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A Perturbation Theory near Convex Hamiltonian Systems*

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I. INTRODUCTION

This paper deals with the following problem. Let $H_\varepsilon(x,p)$ be a family of functions on $\mathbb{R}^n \times \mathbb{R}^n$, depending on the parameter $\varepsilon \in \mathbb{R}^d$. For small ε, we seek periodic solutions to Hamilton's equations,

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial}{\partial p_i} H_\varepsilon(x,p), \\ \frac{dp_i}{dt} &= \frac{-\partial}{\partial x_i} H_\varepsilon(x,p), \end{aligned} \qquad 1 \leqslant i \leqslant \eta \qquad (\mathscr{H}_\varepsilon)$$

and we wish to relate them to the periodic solutions of the unperturbed system:

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial}{\partial p_i} H_0(x,p), \\ \frac{dp_i}{dt} &= \frac{-\partial}{\partial x_i} H_0(x,p), \end{aligned} \qquad 1 \leqslant i \leqslant \eta. \qquad (\mathscr{H}_0)$$

For this problem to have any practical interest, the unperturbed Hamiltonian system $(\mathscr{H}_0)$ must be completely integrable. We then find ourselves dealing with the fundamental problem of perturbation theory, which has given rise to a considerable amount of mathematical developments for more than two centuries, the most recent being the Kolmogorov–Arnold–Moser theorem on invariant tori (see [9] for a survey). In applying this

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result to linear systems, it must be borne in mind that the n closed trajectories corresponding to the normal modes are degenerate rational tori. For this reason, it is often best to treat separately the case when $(\mathscr{H}_0)$ is linear.

Roughly speaking, the two salient facts which emerge are the following:

(a) Closed trajectories for $(\mathscr{H}_0)$ which are isolated on their energy level give rise to closed trajectories for the perturbed problems $(\mathscr{H}_\varepsilon)$. Asymptotic expansions for these can be found by some variant of the Lindstedt–Poincare method, i.e., by killing off secular terms in the kth-order terms by ad hoc conditions on the $(k-1)$th-order terms.

(b) Closed trajectories for $(\mathscr{H}_0)$ which belong to a continuous family within the same energy level (a rational invariant torus, for instance) will in general be destroyed by small perturbations. Only a few will give rise to closed trajectories for $(\mathscr{H}_\varepsilon)$, with $\varepsilon \neq 0$, and it is part of the problem to find those that do.

This paper gives a unified approach to all these problems, based on a single mathematical result (Theorem 8). It is a purely functional-analytic approach, by repeated use of the implicit function theorem, and makes no reference to such classical tools as the Poincare map. It provides us with a complete and detailed justification of the above (a), (b) picture. Moreover, we get some by-products:

(i) In the case of isolated closed trajectories, the asymptotic expansions for closed trajectories of $(\mathscr{H}_\varepsilon)$ have different form when $(\mathscr{H}_0)$ is linear and when $(\mathscr{H}_0)$ is nonlinear (see Section III, A and B).

(ii) The (usually divergent) asymptotic expansions are understood as Taylor series. The reason why there should be no secular terms is given. Moreover the procedure described does not need computation of the $(k+1)$th terms to completely determine the kth terms (Section III, A and B).

(iii) A simple necessary condition for bifurcation from a continuous family of closed trajectories at u_0 is given (Section IV), namely, that the d functionals

$$y \to \int_0^{T_0} (H''_{\varepsilon_i u}(0, u_0(t)), y(t))\, dt$$

be linearly dependent on the space of all functions y such that

$$\dot{y}(t) = \sigma H''_{uu}(0, u_0(t))\, y(t)$$
$$y(0) = y(T_0)$$
$$\int_0^{T_0} (\sigma \dot{u}_0(t), y(t))\, dt = 0.$$

(iv) A new proof is given of the classical theorem by Weinstein on the existence of n distinct closed trajectories near an equilibrium (Section V).

We rely on a global method developed by Clarke [3, 4] and Ekeland [5, 6, 8] to find periodic solutions of Hamiltonian systems. It requires the unperturbed Hamiltonian H_0 to be convex in both variables (x, p), at least in a neighbourhood of the unperturbed orbit under study.

Section II of this paper benefited from the collaboration of J. Blot [2], whom we thank for his help.

II. The Abstract Result

Denote by $u = (x, p)$ the points in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, and by σ the symplectic $2n \times 2n$ matrix

$$\sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Note that $\sigma^* = -\sigma = \sigma^{-1}$. Equation $(\mathscr{H}_\varepsilon)$ can be written as follows:

$$\dot{u} = \sigma H_u'(\varepsilon, u). \qquad (\mathscr{H}_\varepsilon)$$

Here we denote by $H(\varepsilon, u)$, instead of $H_\varepsilon(u)$, the Hamiltonian, by H_u' and H_ε' the vectors with components $\partial H/\partial u_i$ and $\partial H/\partial \varepsilon_j$, by H_{uu}'' and $H_{\varepsilon u}''$ the matrices with coefficients $\partial^2 H/(\partial u_i \partial u_j)$ and $\partial^2 H/(\partial \varepsilon_i \partial u_j)$, and so on.

Throughout this paper, we shall assume that:

$$H: \mathbb{R}^d \times \mathbb{R}^{2n} \to \mathbb{R} \text{ is } C^\infty \qquad \text{(H1)}$$

$$\exists c > 0: (v, H_{uu}''(0, u)\, v) \geqslant c(v, v) \qquad \text{all } v \text{ and } u. \qquad \text{(H2)}$$

In other words, the unperturbed Hamiltonian $H(0, \cdot)$ is strictly convex.

We are interested in local results only, i.e., in what happens near a compact trajectory u_0, or a compact manifold of trajectories, for small ε. For any bounded set K, it will follow from (H2) that the perturbed Hamiltonian $H(\varepsilon, \cdot)$ will also be strictly convex over K, for small enough ε. So there is no loss of generality in strengthening (H2) to:

$$\exists c > 0: (v, H_{uu}''(\varepsilon, u)\, v) \geqslant c(v, v) \qquad \text{all } \varepsilon, v, \text{ and } u. \qquad \text{(H2)}'$$

This second condition obviously implies that for any $\varepsilon \in \mathbb{R}^d$, the map $u \to H(\varepsilon, u)$ is strictly convex, and has quadratic growth at infinity. It follows that, for any $v \in \mathbb{R}^{2n}$, there is a single point $u(\varepsilon, v)$ such that

$$(v, u(\varepsilon, v)) - H(\varepsilon, u(\varepsilon, v)) = \operatorname*{Min}_u \{(v, u) - H(\varepsilon, u)\}.$$

We readily identify $u(\varepsilon, v)$ as the solution to the equation $H_u'(\varepsilon, u) = v$. By the implicit function theorem, it is a C^∞ function of (ε, v), and $u_v' = H_{uu}''(\varepsilon, u)^{-1}$. Plugging it back into the preceding equation, we get a C^∞ function $G(\varepsilon, v)$ defined by

$$G(\varepsilon, v) = \min_u \{(v, u) - H(\varepsilon, u)\}. \tag{1}$$

Differentiating once, we get

$$\begin{aligned} G_v'(\varepsilon, v) &= u(\varepsilon, v) + [v - H_u'(\varepsilon, u)]\, u_v'(\varepsilon, v) \\ &= u(\varepsilon, v). \end{aligned}$$

Differentiating twice, we get

$$\begin{aligned} G_{vv}''(\varepsilon, v) &= u_v'(\varepsilon, v) \\ &= H_{uu}''(\varepsilon, u(\varepsilon, v))^{-1}. \end{aligned}$$

We have proved:

PROPOSITION 1. *Formula* (1) *defines a C^∞ function $G(\varepsilon, v)$, the Legendre transform of $H(\varepsilon, v)$. It is convex with respect to v. The following three conditions are equivalent*:

(a) $G(\varepsilon, v) + H(\varepsilon, u) = (v, u)$,

(b) $v = H_u'(\varepsilon, u)$,

(c) $u = G_v'(\varepsilon, v)$.

If they hold at (u, v), we have

(d) $G_{vv}''(\varepsilon, v)\, H_{uu}''(\varepsilon, u) = I$. ▮

Assumption (H2), combined with condition (d), gives

$$0 \leqslant (G_{vv}''(\varepsilon, v)\, w, w) \leqslant c^{-1} \|w\|^2 \tag{2}$$

so that $v \to G(\varepsilon, v)$ also is a convex function. Using the fact that the problem of minimizing $(v, u) - G(\varepsilon, v)$ in v always has precisely one solution, namely, $v = H_u'(\varepsilon, u)$, it can be shown that G actually is strictly convex.

We now use G to construct a problem in the calculus of variations, the solutions of which will give us periodic projections of $(\mathscr{H}_\varepsilon)$. This method is due to Clarke and Ekeland (see [5]), following an idea of Clarke (see [3]). It can be understood as a dual version of the least action principle.

We first define the ad hoc function space E:

$$E = \left\{ w \in C^r(S^1; \mathbb{R}^{2n}) \;\middle|\; \int_0^1 w(s)\, ds = 0 \right\}.$$

Here $S^1 = \mathbb{R}/\mathbb{Z}$. Equivalently, E is the space of all C^r functions on $[0, 1]$ such that $d^k w/dt^k(0) = d^k w/dt^k(1)$ for $0 \leqslant k \leqslant r$ and $\int_0^1 w(s)\, ds = 0$.

If $w \in E$, its primitives are 1-periodic. We denote by w the one which has mean value zero:

$$(\Pi w)(t) = \int_0^t w(t)\, dt - \int_0^1 \left(\int_0^s w(t)\, dt \right) ds.$$

We will consider Π as a compact linear operator of E into itself. Note that $\sigma\Pi$ is self-adjoint; indeed, integrating by parts, with $\Pi w(1) = \Pi w(0)$ for all w in E, yields

$$\int_0^1 (w_1, \sigma\Pi w_2)\, ds - \int_0^1 (w_2, \sigma\Pi w_1)\, ds = (\Pi s_1, \sigma\Pi w_2)_0^1 = 0.$$

We then define a function Φ on $(0, \infty) \times \mathbb{R}^d \times E$ by

$$\Phi(T, \varepsilon, w) = \int_0^1 ((w, \sigma\Pi w)T/2 - G(\varepsilon, -\sigma w))\, ds.$$

PROPOSITION 2. *The function Φ is C^∞; we have*:

$$\Phi'_T(T, \varepsilon, w) = \int_0^1 \tfrac{1}{2}(w, \sigma\Pi w)\, ds \in \mathbb{R}$$

$$\Phi'_\varepsilon(T, \varepsilon, w) = -\int_0^1 G'_\varepsilon(\varepsilon, -\sigma w)\, ds \in \mathbb{R}^d$$

$$\Phi'_w(T, \varepsilon, w)\, w_1 = \int_0^1 [T(w_1, \sigma\Pi w) - (\sigma G'_v(\varepsilon, -\sigma w), w_1)]\, ds$$

$$\Phi''_{Tw}(T, \varepsilon, w)\, w_1 = \int_0^1 (w_1, \sigma\Pi w)\, ds \in \mathbb{R}$$

$$\Phi''_{\varepsilon w}(T, \varepsilon, w)\, w_1 = \int_0^1 (-\sigma G''_{\varepsilon v}(\varepsilon, -\sigma w), w_1)\, ds \in \mathbb{E}^d$$

$$(\Phi''_{ww}(T, \varepsilon, w)\, w_1, w_2) = \int_0^1 [T(w_1, \sigma\Pi w_2) + (\sigma G''_{vv}(\varepsilon, -\sigma w)\, \sigma w_2, w_1)]\, ds. \quad ∎$$

The proof is left to the reader. We note for future use that if an L^2

function φ is such that $\int_0^1 \phi w\, ds = 0$ for all $w \in E$, then $\int_0^1 \phi \psi ds = 0$ for all $\psi \in L^2$ such that $\int_0^1 \psi ds = 0$, so that ϕ must be constant in $\mathbb{R}^{2n}$. It follows that Φ'_w and Φ''_{Tw} can be identified with elements of E,

$$\Phi'_w(T, \varepsilon, w) = T\sigma\Pi w - \sigma P G'_v(\varepsilon, -\sigma w)$$

$$\Phi''_{Tw}(T, \varepsilon, w) = \sigma\Pi w,$$

$\Phi''_{\varepsilon w}$ with an element of E^d,

$$\Phi''_{\varepsilon w}(T, \varepsilon, w) = -\sigma P G''_{\varepsilon v}(\varepsilon, -\sigma w),$$

and Φ''_{ww} with a linear operator of E into itself,

$$\Phi''_{ww}(T, \varepsilon, w)\, w_1 = T\sigma\Pi w_1 + P\sigma G''_{vv}(\varepsilon, -\sigma w)\, \sigma w_1.$$

Here P is defined as the projection of C^r onto E associated with the natural splitting $C^r = E \oplus \mathbb{R}^{2n}$. In other words,

$$(Pw)(s) = w(s) - \int_0^1 w(s)\, ds.$$

PROPOSITION 3. *The following statements are equivalent*:

(a) $\Phi'_w(T, \varepsilon, w) = 0$, *i.e., w is a critical point of* $\Phi(T, \varepsilon, \cdot)$;

(b) *there is some $\xi \in R^{2n}$ such that $u(t) = T\Pi w(t/T) + \xi$ is a T-periodic solution of* $(\mathscr{H}_\varepsilon)$.

Moreover, we have the relation

$$u(t) = G'_v(\varepsilon, -\sigma w(tT^{-1}), \qquad \text{all} \quad t$$

$$w(t) = \sigma H'_u(\varepsilon, u(sT)), \qquad \text{all} \quad s. \quad \blacksquare$$

Proof. The equation $\Phi'_w(T, \varepsilon, w) = 0$ means that, for some constant $\xi' \in \mathbb{R}^{2n}$, we have

$$T\sigma\Pi w(s) - \sigma G'_v(\varepsilon, -\sigma w(s)) = \xi', \qquad \text{all } s.$$

Using the equivalence (b) $\Leftrightarrow$ (c) in Proposition 1 transforms this into

$$H'_u(\varepsilon, T\Pi w(s) - \sigma\xi') = -\sigma w(s)$$

$$\Pi w(T) = \Pi w(0).$$

The result follows by setting $u(t) = T\Pi w(t/T) - \sigma\xi' T$. $\blacksquare$

From now on it shall be assumed that some T_0-periodic solution $u_0(t)$ of $(\mathscr{H}_0)$ has been found, corresponding to some solution w_0 of $\Phi'_w(T_0, 0, w) = 0$.

We wish to investigate the situation around $(T_0, 0, w_0)$ by applying the inverse function theorem. For this, we need to know more about the derivative of Φ'_w:

$$L = (\Phi''_{Tw}, \Phi''_{\varepsilon w}, \Phi''_{ww}) \in \mathscr{L}(\mathbb{R} \times \mathbb{R}^d \times E, E).$$

PROPOSITION 4. *$L(T, \varepsilon, w)$ is a Fredholm map of index $d+1$.* ∎

Proof. L has been computed in Proposition 2, and can be written as the sum of the map

$$L_1 : (T_1, \varepsilon_1, w_1) \to \sigma \Pi w T_1 - \sigma P G''_{\varepsilon v}(\varepsilon, -\sigma w)\, \varepsilon_1 + T \sigma \Pi w_1$$

which is compact, and the map

$$L_2 : (T_1, \varepsilon_1, w_1) \to \sigma P G''_{vv}(\varepsilon, -\sigma w)\, \sigma w_1 .$$

We claim that the map $L'_2 : w_1 \to \sigma P G''_{vv}(\varepsilon, -\sigma w)\, \sigma w_1$ is an isomorphism of E onto itself. Indeed, the equation $L'_2 w_1 = w_2$ can be written as

$$\sigma G''_{vv}(\varepsilon, -\sigma w(s))\, \sigma w_1(s) = w_2(s) + \xi \qquad \text{with } \xi \in \mathbb{R}^{2n}$$

and inverted pointwise:

$$w_1(s) = -\sigma H''_{uu}(\varepsilon, u(s))(-\sigma w_2(s) - \sigma \xi), \qquad \text{with } u(s) = G'_v(\varepsilon, -\sigma w(s)).$$

It follows that w_1 is just as smooth as w_2. For w_1 to belong to E, all we need to do is to adjust ξ in $\mathbb{R}^d$ so that

$$\int_0^1 w_1(s) = \int_0^1 \sigma H''_{uu}(\varepsilon, u)\, \sigma w_2\, ds + \sigma \left(\int_0^1 H''_{uu}(\varepsilon, u)\, ds \right) \sigma \xi = 0.$$

By assumption, $H''_{uu}(\varepsilon, u(s))$ is positive definite for all s, and so is the integral $\int_0^1 H''_{uu}(\varepsilon, u(s))\, ds$. The equation thus determines ξ uniquely, and L'_2 turns out to be invertible.

It follows that L_2 has range E and kernel R^{d+1}. It thus is Fredholm of index $d+1$, and so is L. ∎

The following proposition implies that $L(T_0, 0, w_0)$ cannot be onto. The ultimate reason for this lies in the fact that the function Φ is invariant by the S^1-action which sends $w(t)$ into $w(t+\Theta)$, and this induces degeneracies in the derivatives.

PROPOSITION 5. *Assume $\Phi'_w(T, \varepsilon, w) = 0$ with $w \neq 0$. Then $\dot{w}$ belongs to E but lies outside the range of $L(T, \varepsilon, w)$. More precisely,*

$$\int_0^1 (\Phi''_{Tw} T_1 + \Phi''_{\varepsilon w} \varepsilon_1 + \Phi''_{ww} w_1, \dot{w})\, ds = 0, \quad \text{all } (T_1, \varepsilon_1, w_1) \in \mathbb{R} \times \mathbb{R}^d \times E. \quad ∎$$

Proof. By Proposition 3, if $\Phi'_w(T, \varepsilon, w) = 0$, then $T\Pi w(t/T) + \xi$ is a solution of $(\mathscr{H}_\varepsilon)$, and hence C^∞. So w really is C^∞, with $w(1) - w(0) = \int_0^1 \dot{w}\,dt = 0$, and $\sigma\dot{w}$ belongs to E.

Computing each term separately, we get

$$\int_0^1 (\Phi''_{Tw}, \dot{w})\,ds = \int_0^1 (\sigma\Pi w, \dot{w})\,ds = -\int_0^1 (w, \sigma w)\,ds = 0$$

$$\int_0^1 (\Phi''_{\varepsilon w}, \dot{w})\,ds = -\int_0^1 (\sigma G''_{\varepsilon v}(\varepsilon, -\sigma w), \dot{w})\,ds$$

$$= -\int_0^1 d/dt[G'_\varepsilon(\varepsilon, -\sigma w(s))]\,ds = 0$$

$$\int_0^1 (\Phi''_{ww} w_1, \dot{w})\,ds = \int_0^1 [T(w_1, \sigma\Pi\dot{w}) + (\sigma G''_{vv}(\varepsilon, -\sigma w)\,\sigma w_1, \dot{w})]\,ds$$

$$= \int_0^1 (w_1, T\sigma w + \sigma G''_{vv}(\varepsilon, -\sigma w)\,\sigma\dot{w})\,ds.$$

Now, since $\Phi'_w(T, \varepsilon, w) = 0$, we have

$$T\sigma\Pi w(s) - \sigma G'_v(\varepsilon, -\sigma w(s)) = \text{constant}.$$

Differentiating with respect to time, we get

$$T\sigma w(s) + \sigma G''_{vv}(\varepsilon, -\sigma w(s))\,\sigma\dot{w}(s) = 0, \qquad \text{all } s.$$

Substituting in the above, we get the desired result. ∎

More generally, we have the following result:

PROPOSITION 6. *Assume $\Phi'_w(T_0, 0, w_0) = 0$, and denote by u_0 the corresponding solution of $(\mathscr{H}_0)$. Let F be a first integral of equation $(\mathscr{H}_0)$. Define a function v by*

$$T_0 v(s) = \sigma F'(u_0(sT_0)).$$

Then v belongs to E, so does $\dot{v}$, and

$$\int_0^1 (\Phi''_{Tw} T_1 + \Phi''_{ww} w_1, \dot{v})\,ds = 0 \qquad \textit{all} \quad (T_1, w_1) \in \mathbb{R} \times E.$$

Proof. Denote by ϕ_θ the flow associated with the differential equation $\dot{u} = \sigma F'_u(u)$. It is certainly well defined in some neighbourhood of the compact curve $u_0(t)$, provided $\theta \in \mathbb{R}$ is small enough. Since it is

Hamiltonian, it will preserve the integral of the 1-form $(\sigma u, du)$ along closed curves:

$$\int_0^{T_0} \frac{1}{2} \left(\sigma\phi_\theta u_0(t), \frac{\partial}{\partial t} \phi_\theta u_0(t)\right) dt = \text{constant}. \qquad (*)$$

Moreover, since F is a first integral of $(\mathscr{H}_0)$, the Hamiltonian flows associated with H_0 and F commute, which implies $\phi_\theta u_\theta(t)$ is still a T_0-periodic solution of $(\mathscr{H}_0)$. Proposition 3 associates with $\phi_\theta u_0$ a function $w_\theta \in E$,

$$w_\theta(s) = \frac{1}{T_0} \frac{\partial}{\partial s} \phi_\theta u_0(sT_0) \qquad (**)$$

which satisfies the equation

$$\Phi'_w(T_0, 0, w_\theta) = 0, \qquad \text{all } \theta.$$

Differentiating with respect to θ at $\theta = 0$, we get

$$\Phi''_{ww}(T_0, 0, w_0) \frac{\partial}{\partial \theta} w_\theta = 0.$$

Computing this derivative from Eq. $(**)$, we get

$$\begin{aligned} \frac{\partial}{\partial \theta} w_\theta(s) &= \frac{1}{T_0} \frac{\partial}{\partial \theta} \frac{\partial}{\partial s} \phi_\theta u_0(sT_0) = \frac{1}{T_0} \frac{\partial}{\partial s} \frac{\partial}{\partial \theta} \phi_\theta u_0(sT_0) \\ &= \frac{1}{T_0} \frac{\partial}{\partial s} \sigma F'_u(u_0(sT_0)) \\ &= \dot{v}(s). \end{aligned}$$

So we get

$$\int_0^1 (\Phi''_{ww} w_1, \dot{v})\, ds = \int_0^1 (\Phi''_{ww} \dot{v}, w_1)\, ds = 0, \qquad \text{all} \quad w_1 \in E.$$

To get the remaining equation, we differentiate Eq. $(*)$ with respect to θ at $\theta = 0$. We get

$$0 = \frac{1}{2} \int_0^{T_0} \left[(-F'_u(u_0), \sigma H'_u(0, u_0)) + \left(\sigma u_0(t), \frac{\partial}{\partial \theta} w_\theta \left(\frac{t}{T_0}\right)\right) \right] dt.$$

The first term is the Poisson bracket of F and H_0, which is zero since F is a first integral of $(\mathscr{H}_0)$. The second term, with $\partial/\partial\theta\, w_\theta = \dot{v}$, gives

$$0 = \tfrac{1}{2}\int_0^{T_0} (\sigma u_0(t), \dot{v}(t/T_0))\, dt$$

$$= T_0/2 \int_0^1 (\sigma u_0(sT_0), \dot{v}(s))\, ds$$

$$= \tfrac{1}{2}\int_0^1 (\sigma \Pi w_0(s), \dot{v}(s))\, ds$$

$$= \tfrac{1}{2}\int_0^1 (\Phi''_{Tw}(s), \dot{v}(s))\, ds. \quad \blacksquare$$

Note that $\dot{v}$ may still be in the range of $L(T_0, 0, w_0)$ because we are missing the last d equations $\Phi''_{\varepsilon w}\dot{v} = 0$.

For the sake of completeness, let us give a few more relations, valid when F is still a first integral of $(\mathscr{H}_\varepsilon)$ for small ε.

PROPOSITION 6 bis. *Assume that F is a first integral of* $(\mathscr{H}_\varepsilon)$ *for small* ε. *We then have*

$$(\sigma F' \circ G'_v(\varepsilon, v), v) = 0 \qquad \textit{all} \quad v \in \mathbb{R}^{2n}$$

$$(\sigma F' \circ G''_{\varepsilon v}(0, v), v) = 0 \qquad \textit{all} \quad v \in \mathbb{R}^{2n}.$$

Proof. The Poisson bracket of F with H has to vanish:

$$(\sigma F'(u), H'_u(\varepsilon, u)) = 0 \qquad \text{all} \quad u \in \mathbb{R}^{2n}.$$

Setting $v = H'_u(\varepsilon, u)$, we get $u = G'_v(\varepsilon, v)$, and the first equation. The second follows by differentiation in ε. $\blacksquare$

We can try another way to estimate the codimension of $L(T_0, 0, w_0)$, by relating Φ''_{ww} to the linearized equation of $(\mathscr{H}_0)$ around u_0. This is done in the following:

PROPOSITION 7. *The following statements are equivalent*:

(a) $\Phi''_{ww}(T_0, 0, w_0)\, w = T_0 \sigma \Pi w + P\sigma G''_{vv}(0, -\sigma w_0)\, \sigma w = 0$,

(b) *there is some* $\xi \in \mathbb{R}^{2n}$ *such that* $y(t) = T_0 \Pi w(t/T_0) + \xi$ *is a* T_0-*periodic solution of the linearized equations*

$$\dot{y} = \sigma H''_{uu}(0, u_0(t))\, y.$$

Proof. Follows immediately from Proposition 3, which we apply to the quadratic, time-dependent Hamiltonian

$$\bar{H}(t, u) = \tfrac{1}{2}(H''_{uu}(0, u_0(t))\, u, u)$$

with Legendre transform

$$\bar{G}(t, v) = \tfrac{1}{2}(G''_{vv}(0, -\sigma w(tT_0^{-1}))\, v, v).$$

The linearized equations can be written as the (non-autonomous) Hamiltonian system $\dot{u} = \sigma \bar{H}'_u(t, u)$, and T_0-periodic solutions correspond to critical points of the quadratic functional:

$$\bar{\Phi}(w) = \int_0^1 (T_0/2\, (w, \sigma \Pi w) - \bar{G}(sT_0, -\sigma w))\, ds.$$

The proof of Proposition 3 carries over to the non-autonomous case. and shows the equation $\bar{\Phi}'_w = 0$ to be equivalent to conditions (a) or (b). ∎

As we have seen when proving Proposition 4, the map $w \to \sigma P G''_{vv}(\varepsilon, -\sigma v)\, w$, denoted by L'_2, is an isomorphism of E onto itself. The map $\Phi''_{ww}(T_0, 0, w_0) = L'_2 + \sigma T_0 \Pi$ is a compact perturbation of L'_2, and so is Fredholm of index zero. We have

$$l = \operatorname{codim} \Phi''_{ww}(T_0, 0, w_0)\, E = \dim \operatorname{Ker} \Phi''_{ww}(T_0, 0, w_0).$$

By Proposition 7, l is the number of linearly independent T_0-periodic solutions of the linearized equations

$$\dot{y} = \sigma H''_{uu}(0, u_0(t))\, y.$$

We know that $y = \dot{u}_0$ is such a solution, so $l \geqslant 1$.

Floquet theory tells us that $l \leqslant m$, where m is the multiplicity of zero (mod $2\pi i$) as a characteristic exponent along the closed trajectory u_0. This multiplicity is at least two, one because the system $(\mathscr{H}_0)$ is autonomous and one because it has H_0 as a first integral. It must be an even number, because the system is Hamiltonian:

$$1 \leqslant l \leqslant m \qquad \text{with} \quad m \text{ even} \geqslant 2.$$

Any further integral of the motion raises m by one, provided it is independent from the preceding ones along u_0. For instance, if $H_0(u) = \tfrac{1}{2}(x_1^2 + p_1^2) + \omega/2(x_2^2 + p_2^2)$, with $n = 2$ and ω irrational, there are two integrals of the motion, $(x_1^2 + p_1^2)$ and $(x_2^2 + p_2^2)$, and one expects $m \geqslant 3$, so $m = 4$. A closer look, however, shows that all closed trajectories lie in the plane $x_1^2+p_1^2=0$, or in the plane $x_2^2+p_2^2=0$, so one of the integrals always degenerates along u_0, and $m=2$.

Let $y_1=\dot{u}_0, y_2, ..., y$ be independent solutions of the linearized problem

$$\dot{y}=\sigma H''_{uu}(0, u_0(t))\, y, \qquad y(0)=y(T_0).$$

Set $z_i(s)=T_0\dot{y}_i(sT_0)$ for $1 \leqslant i \leqslant l$. By Proposition 7, the z_i are a basis of $\operatorname{Ker} \Phi''_{ww}(T_0, 0, w_0)$. Denote by V the linear subspace of E given by

$$V=\left\{ v \in \operatorname{Ker} \Phi''_{ww}(T_0, 0, w_0) \;\middle|\; \begin{array}{l} \int_0^1 (\sigma \Pi w_0, v)\, ds=0 \text{ and} \\ \int_0^1 (\sigma G''_{\varepsilon w}(0, -\sigma w_0), v)\, ds=0 \end{array} \right\}.$$

By Proposition 5, we have $z_1=\dot{w}_0 \in V$, so that

$$1 \leqslant \dim V \leqslant l.$$

By standard Sturm–Liouville theory, v belongs to the range of $\Phi''_{ww}(T_0, 0, w_0)$ in E if and only if $\int_0^1 (w, w)\, ds=0$ for all $w \in \operatorname{Ker} \Phi''_{ww}$. By Proposition 2, the range of $L(T_0, 0, w_0)$ is generated by

$$\sigma \Pi w_0, \qquad \sigma P G''_{\varepsilon v}(\varepsilon, -\sigma w_0), \qquad \Phi''_{ww}(T_0, 0, w_0)\, E.$$

The range of $L(T_0, 0, w_0)$ is related to V by

$$L(T_0, 0, w_0)\, E=\left\{ w \in E \;\middle|\; \int_0^1 (v, w)\, ds=0\ \forall v \in V \right\}.$$

We now state our main result:

THEOREM 8. *There is some neighbourhood* $\mathscr{U}$ *of* $(T_0, 0, w_0)$ *in* $(0, \infty) \times \mathbb{R}^d \times E$ *such that the set* $S \cap \mathscr{U}$ *defined by*

$$S=\{(T, \varepsilon, w) \mid \Phi'_w(T, \varepsilon, w) \in V\}$$

is a $(\dim V+d+1)$*-dimensional* C^∞ *submanifold of* $(0, \infty) \times \mathbb{R}^d \times E$.

Proof. Denote by $\mathrm{R}(L)$ and $\operatorname{Ker}(L)$ the range and the kernel of $L(T_0, 0, w_0)$. Write $(0, \infty) \times \mathbb{R}^d \times E=\operatorname{Ker}(L) \oplus E_0$, and $E=\mathrm{R}(L) \oplus V$. Denote by pr_{K}: $(0, \infty) \times \mathbb{R}^d \times E \rightarrow \operatorname{Ker}\ (L)$ and $\mathrm{pr}_{\mathrm{R}}: E \rightarrow \mathrm{R}(L)$ the corresponding projection. Consider the map

$$\Psi: (0, \infty) \times \mathbb{R}^d \times E \rightarrow \mathrm{R}(L) \times \operatorname{Ker}(L)$$

$$\Psi(T, \varepsilon, w)=(\mathrm{pr}_{\mathrm{R}}\, \Phi'_w(T, \varepsilon, w), \mathrm{pr}_{\mathrm{K}}(T, \varepsilon, w)).$$

By construction, $\Psi'(T_0, 0, w_0)$ is invertible, so that the inverse function theorem applies around this point. The set S is just $\Psi^{-1}(\{0\} \times \mathrm{Ker}(L))$; by the inverse function theorem, $S \cap \mathscr{U}$ must be a submanifold modelled on $\mathrm{Ker}(L)$, which has dimension $\mathrm{Index}(L) + \mathrm{Codim}\ \mathrm{R}(L) = 1 + d + \dim V$. ∎

There are now two cases to consider: $\dim V = 1$ and $\dim V > 1$.

III. Asymptotic Expansions

$\boxed{\dim V = 1}$ This will always be the case when $l = 1$. It will also be the case when $l = 2$, provided

$$\sigma \Pi w_0 \notin \Phi''_{ww}(T_0, 0, w_0)\, E$$

or

$$\sigma P G''_{\varepsilon w}(0, -\sigma w_0) \notin \Phi''_{ww}(T_0, 0, w_0)\, E.$$

More generally, we will have $\dim V = 1$ when $l \leqslant d + 2$ and the $\sigma P G''_{\varepsilon_i w}(0, -\sigma w_0)$, $1 \leqslant i \leqslant d$, $\sigma \Pi w_0$, $\dot{w}_0$, and $\Phi''_{ww}(T_0, 0, w_0)\, E$ span the space E.

We then have a considerable simplification:

Proposition 9. *If* $\dim V = 1$, *there is some neighbourhood* $\mathscr{U}$ *of* $(T_0, 0, w_0)$ *in* $(0, \infty) \times \mathbb{R}^d \times E$ *inside which*

$$\Phi'_w(T, \varepsilon, w) \in V \Leftrightarrow \Phi'_w(T, \varepsilon, w) = 0.$$

Proof. V is spanned by $z_1 = \dot{w}_0$. The equation $\Phi'_w \in V$ means that, for some $(\lambda, \xi) \in \mathbb{R}^{n+1}$,

$$T \sigma \Pi w(t) - \sigma G'_v(\varepsilon, -\sigma w(t)) = \lambda \dot{w}_0(t) + \xi, \qquad \text{all} \quad t.$$

Multiply both sides by $\dot{w}$ and integrate. We get

$$T \int_0^1 (\sigma \Pi w, \dot{w})\, ds - \int_0^1 (\sigma G'_v(\varepsilon, -\sigma w), \dot{w})\, ds = \lambda \int_0^1 (\dot{w}, \dot{w}_0)\, ds.$$

The first term on the left we integrate by parts, and get zero. In the second term we recognize the time derivative of $G(\varepsilon, -\sigma w(t))$, which integrates away to zero. Finally, we get

$$\lambda \int_0^1 (\dot{w}, \dot{w}_0)\, dt = 0.$$

If w is close to w_0 in C^r, $r \geqslant 0$, the integral is strictly positive, and hence λ must be zero. ∎

Let us apply Proposition 9 to the study of *simple* closed trajectories. We say that the chosen periodic orbit u_0 is simple if $m = 2$, i.e., the characteristic exponent 0 has the lowest possible multiplicity along u_0. Since $1 \leqslant l \leqslant m$, either $l = 1$ or $l = 2$. The latter occurs when the unperturbed system is linear and non-resonant; by this we mean that $H_0(u) = \sum_{i=1}^{n} \omega_i/2(x_i^2 + p_i^2)$, with $\omega_i \notin \mathbb{Z}\omega_1$ for $i \neq 1$, and $u_0(t)$ is $2\pi\omega_1^{-1}$-periodic, and so lies in the plane $x_i = p_i = 0$ all $i \neq 1$. The case $l = 1$ occurs when the unperturbed system $(\mathscr{H}_0)$ is nonlinear. We treat both cases separately.

A. *Case $l = 2$*

PROPOSITION 10. *Assume that $l = 2$ and that*:

(a) $\int_0^{T_0} u_0(t) \exp(2i\pi t T_0^{-1})\, dt \neq 0$ *in* $\mathbb{C}^{2n}$,

(b) $\int_0^{T_0} (\dot{u}_0, \sigma u_0)\, dt \neq 0$,

(c) $\exists y_2 : \dot{y}_2 = \sigma H''_{uu}(0, u_0)\, y_2$ *and* $\int_0^{T_0} (H'_u(0, u_0), y_2)\, dt \neq 0$.

Then there are positive numbers α and β, a neighbourhood $\mathscr{U}$ of 0 in $\mathbb{R}^d$, a tubular neighbourhood $\mathscr{V}$ of the path u_0 in $\mathbb{R}^{2n}$, and C^∞ maps $U: S^1 \times \mathscr{U} \times (h_0 - \alpha, h_0 + \alpha) \to \mathbb{R}^{2n}$ and $\theta: \mathscr{U} \times (h_0 - \alpha, h_0 + \alpha) \to \mathbb{R}$ such that

$$\theta(0,..., 0, h_0) = T_0$$

$$U(tT_0^{-1}, 0,..., 0, h_0) = u_0(t)$$

and for any $\varepsilon \in \mathscr{U}$ and h with $|h - h_0| < \alpha$, the curve

$$U(tT^{-1}, \varepsilon_1,..., \varepsilon_d, h) = u(t)$$

with $T = \theta(\varepsilon_1,..., \varepsilon_d, h)$ is a T-periodic trajectory of the Hamiltonian system $(\mathscr{H}_\varepsilon)$ with energy level h:

$$\dot{u}(t) = \sigma H'_u(\varepsilon, u(t))$$

$$u(0) = u(T)$$

$$H(\varepsilon, u(t)) = h, \qquad \text{all } t.$$

Conversely, whenever u is a T-periodic solution of $(\mathscr{H}_\varepsilon)$ with energy level h, provided $\varepsilon \in \mathscr{U}$, $|T - T_0| < \beta$, $|h - h_0| < \alpha$, and $u(t) \in \mathscr{V}$ for all t, then some $\phi \in \mathbb{R}$ can be found such that

$$T = \theta(\varepsilon_1,..., \varepsilon_d, h)$$

$$u(t) = U(tT^{-1} + \phi, \varepsilon_1,..., \varepsilon_d, h).$$

Proof. It is known that $\Phi''_{ww}(T_0, 0, w_0)$ is a Fredholm map of index zero. Split E into $\mathrm{Ker}(\Phi''_{ww}) \oplus \Phi''_{ww}(E)$; then Φ''_{ww} is an isomorphism of $\Phi''_{ww}(E)$ onto itself.

$\Phi''_{ww}(E)$ has codimension $l = 2$. Condition (b) means that there is some $w \in \mathrm{Ker}\, \Phi''_{ww}$, namely, $w = w_0$, such that

$$\int_0^1 (\sigma \Pi w_0, w)\, ds = \int_0^1 1/T_0\, (\sigma u_0(sT_0), \dot{u}_0(sT_0))\, ds \neq 0.$$

It follows that V is one-dimensional and spanned by $z_1 = \dot{w}_0$.

On the other hand, by the Fredholm theory, we have $\sigma \Pi w_0 \notin \Phi''_{ww}(E)$. It follows that the map

$$(T, w) \to (\sigma \Pi w_0)\, T + \Phi''_{ww}(T_0, 0, w_0)\, w$$

of $\mathbb{R} \times \Phi''_{ww}(E)$ into a supplementary subspace of V must be an isomorphism.

By the implicit function theorem, the equation

$$\Phi'_w(T, \varepsilon, w) \in V$$

determines T and the components of w in $\Phi''_{ww}(E)$ in terms of the remaining variables. These are $(\varepsilon_1, ..., \varepsilon_d)$ and the components (ξ, η) of w in $w_0 + \mathrm{Ker}(\Phi''_{ww})$. By this we mean

$$w = w_0 + \xi z_1 + \eta z_2$$

where $z_1 = \dot{w}_0$ and z_2 are independent solutions of $\Phi''_{ww} z = 0$.

By Proposition 9, this means that the equation $\Phi'_w(T, \varepsilon, w) = 0$ can be solved as follows in a neighbourhood of $(T_0, 0, w_0)$:

$$T = T(\varepsilon_1, ..., \varepsilon_d, \xi, \eta) \in \mathbb{R}$$
$$w = W(\varepsilon_1, ..., \varepsilon_d, \xi, \eta) \in E.$$

We now replace ξ and η by more convenient variables h and ϕ. We first define

$$h(\varepsilon, w) = \int_0^1 H(\varepsilon, G'_v(\varepsilon, -\sigma w(s)))\, ds.$$

When $\Phi'_w(T, \varepsilon, w) = 0$, $h(w)$ is the energy level of the solution of $(\mathscr{H}_\varepsilon)$ associated with w. We have

$$\int_0^1 (h'_w(0, w_0), w)\, dt = \int_0^1 (\sigma G''_{vv}(0, -\sigma w_0)\, H'_u(0, u_0), w)\, dt.$$

For any $s \in S^1$, set $w^s(t) = w(s+t)$. If $\Phi'_w(T, \varepsilon, w) = 0$, we have $\Phi'_w(T, \varepsilon, w^s) = 0$ also. We thus have an S^1-action which leaves our equations invariant, and we wish to find a coordinate system adapted to this group-invariance.

To do so, we use assumption (a). Say the first component of $\int_0^1 w_0(s) \exp(2i\pi s)\, ds$ is non-zero. For w near w_0 in E, the complex number

$$\left(\int_0^1 w_0(s) \exp(2i\pi s)\, ds\right)_1 \left(\int_0^1 w(s) \exp(-2i\pi s)\, ds\right)_1$$

has a well-defined argument $\phi(w)$, called the *phase* of w with respect to w_0. It will be checked that $\phi(w^s) = \phi(w) + 2\pi s$.

I now claim that we can use $(\varepsilon_1, ..., \varepsilon_d, h, \phi)$ as a local coordinate system for $\mathbb{R}^d \times \{w_0 + \mathrm{Ker}(\Phi''_{ww})\}$ near $(0, w_0)$. Computing the jacobian at this point gives

$$\frac{D(\varepsilon_1, ..., \varepsilon_d, h, \phi)}{D(\varepsilon_1, ..., \varepsilon_d, \xi, \eta)} = \frac{D(h, \phi)}{D(\xi, \eta)}$$
$$= \begin{vmatrix} (h'(0, w_0), z_1)_E & (\phi'(w_0), z_1)_E \\ (h'(0, w_0), z_2)_E & (\phi'(w_0), z_2)_E \end{vmatrix}.$$

But $z_1 = \dot{w}_0$, so that

$$(h'(0, w_0), z_1) = 0$$
$$(\phi'(w_0), z_1) = d/ds\, \phi(w^s) = d/ds(\phi(w) + 2\pi s) = 2\pi.$$

So the jacobian is $-2\pi(h'(0, w_0), z_2)$, which does not vanish by assumption (c).

The equations now become

$$T = \theta(\varepsilon_1, ..., \varepsilon_d, h, \phi)$$
$$w = w(\varepsilon_1, ..., \varepsilon_d, h, \phi).$$

Using Proposition 3 to translate in terms of u and u_0 the results and the assumptions we have just stated in terms of w and w_0, we get the desired result. ∎

We can also consider U as a C^∞ map from $\mathbb{R}^d \times \mathbb{R}$ into the space $C^r(S^1; \mathbb{R}^{2n})$. It will then have a (possibly divergent) Taylor expansion, the coefficients of which are periodic functions of time. We thus have a theoretical basis for the Lindstedt–Poincaré expansions:

$$T = \sum \varepsilon_1^{p_1} \cdots \varepsilon_d^{p_d} (h - h_0)^Q \, \theta_{p_1, \ldots, p_d, Q}$$

$$u(t) = \sum \varepsilon_1^{p_1} \cdots \varepsilon_d^{p_d} (h - h_0)^Q \, U_{p_1, \ldots, p_d, Q}(tT^{-1})$$

$$U_{p_1, \ldots, p_d, Q}(s) = U_{p_1, \ldots, p_d, Q}(s + 1) \qquad \text{all } s.$$

The last condition (periodicity in s) means that there are no "secular," i.e., non-periodic, terms in the expansion for $u(t)$. Set $U(s) = u(sT)$. The coefficients $\theta_{p_1, \ldots, p_d, Q}$ and the functions $U_{p_1, \ldots, p_d, Q}$ can be computed by substitution into the defining equations

$$\dot{U}(s) = TH'_u(\varepsilon, U(s))$$

$$\int_0^1 H(s, U(s))\, ds = h$$

and formal identification. Indeed, we have just seen that they define U as a smooth function of (ε, h, ϕ) with values in $C^\infty(S^1; \mathbb{R}^{2n})$ (recall that $u(t)$ is really C^∞ by bootstrapping), so that they must determine its Taylor expansion. The parameter ϕ is the phase, and its determination is a matter of convention; one more condition, added to the defining equations, will fix the phase and completely determine the asymptotic expansion.

As an example, let us figure out θ_{10}, θ_{01} and U_{10}, U_{01}, taking $d = 1$ for simplicity. Differentiating the defining equations at $\varepsilon = 0$, $h = h_0$, gives

$$\frac{\partial}{\partial \varepsilon} \dot{U} = \frac{\partial T}{\partial \varepsilon} \sigma H'_u(0, U_0) + T_0 \sigma H''_{u\varepsilon}(0, U_0) + T_0 \sigma H''_{uu}(0, U_0) \frac{\partial U}{\partial \varepsilon} \tag{1}$$

$$\frac{\partial}{\partial h} \dot{U} = \frac{\partial T}{\partial h} \sigma H'_u(0, U_0) + T_0 \sigma H''_{uu}(0, U_0) \frac{\partial U}{\partial h} \tag{2}$$

$$0 = \int_0^1 \left(H'_\varepsilon(0, U_0) + \left(H'_u(0, U_0), \frac{\partial U}{\partial \varepsilon} \right) \right) ds \tag{3}$$

$$1 = \int_0^1 \left(H'_u(0, U_0), \frac{\partial U}{\partial h} \right) ds. \tag{4}$$

We have

$$\frac{\partial T}{\partial \varepsilon} = \theta_{10} \qquad \text{and} \qquad \frac{\partial T}{\partial h} = \theta_{01}$$

$$\frac{\partial U}{\partial \varepsilon} = U_{10} \qquad \text{and} \qquad \frac{\partial U}{\partial h} = U_{01}.$$

Hence two linear, non-homogeneous differential equations for $U_{10}(s)$ and $U_{01}(s)$:

$$\left[\sigma \frac{d}{ds} + T_0 H''_{uu}(0, U_0)\right] U_{10} = -\theta_{10} H'_u(0, U_0) - T_0 H''_{u\varepsilon}(0, U_0) \tag{5}$$

$$\left[\sigma \frac{d}{ds} + T_0 H''_{uu}(0, U_0)\right] U_{01} = -\theta_{01} H'_u(0, U_0). \tag{6}$$

Here U_{10} and U_{01} have to be 1-periodic functions. The operator $[\sigma\, d/ds + T_0 H''_{uu}(0, U_0)]$ with these boundary conditions is self-adjoint, and its inverse is Fredholm of index zero. The assumption $l = 2$, together with Proposition 7, means that its kernel is two-dimensional. By assumption (c) of Proposition 10, the vector $H'_u(0, U_0)$ is not orthogonal to the kernel of this operator, and so cannot belong to its range. Equation (6) then yields

$$\theta_{01} = 0$$
$$U_{01}(s) = \xi y_1(sT) + \eta y_2(sT)$$

where $y_1 = \dot{u}_0$ and y_2 are linearly independent solutions of the linearized equations $\dot{y} = \sigma H''_{uu}(0, u_0)\, y$.

Similarly, Eq. (5) determines θ_{10} by the condition that the right-hand side should be orthogonal to the kernel of the operator,

$$0 = \theta_{10} \int_0^1 (H'_u(0, U_0(s)), y_2(sT))\, ds$$
$$+ T_0 \int_0^1 (H''_{u\varepsilon}(0, U_0(s)), y_2(sT))\, ds$$

and U_{10} is thus determined up to two constants ξ' and η':

$$U_{10}(s) = \bar{U}_{10}(s) + \xi' y_1(sT) + \eta' y_2(sT).$$

Relations (3) and (4) now give

$$-\int_0^1 (H'_\varepsilon(0, U_0) + (H'_u(0, U_0), \bar{U}_{10}))\, ds = \eta' \int_0^1 (H'_u(0, U_0), y_2(sT))\, ds$$

$$1 = \eta \int_0^1 (H'_u(0, U_0), y_2(sT))\, ds.$$

This determines η and η'. As for ξ and ξ', they can be chosen arbitrarily, different choices leading to phase differences in the reference solution $U(s)$. We can for instance, take

$$\xi = 0 \qquad \text{and} \qquad \xi' = 0$$

and the first-order terms $\theta_{01}, \theta_{10}, U_{01}(s), U_{10}(s)$ are now fully determined.

Note that higher-order approximations have not been used in computing the first-order terms; the calculations can be carried out fully within the first-order terms.

B. *Case* $l = 1$

PROPOSITION 11. *Assume that* $l = 1$ *and that*

$$\int_0^{T_0} u_0(t) \exp(2i\pi t/T_0)\, dt \neq 0 \qquad \text{in} \quad \mathbb{C}^{2n}.$$

Then there is a positive number β, *a neighbourhood* $\mathscr{U}$ *of* 0 *in* $\mathbb{R}^d$, *a tubular neighbourhood* $\mathscr{V}$ *of the path* u_0 *in* $\mathbb{R}^{2n}$, *and a* C^∞ *map*

$$U: S^1 \times \mathscr{U} \times (T_0 - \beta, T_0 + \beta) \to \mathbb{R}^{2n}$$

such that

$$U(tT_0^{-1}, 0,..., 0, T_0) = u_0(t)$$

and for any $(\varepsilon_1,..., \varepsilon_d) \in \mathscr{U}$ *and* T *such that* $|T - T_0| < \beta$, *the curve*

$$U(tT^{-1}, \varepsilon_1,..., \varepsilon_d, T) = u(t)$$

is a T-periodic trajectory of the Hamiltonian system

$$\dot{u}(t) = \sigma H_u'(\varepsilon, u(t)). \tag{$\mathscr{H}_\varepsilon$}$$

Conversely, whenever u *is a T-periodic solution of* $(\mathscr{H}_\varepsilon)$, *with* $\varepsilon \in \mathscr{U}, |T - T_0| < \beta$, *and* $u(t)$ *remaining in* $\mathscr{V}$ *for all* t, *then some* $\phi \in \mathbb{R}$ *can be found such that*

$$u(t) = U(tT^{-1} + \phi, \varepsilon_1,..., \varepsilon_d, T).$$

The proof is similar to that of Corollary 1, with obvious modifications. The range of $\Phi_{ww}''(T_0, 0, w_0)$ now has codimension one, and is by itself a supplementary subspace to V. The remaining variables now are $(\varepsilon_1,..., \varepsilon_d)$, T and the one component of w in $\mathrm{Ker}(\Phi_{ww}'')$, namely, ξ, which we interpret as before.

Here again, we may regard U as a C^∞ map from $\mathbb{R}^d \times \mathbb{R}$ into the space

$C^r(S^1; \mathbb{R}^{2n})$ and write its (possibly divergent) Taylor expansion at $(0, T_0)$. We thus get the asymptotic expansion:

$$u(t) = \sum \varepsilon_1^{p_1} \cdots \varepsilon_d^{p_d} (T - T_0)^Q \, U_{p_1, \ldots, p_d, Q}(tT^{-1})$$

$$U_{p_1, \ldots, p_d, Q}(s) = U_{p_1, \ldots, p_d, Q}(s+1), \qquad \text{all } s.$$

The last condition expresses the absence of secular terms. Setting $U(s) = u(sT)$, the functions $U_{p_1, \ldots, p_d, Q}$ can be computed by substitution into the defining equations

$$\dot{U}(s) = T\sigma H_u'(\varepsilon, U(s))$$

and formal identification.

As an example, let us figure out U_{10} and U_{01}, taking $d = 1$ for simplicity. Differentiating the defining equations at $\varepsilon = 0$, $T = T_0$ gives

$$\frac{\partial}{\partial \varepsilon} U = T_0 \sigma H_u''(0, U_0) + T_0 \sigma H_{uu}''(0, U_0) \frac{\partial U}{\partial \varepsilon} \tag{1}$$

$$\frac{\partial}{\partial T} U = \sigma H_u'(\varepsilon, U_0) + T_0 \sigma H_{uu}''(0, U_0) \frac{\partial U}{\partial T}. \tag{2}$$

We rewrite this as

$$\left[\sigma \frac{d}{ds} + T_0 H_{uu}''(0, U_0)\right] U_{10} = -T_0 H_{u\varepsilon}''(0, U_0) \tag{1}$$

$$\left[\sigma \frac{d}{ds} + T_0 H_{uu}''(0, U_0)\right] U_{01} = -H_u'(0, U_0). \tag{2}$$

We know that the kernel of the self-adjoint operator $[\sigma\, d/ds + T_0 H_{uu}''(0, U_0)]$ on 1-periodic functions is one-dimensional and spanned by $\dot{U}_0$, which is orthogonal to the right-hand sides of Eqs. (1) and (2). So we can solve them up to a constant:

$$U_{10} = \bar{U}_{10} + \xi \dot{U}_0$$
$$U_{01} = \bar{U}_{01} + \xi' \dot{U}_0.$$

The constants ξ and ξ' can be chosen freely, different choices corresponding to phase differences in U. Taking $\xi = \xi' = 0$, for instance, fully determines U_{10} and U_{01}.

C. *General Case*

Let us first win some intuition about the two preceding cases by looking at the simple situation where $d = 0$: the system $(\mathscr{H}_0)$ is not perturbed at all.

Propositions 10 (if $l = 2$) and 11 (if $l = 1$) both assert that the closed orbit u_0 will belong to a one-parameter family of closed orbits. But there are essential differences.

When $l = 2$, the period and the orbit are functions of the energy level h, and $\partial T/\partial h(h_0) = 0$. This is typical of linear systems, which feature vector spaces of closed trajectories with the same period.

When $l = 1$, the orbit is given directly as a function of the period. This is typical of nonlinear systems, which can vibrate with any frequency, the frequency of the vibration then being related to its amplitude.

In the case when $l > 2$, with $d \geqslant 0$ again, we may still have dim $V = 1$ if the $\Phi''_{w\varepsilon_i} = -\sigma PG''_{v\varepsilon_i}(0, -\sigma w_i)$ make up for the increase in the codimension of Φ''_{ww}. So Propositions 8 and 9 will still apply, but Propositions 10 and 11 will not.

In order words, if the perturbations are significant enough, there will still be in $\mathbb{R}^d \times \mathbb{R} \times C^r(S^1; \mathbb{R}^{2n})$ a $(d+2)$-dimensional family of periodic solutions, but it will not be possible to take $(\varepsilon_1, \ldots, \varepsilon_d)$ as independent variables.

IV. Bifurcation

dim $V > 1$ This is sure to happen when the multiplicity of zero as a characteristic exponent along u_0 is $l \geqslant 3$ and the number of parameters is $d \leqslant l - 3$.

In contrast to the preceding situation, we cannot assert that the points of the $(\dim V + 1 + d)$-dimensional manifold defined by the equations $\Phi'_w(T, \varepsilon, w) \in V$ all correspond to periodic solutions. What we have done is to reduce the problem to a finite-dimensional one:

Proposition 12. *Assume* $\int_0^{T_0} u_0(t) \exp(2i\pi t T_0^{-1})\, dt \neq 0$. *Then*

$$\dot{u} = \sigma H'_u(\varepsilon, u), \qquad u(0) = u(T) \qquad (\mathscr{H}_\varepsilon)$$

with $(T, \varepsilon, u(sT))$ *near* $(T_0, 0, u_0(sT_0))$ *in* $\mathbb{R} \times \mathbb{R}^d \times C^0(S^1; \mathbb{R}^{2n})$ *reduces to a system of* $(\dim V - 1)$ *equations in* $(\dim V + d)$ *unknowns near a singular point.*

Proof. By Proposition 3, problem $(\mathscr{H}_\varepsilon)$ is equivalent to the equation $\Phi'_w(T, \varepsilon, w) = 0$. Let $z_1 = \dot{w}_0, z_2, \ldots, z_k$ generate V, so that $k = \dim V$. The equation $\Phi'_w = 0$ then can be split in two parts:

$$\Phi'_w(T, \varepsilon, w) = \lambda_1 z_1 + \cdots + \lambda_k z_k$$
$$\lambda_1 = \cdots = \lambda_k = 0.$$

The first part has been studied in Proposition 8, and we have shown that it defines near $(T_0, 0, w_0)$ a manifold of dimension $(k + d + 1)$, which means that T, ε, and w can be expressed in terms of $(k + d + 1)$ independent variables. One of these is the phase, and can be eliminated as we saw in Proposition 10, so that we are left with $(k + d)$ significant variables.

Finally, we multiply both sides of the equation by $\dot{w}$ and we integrate, as in Proposition 9. We get

$$0 = \lambda_1 \int_0^1 (\dot{w}, z_1)\, ds + \cdots + \lambda_k \int_0^1 (\dot{w}, z_k)\, ds.$$

Note that $z_1 = \dot{w}_0$, so that the first coefficient $\int_0^1 (\dot{w}, \dot{w}_0)\, ds$ will not vanish for w close to $\dot{w}_0$ in C^r, $r \geqslant 1$. It follows that the $(k - 1)$ equations

$$\lambda_2 = \cdots = \lambda_k = 0$$

will imply the last one $\lambda_1 = 0$. We have reduced the problem to $(k - 1)$ equations in $(d + k)$ unknowns $(\xi_1, \ldots, \xi_{k+d})$. Since the range of $L(T_0, 0, w_0)$ intersects V at 0 only, all the $\partial\lambda_i/\partial\xi_j$ must vanish at w_0. ∎

One must be careful in interpreting this result. Situations where $\dim V > 1$ will usually arise when there are many integrals of the motion ($d + 1$ or more when $(\mathscr{H}_0)$ is nonlinear, d or more when $(\mathscr{H}_0)$ is linear). But this is precisely the situation when the periodic solutions to $(\mathscr{H}_0)$ come in families depending on two or more parameters, so that the condition $\dim V > 1$ will hold for every trajectory of this family, and it is hard to see how every such trajectory could be somehow "singular," as Proposition 12 seems to suggest. So a closer look is required.

PROPOSITION 13. *Assume u_0 belongs to an l-dimensional family of periodic solutions for $(\mathscr{H}_0)$. In other words, there is a neighbourhood $\mathscr{U}$ of the origin in $\mathbb{R}^l$ and smooth maps $U\colon \mathscr{U} \to C^r(S^1, \mathbb{R}^{2n})$ and $\theta\colon \mathscr{U} \to \mathbb{R}$ such that*

$$U(0) = u_0, \qquad \theta(0) = T_0$$

$U'(0)$ *has rank* l

$$\forall \xi \in \mathscr{U}, \qquad \frac{d}{ds} U(\xi; s) = \theta(\xi)\, \sigma H'(0, U(\xi; s))$$

$$\forall \xi \in \mathscr{U}, \qquad U(\xi; \theta(\xi)) = U(\xi; 0).$$

Denote by Y the space of solutions to the linearized equations $\dot{y} = \sigma H''_{uu}(0, u_0(t))\, y$, $y(0) = y(T_0)$. Assume that

(a) $Y_0 = \{ y \in Y \mid \int_0^{T_0} (\sigma \dot{u}_0, y)\, dt = 0 \}$ *has dimension* $l - 1$,

(b) *the linear functionals* $y \to \int_0^{T_0} (\sigma G''_{\varepsilon_i v}(0, -\sigma u_0), y)\, dt$ *are independent on* Y_0.

Then there is a neighbourhood $\mathscr{N}$ of the origin in $\mathbb{R}^d$, a tubular neighbourhood $\mathscr{V}$ of u_0 in $\mathbb{R}^{2n}$, and some $\beta > 0$ such that, for $\varepsilon \in \mathscr{N}$, $|T - T_0| < \beta$ and $\varepsilon \neq 0$, the problem

$$\dot{u} = \sigma H'_u(\varepsilon, u), \qquad u(0) = u(T) \qquad (\mathscr{H}_\varepsilon)$$

has no solution inside $\mathscr{V}$. ∎

Let us explain this result before we prove it. What it says is that if a trajectory belongs to an l-dimensional family of closed trajectories for the unperturbed system $(\mathscr{H}_0)$, with $l \geqslant d + 2$, it will in general *not* give rise to a periodic solution of the perturbed system $(\mathscr{H}_\varepsilon)$.[1] This phenomenon we already mentioned in the introduction.

Note that we count the phase as one parameter so that $u_0(t)$ belongs to the one-dimensional family $u_0(t + \phi)$, $\phi \in \mathbb{R}/T_0\mathbb{Z}$. It is a well-known fact (found, for instance, in [11]) that in a continuous family of closed trajectories, as described in Proposition 13, the period $T = \theta(\xi)$ depends only on the energy level $h = H(0, U(\xi; t))$. In other words, two trajectories of this family with the same energy have the same period.

We will now give two important cases when condition (a) of Proposition 13 is met.

Lemma 14. *Assume the equations $(\mathscr{H}_0)$ are linear:*

$$H(0, u) = \tfrac{1}{2}(Au, u) \qquad \textit{with} \quad A = A^* \textit{ positive definite}$$

and let l be the number of linearly independent T_0-periodic solutions of $(\mathscr{H}_0)$. Then u_0 belongs to an l-dimensional family of periodic solutions of $(\mathscr{H}_0)$, and condition (a) *holds.*

[1] Note that there are d linear functionals which are supposed to be independent on a space of dimension $l - 1$.

Proof. Since the equations are linear, $(\mathscr{H}_0)$ coincides with the linearized problem:

$$\dot{u} = \sigma A u, \qquad u(0) = u(T_0).$$

So Y has dimension l. Moreover $u_0 \in Y$, so that

$$\int_0^{T_0} (\sigma \dot{u}_0, u_0)\, dt = \int_0^{T_0} (-A u_0, u_0)\, dt \neq 0.$$

It follows that $\sigma \dot{u}_0$ is not orthogonal to Y, and

$$\dim Y_0 = \dim Y - 1 = l - 1. \quad \blacksquare$$

The second case we will deal with is the case when the equations are genuinely non-linear, but completely integrable. Recall (from [1], for instance) that this means that the energy levels are compact, and that there are n first integrals $F_1(u) = H(0, u)$, $F_2(u), \ldots, F_n(u)$. The phase space $\mathbb{R}^{2n}$ then is partitioned into n-dimensional invariant tori, on which the motion is quasi-periodic.

To be more precise, there are new variables $(I_1, \ldots, I_n) \in \mathbb{R}^n$ and $(\phi_1, \ldots, \phi_n) \in (S^1)^n$ such that the equations for the motion become

$$\dot{I}_i = 0, \qquad \dot{\phi}_i = \omega_i(I_1, \ldots, I_n), \qquad 1 \leqslant i \leqslant n.$$

For a T_0-periodic trajectory u_0 to exist on the torus defined by $I_1 = I_1^0, \ldots, I_n = I_n^0$, in short $I = I^0$, all the $\omega_i(I^0)$ must be multiples of T_0^{-1}.

LEMMA 15. *Assume the system $(\mathscr{H}_0)$ is completely integrable, all the $\omega_i(I^0)$ are multiples of T_0^{-1}, and the matrix $((\partial \omega_i / \partial I_j(I^0)))$ has rank n. Then u_0 belongs to an $(n+1)$-dimensional family of periodic solutions to $(\mathscr{H}_0)$ and condition* (a) *holds with $l = n + 1$.*

Proof. Say $\omega_i(I^0) = k_i T_0^{-1}$ for $1 \leqslant i \leqslant n$, the k_i being appropriate integers. Let L be the straight line in $\mathbb{R}^n$ spanned by the vector $(k_1, \ldots, k_n)$. By the inverse function theorem, the map $\omega(I) = (\omega_1(I), \ldots, \omega_n(I))$ is locally invertible near I_0. Since $\omega(I_0) \in L$, there is a neighbourhood $\mathscr{W}$ of I_0 in $\mathbb{R}^n$ such that $\mathscr{W} \cap \omega^{-1}(L)$ is a one-dimensional submanifold.

Let s be a local coordinate on $\omega^{-1}(L)$ near I_0. We have

$$\omega_i(I(s))/k_i = \text{constant} \qquad \text{for } 1 \leqslant i \leqslant n.$$

Call this constant $T(s)^{-1}$. The equation of motion on the torus defined by $I = I(s) \in \mathbb{R}^n$ are

$$\dot{I}_i = 0, \qquad \dot{\phi}_i = \omega_i(I(s)) = k_i T(s)^{-1}$$

so that all solutions are $T(s)$-periodic.

We have found a 1-parameter family of tori, each of these being partitioned into an n-parameter family of periodic solutions. The result is indeed an $(n+1)$-dimensional family of closed trajectories.

Now for condition (a). We first write the linearized equations near u_0:

$$\delta \dot{I}_i = 0, \qquad \delta\dot{\phi}_i = \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial I_j}(I^0)\,\delta I_j$$

where $\delta\phi_i(t)$ belongs to the tangent space to S^1 at $\phi_i(t)$, which is $\mathbb{R}$, So we have here a system of ordinary differential equations in $\mathbb{R}^n$, which has the obvious periodic solutions $\delta I_i(t) = 0$, $\delta\phi_i(t) = c_i$, and none other under our nondegeneracy assumption. Hence

$$\dim Y = n.$$

I now claim that

$$\int_0^{T_0} (\sigma \dot{u}_0, y)\,dt = 0, \qquad \text{all } y \in Y$$

so that $Y = Y_0$ and $\dim Y_0 = \dim Y = n = l - 1$, as announced.

This is easily checked. It is clear from the above that $y(t)$ belongs at any time t to the tangent space to the invariant torus at $u_0(t)$, which is spanned by the vectors $\sigma I_i'(u_0(t))$. We then have

$$\begin{aligned}(\sigma\dot{u}_0(t), y(t)) &= \Big(\sigma\dot{u}_0(t), \sum_{i=1}^{n} \xi_i \sigma I_i'(u_0(t))\Big)\\ &= \sum_{i=1}^{n} \xi_i(\dot{u}_0(t), I_i'(u_0(t)))\\ &= \sum_{i=1}^{n} \xi_i \frac{d}{dt} I_i(u_0(t))\end{aligned}$$

which vanishes because the I_i are first integrals. ∎

We now can proceed to the:

Proof of Proposition 13. By Proposition 8, there is some neighbourhood $\mathscr{W}$ of $(T_0, 0, w_0)$ in $\mathbb{R} \times \mathbb{R}^d \times E$ such that the set $S \cap \mathscr{W}$ defined by

$$S = \{(T, \varepsilon, w) \mid \Phi_w'(T, \varepsilon, w) \in V\}$$

is a $(\dim V + d + 1)$-dimensional submanifold.

By the definition of V and Proposition 7, we have: $z \in V$ if and only if

$z(s) = T_0\dot{y}(sT_0)$, with $y \in Y_0$ and $\int_0^{T_0} (\sigma G''_{\varepsilon v}(0, -\sigma\dot{u}_0), \dot{y})\, dt = 0$ in $\mathbb{R}^d$. Since there are no constant functions in V, we get from (a) and (b)

$$\dim V = \dim Y_0 - d$$
$$= l - 1 - d.$$

So $S \cap \mathscr{W}$ has dimension l.

On the other hand, S must contain the set of solutions of $\Phi'_w(T, \varepsilon, w) = 0$. But we have assumed that u_0 belongs to an l-dimensional family of periodic solutions for $(\mathscr{H}_0)$. Going back to what we mean by this, we see that the maps U and θ satisfy

$$\Phi'_w(\theta(\xi), 0, \theta(\xi)^{-1}\, \dot{U}(\xi)) = 0, \qquad \text{all} \quad \xi \in \mathscr{U}.$$

Since $U'(0)$ has rank l, the image of $\mathscr{U}$ by the map $\xi \to (\theta(\xi), 0, \theta(\xi)^{-1}\, \dot{U}(\xi))$ is an l-dimensional submanifold of $\mathbb{R} \times \mathbb{R}^n \times E$ which is contained in $S \cap \mathscr{W}$. Since the latter also is an l-dimensional submanifold, they must coincide in a neighbourhood of $(T_0, 0, w_0)$. In other words, there is no solution to $\Phi'_w(T, 0, w) = 0$ except those which belong to the l-parameter family we started with. ∎

Let us say that u_0 is a *bifurcating trajectory* if there are sequences $\varepsilon_n \to 0$ (with $\varepsilon_n \neq 0$ for all n), $T_n \to T_0$ and $U_n \to U_0$ in $C^r(S^1; \mathbb{R}^{2n})$ such that

$$\dot{u}_n = T_n \sigma H'_u(\varepsilon_n, u_n), \qquad \text{all } n.$$

Proposition 13 may be put as follows:

COROLLARY 16. *Assume u_0 belongs to an l-dimensional family of periodic solutions for $(\mathscr{H}_0)$ satisfying condition* (a). *If u_0 is a bifurcating trajectory, then the functionals*

$$y \to \int_0^{T_0} (\sigma G''_{\varepsilon v}(0, -\sigma\dot{u}_0), \dot{y})\, dt$$

are linearly dependent on Y_0. ∎

For the sake of simplicity, let us confine ourselves to the case when $d = 1$, so the necessary condition for a bifurcating trajectory becomes

$$\text{(c)} \qquad \int_0^{T} (\sigma G''_{\varepsilon v}(0, -\sigma\dot{u}_0), \dot{y})\, dt = 0, \text{ all } y \in Y_0.$$

Now the bifurcating trajectory u_0 has to be sought in an l-dimensional family of periodic solutions. On the other hand, $\dim Y_0 = l - 1$, so u_0 has to satisfy a system of $l - 1$ equations. The first of these equations, corresponding to $y=y_1=\dot{u}_0$, is identically zero, so we are left with $l-2$ equations only. Under appropriate transversality conditions, we see that the set of bifurcating trajectories will be a two-dimensional subfamily of the original l-dimensional family (phase accounts for one dimension).

We will now show that, if a transversality condition is satisfied, condition (c) is also sufficient for u_0 to be a bifurcating trajectory.

We know u_0 belongs to an l-dimensional family of periodic solutions to $(\mathscr{H}_0)$. As in Proposition 13, we use $\xi=(\xi_1,..., \xi_l)\in\mathscr{U}$ as a local coordinate system for this family near (T_0, u_0), with $T=\theta(\xi)$ and $u(t)=U(\xi, tT^{-1})$. From now on, we will assume the last coordinate ξ_l is the phase, so that

$$T=\theta(\xi_1,..., \xi_{l-1}), \qquad \text{which we denote by } T_\xi$$

$$u(t)=U(\xi_1,..., \xi_{l-1}, (t+\xi_l)\,T^{-1}), \qquad \text{denoted by } u_\xi(t).$$

Assume condition (a) is satisfied at u_0, together with condition (c). We then have

$$V=\{\dot{y}(sT_0)\mid y\in Y_0\}$$

so that $\dim V=l-1$. Let $y_1=\dot{u}_0, y_2,..., y_{l-1}$ span Y_0.

By Proposition 12, solving the equation

$$\Phi'_w(T, \varepsilon, w)=\sum_{j=1}^{l-1} F_j z_j \qquad \text{with } z_j(s)=T_0\dot{y}_j(sT_0) \tag{$\mathscr{F}$}$$

will give the F_j, $1\leqslant j\leqslant l-1$, as smooth functions of $(l+1)$ unknowns which can be chosen to be $(\xi_1,..., \xi_l)$ and a new variable ζ. After the phase ξ_l has been eliminated as a meaningful variable, we are left with $(l-2)$ equations in the l independent variables $(\xi_1,..., \xi_{l-1}, \zeta)$:

$$F_j(\xi_1,..., \xi_{l-1}, \zeta)=0, \qquad 2\leqslant j\leqslant l-1$$

(the equation $F_1=0$ being a consequence of the others), with the set of trivial solutions

$$F_j(\xi_1,..., \xi_{l-1}, 0)=0, \qquad \text{all } \xi\in\mathscr{U} \text{ and } j$$

and the degeneracy conditions

$$\partial F_j/\partial\zeta(0,..., 0, 0)=0, \qquad 2\leqslant j\leqslant l-1.$$

PROPOSITION 17. *Assume u_0 belongs to an l-dimensional family of periodic solutions to $(\mathscr{H}_0)$, and satisfies conditions* (a) *and* (c), *with $d=1$.*

Assume that

$$\forall \xi \neq 0 \qquad \exists j: \frac{\partial F_j}{\partial \zeta}(\xi_1, \ldots, \xi_{l-1}, 0) \neq 0$$

$$\textit{the matrix } \left(\left(\frac{\partial^2 F_j}{\partial \xi_i \partial \zeta}(0, 0) \right) \right) \textit{ has rank } (l-2).$$

Then u_0 belongs to a three-dimensional family of closed trajectories which intersects the original l-dimensional family along a two-dimensional family of bifurcating trajectories (phase included in the dimension count).

Proof. We follow the pattern of Crandall and Rabinowitz (see [7], for instance). Set $F = (F_2, \ldots, F_{l-1})$ and define $\psi(\xi_1, \ldots, \xi_{l-1}, \zeta)$ by

$$\begin{array}{ll} \partial F/\partial\zeta(\xi, 0) & \text{if } \zeta = 0 \\ \zeta^{-1} F(\xi, \zeta) & \text{if } \zeta \neq 0. \end{array}$$

For $\zeta \neq 0$, taking into account the fact that $\partial F/\partial\zeta(0, 0) = 0$, we can also write $\psi(\xi_1, \ldots, \xi_{l-1}, \zeta)$ as

$$\frac{\partial^2 F}{\partial \xi \partial \zeta}(0, 0)\, \xi + \zeta^{-1} R(\xi, \zeta).$$

The remainder $R(\xi, \zeta)$ satisfies $R(0, 0) = R'_\zeta(0, 0) = 0$ and $R''_{\xi\zeta}(0, 0) = 0$. It follows that

$$\psi'_\xi(0, 0) = \frac{\partial^2 F}{\partial \xi \partial \zeta}(0, 0): \mathbb{R}^{l-1} \to \mathbb{R}^{l-2}.$$

This is onto by assumption. But then one of the $(l-2) \times (l-2)$ matrices obtained from $\partial^2 F/(\partial\xi\partial\zeta)(0, 0)$ by deleting one column must be invertible. Say, for instance, the following is:

$$\left(\left(\frac{\partial^2 F_j}{\partial \xi_i \partial \zeta}(0, 0) \right) \right), \qquad 2 \leqslant i, j \leqslant l-1.$$

Then, by the implicit function theorem, the equation $\psi(\xi_1, \ldots, \xi_{l-2}, \xi_{l-1}, \zeta) = 0$ can be uniquely solved near $(0, \ldots, 0)$ with ξ_{l-1} and ζ as independent variables, $(\xi_1, \ldots, \xi_{l-2})$ being smooth functions of ξ_{l-1} and ζ. But $\psi(\xi, \zeta) = 0$ means that $\zeta \neq 0$ (unless $\xi = 0$ also) and $F(\xi, \zeta) = 0$. The result follows. ∎

For practical purposes, it is best to choose ε as the new variable. The variables $(\xi_1,\dots,\xi_l)$ describe the l-dimensional family u_0 belongs to. The functions $F_j(\xi_1,\dots,\xi,\varepsilon)$ are found by solving the equation

$$\Phi'_w(T,\varepsilon,w)=\sum_{j=1}^{l-1}F_jz_j. \qquad (\mathscr{E})$$

Arguing as in Proposition 3, we transform this successively into

$$T\sigma\Pi w-\sigma G'_v(\varepsilon,-\sigma w)=c+\sum_{j=1}^{l-1}F_jz_j$$

$$H'_u(\varepsilon,T\Pi w-\sigma c-\sigma\sum_{j=1}^{l-1}F_jz_j)=-\sigma w.$$

Calling $U(s)$ the second argument of H'_u, and replacing $z_j(s)$ by $T_0\dot{y}_i(sT_0)$, we finally get

$$\begin{aligned}\dot{U}(s)&=T\sigma H'_u(\varepsilon,U(s))-\sigma\sum_{j=1}^{l-1}T_0^2F_j\ddot{y}_j(sT_0)\\ U(0)&=U(1).\end{aligned} \qquad (\mathscr{E})$$

Here $y_1=\dot{u}_0$ and we do not need to compute F_1.

If we are in the non-degeneracy situation of Proposition 17, we will be able to compute the Taylor expansion for the non-trivial branch in terms of (ξ_1,ε). We thus get an asymptotic expansion of the type

$$T=\Sigma\varepsilon^P(\xi_1-\xi_1^0)^Q\theta_{PQ}$$

$$u(t)=\Sigma\varepsilon^P(\xi_1-\xi_1^0)^QU_{P,Q}(tT^{-1})$$

$$U_{P,Q}(s)=U_{P,Q}(s+1).$$

Setting $\varepsilon=0$ and $\xi=\xi_1^0$, we get $T=T_0$ and $u(t)=u_0(t)$. Any solution close to (T_0,u_0) will be written $(T,u(t+\phi))$, where T and $u(t)$ are given above.

Finally, note that the method will require the computation of $G''_{\varepsilon v}(0,-\sigma\dot{u}_0(t))$. This can be done by differentiating with respect to ε the identity

$$G'_v(\varepsilon,H'_u(\varepsilon,u))=u$$

$$G''_{\varepsilon v}(0,H'_u(0,u))+G''_{vv}(0,H'_u(0,u))\,H''_{\varepsilon u}(0,u)=0.$$

But, by differentiating the same equation with respect to u, we get $H''_{uu}(0,u)\,G''_{vv}(0,H'_u(0,u))=I$, and hence

$$G''_{\varepsilon v}(0,H'_u(0,u))=-H''_{uu}(0,u)\,H''_{\varepsilon u}(0,u).$$

So, using the equation $\dot{u}_0 = \sigma H'_u(0, u_0)$:

$$G''_{\varepsilon v}(0, -\sigma\dot{u}_0(t)) = -H''_{uu}(0, u_0(t))^{-1} H''_{\varepsilon u}(0, u_0(t)).$$

The necessary condition for bifurcation now reads (remembering that $\dot{y} = H''_{uu}(0, u_0(t))\, y$): the

$$y \to \int_0^{T_0} (H''_{\varepsilon_i u}(0, u_0(t)), y(t))\, dt$$

are linearly dependent on Y_0.

To illustrate the relevance of these results, we might mention *synchronization theory*. Let $d = n + 1$, and

$$H(\omega, \varepsilon, u) = \sum_{i=1}^{n} \tfrac{1}{2}[x_i^2 + (1 + \omega_i)\, p_i^2] + \varepsilon h(x, p).$$

The perturbation parameters are $(\omega_1, ..., \omega_n, \varepsilon)$ and the unperturbed system is just n uncoupled linear oscillators with the same frequency:

$$H(0, 0, u) = \sum_{i=1}^{n} \tfrac{1}{2}[x_i^2 + p_i^2].$$

If u_0 is a bifurcating trajectory, there will exist near u_0 closed trajectories of the perturbed system with period close to 2π. The physical meaning of this is that the non-linear coupling $\varepsilon h(x, p)$ succeeds in synchronizing the n oscillators, which would by themselves by out of tune with each other, their natural pulsations $(1 + \omega_i)$, $1 \leqslant i \leqslant n$, being distinct.

To reduce this to the case $d = 1$, it is usually done by writing $\omega_i = k_i \varepsilon$, the coefficients $k_1, ..., k_n$ being fixed. The equations then become

$$H(\varepsilon, u) = \sum_{i=1}^{n} \tfrac{1}{2}[x_i^2 + (1 + k_i \varepsilon)\, p_i^2] + \varepsilon h(x, p)$$

and we are in the setting of Proposition 17.

Unfortunately, it will happen in some cases that all closed trajectories in the given T_0-periodic family satisfy condition (c), which becomes useless to find bifurcating trajectories. This happens, for instance, when $k_i = 0$ all i and $h(x, p)$ is a homogeneous polynomial of degree 3. To treat these cases, more refined methods are required. They will be described in the next section.

V. Weinstein's Theorem

We will now prove an existence theorem which is due to Weinstein [13]. This we do by reducing the problem of finding periodic solutions to finding critical points of a function on a finite-dimensional manifold. This is Moser's approach (see [11] and [12]).

THEOREM 18. *Assume $d=1$ and the equations $(\mathscr{H}_0)$ are linear:*

$$H(0,u)=\tfrac{1}{2}(Au,u) \qquad \textit{with } A=A^* \textit{ positive definite.}$$

Let l be the number of linearly independent T_0-periodic solutions of $(\mathscr{H}_0)$. Then, for all ε small enough, and for all $h>0$, the system $(\mathscr{H}_\varepsilon)$ has at least $l/2$ closed trajectories, with periods close to T_0, such that $H(\varepsilon, u(t))=h$. ∎

Note that two periodic solutions differing only by the phase give rise to the same closed trajectory. Note also that l is even, so that $l/2$ is an integer.

We only sketch the proof, since the main ideas have been discussed at length in the preceding sections.

Let Y be the l-dimensional vector space of solutions to $(\mathscr{H}_0)$. Let $\Sigma_0 \subset Y$ be the subset of solutions with energy h:

$$\Sigma_0=\{u\in Y \mid \tfrac{1}{2}(Au(t),u(t))=h\}.$$

Σ_0 is an $(l-1)$-dimensional sphere, invariant by the S^1-action. The map

$$\gamma\colon u\to (T_0, 0, \dot{u}(sT_0))$$

imbeds Σ_0 into $\mathbb{R}\times\mathbb{R}\times E$.

Let us consider a tubular neighbourhood $\mathscr{U}$ of Σ_0 in $\mathbb{R}\times\mathbb{R}\times E$, and an adapted coordinate system $\tilde{w}\to(\pi_0(\tilde{w}),\pi_1(\tilde{w}))$ for $\tilde{w}=(T,\varepsilon,w)\in\mathscr{U}$,

$$\pi_0(\tilde{w})\in\Sigma_0$$

$$\pi_1(\tilde{w})\in F \qquad \text{with} \quad F\oplus\mathbb{R}^{l-1}=\mathbb{R}\times\mathbb{R}\times E$$

$$u\in\Sigma_0\Rightarrow \pi_0\gamma(u)=u \qquad \text{and} \qquad \pi_1\gamma(u)=0.$$

We want to solve the equation $\Phi'_w(\tilde{w})=0$ with $\tilde{w}=(T,\varepsilon,w)\in\mathscr{U}$. This we cannot do directly because the tangent map is not onto. So instead we try to solve the equation

$$\Phi'_w(\tilde{w})\in Z(\pi_0(\tilde{w})). \tag{1}$$

If $y\in\Sigma_0$, we define

$$Z(u)=\left\{\dot{y}(sT_0) \mid y\in Y \text{ and } \int_0^{T_0}(Au,y)\,dt=0\right\}.$$

The transversality conditions are now fulfilled, and Eq. (1) defines a submanifold S of $\mathscr{U}$ (not exactly the S of Theorem 8) which:

(a) has dimension $l+1$, because Z has dimension $l-1$ and Φ'_w is Fredholm with index 2;

(b) contains $\gamma(\Sigma_0)$, the set of trivial solutions to $\Phi'_w(\tilde{w})=0$;

(c) is S^1-invariant.

For $\tilde{w}=(T, \varepsilon, w)$ set

$$\rho(\tilde{w})=\int_0^1 H(\varepsilon, G'_v(\varepsilon, -\sigma w(s)))\, ds$$

$$=\int_0^1 [(-\sigma w(s), G'_v(\varepsilon, -\sigma w(s)))-G(\varepsilon, -\sigma w(s))]\, ds.$$

If $\Phi'_w(\tilde{w})=0$, then $G'_v(\varepsilon, -\sigma w(s))=u(sT)$, where u is a solution of $(\mathscr{H}_\varepsilon)$, and $\rho(\tilde{w})=H(\varepsilon, u(sT))$ is the energy of this solution.

Consider the map

$$\Psi\colon (T, \varepsilon, w)\to(\varepsilon, \rho(\tilde{w}), \pi_0(\tilde{w}))$$

of S into $\mathbb{R}\times\mathbb{R}\times\Sigma_0$. The tangent map is easily seen to be injective, and since both sides have dimension $(l+1)$, it is invertible. It follows (by suitably restricting $\mathscr{U}$) that Ψ is a diffeomorphism of S onto $(-\alpha, \alpha)\times(h-\beta, h+\beta)\times\Sigma_0$, with α and β suitably small.

Set

$$\Gamma_\varepsilon=\Psi^{-1}(\varepsilon, h, \Sigma_0).$$

Then Γ_ε is diffeomorphic to Σ_0. Moreover, it is invariant by the S^1-action. We have $\Gamma_0=\gamma(\Sigma_0)$.

We now go back to our original problem, which was to solve the equation

$$\Phi'_w(\tilde{w})=0. \tag{2}$$

Let us consider the restriction of Φ to Γ_ε: we call it ϕ_ε. Let $\tilde{w}\in S$ be a critical point of ϕ_ε; by definition, we have

$$\Phi'_w(\tilde{w})\in N(\tilde{w}; \Gamma_\varepsilon) \tag{3}$$

where $N(\tilde{w}; \Gamma_\varepsilon)$ is the space of all continuous linear functionals on $\mathbb{R}\times\mathbb{R}\times E$ which vanish on the tangent space to Γ_ε at $\tilde{w}$.

This tangent space is easily identified when $\varepsilon=0$. Indeed, the tangent space to Σ_0 at u_0 is the set of $y\in Y$ such that $\int_0^{T_0}(Au_0, y)\, dt=0$. It follows that the tangent space to $\gamma(\Sigma_0)=\Gamma_0$ at $\tilde{w}_0=\gamma(u_0)$ is

$$T(\tilde{w}_0, \Gamma_0)=\{0\}\times\{0\}\times Z(u_0).$$

We now go back to $\tilde{w}$, a critical point of ϕ_ε on S. Since $\tilde{w}$ belongs to S, Eq. (1) holds, and tells us in effect that $\Phi'_w(\tilde{w})$ can be represented by a vector $\tilde{z}$ in $T(\pi_0(\tilde{w}), \Gamma_0)$:

$$\langle \Phi'_w(\tilde{w}), \tilde{w}_1 \rangle = \int_0^1 (\tilde{z}(s), \tilde{w}_1(s))\, ds.$$

By continuity, $T(\tilde{w}, \Gamma_\varepsilon)$ is close to $T(\pi_0(\tilde{w}), \Gamma_0)$; so no linear functional represented by a non-zero vector in the second space can vanish identically on the first one. Since $\Phi'_w(\tilde{w}) \in N(\tilde{w}; \Gamma_\varepsilon)$, we must have $\Phi'_w(\tilde{w}) = 0$. In other words, Eqs. (1) and (3) together imply equation (2).

The problem is now reduced to finding critical points of ϕ_ε on Γ_ε. Any such critical point $\tilde{w}$ solves $\Phi'_w(\tilde{w}) = 0$, and $\gamma^{-1}(\tilde{w})$ is a periodic solution of $(\mathscr{H}_\varepsilon)$ with energy h.

But Γ_ε is (diffeomorphic to) an $(l-1)$-sphere, with the natural S^1-action (Hopf fibration; note l is even), and ϕ_ε is a smooth S^1-invariant function. By a well-known result, which apparently was first proved by Krasnoselski in [10], such a function has at least $l/2$ distinct critical orbits. This concludes the proof of the theorem. ∎

Weinstein's theorem on periodic orbits near an equilibrium follows immediately:

COROLLARY 19. *Assume* $H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is* C^2, *with* $H(u) = 0$, $H'_u(0) = 0$, *and* $H''_{uu}(0)$ *positive definite. For all* $h > 0$ *sufficiently small, the Hamiltonian system* $\dot{u} \in \sigma H'_u(u)$ *has at least n distinct periodic orbits with energy h.* ∎

This problem reduces to the preceding one by blowing up the situation at the origin. We set

$$H(\varepsilon, u) = \varepsilon^{-2} H(\varepsilon, u) \qquad \text{for} \quad \varepsilon > 0$$

$$H(0, u) = \tfrac{1}{2}(H''(0)\, u, u).$$

The solutions to $\dot{u} = \sigma H(0, u)$ split into k families with periods $T_1, \ldots, T_k$ and dimensions $n_1, \ldots, n_k$. We have $l_1 + \cdots + l_k = 2n$. Care is taken in separating the families, so that two distinct families have no common period.

Theorem 18 then applies separately to each of these families, giving rise to $l_1/2 + \cdots + l_k/2 = n$ trajectories at least. These are distinct within the same family, by Theorem 18, and from one family to another, because they have no common period (by continuity). Hence the result.

References

1. V. Arnold, "Méthodes mathématiques de la mécanique classique," Editions de Moscou 1974; Russian 76.
2. J. Blot, These 3^{eme} cycle, Ceremade, Université Paris-Dauphine, 1981.
3. F. Clarke, Solution périodique des équations hamiltoniennes, *C. R. Acad. Sci. Paris* **287** (1978), 951–952.
4. F. Clarke, Periodic solutions to Hamiltonian inclusions, *J. Differential Equations* **40** (1981), 1–6.
5. F. Clarke and I. Ekeland, Solutions périodiques, de période donnée, des équations hamiltoniennes, *C. R. Acad. Sci. Paris* **287** (1978), 1013–1015.
6. F. Clarke and I. Ekeland, Hamiltonian trajectories having prescribed minimal period, *Comm. Pure Appl. Math.* **33** (1980), 103–116.
7. M. Crandall and P. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.* **8** (1971), 321–370.
8. I. Ekeland, Periodic solutions of Hamiltonian equations and a theorem of P. Rabinowitz, *J. Differential Equations* **34** (1979), 523–534.
9. S. Jorna, Ed., "Topics in Nonlinear Dynamics," American Institute of Physics Conference Proceedings 46, 1978.
10. M. Krasnolselski, On special coverings of finite-dimensional spheres, *Dokl. Akad. Nauk USSR* **103** (1955), 961–964.
11. J. Moser, Regularization of Kepler's problem and the averaging method on a manifold, *Comm. Pure Appl. Math.* **23** (1970), 609–636.
12. J. Moser, Periodic orbits near an equilibrium and a theorem of Alan Weinstein, *Comm. Pure Appl. Math.* **24** (1976), 727–747.
13. A. Weinstein, Normal modes for nonlinear Hamiltonian systems, *Invent. Math.* **20** (1973), 47–57.