Exterior Differential Calculus and Aggregation Theory : a Presentation and Some New Results*

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Abstract

In many economic contexts, a given function can be disaggregated as a linear combination of gradients. Examples include the literature on the characterization of aggregate demand and excess demand (Sonnenschein 1973ab, Debreu 1974), and the model of efficient household behaviour recently proposed by Browning and Chiappori (1994). We show that exterior differential calculus provides very useful tools to address these problems. In particular, we show, using these techniques, that any analytic mapping in $\mathbb{R}^n$ satisfying Walras Law can be locally decomposed as the sum of n individual, utility-maximizing demand functions.

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1 Introduction : aggregation and gradient structures

In many situations, economists are interested in the behavior of aggregate variables that stem from the addition of several elementary demand or supply functions. In turn, each of these elementary components results from some maximizing decision process at the 'individual' level. A standard illustration is the characterization of aggregate market or excess demand in an exchange economy, a problem initially raised by Sonnenschein (1973a,b) and to which a number of author contributed, including Debreu (1974), McFadden et al. (1974), Mantel (1974, 1976, 1977), Diewert (1977) and Geanakoplos and Polemarchakis (1980). Here, agents maximize utility under budget constraint, and individual demands add up to an aggregate demand or excess demand function. This research has recently been extended to incomplete markets by Bottazzi and Hens (1996) and Gottardi and Hens (1995). A different but related example is provided by Browning and Chiappori (1994), who consider the demand function of a two-person household, where each member is characterized by a specific utility function and decisions are only assumed to be Pareto-efficient.

These models share a common feature : they lead to the same type of mathematical problem. Specifically, in all cases, the economic context has the following translation : some given function $\mathbf{X}(\mathbf{p})$, mapping $\mathbb{R}^n_+$ to $\mathbb{R}^n$, can be decomposed as a linear combination of k gradients. Here, k is the number of individuals; $\mathbf{X}(\mathbf{p})$ is the original (aggregate) function; and gradients are the natural mathematical translation of the underlying optimization problem. Formally, $\mathbf{X}(\mathbf{p})$ writes down as :

$$\mathbf{X}(\mathbf{p}) = \lambda_1(\mathbf{p})\, D_{\mathbf{p}}V^1(\mathbf{p}) + ... + \lambda_k(\mathbf{p})\, D_{\mathbf{p}}V^k(\mathbf{p}) \qquad (1)$$

where the $\lambda_i(\mathbf{p})$ and the $V^i(\mathbf{p})$ are scalar functions (V^i being in general interpreted as an indirect utility function), and where $D_{\mathbf{p}}V^i(\mathbf{p})$ is the gradient of $V^i(\mathbf{p})$ at $\mathbf{p}$:

$$D_{\mathbf{p}}V^i(\mathbf{p}) = \left(\frac{\partial V^i}{\partial p_1}, ..., \frac{\partial V^i}{\partial p_n} \right)'$$

Note that, depending on the context, these functions may have to fulfill specific, additional conditions, such as positiveness, monotonicity, (quasi-)convexity, budget constraints and others.

A natural question is then the following : what does (1) imply upon the form of the function $\mathbf{X}$?

From a mathematical point of view, the structure (1) is highly specific. In the first half of this century, Elie Cartan developed a set of concepts, usually referred to as exterior differential calculus (from now on EDC), that proved especially convenient to deal with problems of this type. Surprisingly enough, however, these tools have hardly ever been used in the field of economic theory. As an obvious exception, one must mention a paper by Russell and Farris (1993), showing that Gorman's rank theorem is a consequence of well-known results on Lie groups. More recently, Russell (1994) proposes a measure of 'quasi-rationality' directly based upon EDC. These works, however, only consider individual behavior.

The goal of this paper is twofold. For one thing, we propose a brief but (hopefully) useful summary of some of the main results in EDC. In particular, we describe in some details a very powerful theorem, due to Cartan and Kähler, that (to our knowledge) has never been used so far in economics, though it reveals extremely helpful to solve a large number of problems. The second goal of the paper is to illustrate and substantiate the latter claim by showing how the tools presented can help addressing issues at stake in this literature. In particular, we solve a two of problems that were still open. One is the complete characterization of household demand behaviour, a problem raised by Browning and Chiappori (1994); the other is the decomposi

The structure of our paper is based upon the distinction between 'mathematical' and 'economic' integration of a demand function. The mathematical integration problem can be stated as follows : when can a given function $\mathbf{X}(\mathbf{p})$ be written as a linear combination of gradients, as in (1)? In particular, this approach simply disregards all additional conditions stemming from the specific economic context. Economic integration, on the other hand, takes these restrictions into account; the question, now, is : when can a given function $\mathbf{X}(\mathbf{p})$ be considered as an aggregate demand (or excess demand, or household demand) ? Clearly, the second problem is more difficult, hence presumably more demanding in terms of mathematical sophistication. In some cases, it is however solvable with the help of adequate tools. We show, for instance, that the characterization of market aggregate demand - an issue that has remained open since its formulation by Sonnenschein more than 20 years ago - can indeed be (locally) solved with the help of a strong result in EDC theory, the Cartan-Kähler theorem. In other situations, including Browning-Chiappori's model of efficient household demand, economic integration seems out of reach. But then the 'mathematical integration' approach turns out to provide, at a lower cost, some very useful insights on the nature of the difficulties at stake.

In the next section, we quickly recall some elementary notions of exterior differential calculus. We end the section with a basic result, Pfaff theorem, that provides a simple answer to the mathematical integration problem. We provide two applications of this theorem in section 3. One, quite standard, is a (quick) proof of Slutsky and Antonelli symmetry in the case of a single decision maker. The second is much less trivial. We consider the characterization of 'collective' household demand, as studied by Browning and Chiappori (1995), and show that, leaving aside convexity issues, their main result admits a reciprocal. Section 4 is devoted to the exposition of the Cartan-Kähler theorem. This tool is finally applied, in the last section, to solve the aggregate market demand problem.

2 Exterior differential calculus : an economist's toolkit

In this section, we introduce the basic notions of exterior differential calculus. Our purpose is exclusively pedagogical. At many places, in particular, our presentation is somewhat intuitive, and skips most technicalities, while many precautions are deliberately left aside. For a much more exhaustive and rigorous presentation, the interested reader is referred to Cartan's book (1945), or to the recent treatise by Bryant et al. (1991).

2.1 Linear and differential forms

The basic notion is that of forms. A linear form (or a 1-form) is a linear mapping from some subspace $E \subset \mathbb{R}^n$ to $\mathbb{R}$:

$$\omega \;:\; \xi \in \mathbb{R}^n \mapsto \omega[\xi] = \langle \omega, \xi \rangle = \sum_{i=1}^{n} \omega^i \xi_i$$

The set of linear forms on E is the *dual* E^* of E. A basic example of linear form is the projection $dp_i \;:\; \xi \mapsto \xi_i$, which, to any vector, associates its ith coordinate. These form a basis of E^*; any form ω can be decomposed as :

$$\omega = \sum \omega^i \, dp_i$$

In what follows, we are especially interested in *differential forms*. Consider a smooth manifold U, and let $T_{\mathbf{p}}U$ denote its tangent space at some point $\mathbf{p}$. A differential (1-)form is, for every $\mathbf{p} \in U$, a 1-form $\omega(\mathbf{p})$ on $T_{\mathbf{p}}U$ (with, say, $\langle \omega(\mathbf{p}), \xi \rangle = \sum \omega^i(\mathbf{p})\xi_i$), such that the coefficients $\omega^i(\mathbf{p})$ depend smoothly on $\mathbf{p}$. Note that to any coordinate system, one can associate a canonical bijection between the space of differential forms and the space of vector fields; i.e., to the differential form $\omega(\mathbf{p})$, one can associate the vector field $\bar{\omega}(\mathbf{p})$ whose coordinates at each $\mathbf{p}$ are the $\omega^i(\mathbf{p})$.

As a simple example of a differential form, we may, for any smooth mapping V from E to R, consider the tangent form dV defined at any point $\mathbf{p}$ by :

$$dV(\mathbf{p}) = \sum \frac{\partial V}{\partial p^i}(\mathbf{p}) dp_i$$

so that

$$dV\,(\mathbf{p}) \,:\; \xi \mapsto \langle dV(\mathbf{p}), \xi \rangle = \sum \frac{\partial V}{\partial p_i} \cdot \xi_i$$

Of course, this form is extremely specific, for the following reason. Consider the $(n-1)$-dimensional manifold M defined by

$$M = \{\mathbf{p} \in E \mid V(\mathbf{p}) = a\}$$

where a is a constant. Then, for any $\mathbf{p}$, the form $dV(\mathbf{p})$ - and, as a matter of fact, any form $\omega(\mathbf{p}) = \lambda(\mathbf{p}).dV(\mathbf{p})$ proportional to $dV(\mathbf{p})$ - vanishes upon the tangent space $T_{\mathbf{p}}M$:

$$\forall \mathbf{p} \in M, \, \forall \xi \in T_{\mathbf{p}}M, \;\; \langle \omega(\mathbf{p}), \xi \rangle = 0 \tag{2}$$

The *integration* problem is exactly this. Starting from some given differential form $\omega(\mathbf{p})$, when is it possible to find a manifold M such that, for any $\mathbf{p}$, the restriction of $\omega(\mathbf{p})$ to $T_{\mathbf{p}}M$ is zero ?

This problem, of course, sounds very familiar for any microeconomist. Take any demand function $\mathbf{x}(\mathbf{p})$ (where income is normalized to 1 by homogeneity). Assume it is invertible, let $\mathbf{p}(\mathbf{x})$ denote the inverse demand function, and consider the form $\pi(\mathbf{x}) = \sum p_i(\mathbf{x})dx^i$. Integrating π means that we are looking for a manifold - or, more precisely, for a *foliation* of the space by manifolds - such that, for any $\mathbf{x}$ on the manifold, the vector $\mathbf{p}(\mathbf{x})$ is orthogonal (at $\mathbf{x}$) to the tangent subspace. In terms of consumer theory, each manifold of the foliation will be called an indifference surface, the tangent subspace a budget constraint, and we are imposing that the price vector be orthogonal to the indifference curve at each point, which is the usual first order condition for utility maximization. This is nothing else than the standard integration problem for individual demand functions[1]. In the next section, we investigate this aspect in more details[2].

One point must however be emphasized. When $\omega(\mathbf{p})$ is proportional to some tangent form dV, the manifold M can be found of (maximum) dimension $(n-1)$. But, of course, life is not always that easy. Starting from an arbitrary form, it is in general impossible to find such a $(n-1)$-dimensional manifold. In general, the integration problem is thus to find a manifold M of *maximum dimension* (or minimum codimension) such that $\omega(\mathbf{p})$ vanishes upon $T_{\mathbf{p}}M$. When the minimum codimension is one, the form is *completely integrable*. But, in general, the minimum codimension will be more than 1.

In fact, this has an interesting translation in terms of our initial problem. Assume, indeed, that instead of being proportional to one tangent form dV, $\omega(\mathbf{p})$ is in fact a linear combination of k tangent forms :

$$\omega(\mathbf{p}) = \lambda_1(\mathbf{p})\, dV^1(\mathbf{p}) + ... + \lambda_k(\mathbf{p})\, dV^k(\mathbf{p})$$

Then we can solve the integration problem with a manifold of codimension (at most) k. Indeed, consider the manifold M defined by :

$$M = \left\{\mathbf{p} \in E \mid V^1(\mathbf{p}) = a_1, ..., V^k(\mathbf{p}) = a_k\right\}$$

Clearly, M is of dimension (at least) $n-k$. Also, the tangent space at $\mathbf{p}$ is the intersection of the tangent spaces to the k manifolds $M^i, ..., M^k$ defined by

$$M^i = \left\{\mathbf{p} \in E \mid V^i(\mathbf{p}) = a_i\right\}$$

It follows that (2) is always fulfilled. A reciprocal property will be given later.

[1] Note that additional restrictions must be imposed upon the manifolds, reflecting monotonicity and quasi-concavity of preferences.

[2] Alternatively, one may consider the form $\omega(\mathbf{p}) = \sum p_i(\mathbf{x})dx_i$. Integration will then lead to recovering the indifference surfaces of the indirect utility function.

2.2 Exterior k-forms and exterior product

Before addressing the integration problem in details, we must generalize our basic concept.

An exterior k-form is a mapping $\omega \ : \ (E)^k \to R$ that is :

- (multi)-linear, i.e., linear w.r.t. each vector :

$$\begin{aligned}\omega\left(\mathbf{p}^1,...,\mathbf{p}^{s-1},a\mathbf{y}+b\mathbf{z},\mathbf{p}^{s+1},...,\mathbf{p}^k\right) &= a.\,\omega\left(\mathbf{p}^1,...,\mathbf{p}^{s-1},\mathbf{y},\mathbf{p}^{s+1},...,\mathbf{p}^k\right)\\ &+b.\,\omega\left(\mathbf{p}^1,...,\mathbf{p}^{s-1},\mathbf{z},\mathbf{p}^{s+1},...,\mathbf{p}^k\right)\end{aligned}$$

$$\forall \ \left(\mathbf{p}^1,...,\mathbf{p}^{s-1},\mathbf{y},\mathbf{z},\mathbf{p}^{s+1},...,\mathbf{p}^k\right) \in E^{k+1}, \quad \forall \ (a,b) \in R$$

- antisymmetric, i.e., whose sign is changed when two vectors are permuted :

$$\forall \ \left(\mathbf{p}^1,...,\mathbf{p}^k\right) \in E^k, \ \ \omega(\mathbf{p}^1,...,\mathbf{p}^i,...,\mathbf{p}^j,...,\mathbf{p}^k) = -.\omega\left(\mathbf{p}^1,...,\mathbf{p}^j,...,\mathbf{p}^i,...,\mathbf{p}^k\right)$$

It follows that for any permutation σ of $\{1,...,k\}$:

$$\forall \ \left(\mathbf{p}^1,...,\mathbf{p}^k\right) \in E^k, \quad \omega\left(\mathbf{p}^{\sigma(1)},...,\mathbf{p}^{\sigma(s)},...,\mathbf{p}^{\sigma(k)}\right) = (-1)^{sign(\sigma)}.\omega\left(\mathbf{p}^1,...,\mathbf{p}^s,...,\mathbf{p}^k\right)$$

Note that, if $k=1$, we are back to the definition of linear forms.

Consider, for instance, the case $k=2$. A 2-form is defined by a matrix Ω :

$$\omega(\mathbf{p},\mathbf{q}) = \sum_{i,j} \omega^{i,j} p_i q_j = \mathbf{p}'\Omega\mathbf{q}$$

Additional restrictions are usually imposed upon the matrix Ω. A standard one is symmetry; i.e., $\Omega = \Omega'$. In EDC, on the contrary, since one considers *exterior* forms, *antisymmetry* is imposed. This gives $\Omega = -\Omega'$, i.e. $\omega_{i,j} = -\omega_{j,i}$ for all i,j; hence

$$\omega(\mathbf{p},\mathbf{q}) = \sum_{i<j} \omega^{i,j}(p_i q_j - p_j q_i)$$

Another case of interest is $k=n$, where n is the dimension of the space E. Then the space of exterior n-form is of dimension one, and include the *determinant.* That is, any n-form ω is colinear to the determinant:

$$\omega\left(\mathbf{p}^1,...,\mathbf{p}^n\right) = \lambda \ \det\left(\mathbf{p}^1,...,\mathbf{p}^n\right)$$

Some well known properties of determinant are in fact due exclusively to multilinearity together with antisymmetry, and can thus be generalized to forms of any order. For instance, take any k-form ω, and take k vectors $(\mathbf{p}^1, ..., \mathbf{p}^k)$ that are not linearly independent. Then $\omega(\mathbf{p}^1, ..., \mathbf{p}^k) = 0$[3]. An important consequence is that, *for any* $k > n$ *, any exterior* $k-$*form must be zero.*

2.2.1 Exterior product

The set of forms is an algebra, on which the multiplication, called the *exterior product,* is formally defined by :

Let ω be a k-form, and γ be a ℓ-form, then $\omega \wedge \gamma$ is a $(k+\ell)$-form such that

$$\omega \wedge \gamma \ (\mathbf{p}^1, ..., \mathbf{p}^{k+\ell}) = \sum_\sigma \frac{1}{k!\ell!} (-1)^{sign(\sigma)}.\omega \ \left(\mathbf{p}^{\sigma(1)}, ..., \mathbf{p}^{\sigma(k)}\right).\gamma \ \left(\mathbf{p}^{\sigma(k+1)}, ..., \mathbf{p}^{\sigma(k+\ell)}\right)$$

where the sum is over all permutations σ of $\{1, ..., k+\ell\}$

The formula may seem complex. Note, however, that it satisfies two basic requirements : $\omega \wedge \gamma$ is multilinear and antisymmetric. To grasp the intuition, consider the case of two linear forms ($k = \ell = 1$). Then

$$\omega \wedge \gamma \ (\mathbf{p}, \mathbf{q}) = \omega(\mathbf{p})\gamma(\mathbf{q}) - \omega(\mathbf{q})\gamma(\mathbf{p})$$

Obviously, this is the simplest 2-form related to ω and γ and satisfying the two requirements above.

A few consequences of this definition must be kept in mind :

- Whenever ω is linear (or of odd order), $\omega \wedge \omega = 0$. More generally, let $\omega_1, ..., \omega_s$ be any 1-forms, and consider the product :

  $$\omega_1 \wedge ... \wedge \omega_s$$

  If the forms are linearly dependent, this product is always zero.

- But : whenever ω is a 2-form (or a form of even order), $\omega \wedge \omega \neq 0$

- Finally, for any k-form, $(\omega)^s = \omega \wedge \omega \wedge ... \wedge \omega$ is a $(k.s)$-form. In particular, $(\omega)^s = 0$ as soon as $k.s > n$.

[3]Indeed, one vector (say, $\mathbf{p}_k$) can be decomposed as a linear combination of the others. Multilinearity implies that $\omega(\mathbf{p}_1, ...\mathbf{p}_k)$ writes down as a linear combination of terms like $\omega(\mathbf{p}_1, ...\mathbf{p}_{k-1}, \mathbf{p}_s)$ with $s < k$. But antisymmetry imposes all these terms be zero.

2.3 Exterior differentiation

The final step is the definition of exterior differentiation. Let $\omega(\mathbf{p})$ be a linear form:

$$\omega(\mathbf{p}) = \sum \omega^j(\mathbf{p})\, dp_j$$

To define the exterior differential of $\omega(\mathbf{p})$, we may first remark that the $\omega^j(\mathbf{p})$ are standard functions from E to $\mathbb{R}$. As such, they admit tangent forms :

$$d\omega^j(\mathbf{p}) = \sum_i \frac{\partial\, \omega^j}{\partial p_i} dp_i$$

Then the exterior differential $d\omega(\mathbf{p})$ of $\omega(\mathbf{p})$ is a 2-form defined by :

$$d\omega(\mathbf{p}) = \sum_j d\omega^j(\mathbf{p}) \wedge dp_j = \sum_{i,j} \frac{\partial\, \omega^j}{\partial p_i}\, dp_i \wedge dp_j = \sum_{i< j} (\frac{\partial\, \omega^j}{\partial p_i} - \frac{\partial\, \omega^i}{\partial p_j})\, dp_i \wedge dp_j \tag{3}$$

Note, again, that this formula guarantees that $d\omega(\mathbf{p})$ is bilinear and antisymmetric.

Now, a formal definition would emphasize the fact that this definition does not depend on the particular basis; in particular, one should introduce the notion of 'pullbacks' and provide a general definition. However, before this more formal presentation, we present a very simple geometric intuition of what exterior differentiation may mean.

The key idea is the emphasis put upon the antisymmetric side of the operation. This can be understood in a 2-dimensional setting. In $E = \mathbb{R}^2$, consider a 1-form $\omega(\mathbf{p}) = \omega^1(\mathbf{p})dp_1 + \omega^2(\mathbf{p})dp_2$, where $\mathbf{p} = (p_1, p_2)$. Consider the four points depicted in Figure 1, namely : $A = (p_1, p_2)$, $B = (p_1{+}\delta p_1, p_2)$, $C = (p_1, p_2{+}\delta p_2)$ and $D = (p_1{+}\delta p_1, p_2{+}\delta p_2)$, where the δp_i are 'infinitesimal'. Assume that we want to compute the following expressions :

$$I = \int_\Gamma \omega(\mathbf{p}) d\mathbf{p} and I' = \int_{\Gamma'} \omega(\mathbf{p}) d\mathbf{p}$$

where Γ (resp. Γ') is the infinitesimal curve ABD (resp. ACD).

INSERT HERE FIGURE 1

Using the infinitesimal nature of the δp_i, we can compute

$$\begin{aligned} I &= \int_A^B \omega(\mathbf{p})d\mathbf{p} + \int_B^D \omega(\mathbf{p})d\mathbf{p} \\ &= \omega^1(p_1,p_2)\delta p_1 + \omega^2(p_1+\delta p_1, p_2)\delta p_2 \end{aligned}$$

and

$$\begin{aligned} I' &= \int_A^C \omega(\mathbf{p})d\mathbf{p} + \int_C^D \omega(\mathbf{p})d\mathbf{p} \\ &= \omega^2(p_1,p_2)\delta p_2 + \omega^1(p_1, p_2+\delta p_2)\delta p_1 \end{aligned}$$

What we are interested in is the difference $I - I'$. As it is well known, if ω was equal to the tangent form dV for some smooth function V, this difference would be zero; in fact, the Jacobian matrix $D_{\mathbf{p}}\omega$ would then be symmetric. In general, using first order approximation, we have that :

$$I - I' = \left(\frac{\partial \omega^2}{\partial p_1} - \frac{\partial \omega^1}{\partial p_2}\right)\delta p_1 \delta p_2$$

This is exactly the coefficient of the 2-form $d\omega(\mathbf{p})$, as defined in (3).

Exterior differentiation is a linear operation, and there is a product formula[4]. If α is a differential p-form and ω a differential q-form, we have :

$$\begin{gathered} d\left[\alpha + \omega\right] = d\alpha + d\omega \\ d\left[\alpha \wedge \omega\right] = d\alpha \wedge \omega + (-1)^p\, \alpha \wedge d\omega \end{gathered}$$

A last property that will turn out to be crucial in the sequel is naturalness with respect to pullbacks. To understand this property, take open subsets $V \subset R^q$ and $U \subset R^p$, and $\varphi\ : V \to U$ a smooth (non-linear) mapping. To any smooth $f\ : U \to R$, we can associate $f \circ \varphi$, which is a smooth function on V. Similarly, to df, which is a 1-form on U, we associate $d(f \circ \varphi)$, which is a 1-form on V, called the *pullback* of f. As a particular case, take for f the ith coordinate map $x \to x^i$ on U ; then the pullback of dx^i will be a 1-form on V that we denote by $\varphi^*(dx_i)$.

[4]Exterior differentiation also has integration properties (Stokes' formula). Since we do not need this part of the theory, we shall nor enter into it in the paper.

The pullback φ^*, being defined for the dx^i, is defined for all 1-forms on U by linearity, and for p-forms in the same way. Also, it is *natural* with respect to exterior products and exterior differentiation, in the following sens :

With the preceding definitions :

$$\varphi^*(\alpha \wedge \omega) = (\varphi^*\alpha) \wedge (\varphi^*\omega)$$
$$\varphi^*(d\omega) = d\,\varphi^*(\omega)$$

These results, and the notion of pullback itself, will be important in studying integral manifolds of exterior differential systems in section 4.

2.4 Poincaré's theorem

The construction detailed above has strong implications for the resolution of the type of equations we are interested in. Let us start with a simple problem : what are the conditions for a given exterior form ω to be the tangent form of some given, twice continuously differentiable function V ? An immediate, necessary condition is given by the following result :

Assume $\omega(\mathbf{p}) = dV(\mathbf{p})$ for some V, then :

$$d[\omega(\mathbf{p})] = 0$$

Just note that,

$$d[\omega(\mathbf{p})] = \sum_{i<j} \left(\frac{\partial^2 V}{\partial p_j \partial p_i} - \frac{\partial^2 V}{\partial p_i \partial p_j}\right) dp_i \wedge dp_j = 0$$

This Proposition admits a converse, due to Poincaré, that requires some topological condition upon U (there should be no 'hole' in U). For the sake of simplicity, let us just assume convexity (a sufficient property), and state the following :

Let ω be a differential 2-form on U such that $d\omega = 0$. Assume U is convex. Then there exists a differential 1-form on U, say Ω, such that :

$$\omega = d\,\Omega$$

: see Bryant et al. (1991), ch.

In fact, Poincaré's result is more general, since it applies to k-forms as well. An immediate consequence is the following. Let $\omega^1(\mathbf{p}), ..., \omega^n(\mathbf{p})$ be given functions. Can we find V such that $\omega^i = \partial V / \partial p_i$? The answer is simple. Define the exterior form $\omega(\mathbf{p}) = \sum \omega^i(\mathbf{p}).\ dp_i$. Then, from the previous results, a necessary and sufficient condition is that :

$$d\omega = \sum_{i,j} \frac{\partial\, \omega^i}{\partial p_j} dp_i \wedge dp_j = \sum_{i< j} (\frac{\partial\, \omega^i}{\partial p_j} - \frac{\partial\, \omega^j}{\partial p_i})\, dp_i \wedge dp_j = 0$$

or

$$\frac{\partial\, \omega^i}{\partial p_j} = \frac{\partial\, \omega^j}{\partial p_i} \quad \forall i, j$$

This result is usually referred to as Frobenius theorem.

2.5 Pfaff theorem

Poincaré's theorem provides necessary and sufficient conditions for a form to be a tangent form (or, equivalently, for vector field to be a gradient field). In this case, the integration problem is straightforward, as illustrated above. But, at the same time, these conditions are very strong. We now generalize this result, by giving necessary and sufficient conditions for a form to be a linear combination of k tangent forms. As discussed above, this means that the integration problem can be solved, but only with an integral manifold of dimension (at least) $(n - k)$.

Assume that $\omega(\mathbf{p})$ can be written under the form :

$$\omega(\mathbf{p}) = \sum_{s=1}^{k} \lambda_s(\mathbf{p}).dV^s(\mathbf{p}) \;\; \forall \mathbf{p} \in U \tag{4}$$

A first remark is that this structure has an immediate consequence. Indeed (forgetting the $\mathbf{p}$ for simplicity) :

$$d\omega = \sum_{s=1}^{k} d\lambda_s \wedge dV^s$$

Let us compute the exterior product $\omega \wedge (d\omega)^k = \omega \wedge d\omega \wedge ... \wedge d\omega \wedge d\omega$. We get first:

$$(d\omega)^k = d\lambda_1 \wedge dV^1 \wedge ... \wedge d\lambda_k \wedge dV^k$$

hence

$$
\begin{aligned}
\omega \wedge (d\omega)^k &= \left(\sum_{s=1}^{k} d\lambda_s \wedge dV^s\right) \wedge d\lambda_1 \wedge dV^1 \wedge ... \wedge d\lambda_k \wedge dV^k \\
&= 0
\end{aligned}
$$

since we get the sum of k terms, each of whom includes twice the same 1-form.

In summary, we have a simple characterization : if ω can be written as in (4), then the product $\omega \wedge (d\omega)^k$ must be zero.

This simple necessary condition admits an important converse.

(Pfaff) Let ω be a linear form, U an open, convex set. Let k be such that :

$$
\omega \wedge (d\omega)^{k-1} = \omega \wedge d\omega \wedge ... \wedge d\omega \neq 0, \quad \forall p \in U
$$
$$
\omega \wedge (d\omega)^{k} = \omega \wedge d\omega \wedge ... \wedge d\omega \wedge d\omega = 0, \quad \forall p \in U
$$

Then there exists $2k$ smooth functions V^s and λ_s such that :

- the V^s are linearly independent
- none of the λ_s vanishes on U
- and
$$
\omega(\mathbf{p}) = \sum_{s=1}^{k} \lambda_s(\mathbf{p}).dV^s(\mathbf{p}) \ \ \forall \mathbf{p} \in U
$$

See Bryant et al. (1991), ch. II, §3

In words, Pfaff theorem provides a necessary and sufficient condition for the mathematical integration problem.

3 Mathematical integration : theory and two applications

In this section, we show, on two specific examples, how the tools previously described have very natural applications in consumer theory.

3.1 Maximization under linear constraint

The basic remark is the following. Consider the program that characterizes the behavior of an individual consumer facing a linear budget constraint :

$$V(\mathbf{p}) = \max_{\mathbf{x}} U(\mathbf{x}) \qquad \mathbf{p}.\mathbf{x} = 1 \tag{5}$$

where the utility function U is continuously differentiable and strongly quasi-concave; note that, from now on, income is normalized to 1. Let $\mathbf{x}(\mathbf{p})$ denote the solution to (5). If α denotes the Lagrange multiplier, we have, from the envelope theorem, that :

$$DV(\mathbf{p}) = -\,\alpha(\mathbf{p}).\mathbf{x}(\mathbf{p}) \tag{6}$$

and $\mathbf{x}(\mathbf{p})$ is proportional to the gradient of the indirect utility V. Incidentally, (5) is equivalent to :

$$-U(\mathbf{x}) = \max_{\mathbf{p}} (-V(\mathbf{p})) \qquad \mathbf{p}.\mathbf{x} = 1 \tag{7}$$

which implies, as above, that

$$DU(\mathbf{x}) = \beta(\mathbf{x}).\mathbf{p}(\mathbf{x}) \tag{8}$$

where $\mathbf{p}(\mathbf{x})$ is the inverse demand function and $\beta(\mathbf{x})$ is the associated Lagrange multiplier; so $\mathbf{p}(\mathbf{x})$ is proportional to the gradient of U[5].

So both $\mathbf{p}(\mathbf{x})$ and $\mathbf{x}(\mathbf{p})$ are proportional to a (single) gradient; we have a problem of the type (1) for $m = 1$. Actually, the programs (8) and (5) are exactly similar, so these two functions share exactly the same properties - a fact that has been known at least since Antonelli (1886).

Now, how does EDC enter the picture[6] ? The idea is to define the linear form ω by :

$$\omega(\mathbf{p}) = \sum x^i(\mathbf{p}).dp^i \tag{9}$$

From Pfaff's theorem, we know that $\mathbf{x}(\mathbf{p})$ is proportional to a gradient if and only if $\omega(\mathbf{p})$ satisfies :

$$\omega \wedge d\omega = 0$$

[5] In fact, (8) can also be seen as the first order conditions of (5); this implies that $\beta\left[\mathbf{x}(\mathbf{p})\right] = \alpha(\mathbf{p})$.

[6] For a development on the links between EDC and Slutsky relations and the consequences upon Gorman forms, see Russell and Farris (1993).

which writes down :

$$\forall\ i,j,k, \qquad x_i(\frac{\partial\, x_j}{\partial p_k}-\frac{\partial\, x_k}{\partial p_j})+x_k(\frac{\partial\, x_i}{\partial p_j}-\frac{\partial\, x_j}{\partial p_i})+x_j(\frac{\partial\, x_k}{\partial p_i}-\frac{\partial\, x_i}{\partial p_k})\ =0 \quad (10)$$

We now show that (10) is nothing else than traditional Slutsky relationships. To see why, note, first, that given our normalization (income is equal to 1), the Slutsky matrix writes down :

$$S = D_{\mathbf{p}}\mathbf{x}.\,(I-\mathbf{p}.\mathbf{x}') \quad (11)$$

where $D_{\mathbf{p}}\mathbf{x}$ is the Jacobian matrix of $\mathbf{x}(\mathbf{p})$. Take some fixed $\bar{\mathbf{p}}$. Slutsky symmetry is equivalent to the following : the restriction of $D_{\mathbf{p}}\mathbf{x}(\bar{\mathbf{p}})$ to the hyperplane orthogonal to $\mathbf{x}(\bar{\mathbf{p}})=\bar{\mathbf{x}}$ must be symmetric. Formally :

$$\forall \mathbf{y},\mathbf{z}\bot\bar{\mathbf{x}},\ \ \mathbf{y}'\,(D_{\mathbf{p}}\mathbf{x})\,\mathbf{z} =\ \mathbf{z}'\,(D_{\mathbf{p}}\mathbf{x})\,\mathbf{y} \Leftrightarrow \mathbf{y}'\,(D_{\mathbf{p}}\mathbf{x}-(D_{\mathbf{p}}\mathbf{x})')\,\mathbf{z}=0 \quad (12)$$

Now, it can readily be seen that the vectors

$$\mathbf{y}_k = \begin{pmatrix} 0 \\ x_k(\bar{\mathbf{p}}) \\ 0 \\ -x_i(\bar{\mathbf{p}}) \\ 0 \end{pmatrix}$$

, where $x_k(\bar{\mathbf{p}})$ (resp. -$x_i(\bar{\mathbf{p}})$) occupies the ith (resp. kth) row, form a basis of $\{\mathbf{x}(\bar{\mathbf{p}})\}^{\bot}$. It is thus sufficient to check (12) for any two $\mathbf{y}_j$ and $\mathbf{y}_k$. But this is exactly equivalent to (10).

Incidentally, given the similarity between (8) and (5), the same conclusion applies to the inverse demand function $\mathbf{p}(\mathbf{x})$; i.e., the matrix A defined by

$$A = D_{\mathbf{x}}\mathbf{p}.\,(I-\mathbf{x}.\mathbf{p}')$$

is symmetric. The reader can check that this is equivalent to the symmetry of the Antonelli matrix.

3.2 Collective household demand

As a second (and less trivial) example, we may take Browning-Chiappori's model of household behavior. Consider a two-member household with respective preferences of the form $u^i(\mathbf{x}^1,\mathbf{x}^2,\mathbf{X})$ for $i=1,2$; here, $\mathbf{x}^i$ denotes the vector of member i's private consumption, while $\mathbf{X}$ is a vector of collective consumption for the household. Assume, as in Browning-Chiappori (1994), that the household is characterized by some decision process, which is only assumed to generate Pareto efficient outcomes. Formally, we suppose there exists some C^1 function $\mu(\mathbf{p})$ such that $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X})$ is a solution of :

$$V(\mathbf{p}, \mu) = \max_{\mathbf{x}^1,\mathbf{x}^2,\mathbf{X}} \mu(\mathbf{p}).u^1(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X}) + [1 - \mu(\mathbf{p})] .u^2(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X}) \tag{13}$$

$$\mathbf{p}.(\mathbf{x}^1 + \mathbf{x}^2 + \mathbf{X}) = 1$$

If $\mathbf{x}(\mathbf{p}) = \mathbf{x}^1(\mathbf{p}) + \mathbf{x}^2(\mathbf{p}) + \mathbf{X}(\mathbf{p})$ denotes the household (aggregate) demand function, what conditions does (13) imply upon the form of $\mathbf{x}(\mathbf{p})$? A necessary condition is the following :

("SR1" property, Browning-Chiappori 1994) : If $\mathbf{x}(\mathbf{p})$ is derived from a program like (13), then the Slutsky matrix $S(\mathbf{p}) = D_{\mathbf{p}}\mathbf{x}.\,(I - \mathbf{p}.\mathbf{x}')$ is the sum of a symmetric matrix Σ and a matrix R of rank at most 1

A detailed proof can be found in the original paper. However, it is important for our present purpose to see the core of the argument. This can be summarized as follows. Define, first, the household utility function by :

$$u^H(\mathbf{x}, \mu) = \max_{\mathbf{x}^1,\mathbf{x}^2,\mathbf{X}} \mu.u^1(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X}) + (1 - \mu) .u^2(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X}) \tag{14}$$

$$\mathbf{x}^1 + \mathbf{x}^2 + \mathbf{X} = \mathbf{x}$$

Note that u^H, as a function of $\mathbf{x}$, is indexed by μ. Then (13) can be written as :

$$V(\mathbf{p}, \mu) = \max_{\mathbf{x}} u^H [\mathbf{x}, \mu(\mathbf{p})] \tag{15}$$

$$\mathbf{p}.\mathbf{x} = 1$$

For any fixed μ, we may define the Marshallian demand $\xi(\mathbf{p}, \mu)$ associated to u^H. This is a standard demand function; in particular, it satisfies Slutsky symmetry. Also, it is related to $\mathbf{x}$ by :

$$\mathbf{x}(\mathbf{p}) = \xi [\mathbf{p}, \mu(\mathbf{p})]$$

It follows that :

$$S(\mathbf{p}) = \Sigma(\mathbf{p}) + \mathbf{u}.\mathbf{v}' \tag{16}$$

where $\Sigma(\mathbf{p})$ is the (symmetric) Slutsky matrix associated to ξ, and where $\mathbf{u}$ and $\mathbf{v}$ are n -vectors such that

$$\mathbf{u} = D_{\mu}\xi \;\; and \;\; \mathbf{v}' = D_{\mathbf{p}}\mu'.\,(I - \mathbf{p}.\mathbf{x}') \tag{17}$$

In particular, $R = \mathbf{u}.\mathbf{v}'$ is of rank at most 1.

Now, let us consider the reciprocal property. Assume that some demand function $\mathbf{x}(\mathbf{p})$ satisfies SR1; is it possible to find three functions $\mathbf{x}^1(\mathbf{p}), \mathbf{x}^2(\mathbf{p}), \mathbf{X}(\mathbf{mathbfp})$, two utility functions, $u^1(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X})$ and $u^2(\mathbf{x}^1, \mathbf{x}^2, \mathbf{X})$, and a function $\mu(\mathbf{p})$ such that $\mathbf{x}(\mathbf{p}) = \mathbf{x}^1(\mathbf{p}) + \mathbf{x}^2(\mathbf{p}) + \mathbf{X}(\mathbf{mathbfp})$, where $[\mathbf{x}^1(\mathbf{p}), \mathbf{x}^2(\mathbf{p}), \mathbf{X}(\mathbf{mathbfp})]$ is a solution of (13) ? In other words, is it possible to find, in the collective setting, a result equivalent to the 'integrability' theorem in the unitary case ?

This question is in fact quite difficult. A first remark is that, when a decomposition like (16) exists, it is not unique. In fact, what is determined by the demand function is simply the subspace $M(\mathbf{p})$ spanned by $\mathbf{u}$ and $\mathbf{v}$; whatever the decomposition, the columns of the R matrix will always belong to $M(\mathbf{p})$. But consider the following question : given a 2-dimensional subspace $M(\mathbf{p})$, is it possible to find two vectors $\mathbf{u}(\mathbf{p})$ and $\mathbf{v}(\mathbf{p})$, always belonging to $M(\mathbf{p})$, and some function $\mu(\mathbf{p})$, such that (17) is always fulfilled ? The answer is not easy, given that, in particular, we impose that $\mathbf{v}$ is related to a gradient (which introduces strong conditions). Also, even if we solve this, we still have to construct the function ξ, taking into account not only the Σ matrix but also the form of $\mathbf{u}$. The problem may thus seem quite intricate.

We now show that the EDC point of view provides a spectacular simplification of the problem. In fact, it allows to solve an even more difficult question, since we may impose, in addition, that all goods are publicly consumed :

$$\mathbf{x}^1(\mathbf{p}) = \mathbf{0}, \quad \mathbf{x}^2(\mathbf{p}) = \mathbf{0}, \quad \mathbf{x}(\mathbf{p}) = \mathbf{X}(\mathbf{mathbfp})$$

Under this further restriction, the program becomes :

$$\max_{\mathbf{x}} \mu(\mathbf{p}).u^1(\mathbf{x}) + (1 - \mu(\mathbf{p}))\,.u^2(\mathbf{x}) \tag{18}$$

$$\mathbf{p}.\mathbf{x} = 1$$

Let $V(\mathbf{p})$ denote the corresponding value of the maximand. The envelope theorem states that :

$$D_{\mathbf{p}}V(\mathbf{p}) = -\,\alpha(\mathbf{p}).\mathbf{x}(\mathbf{p}) + \left(u^1(\mathbf{x}(\mathbf{p})) - u^2(\mathbf{x}(\mathbf{p}))\right) D_{\mathbf{p}}\mu(\mathbf{p})$$

which can be written as :

$$\mathbf{x}(\mathbf{p}) = \lambda_1(\mathbf{p}).D_{\mathbf{p}}V(\mathbf{p}) + \lambda_2(\mathbf{p})D_{\mathbf{p}}\mu(\mathbf{p}) \tag{19}$$

The mathematical integration approach leads to considering the following question : when can a given function $\mathbf{x}(\mathbf{p})$ be written under the form (19) ? The answer is strikingly simple, as expressed by the following theorem :

A given, continuously differentiable function $\mathbf{x}(\mathbf{p})$ satisfying $\mathbf{p}.\mathbf{x}(\mathbf{p}) = 1$ can be written under the form (19) if and only if it satisfies SR1.

A complete proof in Appendix. But the important and previously difficult part - the 'if' - is now immediate. Define, as above, the linear form ω by :

$$\omega(\mathbf{p}) = \sum x^i(\mathbf{p}).dp_i$$

From Pfaff's theorem, a necessary and sufficient condition for (19) is that :

$$\omega \wedge d\omega \wedge d\omega = 0 \tag{20}$$

Now, note that SR1 implies the following :

$$d\omega - \omega \wedge a = b \wedge c$$

where a, b and c are 1-forms whose definition is clear. Then

$$d\omega \wedge d\omega = \omega \wedge a \wedge b \wedge c$$

and (20) is fulfilled.

Two remarks can be made at this point. First, SR1 turns out to be necessary and sufficient for mathematical integration. But economic integration requires additional restrictions. Namely, $\lambda_1(\mathbf{p})$ must be negative; and $V(\mathbf{p})$ must be decreasing and quasi-convex. In fact, only the last restriction turns out to be really binding. Browning and Chiappori (1994) find additional restrictions linked to quasi-convexity[7]. But we do not know yet whether these are sufficient for economic integration.

Secondly, we can characterize inverse demand in exactly the same way. Indeed, first order conditions of (18) imply that :

$$\mu\left(\mathbf{p}(\mathbf{mathbfx})\right).D_{\mathbf{x}}u^1(\mathbf{x}) + \left[1 - \mu\left(\mathbf{p}(\mathbf{mathbfx})\right)\right].D_{\mathbf{x}}u^2(\mathbf{x}) = \alpha(\mathbf{p}).\mathbf{p}(\mathbf{x}) \tag{21}$$

and we conclude that :

[7]The restriction of $S(\mathbf{p})$ to the orthogonal of $M(\mathbf{p})$ must be negative.

A given, continuously differentiable function $\mathbf{p}(\mathbf{x})$ satisfying $\mathbf{p}(\mathbf{x}).\mathbf{x} = 1$ can be written under the form (21) if and only if it satisfies SR1, i.e., if the matrix

$$A = D_{\mathbf{x}}\mathbf{p}.\,(I - \mathbf{x}.\mathbf{p}')$$

can be written as the sum of a symmetric matrix and a matrix of rank at most one.

This alternative version leads to a very nice geometric interpretation. Start from a demand function such that SR1 is satisfied, while Slutsky symmetry is not. At some given point $\mathbf{x}$, consider the two indifference surfaces corresponding respectively to u^1 and u^2, and call N their intersection (see Fig. 2).

INSERT HERE FIGURE 2

Take any vector $\mathbf{y}$ that is orthogonal to both gradients $D_{\mathbf{x}}u^1$ and $D_{\mathbf{x}}u^2$; then $\mathbf{y}$ is tangent to N. The relation (21) expresses that $\mathbf{y}$ is orthogonal to the price vector $\mathbf{p}(\mathbf{x})$. From the integration viewpoint, since Slutsky symmetry does not hold, complete integration does not obtain. The integral manifold of maximum dimension is N, which is of dimension $(n-2)$. This has an immediate economic translation : instead of recovering one indifference surface, as in the Slutsky case, we recover the intersection of two indifference surfaces, who respectively correspond to the two agents in the household.

4 Exterior differential systems on manifolds : the Cartan-Kähler theorem

We now present the key result upon which our appraoch relies. This theorem, due to Cartan and Kähler, solves the following, general problem. Given a certain family of differential forms (not necessarily 1-forms, nor even of the same degree), a point $\bar{x}$ and an integer $m \geq 1$, can one find some m-dimensional submanifold M containing $\bar{x}$ and on which all the given forms vanish ?

4.1 An introductory example

As an introduction, let us start from a simple version of our problem, namely the Cauchy-Lipschitz theorem for ordinary differential equations. It states that, given a point $\bar{x} \in \mathbb{R}^{n-1}$ and a C^1 function f, defined from some neighborhood $\mathcal{U}$ of $\bar{x}$ into $\mathbb{R}^{n-1}$, there exists some $\epsilon > 0$ and a C^1 function $\varphi :$ $]-\epsilon, \epsilon\,[\longrightarrow \mathcal{U}$ such that

$$\frac{d\varphi}{dt} = f\left(\varphi(t)\right) \quad \forall t \;\in\,]-\epsilon, \epsilon\,[\tag{22}$$

$$\varphi(0) = \bar{x}$$

It follows that $\frac{d\varphi}{dt}(0) = f(\bar{x})$. If $f(\bar{x}) = 0$, the solution is trivial, $\varphi(t) = \bar{x}$ for all t so we assume that $f(\bar{x})$ does not vanish.

This theorem can be rephrased in a geometric way. Consider the graph M of φ :

$$M = \{\ (t, \varphi(t)\ )\ |\ -\epsilon < t < \epsilon\ \}$$

It is a 1-dimensional submanifold of $]-\epsilon, \epsilon\,[\times \mathcal{U}$ we introduce the 1-forms ω^i defined by :

$$\omega^i = f^i(x)dt - dx^i \quad , \quad 1 \leq i \leq n-1$$

Clearly φ solves the differential equation (22) if and only if the ω^i all vanish on M. More precisely, substituting $x^i = \varphi^i(t)$ into formula (4) yields the pullbacks :

$$\varphi^*\omega^i = \left[f^i\left(\varphi(t)\right) - \frac{d\varphi^i}{dt}(t) \right] dt$$

which vanish if and only if φ solves equation (22).

So the Cauchy-Lipschitz theorem tells us how to find a 1-dimensional submanifold of $\mathbb{R} \times \mathbb{R}^n$ on which certain 1-forms vanish.

4.2 The general problem

The Cauchy-Lipschitz theorem deals with 1-forms of a specific nature. By extension, the fully general problem can formally be stated as follows.

Definition *Let ω^k , $1 \leq k \leq K$, be differential forms on an open subset of $\mathbb{R}^n$, and $M \subset \mathbb{R}^n$ a submanifold. We call M an* integral manifold *of the system :*

$$\omega^1 \;=\; 0\;, ..., \omega^K \;=\; 0 \tag{23}$$

if the pullbacks of the ω^i to M all vanish :

$$\omega_x^k \left(\xi^1, ...\xi^{d_k}\right) \;=\; 0 \quad 1 \leq k \leq K \tag{24}$$

whenever $x \in M$, ω M , ω^k has degree d_k, and $\xi^i \in T_x M$ for $1 \leq i \leq d_k$.

Given a point $\bar{x} \in \mathbb{R}^n$, the Cartan-Kähler theorem will give necessary and sufficient conditions for the existence of an integral manifold containing $\bar{x}$.

Necessary conditions are easy to find. Assume an integral manifold $M \ni \bar{x}$ exists, and let m be its dimension. Then its tangent space at $\bar{x}$, $T_{\bar{x}}M$, is m-dimensional, and all the $\omega_{\bar{x}}^j$ must vanish on $T_{\bar{x}}M$, because of formula (7). Any subspace $E \subset T_{\bar{x}}M$ with this property will be called an integral element of system (6) at $\bar{x}$. The set of all m-dimensional integral elements at $\bar{x}$ will be

$$G_{\bar{x}}^m \;=\; \left\{ E \;\middle|\; \begin{array}{c} E \subset T_{\bar{x}}M \text{ and dim } \; E = m \\ \omega_{\bar{x}}^1, ..., \omega_{\bar{x}}^K \text{ vanish on } E \end{array} \right\}$$

Our first necessary condition is clear :

$$G_{\bar{x}}^m \;\neq\; \oslash \tag{25}$$

4.3 Differential ideals

To get the second one, let us ask a strange question : have we written all the equations ? In other words, does the system :

$$\omega^1 \;=\; 0, ..., \omega^K \;=\; 0 \tag{26}$$

exhibit all the relevant information ?

The answer is negative. To see why, recall that M is a submanifold of $\mathbb{R}^n$, and denote by $\varphi_M \;:\; M \to \mathbb{R}^n$ the standard embedding $\varphi_M(x) \;=\; x$ for all $x \in M$. M is an integral manifold of system (26) if :

$$\varphi_M^* \,\omega^1 \;=\; 0, ..., \varphi_M^* \,\omega^K \;=\; 0 \tag{27}$$

But we know that exterior differentiation is natural with respect to pull-backs, that is, d commutes with φ_M^*. So (27) implies :

$$\varphi_M^*(d\omega^1) \ = \ 0, ..., \varphi_M^*(d\omega^K) \ = \ 0$$

In other words, M is also an integral manifold of the larger system :

$$\begin{cases} \omega^1 \ = \ 0, ..., \ \omega^K = 0 \\ d\omega^1 \ = \ 0, ..., d\omega^K = 0 \end{cases} \tag{28}$$

which is different from (26). So integral elements of (28) will be different from integral elements of (26), and it is not clear which ones we should be working with.

To resolve this quandary, we shall assume that systems (26) and (28) have the same integral elements. In other words, the second equations in (28) must be *algebraic* consequences of the first ones. The precise statement for this is as follows :

Definition. *The family* $\{\omega^k | 1 \leq k \leq K\}$ *is said to generate a differential ideal if there are forms* $\{\alpha_j^k \mid 1 \leq j, k \leq K \}$ *such that :*

$$\forall k, \ d\omega^k \ = \ \Sigma \ \alpha_j^k \wedge \omega^j \tag{29}$$

Our second necessary condition is that the ω^k, $1 \leq k \leq K$, must generate a differential ideal.

Note that if the given family $\left\{\omega^k \ |1 \leq k \leq K\right\}$ does not satisfy this condition, the enlarged family $\left\{\omega^k \ , \ d\omega^k | 1 \leq k \leq K\right\}$ certainly will (because $dd\omega^k = 0$). So the condition that the ω^k must generate a differential ideal can be understood as saying that the enlargement procedure has already taken place.

4.4 A counter example

It turns out that conditions (25) and (29) are almost sufficient. All we have to do is to replace (25) by a slightly stronger condition.

To see that (25) is not sufficient, let us give a simple example. Consider two functions f and g from $\mathbb{R}^{n-1}$ into itself, with $f(0) = g(0) = v \neq 0$. Define α^i and β^i , $1 \leq i \leq n-1$, by

$$\alpha^i \ = \ f^i(x)dt - dx^i$$

$$\beta^i \;=\; g^i(x)dt - dx^i$$

and consider the exterior differential system in $\mathbb{R}^n$:

$$\alpha^i \;=\; 0 \;,\; \beta^i \;=\; 0 \;, \quad 1 \leq i \leq n-1$$

The α^i and the β^i generate a differential ideal, and there is an integral element at 0, namely the line carried by $(1, N)$, so $G_0^1 \neq \oslash$. However, finding an integral manifold of the initial system containing 0 amounts to finding a common solution of the two Cauchy problems :

$$\frac{dx}{dt} \;=\; f(x) \;,\; x(0) \;=\; 0$$

$$\frac{dx}{dt} \;=\; g(x) \;,\; x(0) \;=\; 0$$

which does not exist in general. The problem clearly is that the equality $f(x) = g(x)$ holds at $x = 0$ only. So we need a regularity condition which will exclude such pathological situations - technically, that guarantees that the required equality hold true at *ordinary points*, a concept we now formally define.

4.5 Regularity conditions

If all the $\omega_k(x)$ are 1-forms, the regularity condition is clear enough : the dimension of the space spanned by the $\omega_k(x)$, $1 \leq k \leq K$, in $T_x^*\mathbb{R}^n$, should be constant on a neighborhood of $\bar{x}$ (which is obviously not the case in the counterexample above). Note that, locally, this dimension can only increase, that is, the codimension can only decrease.

If some of the ω_k have higher degree, the regularity condition is more complicated, because several codimensions are involved. However, the idea is the same : all these codimensions should be constant in a neighborhood of $\bar{x}$, which amounts to saying that they should have the lowest possible value at $\bar{x}$. This is expressed in the following.

Pick a point $\bar{x} \in \mathbb{R}^n$; from now on, we work in the tangent space $V = T_x\mathbb{R}^n$. Let $E \subset V$ be an m-dimensional integral element at $\bar{x}$. Pick a basis $\bar{\alpha}_1, ..., \bar{\alpha}_n$ of V^* such that :

$$E \;=\; \{\xi \in V \mid \; <\xi, \bar{\alpha}_i> = 0 \quad \forall i \geq m+1\;\}$$

For $n' \leq n$, denote by $\mathcal{I}(n', d)$ the set of all ordered subsets of $\{1, ..., n'\}$ with d elements. Denote by d_k the degree of ω_k. For every k, writing $\omega_k(\bar{x})$ in the $\bar{\alpha}_i$ basis, we get

$$\omega_k(\bar{x}) \;=\; \sum_{I \in \mathcal{I}(n, d_k)} c_I^k \; \bar{\alpha}_{i_1} \wedge ... \wedge \bar{\alpha}_{i_{d_k}}.$$

In this summation, it is understood that $I = \{i_1, ..., i_{d_k}\}$. Since $\omega^k(\bar{x})$ vanishes on E, each monomial must contain some $\bar{\alpha}_i$ with $i \geq m+1$. Let us single out the monomials containing one such term only. Regrouping and rewriting, we get the expression :

$$\omega^k(\bar{x}) = \sum_{J \in \mathcal{I}(m, d_k - 1)} \bar{\beta}_J^k \wedge \bar{\alpha}_{j_1} \wedge ... \wedge \bar{\alpha}_{j d_k - 1} \; + \; \text{remainder}$$

where $\bar{\beta}_J^k$ is a linear combination of the α_i for $i \geq m+1$; and all the monomials in the remainder contain $\bar{\alpha}_i \wedge \bar{\alpha}_{i'}$ for some $i > i' \geq m+1$.

Define an increasing sequence of linear subspace $H_0^* \subset H_1^* \subset ... \subset H_M^* \subset V^*$ as follows :

$$\begin{aligned} H_m^* &= \text{ Span}[\; \bar{\beta}_J^k \;|1 \leq k \leq K, \; J \in \mathcal{I}(m, d_k - 1)\} \\ H_{m-1}^* &= \text{ Span}[\; \bar{\beta}_J^k \;|1 \leq k \leq K, \; J \in \mathcal{I}(m-1, d_k - 1)\} \\ H_0^* &= \text{ Span}[\; \bar{\beta}_J^k \;|1 \leq k \leq K, \; J \in \mathcal{I}(0, d_k - 1)\} \end{aligned}$$

The latter is just the linear subspace generated by those of the $\omega^k(\bar{x})$ which happen to be 1-forms. We define an increasing sequence of integers $0 \leq c_0(\bar{x}, E) \leq ... \leq c_m(\bar{x}, E) \leq n$ by :

$$c_i(\bar{x}, E) \;=\; \dim H_i^*$$

We are finally able to express Cartan's regularity condition. Denote by $\mathbb{P}^m(\mathbb{R}^n)$ the set of all m-dimensional subspaces of $\mathbb{R}^n$ with the standard (Grassmannian) topology : it is known to be a manifold of dimension $m(n-m)$. Denote by G^m the set of all (x, E) such that E is an m-dimensional integral element at x. Note that G^m is a subset of $\mathbb{R}^n \times \mathbb{P}^m(\mathbb{R}^n)$.

Definition. *Let $(\bar{x}, \bar{E}) \in G^m$ - that is, $\bar{E}$ is an m-dimensional integral element at $\bar{x}$. We say that $(\bar{x}, \bar{E})$ is ordinary if there is some neighborhood $\mathcal{U}$ of $(\bar{x}, \bar{E})$ in $\mathbb{R}^n \times \mathbb{P}^m(\mathbb{R}^n)$ such that $G^m \cap \mathcal{U}$ is a submanifold of codimension*

$$c_0(\bar{x}, \bar{E}) + ... + c_{m-1}(\bar{x}, \bar{E})$$

If all the ω^k are 1-forms, denote by $d(x)$ the dimension of the space spanned by the $\omega^k(x)$. Then $c_i(x,E) = d(x)$ for every i, and $(\bar{x}, \bar{E})$ is ordinary if $G^m \cap \mathcal{U}$ is a submanifold of codimension $md(\bar{x})$ in $\mathbb{R}^n \times \mathbb{P}^m(\mathbb{R}^n)$. This implies that, for every x in a neighborhood of $\bar{x}$, the set of $E \in G_x^m$ (integral elements at x) has codimension $md(\bar{x})$ in $\mathbb{P}^m(\mathbb{R}^n)$. It can be seen directly to have codimension $md(x)$. So $d(x) = d(\bar{x})$ in a neighborhood of $\bar{x}$: this is exactly the regularity condition we wanted for 1-forms.

In the general case, if $(\bar{x}, \bar{E})$ is ordinary, the numbers c_i will also be locally constant :

$$c_i(x,E) \;=\; c_i(\bar{x},\bar{E}) \;=\; c_i \quad \forall (x,E) \in \mathcal{U}.$$

The (non-negative) numbers

$$\begin{aligned} s_0 &= \; c_0 \\ s_i &= \; c_i - c_{i-1} \quad \text{for} \quad 1 \leq i < m \\ s_m &= \; n - m - c_{n-1} \end{aligned}$$

are called the *Cartan characters*. We shall use them later on.

4.6 The main result

We are now in a position to state the Cartan-Kähler theorem :

(Cartan-Kähler) Consider the exterior differential system :

$$\omega^k \;=\; 0 \;, \quad 1 \leq k \leq K \tag{30}$$

Assume that the ω^k are real analytic and that they generate a differential ideal. Let $\bar{x}$ be a point and let $\bar{E}$ be a regular integral element at $\bar{x}$. Then there is a real analytic integral manifold M, containing $\bar{x}$ and such that :

$$T_{\bar{x}}M \;=\; \overline{E} \;. \tag{31}$$

Nothing should come as a surprise in this statement, except the real analyticity. It comes from the very generality of the Cartan-Kähler theorem. Indeed, every system of partial differential equations can be written as an exterior differential system.

To conclude, let us mention the question of uniqueness. There is no uniqueness in the Cartan-Kähler theorem : there may be infinitely many analytic integral manifolds going through $\bar{x}$ and having $\bar{E}$ as a tangent space at $\bar{x}$. However, the theorem describes in a precise way (not given here) the set

$$\mathcal{M}_{\mathcal{U}} = \left\{ M \;\middle|\; \begin{array}{l} M \text{ is an integral manifold} \\ \text{and there exists } (x, E) \in \mathcal{U} \\ \text{such that } x \in M \text{ and } T_x M = E \end{array} \right\},$$

where $\mathcal{U}$ is a suitably chosen neighborhood of $(\bar{x}, \bar{E})$. Loosely speaking, each M in $\mathcal{M}_{\mathcal{U}}$ is completely determined by the (arbitrary) choice of s_m analytic functions of m variables, the s_m being the Cartan character.

5 Application : the characterization of aggregate market demand

5.1 The problem

The main application we draw in this paper from the Cartan-Kähler theorem is the characterization of aggregate market demands. The problem, initially raised by Sonnenschein (1993b), can be stated as follows. Take some continuous mapping $\mathbf{X}(\mathbf{p}) : \mathbb{R}^n \to \mathbb{R}^n$, such that $\mathbf{p}.\mathbf{X}(\mathbf{p}) = n$ (Walras Law). Can we find n individual demand functions $\mathbf{x}^1(\mathbf{p}), ..., \mathbf{x}^n(\mathbf{p})$ such that

$$\mathbf{X}(\mathbf{p}) = \mathbf{x}^1(\mathbf{p}) + ... + \mathbf{x}^n(\mathbf{p}) \tag{32}$$

where $x_i(p)$ is the solution of

$$\begin{aligned} V^{\,i}(\mathbf{p}) &= \max U^i(\mathbf{x}^i) \\ \mathbf{p}.\mathbf{x}^i &= 1 \\ \mathbf{x}^i &\geq 0 \end{aligned} \tag{33}$$

for some well-behaved utility function U^i ?

Sonnenschein's original paper raised both the issue of aggregate demand and that of aggregate excess demand. In the second case, the problem is similar, the difference being that each agent is characterized by some initial *endowment* (instead of income). This, in turn, implies some modifications in the form of the basic equations.

The techniques we now describe actually apply to both problems in exactly the same way. But, as it is well known, the second question - the characterization of excess demand - has been solved by Debreu (1974). In fact, Debreu's contribution solves a more difficult problem than the mere characterization issue. When looking at Debreu's proof, one is stuck by the fact that it does not only prove the existence of a solution - i.e., of a set of preferences and initial endowments that generate, as aggregate excess demand, the initial function. It also provides a way of explicitly constructing such a solution - i.e., each of the n individual demands[8]. This result is quite striking. In many existence theorems of the (mathematical) literature, the result is established "indirectly" (say, by showing that non existence would lead to a contradiction), and the construction of a solution is left aside. Even when an explicit construction obtains, it generally leads to (at best) a numerical algorithm. In Debreu's proof, however, an explicit, analytical solution is provided.

This appears as a very strong result. But, at the same time, one can hardly hope that life will always be so easy. In the case of aggregate market demand - a problem that is still open - looking for an explicit, analytical solution is probably too demanding; it might be the case that such a solution simply does not exist in general. In what follows, we only show existence. While our approach does provide basic insights for the numerical construction of a solution, nothing indicates that the latter will exhibit a simple form. But, of course, that is not needed anyway.

We now tackle the problem. A first remark, due to Sonnenschein, is that, in contrast with the excess demand problem, the characterization of market demand will face complex non-negativity restrictions. In particular, he exhibits a counter-example of a function $\mathbf{X}$ that cannot be globally decomposed as above because of these constraints. However, the local version of the problem remains : is it possible, for any given $\mathbf{p} >> 0$, to find individual demand functions $\mathbf{x}^1(\mathbf{p}), ..., \mathbf{x}^n(\mathbf{p})$, defined on some neighborhood of $\mathbf{p}$, such that (32) and (33) are fulfilled on this neighborhood ?

A result initially demonstrated by Sonnenschein (1973b) and then generalized by Diewert (1977) and Mantel (1977), states that for $n \geq \ell$ any continuous function satisfying Walras Law does, when considered *at some given point* $\bar{\mathbf{p}}$, 'look like' aggregate market demand, in the following sense :

[8] The explicit construction of utility functions has been consequently derived by Geanakoplos (1978).

it is possible to find individual demand functions $\mathbf{x}^1(\mathbf{p}), ..., \mathbf{x}^n(\mathbf{p})$ such that :

$$\mathbf{X}(\bar{\mathbf{p}}) = \sum_i \mathbf{x}^i(\bar{\mathbf{p}}),\ D_{\mathbf{p}}\mathbf{X}(\bar{\mathbf{p}}) = \sum_i D_{\mathbf{p}}\mathbf{x}^i(\bar{\mathbf{p}})$$

In their 1982 survey, Shafer and Sonnenschein ask whether it is possible to go beyond this result, and find the $\mathbf{x}^i(\mathbf{p})$ such that $\mathbf{X}(\mathbf{p})$ coincides with $\sum_i \mathbf{x}^i(\mathbf{p})$ on an open neighborhood of $\bar{\mathbf{p}}$. While Andreu (1983) has demonstrated this property for finite sets of price-income bundles, the continuous version has not yet been established. In what follows, we show that the answer to the question is positive, at least if we assume that the function $\mathbf{X}$ is analytic on such a neighborhood (which implies, in particular, that it is infinitely differentiable).

5.2 The EDC approach

Let us first rephrase the problem in terms of EDC. First, if V^i denote consumer i's indirect utility, we know that utility maximization implies $D_{\mathbf{p}}V^i(\mathbf{p}) = -\alpha_i.\mathbf{x}^i(\mathbf{p})$, where α_i is the Lagrange multiplier. It follows that :

$$\begin{aligned} \mathbf{X}(\mathbf{mathbfp}) &= -\frac{1}{\alpha_1(\mathbf{p})} D_{\mathbf{p}}V^1(\mathbf{p}) - ... - \frac{1}{\alpha_n(\mathbf{p})} D_{\mathbf{p}}V^n(\mathbf{p}) \qquad (34) \\ &= \lambda_1(\mathbf{p})\, D_{\mathbf{p}}V^1(\mathbf{p}) + ... + \lambda_n(\mathbf{p})\ D_{\mathbf{p}}V^n(\mathbf{p}) \end{aligned}$$

and $X(\mathbf{p})$ must be a linear combination of n gradients. In addition :

- the V^i are (quasi) convex and decreasing
- the λ_i are negative
- furthermore, the budget constraint implies :

$$\mathbf{p}.D_{\mathbf{p}}V^i(\mathbf{p}) = 1/\lambda_i \quad \forall i \qquad (35)$$

The problem is thus to find, on a neighborhood of some given $\bar{\mathbf{p}}$, functions $\lambda_1, ..., \lambda_n$ and $V^1, ..., V^n$ satisfying (34) and the set of conditions (35).

We now describe the basic strategy used throughout the proof. Consider the space $E = \{\mathbf{p}, \lambda_1, ..., \lambda_n, \mathbf{\Delta}^1, ..., \mathbf{\Delta}^n\} = \mathbb{R}^{2n+n^2}$(the vector $\mathbf{\Delta}^i$ will later be interpreted as the $D_{\mathbf{p}}V^i$). Clearly, if a solution exists, then the equations

$\lambda_i = \lambda_i(\mathbf{p})$ and $\mathbf{\Delta}^i = \mathbf{\Delta}^i(\mathbf{p})$ define a (n-dimensional) manifold $\mathcal{S}$ in E; and $\mathcal{S}$ is included in the n^2-dimensional manifold $\mathcal{M}$ defined by :

$$\begin{aligned} \mathbf{X}(\mathbf{p}) &= \sum_i \lambda_i \mathbf{\Delta}^i \\ \mathbf{p}.\mathbf{\Delta}^i &= 1/\lambda_i \quad \forall i \end{aligned} \tag{36}$$

Conversely, assume that we have found functions $\lambda_i = \lambda_i(\mathbf{p})$ and $\mathbf{\Delta}^i = \mathbf{\Delta}^i(\mathbf{p})$ such that :

- for every $\mathbf{p}$, $\{\mathbf{p}, \lambda_1(\mathbf{p}), ..., \lambda_n(\mathbf{p}), \mathbf{\Delta}^1(\mathbf{p}), ..., \mathbf{\Delta}^n(\mathbf{p})\}$ belong to $\mathcal{M}$
- for every $i = 1, ..., n$, $\mathbf{\Delta}^i(\mathbf{p})$ satisfies the equation :

$$d\left(\sum_j \mathbf{\Delta}^{ij} dp_j\right) = \sum_j d\mathbf{\Delta}^{ij} \wedge dp_j = \sum_{k<j} \left(\frac{\partial \mathbf{\Delta}^{ij}}{\partial p_k} - \frac{\partial \mathbf{\Delta}^{ik}}{\partial p_j}\right) dp_k \wedge dp_j = 0$$

Then $\mathbf{\Delta}^i(\mathbf{p})$ is the gradient of some function V^i, and the $(V^1, ..., V^n)$ solve the problem. In the language of the previous section, we are looking for an n-dimensional integral manifold of the exterior differential system :

$$\sum_j d\mathbf{\Delta}^{ij} \wedge dp_j = 0 \quad \forall i \tag{37}$$

Finally, the solution must be parametrized by $(p_1, ..., p_n)$. The formal translation of this is :

$$dp_1 \wedge ... \wedge p_n \neq 0 \tag{38}$$

An important remark, here, is that this system is closed, in the sense of the previous section (since all 1-forms involved are already tangent forms). So the condition that the forms constitute a differential ideal is automatically fulfilled.

The idea, now, is to consider (37) and (38) as an exterior differential system to be solved on the manifold $\mathcal{M}$. Following the approach described in the previous section, the proof is in two steps.

- As a first step, one must look for a solution of the *linearized* problem (at some given point $\bar{\mathbf{p}}$). Specifically, choose (arbitrarily) the values (at $\bar{\mathbf{p}}$) of λ_i and $\mathbf{\Delta}^i = D_{\mathbf{p}} V^i$. In particular, one may choose $\lambda_i < 0$, $\mathbf{\Delta}^i << 0$ and $\mathbf{\Delta} = (\mathbf{\Delta}^1, ..., \mathbf{\Delta}^n)$ invertible; if these properties hold at $\bar{\mathbf{p}}$, they will hold by continuity on a neighborhood as well. Now, linearize λ_i and $\mathbf{\Delta}^i$ (as functions of $\mathbf{p}$) around $\bar{\mathbf{p}}$:

$$\frac{\partial \lambda_i}{\partial p_j} = N^i_j$$
$$\frac{\partial \mathbf{\Delta}^i_k}{\partial p_j} = M^i_{k\,j}$$

Solving the linearized problem is equivalent to finding vectors N^i and matrices M^i that satisfy the integration equations, i.e., (37) and (38), plus the equations expressing that λ_i and Δ^i remain on the manifold $\mathcal{M}$ (the latter obtain by differentiating (36)); in addition, we want the V^i to be convex. Formally, we write that :

– $\mathbf{\Delta}^i$ is the gradient of a convex function; this implies that

$$M^i symmetricpositive, i = 1, ..., n$$

– 'the point remains on the manifold', which leads to :

$$D_{\mathbf{p}}\mathbf{X}(\bar{\mathbf{p}}) = \sum_i \left(\mathbf{\Delta}^i D_{\mathbf{p}} \lambda'_i + \lambda_i D_{\mathbf{p}} \mathbf{\Delta}^i\right) = \sum_i \left(\mathbf{\Delta}^i N'_i + \lambda_i M^i\right)$$

$$M^i \mathbf{p} + \mathbf{\Delta}^i = -\frac{1}{\lambda_i^2} N_i \Leftrightarrow N'_i = -\lambda_i^2 (\mathbf{p}' M^i + \mathbf{\Delta}^{i\prime})$$

- the second, and more tricky step is to show that the previous conditions hold true at *ordinary* points. This is crucial in order to go from a solution to the linearized version at each point to a solution to the general, non-linear problem; a move that may not be possible otherwise, as illustrated by the counter-examples in the previous section. Formally, this requirement translates into the fact that the subspaces involved have the 'right' codimension.

Is it possible to find vectors N^i and matrices M^i that satisfy the previous conditions ? The answer is positive; a general proof is in Appendix. Two remarks can be made at this point :

- the technique used in this proof applies not only to the aggregate demand problem, but also to excess demand, and presumably to other problems of the same type. In particular, it is fairly easy to redemonstrate the excess demand theorem, even in the local version due to Geanakoplos and Polemarchakis (1980), which requires only $n-1$ consumers. However, we do not develop this result here; instead, we concentrate upon the (original) proof of the market demand case.
- in the present case, the existence of a solution to the linearized problem (step one above) is in fact a consequence of known results of the literature, due to Sonnenschein (1973b), Diewert (1977) and Mantel (1977). These results, however, are not sufficient for the present purpose, because they do not allow to check the codimension properties of step two. The proof we provide in Appendix does indeed allow to compute the required dimensions.

Once these conditions have been checked, Cartan-Kähler theorem applies. Finally, one gets the following statement :

Consider some open set $\mathcal{U}$ in $\mathbb{R}^n - \{0\}$ and some analytic mapping $\mathbf{X} : \mathcal{U} \to \mathbb{R}^n$ such that $p.X(p) = 1$. For all $\bar{p} \in \mathcal{U}$ and for all $(\bar{\mathbf{x}}_1, ..., \bar{\mathbf{x}}_n) \in \mathbb{R}^{n^2}$ and $(\bar{\lambda}_1, ..., \bar{\lambda}_n) \in \mathbb{R}^n$ that satisfy :

$$\begin{aligned} \bar{\mathbf{x}}_1 + ... + \bar{\mathbf{x}}_n &= \mathbf{X}(\bar{\mathbf{p}}) \\ \forall i, \quad \lambda_i &> 0 \end{aligned}$$

, there exist n functions $U^1, ..., U^n$, where each U_i is defined in some convex neighborhood $\mathcal{U}_i$ of $\bar{\mathbf{x}}_i$ and is analytic and strictly quasi-concave in $\mathcal{U}_i$, n mappings $(\mathbf{x}_1, ..., \mathbf{x}_n)$ and n functions $(\lambda_1, ..., \lambda_n)$, all defined in some neighborhood $\mathcal{V}$ of $\bar{\mathbf{p}}$ and analytic in $\mathcal{V}$, such that, for all $\mathbf{p} \in \mathcal{V}$:

$$\begin{aligned} \mathbf{p}.\mathbf{x}_i(\mathbf{p}) &= 1/n \\ U_i\left(\mathbf{x}_i(\mathbf{p})\right) &= \max\left\{U_i(\mathbf{x}) \mid \mathbf{x} \in \mathcal{U}_i,\ \mathbf{p}.\mathbf{x} \leq 1/n\right\}, \ \ i = 1, ..., n \\ \frac{\partial U_i}{\partial x^j}\left(\mathbf{x}_i(\mathbf{p})\right) &= \lambda_i(\mathbf{p})\, p_j, \ \ i = 1, ..., n, \ \ j = 1, ..., n \\ \sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}) &= \mathbf{x}(\mathbf{p}) \\ \mathbf{x}_i(\bar{\mathbf{p}}) &= \bar{\mathbf{x}}_i\,, \ \ i = 1, ..., n \\ \lambda_i(\bar{\mathbf{p}}) &= \bar{\lambda}_i, \ \ i = 1, ..., n \end{aligned}$$

Note that both the individual demands and the Lagrange multipliers (i.e., each agent's marginal utility of income) can be freely chosen at $\bar{\mathbf{p}}$. In particular, nonnegativity constraints can be forgotten, since one can choose individual demands to be strictly positive at $\bar{\mathbf{p}}$, and they will remain positive in a neighbourhood.

5.3 Mathematical integration

Finally, we may wonder how the conclusion would be modified if we adopt the 'mathematical integration' viewpoint. Note, first, that this leads to disregarding condition (35). This has a simple economic interpretation. Indeed, this relation results from the budget constraint, and more precisely from the fact that income is assumed independent from prices. Suppose, on the contrary, that each individual's income is allowed to be some arbitrary function of prices; i.e., we assume the existence of some functions $\rho_1(\mathbf{p}), ..., \rho_n(\mathbf{p})$ such that $\sum \rho_i(\mathbf{p}) = n$, and such that i's program writes down :

$$\begin{array}{l} MaxU^i(\mathbf{x}^i) \\ \mathbf{p}.\mathbf{x}^i = \rho_i(\mathbf{p}) \end{array}$$

Then condition (35) can be cancelled. Now, if we disregard convexity requirements, how many individuals are necessary to generate an arbitrary aggregate demand $\mathbf{X}(\mathbf{p})$?

The answer can readily be derived from Pfaff theorem. Let k denote the number of consumers. A necessary and sufficient condition for (34) is that :

$$\omega \wedge (d\omega)^k = 0$$

Now, assume $k \geq n/2$. Then $\omega \wedge (d\omega)^k$ is a s-form with $s \geq n+1$ (remember that $d\omega$ is a 2-form). It follows that it is identically zero, so that the condition is always fulfilled. We conclude that only $n/2$ consumers are necessary to solve the 'mathematical integration' problem !

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