k$:

$$\lim_{n\to\infty} \varphi_k(n) = \beta_k \|v\|_k.$$

So, applying Lebesgue's dominated convergence theorem to the series $\sum_k \varphi_k(n)$ and $\sum_k \psi_k(n)$, we get:

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \beta_k \langle y_k^*(n), v_n\rangle_k = \sum_{k=0}^{\infty} \beta_k \|v\|_k,$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \alpha_k \|L(\bar{x})v_n\|_k = \sum_{k=0}^{\infty} \alpha_k \|L(\bar{x})v\|_k.$$

It follows from the above and from (3.26) that:

$$\sum_{k=0}^{\infty} \beta_k \|v\|_k \leqslant A \sum_{k\geqslant 0} \alpha_k \|L(\bar{x})v\|_k = A \sum_{k\geqslant 0} \frac{\beta_{k+d_2}}{m_k'} \|L(\bar{x})v\|_k.$$

Estimating the right-hand side by condition (5), we finally get:

$$\sum_{k=0}^{\infty} \beta_k \|v\|_k \leqslant A \sum_{k\geqslant 0} \beta_{k+d_2} \|v\|_{k+d_2}$$

with $v = F(\bar{x}) - \bar{y} \neq 0$, hence $A \geqslant 1$. Remembering the definition (3.10) of A, this yields:

$$\frac{\sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k}{R' \alpha_{k_0}} = \frac{\sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k}{R' \beta_{k_0+d_2}} m'_{k_0} \geqslant 1$$

which contradicts (3.4). This shows that $F(\bar{x}) - \bar{y}$ cannot be non-zero, and concludes the proof.

References

[1] Serge Alinhac, Patrick Gérard, Opérateurs Pseudo-différentiels et Théorème de Nash–Moser, Interéditions et Éditions du CNRS, Paris, 1991; English translation: Pseudo-Differential Operators and the Nash–Moser Theorem, Grad. Stud. Math., vol. 82, Amer. Math. Soc., Rhode Island, 2000.

[2] Vladimir I. Arnol'd, Small divisors, Dokl. Akad. Nauk CCCP 137 (1961) 255–257; Dokl. Akad. Nauk CCCP 138 (1961) 13–15 (in Russian).

[3] Vladimir I. Arnol'd, Small divisors I, Izvestia Akad. Nauk CCCP 25 (1961) 21–86 (in Russian).

[4] Vladimir I. Arnol'd, Small divisors II, Ouspekhi Math. Nauk 18 (1963) 81–192 (in Russian).

[5] Massimiliano Berti, Philippe Bolle, Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions, Arch. Ration. Mech. Anal. 195 (2010) 609–642.

[6] Nassif Ghoussoub, Duality and Perturbation Methods in Critical Point Theory, Cambridge Tracts in Math., vol. 107, Cambridge Univ. Press, 1993.

[7] Ivar Ekeland, Sur les problèmes variationnels, C. R. Acad. Sci. Paris 275 (1972) 1057–1059; C. R. Acad. Sci. Paris 276 (1973) 1347–1348.

[8] Ivar Ekeland, Nonconvex minimization problems, Bull. Amer. Math. Soc. 1 (1979) 443–474.

[9] Richard Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. (1) 7 (1982) 65–222.

[10] Andrei N. Kolmogorov, On the conservation of quasi-periodic motion for a small variation of the Hamiltonian function, Dokl. Akad. Nauk CCCP 98 (1954) 527–530 (in Russian).

[11] Jürgen Moser, A new technique for the construction of solutions of nonlinear differential equations, Proc. Natl. Acad. Sci. USA 47 (1961) 1824–1831.

[12] Jürgen Moser, A rapidly convergent iteration method and nonlinear differential equations, Ann. Scuola Norm. Sup. Pisa 20 (1966) 266–315, 499–535.

[13] John Nash, The imbedding problem for Riemannian manifolds, Ann. of Math. (2) 63 (1956) 20–63.

[14] R.T. Rockafellar, Conjugate Duality and Optimization, SIAM/CBMS Monograph Ser., vol. 16, SIAM Publications, 1974.

[15] Jacob Schwartz, On Nash's implicit functional theorem, Comm. Pure Appl. Math. 13 (1960) 509–530.

Please cite this article in press as: I. Ekeland, An inverse function theorem in Fréchet spaces, Ann. I. H. Poincaré – AN (2010), doi:10.1016/j.anihpc.2010.11.001