Equilibrium resource management with altruistic overlapping generations*

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Abstract

We imbed a classic fishery model, where the optimal policy follows a Most Rapid Approach Path to a steady state, into an overlapping generations setting. The current generation discounts future generations' utility flows at a rate possibly different from the pure rate of time preference used to discount their own utility flows. The resulting model has non-constant discount rates, leading to time inconsistency. The unique Markov Perfect equilibrium to this model has a striking feature: provided that the current generation has *some* concern for the not-yet born, the equilibrium policy does not depend on the degree of that concern.

JEL: C73, D64, D90, Q01, Q22, C 61

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1 Introduction

To what extent does the degree of our concern for unborn generations affect the equilibrium management of a resource? We address this question using a fishery model, both because of fisheries' intrinsic importance, and also in order to provide focus. However, the research question is central to many resource problems where current decisions have long-lived consequences. These resource problems are intergenerational, but the standard approach studies them using an infinitely lived agent model. That model cannot distinguish between two types of intertemporal transfer, the first between the same agent at different stages of her life, and the second between two different people living at different times. The pure rate of time preference (PRTP) is appropriate for evaluating the first type of transfer, but there should be no presumption that society uses the same discount rate to evaluate the second type of transfer. We imbed a two-parameter discounting model in an overlapping generations (OLG) model. One parameter is the agent's PRTP for their own utility, having the standard interpretation. The second parameter, reflecting intergenerational altruism, measures society's willingness to forgo current utility for the sake of future generations. We address our research question by showing the relation between this altruism parameter and the equilibrium in the natural resource problem.

The Millennium Ecosystem Assessment identifies fisheries as a critical environmental stock (United Nations 2005). Due to overfishing, loss of habitat, and climate change, at least 30% of the world's fisheries are at risk of population collapse (Sumaila et al. 2011). Fisheries support nations' well-being through direct employment in fishing, processing, and ancillary services amounting to over US$ 220 billion annually (Dyck and Sumaila 2010). Fish provide nearly 3.0 billion people with 15 percent of their animal protein needs; including post-catch activities and workers' dependents, marine fisheries support nearly 8% of the world's population (FAO, 2011).

The actual fishery management problem is intergenerational: agents alive today have to decide how much of the stock to retain for future generations. Agents currently alive have a standard optimization problem if they do not care about the not-yet born, or if they discount the future utility flows of the not-yet-born at the same rate as they discount their own future utility. In all other cases, their implied discount rate is non-constant, either decreasing, as with hyperbolic discounting, or increasing. In these cases, the policy trajectory optimal for the current generation is not time consistent.

The current generation cannot reasonably believe that it can choose actions that subsequent generations will implement. We therefore consider a particular class of time consistent equilibria, in which the harvest decision at a point in time is conditioned on the fish stock – the state variable – at that point in time. We obtain a Markov equilibrium to the dynamic game amongst the sequence of policymakers. Each policymaker in this sequence is the representative agent at a point in time. The Markov Perfect Equilibrium (MPE) is a subgame perfect Nash equilibrium to this sequential game: the policy rule chosen by each representative agent is optimal, given her beliefs about the policy rule future generations will use.

For general functional forms, this game provides few insights into our research question. Most natural resource problems, like most optimal control problems in other areas of economics, rely on particular functional forms to provide insights. Perhaps the model most widely used to propose target fishery stocks, and certainly the model most widely used to explain the management problem, is linear in the harvest rate (Clark and Munro 1975; Clark, Clarke and Munro 1979; Clark 2005). With this model, the benefit per unit of harvest is constant and the cost per unit of harvest is a decreasing convex function of the fish stock; harvest costs increase as the stock falls. The equation of motion equals the natural growth rate of biomass minus the harvest. This model provides a plausible, elegant, and easily interpreted recommendation: the stock should be driven as rapidly as possible to a steady state. The solution is "bang-bang", i.e. it involves a Most Rapid Approach Path (Spence and Starrett 1975). The steady state depends on the per unit benefit of harvest, the growth equation, the harvest cost function, and importantly, on the discount rate used to evaluate future benefits.

We imbed this linear-in-control model into the sequential game described above, and obtain a striking conclusion: provided that the current generation has *some* concern for the not-yet born, the equilibrium policy rule, and thus the stock trajectory and the steady state, are all independent of the *degree* of concern for future generations. The steady state depends on the agent's pure rate of time preference and the population growth rate, but not on the altruism parameter. There is one exception to this strong result. If the current generation has literally no concern for future generations (i.e., it discounts those generations' benefits at an infinite rate), then the steady state is different: there is a discontinuity in the equilibrium decision rule, in the limit as the current generation's concern for the not-yet born vanishes.

We use the fishery model throughout, but it is worth emphasizing that our fundamental result applies in all optimization problems with this linear-in-control structure. Spence and Starrett (1975) consider two natural resource applications and three other (non-resource) capital accummuation problems. Their first natural resource application is the fishery model that we also use, and their second application involves a stock pollutant. Our proofs rely on the linear-in-control structure, but not on the fishery context. Therefore, one could also apply our results to stock pollutant and other problems. Discounting assumptions are important in most of these settings. For example, discounting was central to an ongoing debate about climate policy; greenhouse gasses are the quintessential stock pollutants. Important contributions to this debate include Stern, 2006; Nordhaus, 2007, Weitzman, 2007, and Dasgupta 2008. Although the prevailing view is that climate and other environmental policy is very sensitive to discounting assumptions, the evidence for this view is based on numerical results or on special functional forms. The relation between equilibrium environmental policy and discounting is model-dependent. Our model is linear in the decision variable, but the growth function and the extraction cost function are general, and we obtain a complete characterization of the unique equilibrium. Thus, our model provides a novel perspective on the relation between resource policy and discounting.

Our analysis contributes more generally to the literature on non-constant (including hyperbolic) discounting and to the OLG literature. OLG models used to study policy sometimes assume that the policy maker discounts the utility of each generation *back to the time of their birth*, e.g. Calvo and Obstfeld (1988) and more recently Schneider, Traeger, and Winkler (2012). This assumption renders preferences time-consistent, making the model appropriate for normative purposes. However, Calvo and Obstfeld (page 414) recognize that this assumption is "unnatural ...[because]...the planner is concerned with [agents'] welfare from the present time onward". Discounting back to the time of birth is therefore not consistent with a political economy equilibrium in which the social planner at a point in time represents the preferences of agents alive at that time. Our sequential game model, and the focus on Markov perfection, provides an alternative that is consistent with such an equilibrium. All agents alive at a point in time have the same preferences, and care about their own current and future utility flows, and (possibly) those of their successors. They discount these future flows from the current time, irrespective of their date of birth.

2 Preliminaries

Here we review that canonical fishery model that our paper generalizes. In this model, the flow payoff is $u(t) = (p - c(x_t))\, h_t$, where the state variable, x_t, is the biomass of fish, the decreasing convex function $c(x)$ is the unit cost of harvest, p is the constant price; the planner chooses the harvest, h_t.[1] With constant discount rate r, the planner's objective is to maximize the present discounted value of the stream of payoffs,

$$\int_0^\infty e^{-rt} (p - c(x_t))\, h_t. \tag{1}$$

The stock of fish evolves according to

$$\frac{dx(t)}{dt} = \dot{x}_t = f(x_t) - h_t. \tag{2}$$

To avoid uninteresting technical issues, we assume that harvest is bounded below by 0 and bounded above by $\bar{h} < \infty$.

The solution to this optimal control problem sets the harvest level at its maximum or minimum value ($\bar{h}$ or 0) in order to drive the stock as quickly as possible to its steady state level. The steady state is the solution to

$$r = f'(x) - \frac{c'(x)f(x)}{p - c(x)}. \tag{3}$$

Figure 1 shows the absolute value of the elasticity of steady state stock, with respect to the discount rate, for Pacific halibut (solid graph) and Anartic fin whale (dashed graph). The figure uses the Shaefer (i.e., quadratic) growth model and $c(x) = \frac{c}{x}$, with parameter values taken from Clark (1975), chapter 2. The intrinsic growth rate for whales is much lower than for halibut. Because it takes whale stocks longer to recover from low levels, compared to halibut stocks, it seems intuitive that the steady state whale stock would be more sensitive than the halibut stock to the discount rate. A typical value for the discount rate is close to 0.05, so a value $r = 0.2$ represents a high discount rate. For $r < 0.2$, i.e. in the plausible range, the steady state whale stock is indeed more sensitive to the discount rate than is the halibut stock (Clark 1973). Much higher discount rates reverse this ranking. As $r \to \infty$, equation (3) implies that

[1] If the price depends on harvest, $u(t)$ is not linear-in-control. In that case, both the results of the standard model, and our variation with non-constant discounting, do not hold.

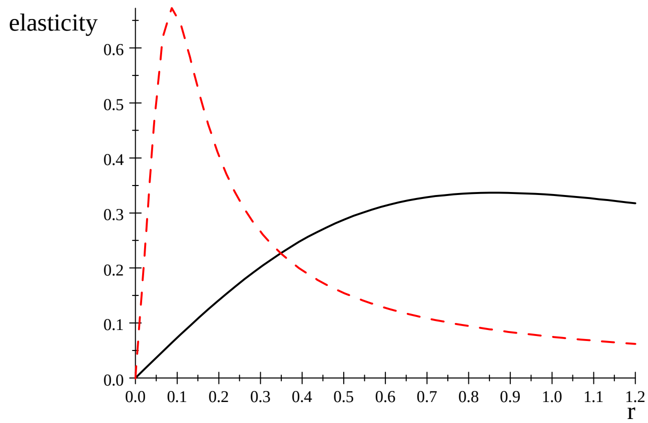

Figure 1: Dashed graph shows (absolute value of) elasticity of steady state stock of Anartic fin whale, with respect to the discount rate; solid graph shows elaticity for Pacific halibut. Unit or time = 1 year.

the steady stock stock converges to the open access level, the solution to $p = c(x)$, and the interest rate elasticity converges to 0, independent of the growth rate.

3 The model

We first discuss the discounting assumptions; for this purpose we do not need to specify the growth function, and it is convenient to use a general flow payoff, $u(t)$. We then specialize to the fishery model in Section 2, except that we replace the exponential discount factor with a generalization having a non-constant discount rate; this generalization includes the constant discount rate as a special case. We then define the equilibrium.

3.1 Discounting

Each measure-zero agent's lifetime is exponentially distributed, with hazard rate (mortality) ω.[2] Agents are born at rate α and die at rate ω, so the population growth rate is $g = \alpha - \omega$. At time t the population size is $N(t)$. The memoryless feature of the exponential distribution means that all agents alive at a point in time have the same probability of dying over any future interval, regardless of their current age. There

[2]Individual agents have random lifetime, but because these agents are measure-zero, there is no aggregate uncertainty (Yaari, 1965; Blanchard, 1985).

are no other sources of age-dependent differences (e.g. private accumulation), so agents alive at a point in time are indistinguishable from each other. *Conditional on both still being alive, two agents born at different times have the same preferences.* At a point in time, there is literally a representative agent, so in this model there is no issue of aggregation of preferences over the agents *currently* alive.[3] The population alive at t can delegate to any agent currently living, the authority to make whatever decisions are available. We refer to that agent as the social planner at time t. The time inconsistency problem that arises in this model makes it necessary to keep in mind that a social planner is indexed by the time that she makes a decision. Thus, we refer to "the social planner at time t" rather than merely "the social planner".

Agents care about their own future expected utility flow, which they discount at a constant pure rate of time preference, δ. Their risk-adjusted discount rate is $\delta+\omega$; the term ω accounts for the mortality risk. To the extent that they are altruistic, they also care about the utility flows of generations that have not yet been born; they discount those agents' utility at the constant rate σ. For altruistic agents, $\sigma < \infty$. We refer to the welfare that an agent obtains from her own future consumption as the selfish component, and the welfare that she obtains from the consumption of her successors as the altruistic component. The sum of these two components comprise the agent's welfare.

We consider only the case where agents have paternalistic altruism: an agent at time t cares about the utility flow received by someone born at time $s > t$, but does not take into account that intervening generations, those born at $t < t' < s$, also care about the agents born at time s. For example suppose that Groucho, Harpo, and Chico live in three successive periods, and have selfish utility, u_G, u_H and u_C, respectively. Harpo cares about himself and about Chico, so his welfare is a function $W_H\left(u_H, u_C\right)$. If Groucho has paternalistic altruism, his welfare is a function $W^{\text{paternal}}\left(u_G, u_H, u_C\right)$. In contrast, if Groucho has pure altruism, he cares about Chico's utility both because he care about Chico, and because he cares about Harpo, who cares about Chico. With pure altruism, his welfare is a function $W^{\text{pure}}\left(u_G, W_H\left(u_H, u_C\right), u_C\right)$ (Ray 1987), (Andreoni 1989), (Saez-Marti and Weibull 2005)[4]

[3]Bergstrom (2006) points out that when agents feel benevolence toward others who share both the costs and the benefits of a public good, it is necessary to count both the "sympathetic costs" as well as the "sympathetic benefits" (those arising from the feeling of benevolence). Here, where currently living agents are identical, the sympathetic costs offset the sympathetic benefits, so benevolent feelings toward other tribal members currently living would not affect the cost benefit calculation.

[4]Karp (2013) shows that, in the case of a constant population ($\alpha = \omega$), an agent with pure al-

Table 1 collects the definitions of the parameters entering the discount factors.

ω	α	g	δ	σ
death rate	birth rate	growth rate $\alpha - \omega$	selfish pure rate of time preference	altruistic discount rate applied to unborn

Table 1: Parameters entering Discount Factor

The aggregate trajectory of profit, or consumption, $\{u(\tau)\}_{\tau=t}^{\infty}$, is endogenous. We assume throughout that $u(\tau)$ is bounded. More significantly, we assume that agents' evaluation of their own flow of utility is linear in *their share* of $u(t)$. In addition, the weight that an agent attaches to the utility flow of future generations is proportional to the number of agents alive at that future time. To state these assumptions more formally, denote $\lambda(t)$ as the share of aggregate flow $u(t)$ that an agent alive at time t obtains. All agents obtain the same share. We restate the assumptions above as

Assumption 1 *Given $u(t)$, $\lambda(t)$, and $N(t)$, (a) an agent's current value of her time t flow is $\lambda(t)\,u(t)$, and (b) the value the planner attaches to the flow of agents born later is $\lambda(t)\,N(t)u(t)$.*

Assumption 1.b states that the weight that a planner attaches to the utility flow of future generations is proportional to the number of agents alive at that future time.

The value of $\lambda(t)$ depends on whether the flow $u(t)$ is a public or a rival (private) good. With public goods, each agent alive at t gets the entire flow $u(t)$; there is no need to share, so $\lambda(t) \equiv 1$. Growth has no effect on the selfish component of welfare. However, a growing population ($g > 0$) increases the weight that an agent today puts on future flows due to the altruistic component of welfare. There will be more people in the future to enjoy those flows, and the presence of those additional people does not reduce the share received by the person alive today. Thus, with a public good, higher growth makes future flows more valuable to an agent currently alive. In the model where $u(t)$ is a private good, the assumption that each agent obtains the same share means that $\lambda(t) \equiv \frac{1}{N(t)}$. Here, the altruistic value that a person today attaches to a future flow is $\lambda(t)\,N(t)u(t) = u(t)$, i.e. it does not depend on growth. However, the flow that a person alive today receives in the future is $\frac{u(t)}{N(t)}$, which decreases with growth: growth decreases the weight put on future flows in the selfish component of

truism and parameters (δ, σ, α) has the same preferences as an agent with paternalistic altruism and parameters $(\delta, \sigma + \alpha, \alpha)$: the two models are isomorphic (for $g = 0$).

welfare. In summary, if $u(t)$ is a public good, higher growth increases the value that an agent today attaches to a future flow. Growth has the opposite effect if $u(t)$ is a private good.

In the case of constant population ($g = 0$) the two models are obviously equivalent. For non-constant population, the models of public and private goods are isomorphic. That isomorphism, described in Proposition 1, is due to the linearity embodied in Assumption 1.

In this stationary model, the discount factor applied at calendar time t to a future utility flow at $t + \tau$ depends on τ but not t. Our goal is to find discount functions $D^{\text{public}}(\tau)$ and $D^{\text{private}}(\tau)$ that aggregate the trajectory $\{u(\tau)\}_{\tau=t}^{\infty}$, yielding welfare for the representative agent alive at t as $W^j(t) = \int_t^{\infty} D^j(\tau - t)\, u(\tau)\, d\tau$ for $j \in \{\text{public, private}\}$. The functions $D^j(\tau)$ are weighted sums of two exponentials. Many papers use the sum of exponentials to represent non-constant discounting (Li and Lofgren 2000), (Gollier and Weitzman 2010), (Zuber 2010), and (Jackson and Yariv 2011). In these papers, the convex combination of exponentials results from aggregating different discount rates; for example, different agents might want to use different discount factors to evaluate future flows, and the decisionmaker takes a weighted sum of their preferences; or the decisionmaker may be uncertain about the correct discount rate, and therefore takes the expectation of the associated discount factors. In both of these cases, at t the social planner's discount factor is a convex combination of the different possible discount factors. Ekeland and Lazrak (2010) provide a different rationale for this form of discounting. They show that an OLG model with paternalistically altruistic agents who consume a public good and have exponentially distributed lifetime, induces a discount factor that is a weighted sum (but not necessarily a convex combination) of exponentials. We extend their result by considering private as well as public goods, and demonstrating the isomorphism between the two. For this purpose, we define

$$\tilde{\sigma} \equiv \sigma + g \quad \text{and} \quad \tilde{\delta} \equiv \delta + g \tag{4}$$

and

$$\sigma^{j,\text{inf}} = \left\{ \begin{array}{c} g \text{ for } j = \text{ public} \\ 0 \text{ for } j = \text{ private} \end{array} \right\}, \tag{5}$$

Proposition 1 *Suppose that agents are paternalistically altruistic and have exponentially distributed lifetime, that $\sigma \neq \alpha + \delta$ and that $\sigma > \sigma^{j,inf}$ for the two cases, $j =$ public and $j =$private. (i) If agents consume a public good, then the discount factor applied* *by the representative agent at any time to utility t periods in the future, $D^{public}(t)$, is*

$$D^{public}(t) = e^{-(\delta+\omega)t}\frac{\sigma-\delta}{\sigma-\alpha-\delta} + \left(\frac{\alpha}{\alpha+\delta-\sigma}\right)e^{-(\sigma-g)t}. \tag{6}$$

(ii) If agents consume a private good, $u(t)$, and each agent alive at a point in time obtains an equal share of that flow, then the discount factor applied by the representative agent at any time to utility t periods in the future, $D^{private}(t)$, has the same form as the expression for $D^{public}(t)$ in equation (6), except that $\tilde{\sigma}$ replaces σ and $\tilde{\delta}$ replaces δ.

Appendix A collects all proofs.[5] For this discount function, welfare converges for any bounded trajectory $\{u(\tau)\}_{\tau=t}^{\infty}$.

In view of the isomorphism described in Proposition 1, we can analyze both the models with the private or the public good by analyzing just one model. Hereafter (merely to conserve notation), we define the right side of equation (6) as "the" discount factor, and write it as $D(t)$ (with no superscript); we use superscript j for "public" and "private" only where needed for clarity.

The discount rate corresponding to the discount function $D(t)$, $r(t)$, and its time derivative, are[6]

$$\begin{aligned} r(t) &= -\frac{\frac{dD}{dt}}{D} = \frac{-(\omega+\delta)(\sigma-\delta)+\alpha(\sigma-g)e^{-t(\sigma-\delta-\alpha)}}{(\delta-\sigma)+\alpha e^{-t(\sigma-\delta-\alpha)}} \\ \frac{dr}{dt} &= \alpha e^{-t(\sigma-\delta-\alpha)}\frac{\sigma-\delta}{\left(\delta-\sigma+\alpha e^{-t(\sigma-\delta-\alpha)}\right)^2}(\sigma-\delta-\alpha)^2. \end{aligned} \tag{7}$$

The discount rate is constant for two values of σ:[7]

$$\begin{aligned} &\text{for } \sigma=\delta,\ r^j(t) = \begin{cases} \delta - g \text{ if } j = \text{public} \\ \tilde{\delta} - g = \delta \text{ if } j = \text{private} \end{cases} \\ &\text{for } \sigma=\infty,\ r^j(t) = \begin{cases} \delta + \omega \text{ if } j = \text{public} \\ \tilde{\delta} + \omega = \delta + \alpha \text{ if } j = \text{private.} \end{cases} \end{aligned} \tag{8}$$

For $\sigma \notin \{\delta, \infty\}$, the discount rate is decreasing if $\sigma < \delta$ and increasing if $\sigma > \delta$. The

[5] We adopt the assumption that $\sigma \neq \alpha+\delta$ only to avoid uninterestsing special cases. Remark 1, following the proof of the proposition, shows that in the limiting case $\sigma = \alpha+\delta$, the discount factor in equation (6) equals $e^{-(\delta+\omega)t}(1+\alpha t)$; the corresponding discount rate is $\frac{1}{t\alpha+1}(\delta-\alpha+\omega+t\alpha\delta+t\alpha\omega)$.

[6] Equation (7) assumes $\sigma \neq \delta+\alpha$. The footnote below Proposition 1 gives the discount factor and rate for the case $\sigma = \delta+\alpha$. This discount rate is used in equations below to handle the case $\sigma = \delta+\alpha$.

[7] The models with $\sigma = \infty$ and $\alpha = 0$ are identical. The outcome is the same if agents do not care about future generations ($\sigma = \infty$) or if future generations do not exist ($\alpha = 0$).

initial value of the discount rate is

$$r^j(0) = \begin{cases} \delta - g \text{ if } j = \text{public} \\ \tilde{\delta} - g = \delta \text{ if } j = \text{private.} \end{cases} \tag{9}$$

Define $r_\infty = \lim_{t\to\infty} r(t)$, the asymptotic discount rate. The signs of the derivatives and the asymptotic values of the discount rate are

$$\begin{aligned} &\text{for } \delta + \alpha \leq \sigma < \infty\text{: } \tfrac{d^j r}{dt} > 0; \quad r^j_\infty = \begin{cases} \delta + \omega \text{ if } j = \text{public} \\ \tilde{\delta} + \omega = \delta + \alpha \text{ if } j = \text{private.} \end{cases} \\ &\text{for } \delta < \sigma \leq \delta + \alpha\text{: } \tfrac{dr^j}{dt} > 0; \quad r^j_\infty = \begin{cases} \sigma - g \text{ if } j = \text{public} \\ \tilde{\sigma} - g = \sigma \text{ if } j = \text{private.} \end{cases} \\ &\text{for } \sigma^{j,\,\min} < \sigma < \delta\text{: } \tfrac{dr^j}{dt} < 0; \quad r^j_\infty = \begin{cases} \sigma - g \text{ if } j = \text{public} \\ \tilde{\sigma} - g = \sigma \text{ if } j = \text{private.} \end{cases} . \end{aligned} \tag{10}$$

The case $\sigma < \delta$ corresponds to hyperbolic discounting, with the discount rate converging to $\sigma - g$ for a public good, and to σ for a private good. The case $\sigma > \delta$ corresponds to an increasing discount rate. For $\sigma > \delta + \alpha$, this rate converges to $\delta + \omega$ for a public good and to $\delta + \alpha$ for a private good. For $\delta + \alpha > \sigma > \delta$, this rate converges to $\sigma - g$ for a public good and to σ for a private good.

Equations (8) – (10) show, for different values of σ, how the representative agents' discount rate depends on whether the flow u is a public or a private good. For example, if agents discount future generations' and their own utility at the same rate ($\sigma = \delta$), a growth in population ($g > 0$) lowers the discount rate in the case of public goods, because there will be more people to enjoy the good. With a private good, growth has no effect on the discount rate (for $\sigma = \delta$) because the benefit of having additional people enjoy the flow exactly offsets the fact that each person has a smaller share; of course, this result is due to the linearity built in to Assumption 1.

Now consider the case $\sigma > \delta + \alpha$, where, for a public good, the asymptotic (as $t \to \infty$) value of the discount rate equals the pure rate of time preference adjusted for mortality risk, $\delta + \omega$; in contrast, for a private good, the asymptotic rate equals the pure rate of time preference plus the birth rate, $\delta + \alpha$. To understand this difference, consider the limiting case where $\alpha = 0$, while $\omega > 0$, so the population shrinks. Because σ is larger than $\delta + \alpha$, the effect of altruism vanishes in the long run. For a public good, only the mortality risk (along with, of course, the pure rate of time preference) matters

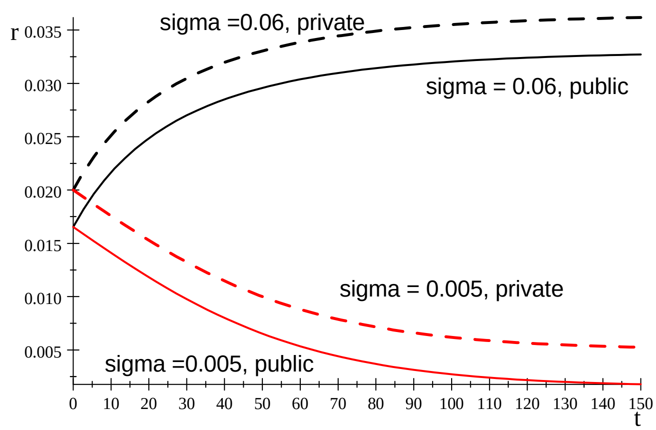

Figure 2: The discount rates for private (dashed) and public (solid) goods, for $\sigma = 0.06$ (increasing) and $\sigma = 0.005$ (decreasing).

in the long run. In contrast, for a private good, the shrinking population means that an agent expects to enjoy an increasing share of the good. This increasing share exactly offsets the mortality risk when $\alpha = 0$, so the asymptotic discount rate equals δ.

In order to illustrate the different discount rate trajectories, let the unit of time be a year and set $\delta = 0.02$, an annual pure rate of time preference of 2%. Let $\omega = 0.013$, corresponding to an expected lifetime of 77 years, and choose the growth rate of $g = \frac{1}{200} \ln 2$, so that population doubles every 200 years. Figure 2 shows the graphs of r for these parameter values, for two values of σ and for the cases of both a public and a private good. The labels on the curves indicate the values of σ: $\sigma = 0.06$ for the two increasing curves, and $\sigma = 0.005$ for the two decreasing curves. The solid curves correspond to the discount rates in the case of a public good, and the dashed curves correspond to the discount rates in the case of a private good.

With one exception, the discount rate is continuous in parameters. This exception plays an important role in our chief result, so we note it here. For finite σ, the initial discount rate, $r^j(0)$, is independent of σ (see equation (9)); for $\sigma = \infty$, the discount rate is constant (see the last part of equation (8)). The constants given by the second line of equation (8) and by equation (9) are not equal, except in the special case where $\alpha = 0$. For large but finite σ, the discount rate begins at $r^j(0)$ and rises rapidly to its asymptotic value (as $t \to \infty$); for $\sigma = \infty$, the discount rate *begins* at its asymptotic value. Thus, there is a discontinuity in the discount rate at $t = 0$ for $\sigma = \infty$.

3.2 Payoff and constraint

The flow payoff, $u(t) = (p - c(x_t))\, h_t$, and the constraint $\frac{dx}{dt} = f(x) - h$, are the same as in Section 2, but the discount factor, $D(s)$, replaces the exponential discount factor e^{-rs}. The welfare of the agents alive at time t is the present discounted value of their selfish and altruistic flow of payoff,

$$\int_0^\infty D(s)\,(p - c(x_{t+s}))\, h_{t+s} ds. \tag{11}$$

If harvest rights are auctioned and the revenue from the sale is used to produce a public good, then the model of a public good is appropriate. If, instead, each agent alive at a point in time has an equal chance of obtaining this revenue, or if the revenue is returned in equal shares to all members of the population alive at a point in time, then the model of a private good is appropriate. In view of Proposition 1, we do not need to decide which model is more relevant. We use the discount factor in equation (6) and the discount rate in equation (7), corresponding to a public good. To obtain the model of the private good, we merely replace σ and δ by $\tilde{\sigma} = \sigma + g$ and $\tilde{\delta} = \delta + g$.

3.3 The equilibrium

For $\sigma \notin \{\delta, \infty\}$ the discount rate is non-constant, so a program that maximizes expression (11) subject to equation (2) is time inconsistent. We obtain a time consistent equilibrium by modelling the decision problem as a sequential game amongst agents who make decisions at different points in time. The agent at time t chooses the current harvest rate, taking as given the current state variable, x_t, under the belief that decisions at time $t + s$, for all $s > 0$, are given by a function $\chi(x_{t+s})$. We look for a symmetric, stationary, pure strategy Nash equilibrium to this game, a function $\chi(x)$ such that $h_t = \chi(x_t)$ is the optimal action for the agent at time t given the state variable x_t, when this agent believes that future actions will be $h_{t+s} = \chi(x_{t+s})$. These beliefs are confirmed in equilibrium for any possible subgame (any realization of x_{t+s}). That is, we obtain a MPE.

Karp (2007) studies the MPE for a more general class of games by taking the limit of a discrete stage infinite horizon game. In that game, each stage lasts for ε units of time, and the discount rate for the first S periods can take arbitrary values, but is constant for period $S + 1$, $S + 2 ... \infty$. The integral in expression (11) is replaced by an infinite sum, and the differential equation (2) is replaced by a difference equation. Harris and

Laibson (2001) obtain the generalized Hamilton-Jacobi-Bellman (HJB) Equation for the case $S = 2$, which corresponds to Laibson (1997)'s β, δ model of quasi-hyperbolic discounting. Their methods are easily extended to obtain the generalized HJB equation for the case of arbitrary finite S. Let $T = S\varepsilon$, the amount of time (as distinct from the number of periods) during which the discount rate may be nonconstant. Taking the formal limit of the discrete time generalized HJB equation as $\varepsilon \to 0$, holding T constant, gives the generalized continuous time HJB equation when the discount rate is allowed to be any function of time for $0 \leq t \leq T$, and is constant after T. One then takes the formal limit of that equation as $T \to \infty$.

Ekeland and Lazrak (2010) take a different route to studying this problem. They begin with the continuous time problem with arbitrary discounting function $r(s)$. At any time t, the agent is allowed to choose a policy over $(t, t + \varepsilon)$, taking as given the decision rule that will be used after $t + \varepsilon$. They obtain the necessary and sufficient condition for this agent's problem and then take the limit as $\varepsilon \to 0$. The two approaches lead to the same generalized HJB equation. Karp (2007) interprets this equation as the standard HJB equation for a "fictitious" optimal control problem: solving one is equivalent to solving the other. In the case at hand, solving the fictitious control problem turns out to be easier and more transparent than solving the generalized HJB equation, and we proceed to do so in the next section.

4 Results

We first explain the methods used to obtain a MPE and then characterize the unique equilibrium.

4.1 Obtaining the MPE

Using Proposition 1 and Remark 1 of Karp (2007), we obtain the MPE to our problem by solving the necessary conditions to the optimal control problem

$$\begin{gathered} J(x_t) = \max \int_0^\infty e^{-r_\infty \tau} \left[(p - c(x_{t+\tau}))\, h_{t+\tau} - K\left(x_{t+\tau}\right) \right] d\tau \\ \text{subject to } \dot{x}_s = f(x_s) - h_s,\ x_t \text{ given.} \end{gathered} \tag{12}$$

(Recall that $r_\infty = \lim_{t\to\infty} r(t)$.) Denote $\chi(x)$ as a (not necessarily unique) MPE decision rule, and define $U(x) := (p - c(x))\, \chi(x)$ as the flow of payoff under this decision rule, given the state variable x. The function $K(x)$ is

$$K(x_t) = \int_0^\infty D(\tau)\left(r(\tau) - r_\infty\right) U\left(x_{t+\tau}\right) d\tau, \tag{13}$$

where $x_{t+\tau}$ is the solution to equation (2) given initial condition x_t and given $h_{t+s} = \chi(x_{t+s})$ for $s \geq 0$. We refer to the optimization problem (12) and the definition (13) as the "fictitious control problem". We use the necessary conditions to this problem to obtain a MPE to the game.

The validity of this approach requires that the value function $J(x)$ and the function $K(s)$ are differentiable. We verify differentiability in Lemma 1 below.[8] We obtain a MPE by solving the necessary conditions to a control problem with constant discount rate r_∞. The integrand in this control problem equals the integrand in the original game, minus the function $K(x)$. That function depends on the MPE decision rule, $\chi(x)$. In general, replacing the original game by the fictitious control problem does not seem to have advanced matters much, because it appears that we need to know the function $K(x)$ to solve the control problem, and $K(x)$ depends on the unknown MPE decision rule. In addition, in general we can not give an intuitive meaning to the function $K(x)$. For the problem at hand, however, there is a simple solution to the problem, and an intuitive interpretation of $K(x)$.

The simplicity arises because the fictitious control problem is linear in the control variable, harvest. For *any* policy rule that results in a differentiable $J(x)$ and $K(x)$, the optimal decision must be on either boundary, $h = 0$ or $h = \bar{h}$, unless a particular function (the "switching function"), defined below, vanishes. The linearity makes this problem tractable.

The asymptotic discount rate, r_∞, takes two possible values, depending on whether $\delta + \alpha < \sigma < \infty$ or $\sigma < \delta + \alpha$. We consider these two cases separately, because the parameter r_∞ is used to discount the payoff in the fictitious control problem, and it also appears in the definition of $K(x)$.

For $\delta + \alpha < \sigma < \infty$, the asymptotic discount rate is $r_\infty = \delta + \omega$. Some calculations establish

$$D(t)\left(r(t) - r_\infty\right) = -\alpha e^{-t(\sigma - g)},$$

[8] Karp (2007) assumes at the outset that the policy rule $\chi(x)$ is differentiable, but that assumption is needed only later in his paper, not for Proposition 1 and Remark 1, which are all that we rely on. However, differentiability of the functions $J(x)$ and $K(x)$ are required. Similarly, Ekeland and Lazrak (2010) assume that the policy rule is differentiable, but an extension of their argument shows that in the current problem, differentiability of $\chi(x)$ is not required.

which implies

$$-K(x_t) = \alpha \int_0^\infty e^{-\tau(\sigma-g)} U(x_{t+\tau})\, d\tau. \tag{14}$$

Here, $-K$ is an annuity, which if received in perpetuity and discounted at the birth rate α, equals the present discounted stream of the future payoff, discounted at $\sigma - g$, the altruistic discount rate minus the growth rate. The fictitious control problem includes this annuity in the flow payoff.

For $\sigma < \delta + \alpha$, $r_\infty = \sigma - g$. In this case,

$$D(t)(r(t) - r_\infty) = -(\sigma - \delta) e^{-t(\omega+\delta)},$$

which implies

$$-K(x_t) = (\sigma - \delta) \int_0^\infty e^{-\tau(\delta+\omega)} U(x_{t+\tau})\, d\tau. \tag{15}$$

Here, $-K$ is an annuity, which if received in perpetuity and discounted at the rate $\sigma - \delta$, equals the present discounted stream of the future payoff, discounted at the risk adjusted rate $\delta + \omega$. Again, this annuity is part of the flow payoff in the fictitious control problem.

4.2 Equilibrium results

To avoid a taxonomy, we adopt the following assumptions regarding equation (3):

Assumption 2 *For all $r \geq 0$ there exists a unique solution to equation (3), decreasing in r.*

Assumption 3 *The growth function $f(x)$ is concave with $f(0) = 0$ and $f'(0) > 0$.*

Assumption 4 *The value of x below which profits are negative, defined as $x_{\min}$, is positive and $f(x_{\min}) - \bar{h} < 0$.*[9]

Assumption 2 implies that in the standard constant discounting problem, a larger discount rate lowers the steady state stock, thereby lowering the steady state flow of profit. Assumption 3 excludes the possibility of "critical depensation", the situation where for sufficiently small initial conditions, the resource is doomed to extinction even in the absence of harvest. Assumption 4 means that although it is feasible to drive the

[9] If $f'(x) - \frac{c'(x)f(x)}{p-c(x)}$ is a decreasing function, and if there exists a carrying capacity x^c at which $f(x^c) = 0 > f'(x^c)$, then Assumption 4 implies Assumption 2.

stock below $x_{\min}$, it is never part of an equilibrium strategy to do so. Therefore, the non-negativity constraint on the stock is not binding.

The current value Hamiltonian for the fictitious control problem is

$$H = (p - c(x) - \psi)\, h - K(x) + \psi f(x)$$

where ψ is the current value costate variable, and the function $(p - c(x) - \psi)$ is known as the switching function. The costate equation is

$$\dot{\psi} = (r_\infty - f'(x))\, \psi + c'(x) h^* + K'(x), \tag{16}$$

where h^* is the optimal control. ("Optimal" for the fictitious control problem, or "equilibrium" for the sequential game.) Due to the linearity in h of the Hamiltonian, an optimal harvest rate must be on the boundary unless the switching function is 0. The harvest rate can be at an interior value for an interval of time (with positive measure) if and only if the switching function is identically 0 during that interval. Differentiating this identity with respect to time and using equations (2) and (16) imply that the switching function is identically 0 if and only if x is a solution to

$$r_\infty = f'(x) - \frac{c'(x)f(x) + K'(x)}{p - c(x)}. \tag{17}$$

Equations (3) and (17) have the same form, apart from the presence of $K'(x)$ on the right side of the latter.

The following proposition summarizes our main result

Proposition 2 *We maintain Assumptions 1 - 4 and require that $\sigma \neq \delta + \alpha$ and $\sigma > \sigma^{j,\, inf}$ (defined in equation (5)) for j = public, private. (i) Within the class of pure strategy equilibria that generate differentiable value functions, the unique MPE to the game amongst the sequence of representative agents, is to follow a most rapid approach path (MRAP) to drive the stock of fish to a level x^*, and thereafter to maintain that stock by harvesting at rate $f(x^*)$:*

$$h^* = \begin{cases} 0 \text{ for } x < x^* \\ f(x^*) \text{ for } x = x^* \\ \bar{h} \text{ for } x > x^*. \end{cases} \tag{18}$$

(iia) For the case of a public good and $\sigma < \infty$, the steady state is the solution to equation (3) with $r = \delta - g$. (iib) For the case of a public good and $\sigma = \infty$, the steady state is the solution to equation (3) with $r = \delta + \omega$. (iiia) For the case of a private good and $\sigma < \infty$, the steady state is the solution to equation (3) with $r = \delta$. (iiib) For the case of a private good and $\sigma = \infty$, the steady state is the solution to equation (3) with $r = \delta + \alpha$.

Note that for all cases (both public and private goods, and regardless of whether σ is finite), the discount rate that determines the steady state equals the *initial* value of the planner's discount rate, $r(0)$. This result does not, of course, imply that the planner is myopic. Today's planner's optimal decision depends on actions that future planners take.

The proof requires establishing that $K'(x)$ exists and finding its value at the steady state. Under the policy in equation (18), it is obvious that $K'(x)$ exists for $x \neq x^*$. We need only show that the left and right derivatives are equal at $x = x^*$, the point at which there is a discontinuous change in the harvest rate. We also need to evaluate that derivative. This result makes the proof of Proposition 2 transparent, so we present it in the text:

Lemma 1 *Under the policy given in equation (18), for $\sigma < \infty$, the derivative $K'(x^*)$ is*

$$K'(x^*) = \begin{cases} -\alpha\left(p - c\left(x^*\right)\right) \text{ for } \sigma > \delta + \alpha \\ \\ -\left(\sigma - \delta\right)\left(p - c\left(x^*\right)\right) \text{ for } \sigma < \delta + \alpha \end{cases} \tag{19}$$

(The lemma states a "smooth pasting" condition, a phenomena that appears in many contexts, e.g. stochastic control.) With this lemma, equation (17), and the formulae for r_∞ in equation (10), parts (ii) and (iii) of Proposition 2 are established by a short calculation. This procedure demonstrates the potential usefulness of our method of studying a class of OLG models.

Implication of and intuition for Proposition 2 In the interest of brevity, consider the case of public goods. Proposition 2.ii states that for $\sigma < \infty$, the MPE steady state stock corresponds to the level chosen by a planner with a constant discount rate equal to $\delta - g = \delta + \omega - \alpha$. This steady state stock, and the corresponding utility flow, exceeds that of the planner with constant discount rate $\delta + \omega$; the latter corresponds to the game in which $\sigma = \infty$. Thus, not surprisingly, the steady state flow of utility is higher for $\sigma < \infty$ compared to $\sigma = \infty$. The noteworthy result is that for $\sigma < \infty$, the outcome is invariant with respect to σ. Provided that agents have some concern for unborn generations, the degree of their concern does not affect equilibrium actions.

We provide intuition for this invariance in three steps. Step 1 reiterates the immediate consequence of the linearity of the model. Step 2 notes that this consequence implies that actions are "weak strategic substitutes": if a positive measure of planners deviate in one direction from their equilibrium action, this causes a positive measure of subsequent planners to change their actions in the other direction, and all other planners to not change their actions (thus the modifier "weak"). (See Jun and Vives (2004) for a recent discussion of strategic substitutability in dynamic games.) The third step explains why a change in σ has offsetting effects on the incentives of each planner.

Step 1 The fact that the current generation's problem is linear in its control *regardless of the Markovian policies that future generations use* is a direct consequence of our assumption that the flow payoff and equation of motion are linear in the harvest. This linearity implies that each agent's action is always at the boundary of its feasible set, unless the state variable is on the singular arc. In the one-state variable model, this singular arc is a point, and equals the steady state; thus, any MPE involves a MRAP to the steady state. Planner t knows exactly how her successors would respond to her deviation from equilibrium: they would follow a MRAP to the steady state. Here there is no possibility of indeterminacy of beliefs about the consequence of out-of-equilibrium play. This determinacy of beliefs, together with mild assumptions on the primitives of the model, and our restriction to the class of equilibria that yield differentiable value and annuity functions (J and $-K$, respectively) insure that the steady state, and thus the entire equilibrium, is unique for a given value of sigma. These facts and assumptions do not, however, explain the invariance of the equilibrium with respect to σ.

Step 2 The fact that the equilibrium policy is a MRAP means that, for given σ, the policy is a non-decreasing step function in x (with an isolated point at the step). Thus, actions are weak strategic substitutes, defined above. For example, if the stock is above the steady state, the only feasible deviation from equilibrium is to reduce extraction. A feasible deviation by a small positive measure of agents, when the stock exceeds the steady state, causes some future agents to harvest more than they would have, and all other future agents to harvest the same amount as they would have, absent the deviation. A parallel explanation applies if the stock is below the steady state. If the stock is at the steady state, deviations in either direction are feasible, and again any deviation causes some subsequent planners to change their decision in the opposite direction. The direction of change of future decisions is weakly the opposite of the direction of a deviation.

Step 3 In the interest of brevity, we consider the invariance with respect to σ, for $\sigma \in \left(\sigma^{\text{min, public}}, \delta + \alpha\right)$, i.e., for the case of hyperbolic discounting, where the long run discount rate is less than the short run discount rate. (A symmetric argument applies if $\sigma \in (\delta + \alpha, \infty)$.) If today's planner could bind her successors to a sequence of actions, she would like to increase her own extraction and instruct her successors to extract less than their own first-best (commitment) level. Today's planner cannot exert this direct control over her successors in the MPE; there, her only means of influencing successors is to change the stock she bequeaths them. Because actions are weak strategic substitutes, the desire to influence successors (the strategic incentive) encourages the planner today to increase her extraction. However, the fact that in equilibrium her successors extract more than she would like them to, means that she considers the future stock too low.[10] In order to compensate, she tends to reduce her extraction. Thus, the planner today has competing incentives: the strategic desire encourages higher current extraction (as a means of influencing successors), but their relatively low future savings encourages high current savings (low extraction). A change in σ makes the problem either more or less similar to the problem of constant discounting (where $\sigma = \delta$) but in either case the change magnifies or diminishes both of the conflicting incentives. In the linear-in-control model, the change in incentives exactly cancel, causing the equilibrium to be invariant to σ.

This intuition uses the fact that there are offsetting incentives regarding current harvest, and that a change in σ reinforces or diminishes both incentives. We do not claim to have economic intuition for why these changes *exactly* cancel.[11] However, the mathematical explanation for this relation is straightforward. For $\sigma \in \left(\sigma^{\text{min, public}}, \delta + \alpha\right)$,

[10]Denote the "full commitment" trajectory as the one chosen by the planner who can commit all of her successors to a particular extraction profile. This problem is also linear in the state, so has a MRAP to a singular arc. However, this problem is non-stationary, due to the non-constant discounting. The singular arc is then a curve giving the stock as a function of time (from the current time). The asymptotic stock as $t \to \infty$, equals the steady state corresponding to $r(\infty)$. For the hyperbolic discounting case assumed here, that asymptotic stock is higher than the stock generated by the constant discount rate $r(0)$.

[11]An analogy may be useful here. We know that a higher current interest rate increases current consumption via the income effect, and decreases current consumption via the substitution effect. It is not surprising that there is some utility function for which the two effects cancel. The fact that this special utility function is logarithmic in current consumption has a simple mathematical explanation, but not a strictly economic one.

the right side of equation (17) is $r_\infty = \sigma - g$, and the left side contains the term $-\frac{K'(x^*)}{p-c(x^*)} = \sigma - \delta$. The difference between these terms is $\delta - g$, which is independent of σ. A parallel argument holds for $r \in (\delta + \alpha, \infty)$.

5 Discussion

This research seeks to improve our understanding of the extent to which the degree of concern for unborn generations affects equilibrium management of a resource. This issue is at the heart of much environmental and resource economics.

Most environment/resource models are based on the infinitely lived agent model, where future utility flows are discounted using a (typically constant) pure rate of time preference. The well-understood drawback of this approach is that it treats agents living at widely different times as the same individual. In fact, our own future utility flows and those of the not-yet born belong to different categories, and there is no reason to apply the same degree of impatience to discount them. An overlapping generations model, in which current generations might discount their own future utility flows and those of successive generations at different rates avoids this conflation of distinct categories, but typically leads to the problem of time inconsistency of optimal programs. Because it is unreasonable for people living today to believe that they can choose policies that will be in effect generations from now, we replace the optimization problem usually used to study resource issues with a sequential game amongst planners; planner t represents the agents alive at time t.

Imbedding a linear-in-control fishery model in an OLG setting, we obtain a striking conclusion: provided that agents have some concern for future generations, their *degree* of concern has no affect on equilibrium resource management. The intuition for this result is that in a world with non-constant discount rates, a planner at a point in time faces conflicting incentives. If future planners save too little, from the perspective of the current planner, then the current planner has an incentive to save more. The current planner also has a strategic incentive that induces her to alter her own savings with a view to influencing her successors. When actions are strategic substitutes, this strategic incentive encourages the current planner to save less. The value of the parameter (σ) that measures concern for future generations also determines the dissimilarity between the sequential game and the control problem with constant discounting. A change in this parameter either increases or diminishes both of the conflicting incentives. For the linear-in-control model, the two changes in incentives exactly cancel. Thus, the change in concern for future generations has no effect on the equilibrium.

This invariance property is a consequence of the assumption that the flow payoff and the equation of motion are linear in the decision variable.[12] In a more general setting, we know that a change in the concern for future generations does affect equilibrium outcomes. Our results are significant for at least three reasons. First, and most importantly, they provide a striking contrast to the prevailing view on the importance of altruism, and they do so in a setting that is simple enough to understand (nearly) completely the forces at work. They thus improve our intuition about the equilibrium consequences, to resource management, of changes in concern for future generations. Second, the linear-in-control model is central to fishery economics, and important to resource economics in general. We have revealed a fundamental feature of this model. Third, the results show how methods that were developed to study non-constant discounting can be used to study OLG models.

[12] It might be tempting to ignore this work on the grounds that the linear-in-control model is not general. We note that many insights into difficult problems are generated using specific functional forms, the log-linear and linear-quadratic dynamic models being the most obvious examples.

A Supplementary material

Appendix A.1 contains routine calculations needed to establish Proposition 1. Appendix A.2 proves our main result, and is not routine.

A.1 Proof and discussion of Proposition 1

Proof. To establish the first part of the proposition, we begin with the observation that because the $N_0 e^{gt}\alpha dt$ agents born during the interval $(t, t+dt)$ die at rate ω, and because they discount their own utility at rate δ, the present discounted value of their selfish payoff from the program $\{u(s)\}_{s=t}^{\infty}$ is (with $N \equiv N_0$)

$$Ne^{gt}\alpha dt \int_t^{\infty} e^{-\delta(s-t)} e^{-\omega(s-t)} u(s)\, ds. \tag{20}$$

We assume that all integrals converge, and then confirm that this assumption is satisfied by the two assumptions $\sigma > g$ and $\sigma \neq \alpha + \delta$. The representative agent alive at time 0 discounts the payoff of generations born in the future at rate σ, so this representative agent's altruistic value of the selfish utility received by the agents born during $(t, t+dt)$ is

$$Ne^{(g-\sigma)t}\alpha dt \int_t^{\infty} e^{-\delta(s-t)} e^{-\omega(s-t)} u(s)\, ds.$$

The current representative agent's value of the selfish utility received by all agents who will be born in the future is therefore the integral of this expression,

$$\begin{gathered} N\alpha \int_0^{\infty} e^{(g-\sigma)t} \left(\int_t^{\infty} e^{-\delta(s-t)} e^{-\omega(s-t)} u(s)\, ds \right) dt = \\ N\alpha \int_0^{\infty} e^{-(\omega+\delta)s} u(s) \left(\frac{e^{(\alpha-\sigma+\delta)s}-1}{\alpha-\sigma+\delta} \right) ds. \end{gathered}$$

The equality follows from changing the order of integration and simplifying. Note that $\frac{e^{(\alpha-\sigma+\delta)s}-1}{\alpha-\sigma+\delta} > 0$ for $s > 0$, so the altruistic component places a positive weight on future generations' utility flows. For bounded $u(s)$, this integral converges if $\omega + \delta > 0$ (which always hold, because ω and δ are positive) and if $\omega + \delta - \alpha + \sigma - \delta = \sigma - g > 0$, which holds by assumption.

The current representative agent discounts the future utility of those currently alive at rate δ and knows that these agents die at rate ω, so her risk-adjusted discount rate for them is $\delta+\omega$. Her (selfish) valuation of their lifetime welfare is therefore

$$N\int_0^\infty e^{-(\delta+\omega)s}u(s)ds.$$

The representative agent's total welfare is the sum of welfare attributed to the utility of the agents who will be born in the future (the altruistic component), and of the agents who are currently alive (the selfish component):

$$N\int_0^\infty \left[e^{-(\omega+\delta)s}\alpha\left(\frac{e^{(\alpha-\sigma+\delta)s}-1}{\alpha-\sigma+\delta}\right)+e^{-(\delta+\omega)s}\right]u\left(s\right)ds.$$

The discount factor for the time t utility flow is

$$D(t):=e^{-(\omega+\delta)t}\alpha\left(\frac{e^{(\alpha-\sigma+\delta)t}-1}{\alpha-\sigma+\delta}\right)+e^{-(\delta+\omega)t}. \tag{21}$$

Simplifying the right side of this equation yields equation (6). The discount factor is the sum of exponentials, each of which converges to 0 as $t\rightarrow\infty$, in view of the assumption $\sigma>g$. Because the utility flows are also bounded (by assumption), all integrals above converge.

We now prove the second part of the proposition. With a private good, the division of a profit flow is a zero sum game. Our assumption that agents receive equal shares at every point means that each receives the share $\frac{1}{N(t)}$. Denote the population at time 0 as N, so the population at time t is Ne^{gt}. There are $Ne^{gt}\alpha dt$ agents born during the interval $(t,t+dt)$. At time $s\geq t$ each of these obtains the flow $\frac{u(s)}{Ne^{gs}}$. The aggregate selfish life-time welfare of these agents is

$$Ne^{gt}\alpha dt\int_t^\infty e^{-\delta(s-t)}e^{-\omega(s-t)}\frac{u\left(s\right)}{Ne^{gs}}ds. \tag{22}$$

Expression (20) and (22) differ because the latter equation assumes that the sum of shares equals 1, whereas the former assumes that each agent's share is 1, so that their sum is $N(s)$. This difference reflects the difference between a private and a public good.

The representative agent (who aggregates the preferences of her generation) alive at time 0 discounts the selfish payoff of generations born in the future at rate σ, so this representative agent's altruistic value of the selfish utility received by the agents born during $(t, t+ds)$ is[13]

$$e^{-\sigma t} N e^{gt} \alpha dt \int_t^\infty e^{-\delta(s-t)} e^{-\omega(s-t)} \frac{u(s)}{N e^{gs}} ds.$$

The current representative agent's altruistic value of the direct utility received by all agents who will be born in the future is therefore the integral of this expression,

$$\begin{aligned}
&\textstyle\int_0^\infty e^{-\sigma t} N e^{gt} \alpha dt \left(\int_t^\infty e^{-\delta(s-t)} e^{-\omega(s-t)} \frac{u(s)}{N e^{gs}} ds \right) \\
&= \alpha \textstyle\int_0^\infty e^{-(g+\omega+\delta)s} u(s) \left(\int_0^s e^{-(\sigma-g-\delta-\omega)t} dt \right) ds \\
&= \textstyle\int_0^\infty \alpha \frac{1-e^{-(\sigma-\alpha-\delta)s}}{\sigma-\alpha-\delta} e^{-(g+\omega+\delta)s} u(s) ds
\end{aligned}$$

The first equality follows from changing the order of integration and the second follows from integrating. Note that $\frac{1-e^{-(\sigma-\alpha-\delta)s}}{\sigma-\alpha-\delta} > 0$ for $s > 0$, so the altruistic component places a positive weight on future generations' utility flows. This discount factor also converges to 0 as $t \to \infty$ if and only if both $g + \omega + \delta = \alpha + \delta > 0$ (which is true because both α and δ are positive) and if $\sigma - \alpha - \delta + g + \omega + \delta = \sigma > 0$, which is true by assumption.

The current representative agent's aggregation of her generation's preferences attributed to their selfish welfare is

$$N \int_0^\infty e^{-(\delta+\omega)s} \frac{u(s)}{e^{gs} N} ds = \int_0^\infty e^{-(\alpha+\delta)s} u(s) ds.$$

Here also the discount factor is positive and converges to 0 because $\alpha + \delta > 0$. The total welfare is the sum of the altruistic and the selfish components:

$$\int_0^\infty \left(\alpha \frac{1 - e^{-(\sigma-\alpha-\delta)s}}{\sigma - \alpha - \delta} e^{-(g+\omega+\delta)s} + e^{-(\alpha+\delta)s} \right) u(s) ds.$$

The discount factor is

$$\begin{aligned}
D(t) &= \alpha \tfrac{1-e^{-(\sigma-\alpha-\delta)s}}{\sigma-\alpha-\delta} e^{-(g+\omega+\delta)s} + e^{-(\alpha+\delta)s} \\
&= \tfrac{\sigma-\delta}{\sigma-\alpha-\delta} e^{-(\alpha+\delta)s} - \alpha \tfrac{e^{-\sigma s}}{\sigma-\alpha-\delta}
\end{aligned} \tag{23}$$

[13] This expression equals the *aggregate* value that *all* agents alive at time 0 attribute to the selfish welfare of agents born during $(t, t+dt)$. It is not the value that a single agent alive at time 0 attributes to this welfare; if it were, we would have to multiply it by N to obtain the aggregate value.

Using the definitions in equation (4), we can write this discount factor as

$$D^{\text{private}}(t) = e^{-\left(\tilde{\delta}+\omega\right)t} \frac{\tilde{\sigma} - \tilde{\delta}}{\tilde{\sigma} - \alpha - \tilde{\delta}} + \left(\frac{\alpha}{\alpha + \tilde{\delta} - \tilde{\sigma}}\right) e^{-(\tilde{\sigma}-g)t}$$

■

Remark 1 *Write the discount function in equation (6) as*

$$\frac{1}{\sigma - \alpha - \delta} \left(e^{-(\delta+\omega)t} (\sigma - \delta) - \alpha e^{-(\sigma-g)t}\right).$$

Both the numerator and denominator approach 0 as $\sigma \to \alpha + \delta$. Using L'Hopital's rule, we have the discount function at $\sigma = \alpha + \delta$

$$D^{public}(t) = e^{-(\delta+\omega)t} (1 + \alpha t).$$

With this discount function, it is easy to establish that the payoff $\int_0^\infty D^{public}(t)u(t) < \infty$ for bounded $u(t)$.

A.2 Proof of Proposition 2

We first prove Lemma 1, then state and prove a second lemma, and then prove Proposition 2.

Proof. (Lemma **1**) We provide details for the case $\sigma > \delta + \alpha$, where $r_\infty = \delta + \omega$. Define $P(\epsilon)$ as the amount of time it takes the state variable to move from $x^* + \varepsilon$ to x^* using the control rule in equation (18); ε may be either positive or negative, but is small. With this definition, the control rule (18), and equation (14), we have

$$\begin{array}{c} -K(x^* + \varepsilon) = \\ \alpha \left(h^* \int_0^{P(\varepsilon)} e^{-\tau(\sigma-g)} \left(p - c\left(x_{t+\tau}\right)\right) d\tau + \left(p - c\left(x^*\right)\right) f\left(x^*\right) \int_{P(\varepsilon)}^\infty e^{-\tau(\sigma-g)} d\tau\right) \end{array}. \tag{24}$$

The first integral on the right side is the contribution to $-K$ of the flow payoff during the approach to the steady state value x^*; the second integral equals the contribution due to the steady state flow payoff.

We want to show that the left and right derivatives are equal, i.e. $\lim_{\varepsilon \to 0} \frac{dK(x^*+\varepsilon)}{d\varepsilon}$ has the same value regardless of whether ε approaches 0 from above or below. Consider the case where $\varepsilon > 0$, so $h^* = \bar{h}$ over $[0, P)$. Integrating equation (2) we have $-\varepsilon =$

$\int_t^{t+P} dx = \int_0^P \left(f\left(x_{t+\tau}\right) - \bar{h}\right) d\tau$. (The first term is $-\varepsilon$ because here $\varepsilon > 0$, so $x_{t+P} = x^* < x^* + \varepsilon = x_t$.) In this case,

$$\frac{dP}{d\varepsilon} = \frac{-1}{f(x^*) - \bar{h}}. \tag{25}$$

Using equations (24) and (25) we have

$$\lim_{\varepsilon \to 0^+} \frac{dK(x^* + \varepsilon)}{d\varepsilon} = -\alpha\left(p - c\left(x^*\right)\right)\left(\bar{h} - f\left(x^*\right)\right)\frac{1}{f(x^*) - \bar{h}} = -\alpha\left(p - c\left(x^*\right)\right).$$

Now consider the case where $\varepsilon < 0$, so $h^* = 0$ over $[0, P)$. Here, $-\varepsilon = \int_t^{t+P} dx = \int_0^P f\left(x_{t+\tau}\right) d\tau$, and

$$\frac{dP}{d\varepsilon} = \frac{-1}{f(x^*)}. \tag{26}$$

Using equation (24) and (26) we have

$$\lim_{\varepsilon \to 0^-} \frac{dK(x^* + \varepsilon)}{d\varepsilon} = -\alpha\left(p - c\left(x^*\right)\right)\left(0 - f\left(x^*\right)\right)\frac{-1}{f(x^*)} = -\alpha\left(p - c\left(x^*\right)\right).$$

Thus, the left and right derivatives are equal, as shown in the first line of equation (19).

The argument for $\sigma < \delta + \alpha$ parallels the above. In this case, using the control rule (18), and equation (15), we have

$$-K\left(x^* + \varepsilon\right) =$$
$$(\sigma - \delta)\left(h^* \int_0^{P(\varepsilon)} e^{-\tau(\delta+\omega)}\left(p - c\left(x_{t+\tau}\right)\right) d\tau + \left(p - c\left(x^*\right)\right) f\left(x^*\right) \int_{P(\varepsilon)}^{\infty} e^{-\tau(\delta+\omega)} d\tau\right).$$

Equation (25) still applies for $\varepsilon > 0$ and equation (26) for $\varepsilon < 0$. We have

$$\lim_{\varepsilon \to 0^+} \tfrac{dK(x^*+\varepsilon)}{d\varepsilon} =$$
$$-\left(\sigma - \delta\right)\left(p - c\left(x^*\right)\right)\left(\bar{h} - f\left(x^*\right)\right)\left(\tfrac{-1}{f(x^*) - \bar{h}}\right) = -\left(\sigma - \delta\right)\left(p - c\left(x^*\right)\right).$$

A similar argument shows that the left derivative $\lim_{\varepsilon \to 0^-} \frac{dK(x^*+\varepsilon)}{d\varepsilon}$, has the same value, shown in the second line of equation (19).

We note that an argument that parallels the proof of this lemma shows that the value function is differentiable. Therefore, the costate variable, ψ can be written as a continuous function of x, $\psi = \psi(x)$; the costate variable equals $J'(x)$. ■

Statement and proof of a second lemma

Lemma 2 *A trajectory that drives the resource to the point where it is not economically viable and thereafter keeps it at that level is not an equilibrium.*

In a sense, this result is obvious, but we need it in order to prove Proposition 2.

Proof. We use a proof by contradiction. Define T as the date at which the resource reaches $x_{\min}$ in the candidate that contradicts the Lemma. At T the fictitious control problem effectively ends; there is no scrap value, so the continuation payoff at T is 0. In addition, $K(x_{\min}) = 0$ from the definition of $K(S)$ and the fact that the equilibrium flow payoff for $t > T$ is identically 0 in this candidate. In the fictitious control problem, a necessary condition for a program that drives the stock to $x_{\min}$ is that the Hamiltonian vanish at T:

$$\begin{aligned} H(T) &= \left[(p - c(x) - \psi) h - K(x) + \psi f(x)\right]_{|x=x(T)=x_{\min}} \\ &= \left[(p - c(x) - \psi) \bar{h} - K(x) + \psi f(x)\right]_{|x=x(T)=x_{\min}} = 0 \Longrightarrow \\ \psi(T) &\left(f(T) - \bar{h}\right) = 0 \Longrightarrow \psi(T) = 0 \Longrightarrow p - c(x) - \psi(T) = 0. \end{aligned} \tag{27}$$

(An obvious abuse of notation replaces the arguments $x_T = x_{\min}$ by T.) The third line of equation (27) follows from the fact that $K(x_{\min}) = 0$ and Assumption 4, which states $f(x_{\min}) - \bar{h} < 0$. Thus, the switching function is 0 at $x = x_{\min}$. In order for the hypothesized trajectory to be optimal, the switching function must be positive for larger values of x. That is, the switching function must approach 0 from above, as $t \to T$. Consequently, the time derivative of the switching function must be non-positive at $t = T$.

We need to know whether the switching function, $\pi(S) - \psi$, approaches 0 from above or below as $t \to T$. Consider the case where $\sigma > \delta + \alpha$ where $r_\infty = \delta + \omega$. Using equation (14),

$$K'(x) = \frac{(\sigma - g) K + \alpha (p - c(x) h)}{f(x) - h}.$$

Substitute this equation into the costate equation (16) to write the time derivative of the switching function, on the candidate equilibrium, in the neighborhood of T (where $h = \bar{h}$):

$$\frac{d(p-c(x)-\psi)}{dt} =$$

$$-c'(x)\left(f(x) - \bar{h}\right) - \left[(\delta + \omega - f'(x))\psi(x) + c'(x)\bar{h} + \frac{(\sigma-g)K+\alpha(p-c(x)h)}{f(x)-h}\right]$$

Evaluating the right side of this equation at $x = x_{\min}$, the right side simplifies to $-c'(x_{\min}) f(x_{\min}) > 0$. This inequality is our contradiction, because our hypothesis requires that the time derivative of the switching function is non-positive at $x = x_{\min}$. A parallel argument deals with the case where $\sigma < \delta + \alpha$. ■

Proof. (Proposition **2)** We provide the argument for the case of a public good, and then use Proposition 1 to obtain the results for the case of a private good (part iii). The Markov assumption means that at any value of the stock, the equilibrium harvest does not depend on whether the stock has approached this value from above or from below. In a model with a single state variable, pure strategy Markov equilibrium trajectories cannot cycle. In view of the linearity of the problem, harvest takes a boundary value unless x satisfies equation (17). We proceed under the hypothesis that the control rule is a MRAP of the form of equation (18), and we then verify this hypothesis.

For $\infty > \sigma > \delta + \alpha$, equation (10) states that $r_\infty = \delta + \omega$. Using this equality and the first line of equation (19) implies that x^* is the solution to

$$\begin{aligned} \delta + \omega &= f'(x) - \frac{c'(x)f(x) - \alpha\,(p - c\,(x))}{p - c(x)} \Longrightarrow \\ \delta + \omega - \alpha &= \delta - g = f'(x) - \frac{c'(x)f(x)}{p - c(x)}. \end{aligned} \tag{28}$$

For $\sigma < \delta + \alpha$, equation (10) states that $r_\infty = \sigma - g$. Using this value of r_∞ and the second line of equation (19) implies that x^* is the solution to

$$\sigma - g = f'(x) - \frac{c'(x)f(x) - (\sigma - \delta)\,(p - c\,(x))}{p - c(x)},$$

which reproduces equation (28). By Assumption 2, there is a unique solution to this equation.

We now confirm that the control rule must be the MRAP in equation (18) with x^* equal to the solution to equation (28). The hypothesis that the control rule is a MRAP is equivalent to the claim that the graph of the switching function, $(p - c(x) - \psi\,(x))$ is negative for $x < x^*$ and positive for $x > x^*$. A proof by contradiction establishes this claim.

We first show that $(p - c(x) - \psi\,(x))$ is negative for $x < x^*$. The switching function cannot be positive in the neighborhood of $x = x_{\min}$, the largest stock value at which average profits are 0. If the switching function were positive in this neighborhood, then for values of x close to $x_{\min}$ the solution to the fictitious control problem would drive the stock of fish to $x_{\min}$ and then harvest at the rate that keeps it there, $f(x_{\min})$. Lemma 2 establishes that this is not an equilibrium outcome. (This outcome results in a zero flow of profit at the steady state $x_{\min}$. Any deviation involving lower harvest for an interval of time results in positive profits. Therefore, the proposal to drive x to $x_{\min}$ can not solve the fictitious control problem, and hence is not a MPE.) Therefore, if the switching function is positive for any $x < x^*$ it must cross the x axis at some $x < x^*$, so that the switching function is negative in the neighborhood of $x_{\min}$. This multiple crossing violates Assumption 2, which requires a unique solution to equation (3).

A similar argument establishes that We first show that $(p - c(x) - \psi(x))$ is positive for $x > x^*$. In the neighborhood of the carrying capacity (defined as the value of $x > 0$ at which $f(x) = 0$), the switching function is not negative. If it were negative, then for fish stocks sufficiently close to the carrying capacity, it would be a MPE to extract nothing forever. That cannot be an equilibrium, because any deviation gives a higher payoff. Therefore, if for any $x > x^*$ the switching function is negative, it must cross the x axis again from below, at a point where $x > x^*$. That possibility violates Assumption 2, which requires a unique solution to equation (3).

If $\sigma = \delta$ then the original problem involves the constant discount rate $\delta - g$. The result is a standard problem, for which the solution is well-known: follow a MRAP until reaching x^*, the solution to equation (28). The only remaining case is where $\sigma = \infty$, where again we have a standard problem, but here the constant discount rate is $\delta + \omega$. The solution is to follow a MRAP to x^*, the solution to equation (3) with $r = \delta + \omega$.

We obtain part (iii) of the proposition by invoking Proposition 1. This completes the proof. ■

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