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On the Variational Principle

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The variational principle states that if a differentiable functional F attains its minimum at some point $\bar{u}$, then $F'(\bar{u}) = 0$; it has proved a valuable tool for studying partial differential equations. This paper shows that if a differentiable function F has a finite lower bound (although it need not attain it), then, for every $\epsilon > 0$, there exists some point u_ϵ where $\|F'(u_\epsilon)\|^* \leqslant \epsilon$, i.e., its derivative can be made arbitrarily small. Applications are given to Plateau's problem, to partial differential equations, to nonlinear eigenvalues, to geodesics on infinite-dimensional manifolds, and to control theory.

1. A General Result

Let V be a complete metric space, the distance of two points $u, v \in V$ being denoted by $d(u, v)$. Let F: $V \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, not identically $+\infty$. In other words, $+\infty$ is allowed as a value for F, but at some point v_0, $F(v_0)$ is finite.

Suppose now F is bounded from below:

$$\inf F > -\infty. \tag{1.1}$$

As no compacity assumptions are made, there need be no point where this infimum is attained. But of course, for every $\epsilon > 0$, there is some point $u \in V$ such that:

$$\inf F \leqslant F(u) \leqslant \inf F + \epsilon. \tag{1.2}$$

THEOREM 1.1. *Let V be a complete metric space, and F: $V \to \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. For every point $u \in V$ satisfying* (1.2) *and every $\lambda > 0$, there exists some point $v \in V$ such that*

$$F(v) \leqslant F(u), \tag{1.3}$$

$$d(u, v) \leqslant \lambda, \tag{1.4}$$

$$\forall w \neq v, \qquad F(w) > F(v) - (\epsilon/\lambda)\, d(v, w). \tag{1.5}$$

The proof of this theorem is based on a device due to Bishop and Phelps [4]. Brønsted and Rockafellar [6] have used it to obtain subdifferentiability properties for convex functions on Banach spaces, and Browder [7] has applied it to nonconvex subsets of Banach spaces. Let $\alpha > 0$ be given, and define an ordering on $V \times \mathbb{R}$ by

$$(v_1, a_1) \prec (v_2, a_2) \text{ iff: } (a_2 - a_1) + \alpha d(v_1, v_2) \leqslant 0. \tag{1.6}$$

This relation is easily seen to be reflexive, antisymetric and transitive. It is also seen to be continuous, in the sense that, for every $(v_1, a_1) \in V \times \mathbb{R}$, the set $\{(v, a) \mid (v, a) \succ (v_1, a_1)\}$ is closed in $V \times \mathbb{R}$. We proceed to show that every closed subset of $V \times \mathbb{R}$ has a maximal element, provided it is "bounded from below."

LEMMA 1.2. *Let S be a closed subset of $V \times R$ such that:*

$$\exists m \in \mathbb{R}: (v, a) \in S \Rightarrow a \geqslant m. \tag{1.7}$$

Then, for every $(v_1, a_1) \in S$, there exists for the ordering $\prec$ an element $(\bar{v}, \bar{a}) \in S$ which is maximal and greater than (v_1, a_1).

Proof. Let us define inductively a sequence $(v_n, a_n) \in S$, $n \in N$, starting with (v_1, a_1). Suppose (v_n, a_n) is known. Denote

$$S_n = \{(v, a) \in S \mid (v, a) \succ (v_n, a_n)\} \tag{1.8}$$

$$m_n = \inf\{a \in \mathbb{R} \mid (v, a) \in S_n\}. \tag{1.9}$$

Clearly, $m_n \geqslant m$. Define now (v_{n+1}, a_{n+1}) to be any point of S_n such that

$$a_n - a_{n+1} \geqslant \tfrac{1}{2}(a_n - m_n). \tag{1.10}$$

All the sets S_n are closed nonempty, and $S_{n+1} \subset S_n$ for every n. Moreover, we get from (1.10)

$$| a_{n+1} - m_{n+1} | \leqslant \tfrac{1}{2} | a_n - m_n | \leqslant (1/2^n) | a_1 - m | . \tag{1.11}$$

Hence, for every $(v, a) \in S_{n+1}$, we get, using (1.8):

$$| a_{n+1} - a | \leqslant (1/2^n) | a_1 - m | \tag{1.12}$$

$$d(v_{n+1}, v) \leqslant (1/2^n)(1/\alpha) | a_1 - m | . \tag{1.13}$$

Which proves that the diameter of S_n goes to zero as $n \to \infty$. As $V \times R$ is metric complete, the sets S_n have one point $(\bar{v}, \bar{a})$ in common:

$$\{(\bar{v}, \bar{a})\} = \bigcap_{n=1}^{\infty} S_n . \tag{1.14}$$

By definition, $(\bar{v}, \bar{a}) \succ (v_n, a_n)$ for every n, in particular for $n = 1$. Suppose now there exists some $(\tilde{v}, \tilde{a}) \in S$ greater than $(\bar{v}, \bar{a})$. By transitivity, one gets $(\tilde{v}, \tilde{a}) \succ (v_n, a_n)$ for every n, i.e., $(\tilde{v}, \tilde{a}) \in \bigcap_{n=1}^{\infty} S_n$, hence $(\tilde{v}, \tilde{a}) = (\bar{v}, \bar{a})$. This proves $(\bar{v}. \bar{a})$ is indeed maximal.[1] ∎

We now proceed easily to prove Theorem 1.1. Take S to be the epigraph of F

$$S = \{(v, a) \mid v \in V, a \geqslant F(v)\}. \tag{1.15}$$

It is a closed subset of $V \times \mathbb{R}$, as F is l.s.c. Take $\alpha = \epsilon/\lambda$, and (v_1, a_1) to be $(u, F(u))$. Apply Lemma 1.2 to obtain a maximal element (v, a) in S satisfying:

$$(v, a) \succ (u, F(u)). \tag{1.16}$$

As $(v, a) \in S$, we have also $(v, F(v)) \succ (v, a)$. Since (v, a) is maximal, $a = F(v)$. The maximality can be written

$$(w, b) \in S \Rightarrow (b - F(v)) + (\epsilon/\lambda)\, d(w, v) > 0, \tag{1.17}$$

unless $w = v$ and $b = F(v)$.

Taking $b = F(w)$ yields (1.5). Now, going back to (1.16), we get

$$(F(v) - F(u)) + (\epsilon/\lambda)\, d(v, u) \leqslant 0. \tag{1.18}$$

Hence, of course, $F(v) \leqslant F(u)$. Thanks to (1.2), we must have $F(v) \geqslant F(u) - \epsilon$. Writing it into (1.18), we get $(\epsilon/\lambda)\, d(v, u) \leqslant \epsilon$, which is (1.4) and ends the proof. We shall now apply Theorem 1.1 in different settings.

2. Gateaux-Differentiable Functions on Banach Spaces

From now on V will be a Banach space, V^* its topological dual. The canonical bilinear form on $V \times V^*$ will be denoted by brackets $\langle \cdot, \cdot \rangle$, the norm of V by $\| \cdot \|$, the dual norm of V^* by $\| \cdot \|^*$. Recall that a function $F\colon V \to \mathbb{R} \cup \{+\infty\}$ is called *Gâteaux-differentiable* (respectively, *Fréchet-differentiable*) if, at every point u_0 with $F(u_0) < +\infty$, there exists a continuous linear functional $F'(u_0) \in V^*$ such that, for every $v \in V$:

$$(d/dt) F(u_0 + tv)|_{t=0} = \langle F'(u_0), v \rangle \tag{2.1}$$

(respectively,

$$F(u_0 + v) = F(u_0) + \langle F'(u_0), v \rangle + \epsilon(v) \| v \|,$$

where $\epsilon(v) \to 0$ in V as $\| v \| \to 0$).

[1] The author is indebted to J. M. Lasry for a helpful comment enabling him to get rid of Zorn's Lemma in this proof.

Clearly, the Fréchet-differentiability of F implies that F is Gateaux-differentiable; moreover, the domain of F, i.e., the subset of V where it is finite, must be open. Here is an important case, where the converse is true.

DEFINITION 2.1. Let F be a Gateaux-differentiable function with open domain, such that $u \mapsto F'(u)$ is a continuous mapping from the domain of F into V^*. Then F is also continuously Frechet-differentiable, and is called a C^1 function.

Proof. Under our hypothesis, we have to prove that the Gateaux-derivative $F'(u)$ is in fact a Fréchet-derivative. Take any point u_0 in the domain of F and some $\eta > 0$ such that the ball of radius η around u_0 is contained in the domain of F. For every $v \in V$ with $\| v \| \leqslant \eta$, there exists some $\theta \in [0, 1]$ such that

$$F(u_0 + v) - F(u_0) = (d/dt) F(u_0 + tv)|_{t=\theta} \,.$$

Using the Gateaux-differentiability

$$\begin{aligned} F(u_0 + v) - F(u_0) &= \langle F'(u_0 + \theta v), v\rangle \\ &= \langle F'(u_0), v\rangle + \langle F'(u_0 + \theta v) - F'(u_0), v\rangle. \end{aligned} \tag{2.2}$$

For every $\epsilon > 0$, we can take $\eta > 0$ small enough so that $\| u - u_0 \| \leqslant \eta$ implies $\| F'(u_0) - F'(u)\|^* \leqslant \epsilon$. Taking $u_0 + \theta v$ as u in formula (2.2), we get

$$| F(u_0 + v) - F(u_0) - \langle F'(u_0), v\rangle| \leqslant \epsilon \, \| v \|$$

which indeed proves Fréchet-differentiability. ∎

In this setting, Theorem 1.1 becomes

THEOREM 2.2. *Let V be a Banach space, and $F\colon V \to \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, Gâteaux-differentiable and such that*

$$-\infty < \inf F < +\infty.$$

Then, for every $\epsilon > 0$, every $u \in V$ such that $F(u) \leqslant \inf F + \epsilon$, every $\lambda > 0$, there exists $v \in V$ such that:

$$F(v) \leqslant F(u), \tag{2.3}$$

$$\| v - u \| < \lambda, \tag{2.4}$$

$$\| F'(v)\|^* \leqslant \epsilon/\lambda. \tag{2.5}$$

Proof. It is a straightforward application of Theorem 1.1. Inequality (1.5) gives us, for every $w \in V$ and every $t \geqslant 0$

$$F(v + tw) \geqslant F(v) - (\epsilon/\lambda)\, t \,\| w \|, \tag{2.6}$$

$$(F(v + tw) - F(v))/t \geqslant -(\epsilon/\lambda) \| w \|. \tag{2.7}$$

Letting $t \to 0$, we obtain

$$(d/dt) F(v + tw)|_{t=0} \geqslant -(\epsilon/\lambda) \| w \|. \tag{2.8}$$

Hence, through (2.1)

$$\langle F'(v), w\rangle \geqslant -(\epsilon/\lambda) \| w \|. \tag{2.9}$$

The inequality (2.9), holding for every $w \in V$, means that

$$\| F'(v)\|^* \leqslant \epsilon/\lambda.$$ ∎

COROLLARY 2.3. *For every $\epsilon > 0$, there exists some point v_ϵ such that*

$$F(v_\epsilon) - \inf F \leqslant \epsilon^2 \tag{2.10}$$

$$\| F'(v_\epsilon)\|^* \leqslant \epsilon. \tag{2.11}$$

Proof. Just take ϵ^2 instead of ϵ and ϵ instead of λ in the preceding theorem. ∎

We can view the preceding corollary as telling us that the equation $F'(v) = 0$, although it need have no solution, alway has "approximate solutions," i.e., there exists a sequence u_n such that $\| F'(u_n)\|^* \to 0$ as $n \to \infty$. The cluster points of such sequences have been studied elsewhere [13], [14]. Let us just draw some easy consequences of Corollary 2.3.

COROLLARY 2.4. *Suppose further that there exist constants $k > 0$ and c such that:*

$$\forall v \in V, \qquad F(v) \geqslant k \| v \| + c. \tag{2.12}$$

Then, the range $F'(V)$ is dense in kB^*, where B^* is the closed unit ball of V^*.

Proof. Take $u^* \in V^*$ with $\| u^* \| < k$. It suffices to prove that, for every $\epsilon > 0$, there exists $u_\epsilon \in V$ such that $\| F'(u_\epsilon) - u^* \|^* \leqslant \epsilon$.

Consider the function G on V defined by

$$G(v) = F(v) - \langle v, u^*\rangle. \tag{2.13}$$

Clearly, G is Gâteaux-differentiable, with $G'(v) = F'(v) - u^*$, and l.s.c. Moreover

$$\begin{aligned}\inf G &= \inf_{v\in V}\{F(v) - \langle v, u^*\rangle\}\\ &\geqslant \inf_{v\in V}\{k\,\|v\| - \langle v, u^*\rangle\} + c\\ &\geqslant \inf_{v\in V}\{(k - \|u^*\|^*)\,\|v\|\} + c\\ &\geqslant c > -\infty.\end{aligned} \tag{2.14}$$

Hence G satisfies all assumptions of Corollary 2.3, and there must exist some point $u_\epsilon \in V$ such that $\| G'(u_\epsilon)\|^* \leqslant \epsilon$. This means

$$\| F'(u_\epsilon) - u^* \|^* \leqslant \epsilon. \quad \blacksquare \tag{2.15}$$

COROLLARY 2.5. *Suppose further that there exists some continuous function*

$$\Phi\colon \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$$

such that $\Phi(t)/t \to \infty$ *as* $t \to \infty$, *and*

$$\forall v \in V, \qquad F(v) \geqslant \Phi(\| v \|). \tag{2.16}$$

Then, the range $F'(V)$ *is dense in* V^*.

Proof. Indeed, for every $k > 0$ there exists some $c \in \mathbb{R}$ such that F satisfies (2.12). Hence $F'(V)$ is dense in every closed ball of V^*. $\blacksquare$

3. Optimization Problems with Regular Constraints

We now take V to be a Banach space, $F\colon V \to \mathbb{R}$ a Fréchet-differentiable function, $G_i\colon V \to \mathbb{R}$, $1 \leqslant i \leqslant m$, m continuously Fréchet-differentiable functions, i.e., C^1-functions. We single out the first p of the G_i's, and we consider the constrained optimization problem

$$\begin{aligned}&\inf F(v)\\ &G_i(v) = 0 \qquad 1 \leqslant i \leqslant p\\ &G_i(v) \geqslant 0 \qquad p+1 \leqslant i \leqslant m.\end{aligned} \tag{3.1}$$

We will denote by $\mathscr{C}$ the *feasible set*

$$\mathscr{C} = \{v \in V \mid G_i(v) = 0\ \forall i \in \{1, p\},\ G_i(v) \geqslant 0\ \forall i \in \{p+1, m\}\}, \tag{3.2}$$

and by $I(v)$ the set of *saturated constraints* at a feasible point $v \in \mathscr{C}$

$$i \in I(v) \Leftrightarrow G_i(v) = 0. \tag{3.3}$$

We can now state our *regularity assumption*, which is of quite a standard type:

$$\forall v \in \mathscr{C}, \textit{ the } \{G_i'(v) \mid i \in I(v)\} \textit{ are linearly independent.} \tag{3.4}$$

It is clear that problem (3.1) is highly nonlinear, and as such can dream of no solution in a Banach space. Nevertheless, we can find points which are "almost" optimal and which "almost" satisfy the necessary conditions for optimality.

THEOREM 3.1. *Suppose F is Fréchet-differentiable, and the G_i's are C^1-functions satisfying regularity assumption* (3.4). *Suppose moreover F is bounded below on the feasible set*

$$\inf_{v \in \mathscr{C}} F(v) > -\infty. \tag{3.5}$$

Then, for every $\epsilon > 0$, there exists some point v_ϵ such that

$$v_\epsilon \in \mathscr{C} \qquad \textit{and} \qquad F(v_\epsilon) \leqslant \inf_{v \in \mathscr{C}} F(v) + \epsilon^2, \tag{3.6}$$

there exists real numbers $\lambda_1, \dots, \lambda_m$ such that:

$$\left.\begin{aligned} &\lambda_i \geqslant 0 \qquad \forall_i \in \{p+1, m\}, \\ &\lambda_i = 0 \qquad \textit{if} \qquad G_i(v) \neq 0, \\ &\left\| F'(v_\epsilon) - \sum_{i=1}^m \lambda_i G_i'(v_\epsilon) \right\|^* \leqslant \epsilon. \end{aligned}\right\} \tag{3.7}$$

Define a function $\bar{F}$: $V \to \mathbb{R} \cup \{+ y\}$ *by*

$$\begin{aligned} \bar{F}(v) &= +\infty \qquad \text{if} \qquad v \notin \mathscr{C} \\ \bar{F}(v) &= F(v) \qquad \text{if} \qquad v \in \mathscr{C}. \end{aligned} \tag{3.8}$$

It is l.s.c. and bounded from below. Applying Theorem 1.1 we get a point v_ϵ such that

$$\bar{F}(v_\epsilon) \leqslant \inf \bar{F} + \epsilon^2 \tag{3.9}$$

$$\forall w \neq v, \qquad \bar{F}(w) > \bar{F}(v) - \epsilon \| v - w \| . \tag{3.10}$$

Hence, using (3.8),

$$v_\epsilon \in \mathscr{C} \text{ and } F(v_\epsilon) \leqslant \inf_{v \in \mathscr{C}} F(v) + \epsilon^2.$$

$$\forall w \in \mathscr{C}, \qquad F(w) \geqslant F(v) - \epsilon \| u - v \| . \tag{3.12}$$

The rest of the proof proceeds by two steps.

LEMMA 3.2. *Let h be a vector in V such that*

$$\langle G_i'(v_\epsilon), h \rangle = 0, \qquad \forall i \in \{1,..., p\} \tag{3.13}$$

$$\langle G_i'(v_\epsilon), h \rangle \geqslant 0, \qquad \forall i \in \{p+1,..., m\} \cap I(v_\epsilon). \tag{3.14}$$

Then

$$\langle F'(v), h \rangle \geqslant -\epsilon \| h \| . \tag{3.15}$$

Proof. Let h be a vector in V satisfying (3.13) and (3.14). By a standard argument using assumption (3.4) and the implicit function theorem, there exists some C^1-curve u: $[0, \tau] \to \mathscr{C}$ such that

$$u(0) = v_\epsilon \qquad \text{and} \qquad (du/dt)(0) = h. \tag{3.16}$$

From (3.12) we get

$$\forall t \in [0, \tau], \qquad F(u(t)) - F(u(0)) \geqslant - \epsilon \| u(t) - u(0) \| . \tag{3.17}$$

Dividing by t and letting t go to zero, we obtain, using (3.16) again

$$\langle F'(v_\epsilon), h \rangle \geqslant -\epsilon \| h \| . \qquad \blacksquare \tag{3.18}$$

LEMMA 3.3. *Let u_i^*, $1 \leqslant i \leqslant p$, v_j^*, $1 \leqslant j \leqslant q$, and w^* be linear functionals V such that:*

$$\langle u_i^*, h \rangle = 0 \qquad \text{for } 1 \leqslant i \leqslant p,$$

$$\langle v_j^*, h \rangle \geqslant 0 \qquad \text{for } 1 \leqslant j \leqslant q,$$

imply

$$\langle w^*, h \rangle \geqslant -\epsilon \| h \| . \tag{3.19}$$

Then there exist p real numbers λ_i, $1 \leqslant i \leqslant p$, and q nonnegative numbers, μ_j, $1 \leqslant j \leqslant q$, such that

$$\left\| w^* - \sum_{i=1}^{p} \lambda_i u_i^* - \sum_{j=1}^{q} \mu_j v_j^* \right\|^* < \epsilon. \tag{3.20}$$

Proof. It is a variant of the celebrated Farkas–Minkowski lemma. Consider in V^* the convex cone Γ

$$\Gamma = \left\{ \sum_{i=1}^{p} \lambda_i u_i^* + \sum_{j=1}^{q} \mu_j v_j^* \;\middle|\; \lambda_i \in \mathbb{R}, \mu_j \geqslant 0 \right\} \tag{3.21}$$

and the convex set Ω:

$$\Omega = \Gamma + \epsilon B^*. \tag{3.22}$$

Where B^* denote the unit ball of V^*. It is well-known that Γ is closed and B^* compact in the weak-* topology $\sigma(V^*, V)$. Hence Ω is closed in that topology.

Suppose now (3.20) is not true. This means that the set Ω does not contain w^*. As it is convex closed in the weak-* topology, we may use the Hahn–Banach separation theorem to get some vector $h \in V$ and some number $\alpha \in \mathbb{R}$ such that

$$\langle w^*, h \rangle < \alpha \tag{3.23}$$

$$\langle x^*, h \rangle \geqslant \alpha, \qquad \forall x^* \in \Omega. \tag{3.24}$$

Write (3.24) in another way

$$\langle y^*, h \rangle + \epsilon \langle z^*, h \rangle \geqslant \alpha, \qquad \forall y^* \in \Gamma, \qquad \forall z^* \in B^*, \tag{3.25}$$

$$\langle y^*, h \rangle \geqslant \alpha + \epsilon \sup_{z^* \in B^*} \langle z^*, -h \rangle, \qquad \forall y^* \in \Gamma. \tag{3.26}$$

This supremum is known to be $\| h \|$. Hence,

$$\langle y^*, h \rangle \geqslant \alpha + \epsilon \| h \|, \qquad \forall y^* \in \Gamma. \tag{3.27}$$

We now use the fact that Γ is made of lines or half-lines. Take $y^* = t u_i^*$, where $1 \leqslant i \leqslant p$ and t is any real number; we get

$$t \langle u_i^*, h \rangle \geqslant \alpha + \epsilon \| h \|, \qquad \forall t \in \mathbb{R}. \tag{3.28}$$

Hence,

$$\langle u_i^*, h \rangle = 0, \qquad 1 \leqslant i \leqslant p. \tag{3.29}$$

Now take $y^* = t v_j^*$, where $1 \leqslant j \leqslant q$ and t is any nonnegative number; we get

$$t \langle v_j^*, h \rangle \geqslant \alpha + \epsilon \| h \|, \qquad \forall t \geqslant 0. \tag{3.30}$$

Hence,

$$\langle v_j, h \rangle \geqslant 0, \qquad 1 \leqslant j \leqslant q. \tag{3.31}$$

At last, take $y^* = 0$. We get from (3.27)

$$\alpha \leqslant -\epsilon \| h \| . \tag{3.32}$$

Formula (3.23) then yields

$$\langle w^*, h \rangle < -\epsilon \| h \| . \tag{3.33}$$

Put now together formulas (3.29), (3.31), and (3.33). They show the vector h to be a counter-example to the assumptions of the lemma. In other words, if the conclusion (3.20) is false, then so is the hypothesis. This proves the lemma. ∎

Theorem 3.1 now follows by putting together Lemmata 3.2 and 3.3. As a simple corollary (case of one constraint) we get a nonlinear eigenvalue theorem.

COROLLARY 3.4. *Let F be a Fréchet-differentiable function and G a C^1-function such that*

$$G(v) = 0 \Rightarrow G'(v) \neq 0. \tag{3.34}$$

Suppose moreover that

$$\exists m \in \mathbb{R}\colon G(v) = 0 \Rightarrow F(v) \geqslant m. \tag{3.35}$$

Then, for every $\epsilon > 0$, there exist some point v_ϵ and some $\lambda_\epsilon \in \mathbb{R}$ such that:

$$G(v_\epsilon) = 0 \tag{3.36}$$

$$\| F'(v_\epsilon) - \lambda_\epsilon G'(v_\epsilon) \|^* \leqslant \epsilon. \tag{3.37}$$

If V is a Hilbert space, we can identify V and V^* in the usual way, and (3.37) then means that the distance of the gradient of F at v_ϵ to the one-dimensional subspace generated by the gradient of G at v_ϵ is not greater than ϵ.

4. EXAMPLES

A. *Minimal Hypersurfaces (Plateau's Problem)*

Let Ω be an open bounded subset of $\mathbb{R}^n$, with regular boundary. As usual we denote by $W^{1,1}(\Omega)$ the Sobolev space of L^1-functions whose first derivatives are also L^1, and by $W_0^{1,1}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in this space. We interpret $W_0^{1,1}(\Omega)$ as the set of $W^{1,1}$-functions vanishing on the boundary, and we state a weak form of Plateau's problem

$$\begin{gathered}\inf \int_\Omega (1 + |\operatorname{grad} v(x)|^2\, dx)^{1/2} \\ v - v_0 \in W_0^{1,1}(\Omega).\end{gathered} \tag{4.1}$$

It is well known that this problem has no solution in general, except if Ω is convex, which we do not assume. An explanation is to be found in the fact that $W^{1,1}(\Omega)$ is not reflexive, and hence its unit ball is not weakly compact. We now proceed to prove that we can perturb the problem as little as we want to get an optimal solution.

Problem (4.1) can be stated differently:

$$\begin{gathered}\inf \int_\Omega (1 + |\operatorname{grad} u(x) + \operatorname{grad} v_0(x)|^2\, dx)^{1/2} \\ u \in W_0^{1,2}(\Omega).\end{gathered} \tag{4.2}$$

Denote by F the function to be minimized on $W_0^{1,1}(\Omega)$:

$$F(u) = \int_\Omega (1 + |\operatorname{grad} u(x) + \operatorname{grad} v_0(x)|^2\, dx)^{1/2}. \tag{4.3}$$

It is well known that this function is convex, continuous, Gâteaux-differentiable with derivative

$$F'(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\operatorname{grad} u + \operatorname{grad} v_0}{(1 + |\operatorname{grad} u + \operatorname{grad} v_0|^2)^{1/2}}, \tag{4.4}$$

all derivatives on the right-hand side to be taken in the sense of distributions. Of course $F'(u) \in W^{-1,\infty}(\Omega)$, which is both the dual of $W_0^{1,1}(\Omega)$ and the Sobolev space of distributions which can be obtained from L^∞-functions by first order differentiation.

Moreover, for every $u \in W_0^{1,1}(\Omega)$, we have

$$\begin{gathered}\int_\Omega (1 + |\operatorname{grad} u(x) + \operatorname{grad} v_0(x)|^2\, dx)^{1/2} \\ \geqslant \int_\Omega |\operatorname{grad} u(x)|\, dx - \int_\Omega |\operatorname{grad} v_0(x)|\, dx.\end{gathered} \tag{4.5}$$

The Poincaré inequality yields some $k > 0$ such that

$$\forall u \in W_0^{1,1}(\Omega), \qquad \int_\Omega |\operatorname{grad} u(x)|\, dx \geqslant k \| u \|_{W^{1,1}}. \tag{4.6}$$

Hence, (4.5) becomes

$$F(u) \geqslant k \| u \| - \text{const.} \tag{4.7}$$

We may now apply Corollary 2.3. There exists in kB^*, where B^* is the unit ball of $W^{-1,\infty}(\Omega)$, a dense subset S such that, for every $T \in S$, the equation $F'(u) = T$ has some solution u in $W^{1,1}(\Omega)$. For every $T \in W^{1,1}(\Omega)$, define the perturbed function F_T:

$$F_T(u) = F(u) - \langle T, u \rangle. \tag{4.8}$$

Then, for every $T \in S$, there is some point u_T where

$$F_T'(u_T) = 0. \tag{4.9}$$

But F_T is convex and even strictly convex. Hence (4.9) is a necessary and sufficient condition for optimality: the point u_T is the unique minimum of F_T on $W_0^{1,1}$. Let us state our results together.

Proposition 4.1. *There exists in $W_0^{-1,1}(\Omega)$ a neighbourhood of the origin, and a dense subset $\mathscr{T}$ in this neighbourhood, such that, for every $T \in \mathscr{T}$, the perturbed minimal hypersurface equation*

$$\begin{gathered} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\operatorname{grad} v}{(1 + |\operatorname{grad} v|^2)^{1/2}} = T \\ v - v_0 \in W_0^{1,1}(\Omega) \end{gathered} \tag{4.10}$$

and the perturbed Plateau's problem

$$\begin{gathered} \inf \int_\Omega (1 + |\operatorname{grad} v|^2 \, dx)^{1/2} - \langle T, v \rangle \\ v \in v_0 + W_0^{1,1}(\Omega) \end{gathered} \tag{4.11}$$

both have a unique solution.

B. *Partial Differential Equations*

Let Ω be a bounded open subset of $\mathbb{R}^n$, and $p \in]1, +\infty[$ a given constant. Denote by $W_0^{1,p}(\Omega)$ the corresponding Sobolev space (i.e., the Banach space of L^p-functions with L^p-first derivatives and zero trace on the boundary).

Consider a borelian function $f: \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that, for almost every $x \in \Omega$

$$f(x, \xi) \geqslant 0, \qquad \forall \xi \in \mathbb{R}^n, \tag{4.12}$$

$$\xi \mapsto f(x, \xi) \text{ is a } C^1\text{-function}, \tag{4.13}$$

$$|f_\xi'(x, \xi)| \leqslant a + b\,|\,\xi\,|^{p-1}, \qquad \forall \xi \in \mathbb{R}^n, \tag{4.14}$$

where a and b are given constants.

Make one more easy assumption

$$\exists v_0 \in W_0^{1,p}(\Omega): \int f(x, \operatorname{grad} v_0(x))\, dx < +\infty. \tag{4.15}$$

We now define a nontrivial function F on $W_0^{1,p}(\Omega)$ by

$$\forall v \in W_0^{1,p}(\Omega), \qquad F(v) = \int_\Omega f(x, \operatorname{grad} v(x))\, dx. \tag{4.16}$$

LEMMA 4.2. *F is a C^1-function on $W_0^{1,p}(\Omega)$, finite everywhere.*

Proof. We first prove that F is Gâteaux-differentiable. Take any point u_0 where $F(u_0)$ is finite. For every $v \in W_0^{1,p}(\Omega)$, consider the function

$$t \mapsto F(u_0 + tv) = \int_\Omega f(x, \operatorname{grad} u_0(x) + t \operatorname{grad} v(x))\, dx. \tag{4.17}$$

For $0 \leqslant t \leqslant 1$, formula (4.14) yields

$$\begin{aligned} &|(\partial/\partial t) f(x, \operatorname{grad} u_0(x) + t \operatorname{grad} v(x))| \\ &\quad = \langle \operatorname{grad} v(x), f_\xi'(x, \operatorname{grad} u_0(x) + t \operatorname{grad} v(x))\rangle \\ &\quad \leqslant |\operatorname{grad} v(x)|\,(a + b\,|\operatorname{grad} u_0(x) + t \operatorname{grad} v(x)|^{p-1}). \end{aligned} \tag{4.18}$$

Suppose first $1 < p \leqslant 2$. Then $(\rho + \sigma)^{p-1} \leqslant \rho^{p-1} + \sigma^{p-1}$ for every non-negative real numbers ρ and σ, and (4.18) yields

$$\begin{aligned} &|(\partial/\partial t) f(x, \operatorname{grad} u_0(x) + t \operatorname{grad} v(x))| \\ &\quad \leqslant |\operatorname{grad} v(x)|\,(a + b\,|\operatorname{grad} u_0(x)|^{p-1} + b\,|\operatorname{grad} v(x)|^{p-1}). \end{aligned} \tag{4.19}$$

Suppose now $p \geqslant 2$. Then $(\rho + \sigma)^{p-1} \leqslant 2^{p-2}(\rho^{p-1} + \sigma^{p-1})$ for every non-negative real numbers ρ and σ, and (4.18) yields

$$\begin{aligned} &|(\partial/\partial t) f(x, \operatorname{grad} u_0(x) + t \operatorname{grad} v(x))| \\ &\quad \leqslant |\operatorname{grad} v(x)|\,(a + 2^{p-2}b\,|\operatorname{grad} u_0(x)|^{p-1} + 2^{p-2}b\,|\operatorname{grad} v(x)|^{p-1}). \end{aligned} \tag{4.20}$$

In both cases, we get an estimate

$$|(\partial/\partial t) f(x, \operatorname{grad} u_0(x) + t \operatorname{grad} v(x))| \leqslant g(x) \tag{4.21}$$

where $g \in L^1(\Omega)$.

It is well known that condition (4.21) enables us to differentiate (4.17), to get

$$F(u_0 + tv) < +\infty, \qquad \forall t \in [0, 1], \tag{4.22}$$

$$\frac{d}{dt} F(u_0 + tv)\Big|_{t=0} = \int_\Omega \langle \operatorname{grad} v(x), f'(x, \operatorname{grad} u_0(x))\rangle \, dx. \tag{4.23}$$

For any $u \in W_0^{1,p}(\Omega)$, taking $u_0 = v_0$ and $tv = u - u_0$ in (4.22) yields $F(u) < + y$; hence F is finite everywhere. The right-hand side of (4.23) shows the distribution:

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial \xi_i}(x, \operatorname{grad} u_0(x))\right] \in W^{-1,q}(\Omega). \tag{4.24}$$

$(1/p + 1/q = 1)$ operating on $v \in W_0^{1,p}(\Omega)$. Hence F is Gâteaux-differentiable, with derivative

$$F'(u_0) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\frac{\partial f}{\partial \xi_i}(x, \operatorname{grad} u_0(x))\right]. \tag{4.25}$$

It remains to prove the continuity of the mapping $u \to F'(u)$. A theorem of Krasnoselski [17, Chapt. I, Theorem 2.1] states that, under hypothesis (4.14), the mapping

$$u \mapsto (\partial f/\partial \xi_i)(\cdot, \operatorname{grad} u(\cdot)) \tag{4.26}$$

is continuous from $L^p(\Omega)$ into $L^q(\Omega)$ $(1/p + 1/q = 1)$. Moreover, the mapping

$$h \mapsto \partial h/\partial x_i \tag{4.27}$$

is continuous from $L^q(\Omega)$ into $W^{-1,q}(\Omega)$. Hence (4.25), which arises through combining (4.26) and (4.27) and summing over i, must be continuous from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$. ∎

We now have a straightforward application of Corollary 2.3. We can also apply Corollary 3.4, defining G by

$$\forall v \in W_0^{1,p}(\Omega), \qquad G(v) = \frac{1}{p}\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial v(x)}{\partial x_i}\right|^p dx - \alpha, \tag{4.28}$$

where α is a positive constant. Indeed, it follows from Lemma 4.2. that G is a C^1-function with derivative

$$G'(v) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i}\right), \tag{4.29}$$

$$\forall w \in W_0^{1,p}(\Omega), \qquad \langle G'(v), w\rangle = \sum_{i=1}^{n} \int_{\Omega} \left|\frac{\partial v}{\partial x_i}\right|^{p-2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i}\, dx. \tag{4.30}$$

Taking $w = v$ in (4.30), we see that $G'(v) = 0$ implies all first derivatives $\partial v/\partial x_i$ are zero in $L^p(\Omega)$, and since $v \in W_0^{1,p}(\Omega)$ (is null on the boundary), v must be zero in $W_0^{1,p}(\Omega)$, hence $G(v) = -\alpha \neq 0$. This proves Assumption (3.34) and shows Corollary 3.4 is applicable. We get

PROPOSITION 4.3. *Let there be given* $p \in]1, +\infty[$ *and* $f: \Omega \times \mathbb{R}^n \to \mathbb{R}$ *a borelian function satisfying Assumption* (4.12), (4.13), (4.14). *Then*

(a) *for every* $\epsilon > 0$, *there exists some function* $u_\epsilon \in W_0^{1,p}(\Omega)$ *such that*

$$\left\| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial \xi_i}(\cdot, \operatorname{grad} u_\epsilon(\cdot)) \right\|_{W^{-1,q}} \leqslant \epsilon. \tag{4.31}$$

(b) *for every* $\epsilon > 0$ *and* $\alpha > 0$, *there exist some real number* λ *and some function* $v_\epsilon \in W_0^{1,p}(\Omega)$ *such that:*

$$\left\| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial \xi_i}(\cdot, \operatorname{grad} v_\epsilon(\cdot)) - \lambda \sum_{i=1}^{n} \left|\frac{\partial v_\epsilon}{\partial x_i}\right|^{p-2} \frac{\partial v_\epsilon}{\partial x_i} \right\|_{W^{-1,q}} \leqslant \epsilon, \tag{4.32}$$

$$\int_{\Omega} \sum_{i=1}^{n} \left|\frac{\partial v_\epsilon}{\partial x_i}\right|^{p} dx = \alpha. \tag{4.33}$$

5. C^1 FUNCTIONS ON COMPLETE RIEMANIANN MANIFOLDS

Let M be a complete riemannian manifold. The finite-dimensional case is common knowledge, and we refer the reader to Lang [18], Eells [12], Ebin [9] for treatment of the infinite-dimensional case. Let us just review the essential features. M is a smooth (C^∞) manifold modelled on some Hilbert space H, and for every $p \in M$, we are given on the tangent space $TM_p \simeq H$ a positive definite symmetric bilinear form $\langle\cdot, \cdot\rangle_p$, depending smoothly on p, and such that $\|\cdot\|_p$ is equivalent to the norm of H. We shall denote by ∇ the Levi–Civita connection on M, i.e., the unique bilinear mapping

$$\nabla: C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM),$$
$$(X, Y) \to \nabla_X Y,$$

such that

$$\nabla_{fX} gY = f[(X \cdot g)\, Y + g\nabla_X Y],$$
$$Z \cdot \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle,$$
$$\nabla_X Y - \nabla_Y X = [X, Y],$$

for any vector fields X, Y, Z in $C^\infty(TM)$ and any real functions on M, f, g in $C^\infty(M, \mathbb{R})$.

Recall that the riemannian structure defines on M a metric d which is compatible with its manifold topology; namely, given any two points p, q in the same component of M, define their distance as

$$d(p, q) = \inf \left\{ \int_0^1 \| \dot{c}(t) \|\, dt \mid c \in C^\infty([0, 1]; M), c(0) = p, c(1) = q \right\}.$$

The manifold M is assumed to be complete for this metric. This implies in particular that any geodesic, i.e., any C^∞ path c such that $\nabla_{\dot{c}}\dot{c} = 0$, can be extended indefinitely, and is in fact a C^∞ mapping from $[0, +\infty[$ into M. Hence, for every point $p \in M$, we can define the exponential mapping $\exp_p :$ $TM_p \to M$ by $\exp_p x = c(1)$, where c is the unique geodesic such that $c(0) = p$ and $\dot{c}(0) = x$. The Hopf–Rinow theorem states that, if M is finite-dimensional, any two points p and q of M can be joined by a geodesic of minimum length. This still holds in certain cases where M is a Sobolev manifold of fibre bundle sections (see [8] and [26]). But it is easily seen not to hold any more when M is an infinite-dimensional ellipsoid in Hilbert space and p and q are axis points (see [16] and [19]). The related question, whether the exponential mapping is surjective in general, is still unanswered.

We now state the main result of this section, an easy consequence of Theorem 1.1, before applying it in the next section to manifolds of fiber bundle sections.

Proposition 5.1. *Let F be a C^1 function on a complete riemannian manifold M. If F is bounded from below, then, for every $\epsilon \geqslant 0$ there exists some point $p_\epsilon \in M$ such that*

$$F(p_\epsilon) \leqslant \inf F + \epsilon^2 \tag{5.1}$$

$$\| \operatorname{grad} F(p_\epsilon) \|_{p_\epsilon} \leqslant \epsilon. \tag{5.2}$$

Proof. Using Theorem 1.1, we see there exists a point $p_\epsilon \in M$ satisfying (5.1) and

$$\forall p \in M, \qquad F(p_\epsilon) \geqslant F(p) - \epsilon d(p, p_\epsilon). \tag{5.3}$$

Recall the exponential mapping $\exp_{p_\epsilon}$ induces a C^∞-diffeomorphism from some open subset $\mathscr{U} \ni 0$ of TM_{p_ϵ} onto some open subset $\mathscr{V} \ni p_\epsilon$ of M. Furthermore:

$$T \exp_{p_\epsilon}(0) = \text{Identity in } TM_p, \tag{5.4}$$

$$\forall x \in \mathscr{U}, \qquad d(p_\epsilon, \exp_{p_\epsilon} x) = \| x \|_{p_\epsilon}. \tag{5.5}$$

Now (5 3) implies

$$\forall x \in \mathscr{U}, \qquad F \circ \exp_{p_\epsilon} x \geqslant F \circ \exp_{p_\epsilon} 0 - \epsilon \| x \|_{p_\epsilon}, \tag{5.6}$$

$$\forall x \in \mathscr{U}, \qquad \forall t > 0, \qquad (F \circ \exp_{p_\epsilon} tx - F \circ \exp_{p_\epsilon} 0)/t \geqslant -\epsilon \| x \|_{p_\epsilon}. \tag{5.7}$$

Letting $t \to 0$, we obtain

$$\forall x \in \mathscr{U}, \qquad T(F \circ \exp_{p_\epsilon})(0)\, x \geqslant -\epsilon \| x \|_{p_\epsilon}. \tag{5.8}$$

As $\mathscr{U}$ is an open subset around 0, we get

$$\| T(F \circ \exp_{p_\epsilon})(0) \|_{p_\epsilon}^* \leqslant \epsilon. \tag{5.9}$$

But

$$T(F \circ \exp_{p_\epsilon})(0) = TF(\exp_{p_\epsilon} 0) = TF(p_\epsilon),$$

due to (5 4), so that (5 9) becomes

$$\| TF(p_\epsilon) \|_{p_\epsilon}^* \leqslant \epsilon. \tag{5.10}$$

Hence, identifying as usual the Hilbert space TM_{p_ϵ} with its dual through $\langle \cdot, \cdot \rangle_{p_\epsilon}$,

$$\| \operatorname{grad} F(p_\epsilon) \|_{p_\epsilon} \leqslant \epsilon. \qquad \blacksquare \tag{5.11}$$

This proposition sheds a new light on the celebrated condition (C) that R. Palais and S. Smale introduced in their work on generalized Morse theory (see [23], [27], [22]). Recall this condition

(C) *if S is a subset of M on which $|F|$ is bounded but on which $\| \operatorname{grad} F \|$ is not bounded away from zero, then there is a critical point of F in the closure of S*

We now get, as a corollary of Proposition 5.1, the

Palais-Smale Existence Theorem. *Assume M is a complete riemannian manifold, and F is a C^1 function on M satisfying condition* (C) *If F is bounded from below on M, then F assumes its greatest lower bound on M*

Proof. Proposition 5.1 tells us there is a sequence $(p_n)_{n\in\mathbb{N}}$ in M such that

$$F(p_n) \to \inf F \tag{5.12}$$

$$\| \operatorname{grad} F(p_n)\|_{p_n} \to 0. \tag{5.13}$$

Applying condition (C) to the subset $S = \bigcup_{n\in\mathbb{N}} \{p_n\}$ we see there is a critical point $\bar{p}$ of F in the closure of S. This just means that the sequence $(p_n)_{n\in\mathbb{N}}$ has a cluster point $\bar{p}$ such that $F'(\bar{p}) = 0$ As F is C^1, we get

$$F(\bar{p}) = \lim_{n\to\infty} F(p_n) = \inf F. \qquad \blacksquare \tag{5.14}$$

6. Geodesics[1]

As before, we shall denote by M a connected complete riemannian manifold, by p and q two points of M. Several equivalent definitions of the Sobolev manifold $L_1^2([0, 1]; M)$ have been given; see [20, 21] and [9]. Let us proceed in the following way. If $c: [0, 1] \to M$ is a continuous path, find in $[0, 1]$ a finite number of partition points $a_0 = 0, a_1, \ldots, a_{n-1}, a_n = 1$, such that for $1 \leqslant i \leqslant n$, the image of the ith subinterval $c([a_{i-1}, a_i])$ is contained in the domain $\mathscr{V}_i$ of some chart $\varphi_i: \mathscr{V}_i \to H$. Then c belongs to $L_1^2([0, 1], M)$ if and only if $(d/dt)\, \varphi_i \circ c \in L^2([a_{i-1}, a_i]; H)$ for $1 \leqslant i \leqslant n$. Moreover, we may define a neighbourhood of c in $L_1^2([0, 1], M)$ as the set of continuons paths c' such that $c'([a_{i-1}, a_i]) \subset \mathscr{V}_i$ for $1 \leqslant i \leqslant n$ and

$$\sup_{\substack{1\leqslant i\leqslant n\\ a_{i-1}\leqslant t\leqslant a_i}} \| \varphi_i \circ c(t) - \varphi_i \circ c'(t)\|_H + \sum_{i=1}^{n} \left[\int_{a_{i-1}}^{a_i} \left\| \frac{d}{dt}\varphi_i \circ c - \frac{d}{dt}\varphi_i \circ c' \right\|_H^2 dt\right]^{1/2}$$
$$\leqslant \epsilon. \tag{6.1}$$

We endow $L_1^2([0, 1]; M)$ with the topology defined by all such neighbourhoods, for $\epsilon > 0$, and we state a preliminary lemma:

Lemma 6.1. *C^∞ paths are dense among L_1^2 paths starting at p and ending at q.*

Proof. Let $c \in L_1^2([0, 1]; M)$ such that $c(0) = p$ and $c(1) = q$. The lemma states that, for every $\epsilon > 0$, there exists a C^∞ path c' such that $c'(0) = p$ and $c'(1) = q$ which verifies (6.1).

[1] The author wishes to thank J. P. Penot for a stimulating conversation and for communication of his unpublished work.

For every subinterval $[a_{i-1}, a_i]$ and every $k \in \mathbb{N}$, define:

$$N_k = [a_{i-1}, a_{i-1} + 1/k] \cup [a_i - 1/k, a_i].$$

We have

$$\int_{N_k} \left\| \frac{d}{dt} \varphi_i \circ c(t) \right\|_H dt \to 0 \quad \text{as} \quad k \to \infty, \tag{6.2}$$

$$\int_{N_k} \left\| \frac{d}{dt} \varphi_i \circ c(t) \right\|_H^2 dt \to 0 \quad \text{as} \quad k \to \infty. \tag{6.3}$$

Define a function $\theta_k: [a_{i-1}, a_i] \to H$ by $\theta_k(a_{i-1}) = \varphi_i \circ c(a_{i-1})$, and

$$\frac{d}{dt} \theta_k(t) = 0 \quad \text{if} \quad t \in N_k$$

$$\frac{d}{dt} \theta_k(t) = \frac{d}{dt} \varphi_i \circ c(t) + \frac{1}{a_i - a_{i-1} - 2/k} \int_{N_k} \frac{d}{dt} \varphi_i \circ c(t)\, dt \quad \text{if} \quad t \notin N_k. \tag{6.4}$$

It is clear from (6.2) and (6.3) that $(d/dt)\, \theta_k \to (d/dt)\, \varphi_i \circ c$ in $L^2([a_{i-1}, a_i]; H)$ as $k \to \infty$. Moreover,

$$\theta_k(a_i) = \theta_k(a_{i-1}) + \int_{a-1}^{a_i} \frac{d}{dt} \theta_k(t)\, dt,$$

$$\theta_k(a_i) = \varphi_i \circ c(a_{i-1}) + \int_{a_{i-1+1/k}}^{a_i - 1/k} \frac{d}{dt} \varphi_i \circ c(t)\, dt + \int_{N_k} \frac{d}{dt} \varphi_i \circ c(t)\, dt, \tag{6.5}$$

$$\theta_k(a_i) = \varphi_i \circ c(a_i).$$

It follows from the usual Sobolev inequalities, or even from the Ascoli theorem, that θ_k is continuous and converges uniformly to $\varphi_i \circ c$ as $k \to \infty$ We choose k large enough so that,

$$\sup_{a_{i-1} \leqslant t \leqslant a_i} \left\| \theta_k(t) - \varphi_i \circ c(t) \right\|_H + \left[\int_{a_{i-1}}^{a_i} \left\| \frac{d}{dt} \varphi_i \circ c - \frac{d}{dt} \theta_k \right\|_H^2 dt \right]^{1/2} \leqslant \frac{\epsilon}{2n}. \tag{6.6}$$

We now smooth down θ_k by convolution. Extend θ_k by $\theta_k(a_{i-1})$ to the left and $\theta_k(a_i)$ to the right. Using distribution theory, we can find a nonnegative

C^∞ function $\rho: \mathbb{R} \to \mathbb{R}$, which is zero outside $[-1/2k, 1/2k]$, and satisfies:

$$\int_{\mathbb{R}} \rho \, dt = 1, \tag{6.7}$$

$$\sup_{a_{i-1} \leqslant t \leqslant a_i} \| \rho * \theta_k(t) - \theta_k(t) \|_H \leqslant \frac{\epsilon}{4n}, \tag{6.8}$$

$$\left\| \frac{d}{dt} \rho * \theta_k - \frac{d}{dt} \theta_k \right\|_H^2 \leqslant \frac{\epsilon}{4n} \qquad \text{in} \qquad L^2([a_{i-1}, a_i]; H). \tag{6.9}$$

Thanks to (6.7), we have

$$\rho * \theta_k(t) = \varphi_i \circ c(a_{i-1}) \qquad \text{for} \qquad a_{i-1} \leqslant t \leqslant a_{i-1} + 1/2k, \tag{6.10}$$

$$\rho * \theta_k(t) = \varphi_i \circ c(a_i) \qquad \text{for } a_i - 1/2k \leqslant t \leqslant a_i. \tag{6.11}$$

Adding (6.6), (6.8), and (6.9), we obtain

$$\sup_{a_{i-1} \leqslant t \leqslant a_i} \| \rho * \theta_k(t) - \varphi_i \circ c(t) \|_H + \left[\int_{a_{i-1}}^{a_i} \left\| \frac{d}{dt} \rho * \theta_k - \frac{d}{dt} \varphi_i \circ c \right\|_H^2 dt \right]^{1/2} \leqslant \frac{\epsilon}{n}. \tag{6.12}$$

We now set $c' = \varphi_i^{-1} \circ (\rho * \theta_k)$. The path $c': [0, 1] \to M$ is well defined, and is C^∞ on every subinterval $[a_{i-1}, a_i]$, $1 \leqslant i \leqslant n$. The pieces fit together because c' is locally constant at the partition points a_i ((6.10) and (6.11)). It only remains to add the n inequalities (6.12) for $1 \leqslant i \leqslant n$ to obtain (6.1). ∎

Convergence in $L_1^2([0, 1]; M)$ implies uniform convergence in $C^0([0, 1]; M)$. Recall that there is a C^∞-manifold structure, and even a riemannian structure, on $L_1^2([0, 1]; M)$ compatible with its topology [28, 15, 25, 26]. For every $c \in L_1^2([0, 1]; M)$, the tangent space $TL_1^2([0, 1]; M)_c$ is canonically isomorphic to $L_1^2(TM_c)$, so that a tangent vector to $L_1^2([0, 1]; M)$ at c is an L_1^2 path ξ in TM over c:

$$\xi: t \mapsto \xi(t) \qquad \text{where} \qquad \xi(t) \in TM_{c(t)}. \tag{6.13}$$

The riemannian inner product of two tangent vectors ξ and ξ' in $L_1^2(TM_c)$ is given by:

$$\langle \xi, \xi' \rangle_c = \int_0^1 \langle \xi(t), \xi'(t) \rangle_{c(t)} \, dt + \int_0^1 \langle \nabla_{\dot{c}(t)} \xi(t), \nabla_{\dot{c}(t)} \xi'(t) \rangle_{c(t)} \, dt. \tag{6.14}$$

The associated riemannian metric will be denoted by δ, and $L_1^2([0, 1]; M)$ can be shown to be complete as a metric space. From now on, we shall always consider $L_1^2([0, 1]; M)$ as a complete riemannian manifold.

LEMMA 6.2. *Set*

$$V = \{c \in L_1^2([0, 1]; M) \mid c(0) = p, c(1) = q\}$$

Then V is a closed C^∞ submanifold of $L_1^2([0, 1]; M)$.

Proof. Consider the evaluation mapping:

$$ev\colon c \mapsto (c(0), c(1))$$

$$ev\colon L_1^2([0, 1]; M) \to M \times M.$$

It is clearly C^∞, the tangent map at c being $\xi \mapsto (\xi(0), \xi(1))$ from $L_1^2(TM_c)$ into $TM_{c(0)} \times TM_{c(1)}$. It is easily seen to be surjective, and its kernel splits. This proves that the evaluation mapping is transversal to (p, q), and hence that $V = ev^{-1}(p, q)$ is a closed C^∞ submanifold of $L_1^2([0, 1]; M)$ [5, 1]). We can even express the tangent space TV_c:

$$TV_c = \{\xi \in L_1^2(TM_c) \mid \xi(0) = 0, \xi(1) = 0\}. \qquad \blacksquare \qquad (6.16)$$

We define the riemannian inner product of two vectors ξ and ξ' in TV_c by:

$$\langle \xi, \xi' \rangle_c^0 = \int_0^1 \langle \nabla_{\dot c(t)} \xi(t), \nabla_{\dot c(t)} \xi'(t) \rangle_{c(t)}\, dt, \qquad (6.17)$$

and we denote by δ^0 the associated riemannian metric.

We easily check that, for $\xi \in TV_c$,

$$\begin{aligned} \| \xi(t) \|_{c(t)}^2 &= \int_0^t \frac{d}{ds} \langle \xi(s), \xi(s) \rangle_{c(s)}\, ds \\ &= \int_0^t 2 \langle \nabla_{\dot c(s)} \xi(s), \xi(s) \rangle_{c(s)}\, ds \\ &\leqslant 2 \| \xi \|_{L^2} \| \nabla \xi \|_{L^2}. \end{aligned} \qquad (6.18)$$

Integrating over [0, 1], we get

$$\| \xi \|_{L^2}^2 \leqslant 2 \| \xi \|_{L^2} \| \nabla_{\dot c} \xi \|_{L^2}$$

or

$$\| \xi \|_{L^2} \leqslant 2 \| \nabla_{\dot c} \xi \|_{L^2}.$$

Comparing (6.14) and (6.17), we get, for every $\xi \in TV_c$:

$$\| \xi \|_c^0 \leqslant \| \xi \|_c \leqslant 3 \| \xi \|_c^0. \qquad (6.19)$$

Hence, of course, for every paths c and c' in V,

$$\delta^0(c, c') \leqslant \delta(c, c') \leqslant 3\delta^0(c, c^0). \tag{6.20}$$

As V is closed in $L_1^2([0, 1]; M)$, it is δ-complete, and from (6.20), we see that it also is δ^0-complete. Hence V, endowed with the riemannian structure defined by (6.17), is a complete riemannian manifold.

Consider now the energy function. $F\colon V \to \mathbb{R}$ defined by

$$F(c) = \int_0^1 \| \dot{c}(t)\|_{c(t)}^2 \, dt. \tag{6.16}$$

This function has been extensively studied by several authors [20, 22, 15]. It is C^∞, and its minima, whenever they exist, are the geodesics of minimal length joining p and q. We state the main result of this section:

THEOREM 6.3. *For every $\epsilon > 0$, there exists a C^∞ path c_ϵ joining p and q and a vector $x_\epsilon \in TM_p$ such that*

$$\int_0^1 \| \dot{c}_\epsilon(t)\|_{c_\epsilon(t)}^2 \, dt \leqslant \inf_{c \in V} \int_0^1 \| \dot{c}(t)\|_{c(t)}^2 \, dt + \epsilon, \tag{6.17}$$

$$\int_0^1 \| \dot{c}_\epsilon(t) - \tilde{x}_\epsilon(t)\|_{c_\epsilon(t)}^2 \, dt \leqslant \epsilon, \tag{6.18}$$

where $\tilde{x}_\epsilon(t)$ is obtained from x_ϵ by parallel translation along c_ϵ.

Proof. Let $\eta > 0$ be given. From Proposition 5.1, we can find a path $c' \in L_1^2([0, 1]; M)$ joining p and q such that

$$F(c') \leqslant \inf_{c \in V} F(c) + \eta^2, \tag{6.19}$$

$$\forall \xi \in TV_{c'}, \qquad |\langle \operatorname{grad} F(c'), \xi\rangle| \leqslant \eta \, \| \xi \|_{c'}^0 . \tag{6.20}$$

Using Lemma 6.1, we can find a C^∞ path c joining p and q, near enough c' for

$$| F(c) - F(c')| \leqslant \eta^2 \text{ and } | \, \| \operatorname{grad} F(c') \|_{c'} - \| \operatorname{grad} F(c) \|_c \, | \leqslant \eta$$

to hold. Hence we get

$$F(c) \leqslant \inf_V F + 2\eta^2, \tag{6.21}$$

$$\forall \xi \in TV_c, \qquad |\langle \operatorname{grad} F(c), \xi\rangle| \leqslant 2\eta \, \| \xi \|_c^0 . \tag{6.22}$$

Let us first take care of condition (6.22). For any $\xi \in TV_c$, define $\widetilde{\exp}_c\,\xi \in V$ by

$$\forall t \in [0, 1], \qquad (\widetilde{\exp}_c\,\xi)\,(t) = \exp_{c(t)}\,\xi(t). \tag{6.23}$$

This is clearly a C^∞ map, the derivative of which at zero is the identity in TV_c, so that

$$\forall \xi \in TV_c\,, \qquad |\langle \operatorname{grad}(F \circ \widetilde{\exp}_c)\,(0), \xi\rangle| \leqslant 2\eta \,\|\, \xi \,\|_c^0\,. \tag{6.24}$$

For every $\xi \in TV_c$, i.e., every $\xi \in L_1^2(TM_c)$ such that $\xi(0) = 0$ and $\xi(1) = 0$, the left-hand side can be expressed in another way:

$$\begin{aligned} &\langle \operatorname{grad}(F \circ \widetilde{\exp}_c)\,(0), \xi\rangle \\ &\quad = \frac{\partial}{\partial\alpha} F \circ \widetilde{\exp}_c(\alpha\xi)\,\Big|_{\alpha=0} \qquad\qquad (6.25)\\ &\quad = \frac{\partial}{\partial\alpha} \int_0^1 \left\langle \frac{\partial}{\partial t} \exp_{c(t)}\,\alpha\xi(t), \frac{\partial}{\partial t} \exp_{c(t)}\,\alpha\xi(t) \right\rangle_{c(t)} dt\,\Big|_{\alpha=0}\,. \end{aligned}$$

The derivative $\partial/\partial\alpha \,\|\, \partial/\partial t \exp_{c(t)}\,\alpha\xi(t)\|_{c(t)}^2$ is a continuous function of both α and t, which vary in compact sets. Hence,

$$\frac{\partial}{\partial\alpha} \int_0^1 \left\| \frac{\partial}{\partial t} \exp_{c(t)}\,\alpha\xi(t) \right\|_{c(t)}^2 dt = \int_0^1 \frac{\partial}{\partial\alpha} \left\| \frac{\partial}{\partial t} \exp_{c(t)}\,\alpha\xi(t) \right\|_{c(t)}^2 dt. \tag{6.26}$$

Using the Levi–Civita connection, for every $t \in [0, 1]$:

$$\begin{aligned} &\frac{\partial}{\partial\alpha} \left\langle \frac{\partial}{\partial t} \exp_{c(t)}\,\alpha\xi(t), \frac{\partial}{\partial t} \exp_{c(t)}\,\alpha\xi(t) \right\rangle_{c(t)} \Big|_{\alpha=0} \\ &\quad = \xi(t) \cdot \langle \dot c(t), \dot c(t)\rangle_{c(t)} \\ &\quad = 2\langle \nabla\xi_{(t)}\dot c(t), \dot c(t)\rangle_{c(t)} \qquad\qquad (6.27)\\ &\quad = 2\langle [\xi(t), \dot c(t)] + \nabla_{\dot c(t)}\xi(t), \dot c(t)\rangle_{c(t)} \\ &\quad = 2\langle \nabla_{\dot c(t)}\xi(t), \dot c(t)\rangle_{c(t)}\,. \end{aligned}$$

Let us now state (6.24) again

$$\begin{aligned} \forall \xi \in TV_c\,, \qquad &2\int_0^1 \langle \nabla_{\dot c(t)}\xi(t), \dot c(t)\rangle_{c(t)}\,dt \\ &\leqslant 2\eta \left[\int_0^1 \langle \nabla_{\dot c(t)}\xi(t), \nabla_{\dot c(t)}\xi(t)\rangle_{c(t)}\,dt\right]^{1/2}. \end{aligned} \tag{6.28}$$

Consider the mapping Der: $\xi \mapsto \nabla_{\dot c}\xi$, which sends $L_1^2(TM_c)$ into $L^2(TM_c)$.

We seek a characterization of $\mathrm{Der}(TV_c)$. It is easy to check that, whenever $\omega \in L^2(TM_c)$, the solution ξ_ω of the linear differential equation $\nabla_{\dot{c}(t)}\xi(t) = \omega(t)$ such that $\xi_\omega(0) = 0$ belongs to $L_1^2(TM_c)$. It remains to get $\xi_\omega(1) = 0$.

For any $x \in TM_p$, denote by $\tilde{x}(t) \in TM_{c(t)}$ the vector obtained from x by parallel translation along c, i.e., the solution of the differential equation $\nabla_{\dot{c}(t)}\tilde{x}(t) = 0$ such that $\tilde{x}(0) = x$. It follows from the definition that, for any $\omega \in L^2(TM_c)$,

$$\int_0^1 \langle \omega(t), \tilde{x}(t) \rangle_{c(t)} \, dt = \langle \xi_\omega(1), \tilde{x}(1) \rangle. \tag{6.29}$$

The mapping $x \mapsto \tilde{x}(1)$ from TM_p to TM_q is easily seen to be surjective, so that $\xi_\omega(1) = 0$ if and only if (6.29) vanishes for all $x \in TM_p$. In other words, $\omega \in \mathrm{Der}(TV_c)$ if and only if

$$\forall x \in TM_p, \qquad \int_0^1 \langle \omega(t), \tilde{x}(t) \rangle_{c(t)} \, dt = 0. \tag{6.30}$$

Denote by E the vector space of paths $\tilde{x}\colon t \to \tilde{x}(t)$ for all $x \in TM_p$. It is a closed subspace of $L^2(TM_c)$, so we can state (6.28) in the following way:

$$\omega \in E^\perp \Rightarrow \langle \omega, \dot{c} \rangle \leqslant \eta \, \| \omega \| \qquad \text{in } L^2(TM_c). \tag{6.31}$$

Now $\dot{c}$ can be written as $\tilde{x} + \zeta$, where $\tilde{x} \in E$ and $\zeta \in E^\perp$, so that (6.31) becomes $\| \zeta \| \leqslant \eta$. In other words, there exists some vector $x \in TM_p$ such that the associated path $\tilde{x} \in C^\infty(TM_c)$ satisfies

$$\| \dot{c} - \tilde{x} \| \leqslant \eta \qquad \text{in } L^2(TM_c), \tag{6.32}$$

$$\int_0^1 \| \dot{c}(t) - \tilde{x}(t) \|_{c(t)}^2 \, dt \leqslant \eta^2. \tag{6.33}$$

Taking η small enough for $2\eta^2$ to be less than ϵ, we obtain (6.17) and (6.18) from (6.32) and (6.33). ∎

Theorem 6.3 states that any two points p and q of M can be joined by a C^∞ path whose length is "almost" minimum and which is "almost" a geodesic. Indeed, condition (6.18) states that the velocity along this path is "almost" constant, i.e., that it can "almost" be obtained by parallel translation of a fixed vector of TM_p. When M is finite dimensional, it is well-known that F satisfies condition (C) of Palais–Smale, which yields the Hopf–Rinow theorem again. In the general case, it seems (although this author has not been able to prove it) that the endpoint q_ϵ of the geodesic starting at p_ϵ with velocity x_ϵ should converge towards the endpoint q of c_ϵ as $\epsilon \to 0$.

7. The Pontryagin Maximum Principle

Let us now switch over to control theory to give one last application of Theorem 1.1. We refer the reader back to the treatise of Pallu de la Barrière [24] for classical results and notations.

Consider a system governed by the equation

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t), u(t), t), \qquad \text{a.e.} \\ x(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{7.1}$$

where $x(t) \in \mathbb{R}^n$ describes the state of the system, $u(t)$ is the control at time t, and belongs to some compact metrizable set K. We prescribe a time $T > 0$ and assume that:

(a) f and $f_x' = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ are continuous functions over $R^n \times K \times [0, T]$

(b) $\langle x, f(t, x, u)\rangle \leqslant c(1 + \| x \|^2)$ for some constant c.

Let a measurable control u: $[0, T] \to K$ be given. Condition (a) ensures that there exists a unique solution X of the differential Equation (7.1) on a time interval $[0, \tau]$ small enough. Through use of Gronwall's inequality, condition (b) becomes

$$\| x(t)\|^2 \leqslant (\| x_0 \|^2 + 2cT)\, e^{2cT}, \tag{7.2}$$

and hence ensures existence of the solution on the whole time interval $[0, T]$. Moreover, (7.2) yields

$$\| dx(t)/dt \| \leqslant \max\{f(t, x, u) \mid (t, x, u) \in [0, T] \times B \times K\}, \tag{7.3}$$

where B denotes the ball of radius $(\| x_0 \|^2 + 2cT)^{1/2}\, e^{cT}$. Applying Ascoli's theorem, we see that the family of all trajectories X of the control system (7.1) is equicontinuous and bounded, and hence relatively compact in the uniform topology.

We are given a C^1 function g: $\mathbb{R}^n \to \mathbb{R}$, and we seek some measurable control u such that the corresponding trajectory x minimizes $g(x(T))$ among all solutions of (7.1).

Theorem 7.1. *For every $\epsilon > 0$, there exists a measurable control u_ϵ, the corresponding trajectory being x_ϵ, such that*

$$g(x_\epsilon(T)) \leqslant \inf g(x(T)) + \epsilon \tag{7.4}$$

$$\langle f(x_\epsilon(t), u_\epsilon(t), t), p_\epsilon(t)\rangle \leqslant \min_{u\in K}\langle f(x_\epsilon(t), u, t), p_\epsilon(t)\rangle \times \epsilon, \qquad \text{a.e.}, \tag{7.5}$$

where p_ϵ is the solution of the linear differential equation

$$\begin{aligned} dp_\epsilon(t)/dt &= -{}^t f_x'(x_\epsilon(t), u_\epsilon(t), t) \cdot p_\epsilon(t) \\ p_\epsilon(T) &= g'(x_\epsilon(T)). \end{aligned} \tag{7.6}$$

Whenever we can take $\epsilon = 0$ in (7.4), we can also take $\epsilon = 0$ in (7.5). In other words, whenever there exists an optimal control, it satisfies the Pontryagin maximum principle. However, our theorem holds even when there is no optimal solution. We prove it in several steps.

First, we denote by $\mathscr{U}$ the set of measurable controls $u\colon [0, T] \to K$, endowed with the following metric

$$\delta(u_1, u_2) = \operatorname{meas}\{t \in [0, T] \mid u_1(t) \neq u_2(t)\}. \tag{7.7}$$

LEMMA 7.2. *$\mathscr{U}$ is a complete metric space.*

Proof. Let us first check that δ is a distance. Take any u_1, u_2, u_3 in $\mathscr{U}$:

$$\{t \mid u_1(t) \neq u_2(t)\} \subset \{t \mid u_1(t) \neq u_3(t)\} \cup \{t \mid u_3(t) \neq u_2(t)\}, \tag{7.8}$$

$$\operatorname{meas}\{t \mid u_1(t) \neq u_2(t)\} \leqslant \operatorname{meas}\{t \mid u_1(t) \neq u_3(t)\} + \operatorname{meas}\{t \mid u_3(t) \neq u_2(t)\}, \tag{7.9}$$

$$\delta(u_1, u_2) \leqslant \delta(u_1, u_3) + \delta(u_3, u_2). \tag{7.10}$$

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{U}$. We can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\delta(u_{n_k}, u_{n_{k+1}}) \leqslant 1/2^k$, and we will show that this subsequence converges. Indeed, set

$$A_k = \bigcup_{p\geqslant k} \{t \mid u_{n_p}(t) \neq u_{n_{p+1}}(t)\}. \tag{7.11}$$

We have

$$\operatorname{meas} A_k = \sum_{p=k}^{\infty} \frac{1}{2^p} = \frac{1}{2^{k-1}}, \qquad \text{and} \qquad A_k \supset A_{k+1}.$$

Define $\bar{u} \in \mathscr{U}$ by

$$\forall t \notin A_k, \qquad \bar{u}(t) = u_{n_k}(t). \tag{7.12}$$

By definition, the subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converges to $\bar{u}$. As the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to $\bar{u}$ as a whole. ∎

LEMMA 7.3. *The mapping $F\colon u \mapsto g(x(T))$, where $u \in \mathscr{U}$, x is the corresponding solution of* (7.1), *is continuous over $\mathscr{U}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence converging towards $\bar{u}$ in $\mathcal{U}$. The sequence of trajectories $(x_n)_{n\in\mathbb{N}}$ is relatively compact, hence there exists a subsequence x_k which converges uniformly to $\bar{x}$. It remains to prove that $\bar{x}$ is the trajectory associated with $\bar{u}$.

For that, write Eq. (7.1) differently

$$x_k(t) = x_0 + \int_0^t f(x_k(s), u_k(s), d))\, ds. \tag{7.13}$$

Now, as $k \to \infty$, $x_k \to \bar{x}$ uniformly, $u_k \to \bar{u}$ a.e., and the integrand remains bounded by (7.3). We can apply the Lebesgue convergence theorem, which yields

$$\bar{x}(t) = x_0 + \int_0^t f(\bar{x}(s), \bar{u}(s), s)\, ds. \quad \blacksquare \tag{7.14}$$

We now are in a position to apply Theorem 1.1. We get a measurable control $u_\epsilon \in \mathcal{U}$ such that

$$F(u_\epsilon) \leqslant \inf F + \epsilon^2, \tag{7.15}$$

$$\forall u \in \mathcal{U}, \qquad F(u) \geqslant F(u_\epsilon) - \epsilon\delta(u, u_\epsilon), \tag{7.16}$$

the corresponding trajectory being x_ϵ given by

$$\begin{aligned} &\frac{dx_\epsilon}{dt}(t) = f(x_\epsilon(t), u_\epsilon(t), t) \qquad \text{a.e.,} \\ &x_\epsilon(0) = x_0 . \end{aligned} \tag{7.17}$$

Take $t_0 \in]0, T[$ where the equality holds, $u_0 \in K$, and define $v_\tau \in \mathcal{U}$ for every $\tau \geqslant 0$ in the following way:

$$\begin{aligned} &v_\tau(t) = u_0 \qquad \text{if} \qquad t \in [0, T] \cap]t_0 - \tau, t_0[, \\ &v_\tau(t) = u_\epsilon(t) \qquad \text{if} \qquad t \notin [0, T] \cap]t_0 - \tau, t_0[. \end{aligned} \tag{7.18}$$

Clearly, $\delta(v_\tau , u_\tau) = \tau$, when τ is small enough. Denoting by x_τ the corresponding trajectory, we shall prove in Lemma 7.4, that

$$\frac{d}{d\tau} g(x_\tau(T)) \Big|_{\tau=0} = \langle f(x_\epsilon(t_0), u_0 , t_0) - f(x_\epsilon(t_0), u_\epsilon(t_0), t_0), p_\epsilon(t_0)\rangle. \tag{7.19}$$

But (7.16) yields

$$g(x_\tau(T)) - g(x_\epsilon(T)) \geqslant -\epsilon\tau \qquad \text{for } \tau \geqslant 0. \tag{7.20}$$

Putting (7.19) and (7.20) together, we get

$$\langle f(x_\epsilon(t_0), u_0, t_0) - f(x_\epsilon(t_0), u_\epsilon(t_0), t_0), p_\epsilon(t_0)\rangle \geqslant -\epsilon, \tag{7.21}$$

which ends the proof, because u_0 is any point of K and t_0 is any point of $]0, T[$ where equality holds in (7.17).

LEMMA 7.4.

$$\frac{d}{d\tau} g(x_\tau(T))\Big|_{\tau=0} = \langle f(x_\epsilon(t_0), u_0, t_0) - f(x_\epsilon(t_0), u_\epsilon(t_0), t_0), p_\epsilon(t_0)\rangle.$$

Proof. This is a classical result, which can be found, for instance, in [24]. We sketch the proof here for the reader's convenience. Write

$$\begin{aligned} x_\tau(t_0) &= x_\epsilon(t_0 - \tau) + \int_{t_0-\tau}^{t_0} f(x_\tau(s), u_0, s)\, ds \\ &= x_\epsilon(t_0) - \tau(dx_\epsilon/dt)(t_0) + \tau f(x_\epsilon(t_0), u_0, t_0) + O(\tau) \\ &= x_\epsilon(t_0) - \tau(f(x_\epsilon(t_0), u_\epsilon(t_0), t_0) - f(x_\epsilon(t_0), u_0, t_0)) + O(\tau), \end{aligned} \tag{7.22}$$

which yields

$$(d/d\tau)\, x_\tau(t_0)|_{\tau=0} = f(x_\epsilon(t_0), u_0, t_0) - f(x_\epsilon(t_0), u_\epsilon(t_0), t_0). \tag{7.23}$$

Hence,

$$(d/d\tau)\, x_\tau(T)|_{\tau=0} = R(T, t_0)\, [f(x_\epsilon(t_0), u_0, t_0) - f(x_\epsilon(t_0), u(t_0), t_0)], \tag{7.24}$$

where $R(T, T_0)$ is the resolvent of the linearized equation

$$(d\xi/dt)(t) = f'(x_\epsilon(t), u_\epsilon(t), t) \cdot \xi(t). \tag{7.25}$$

We now have:

$$\begin{aligned} &(d/d\tau)\, g(x_\tau(T))|_{\tau=0} \\ &\quad = \langle g'(x_\epsilon(T)), (d/d\tau)\, x_\tau(T)|_{\tau=0}\rangle \\ &\quad = \langle g'(x_\epsilon(T)), R(T, t_0)\, [f(x_\epsilon(t_0), u_0, t_0) - f(x_\epsilon(t_0), u_\epsilon(t_0), t_0)]\rangle \\ &\quad = \langle {}^tR(T, t_0)\, g'(x_\epsilon(T)), [f(x_\epsilon(t_0), u_0, t_0) - f(x_\epsilon(t_0), u_\epsilon(t_0), t_0)]\rangle. \end{aligned} \tag{7.26}$$

But ${}^tR(T, T_0)\, g'(x_\epsilon(T))$ is just $p_\epsilon(t_0)$, where p_ϵ is the solution of (7.6). Hence the result. ∎

8. Conclusion

In some respects the results of Section 2 are to be compared with papers of Asplund [2] and Edelstein [10, 11]; see also Baranger [3]. These authors exhibit a class of nonconvex optimization problems which acquire solutions for a dense set of linear perturbations. The method applied here is quite different, and uses much weaker assumptions on V.

In his book [17], Krasnoselski has proved a result similar to Corollary 3.4; namely, every completely continuous operator acting in an infinite-dimensional Banach space has approximate eigenvectors (Chap. 4, Sect. 1).

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