Optimal pits and optimal transportation

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Abstract

In open pit mining, one must dig a pit, that is, excavate the upper layers of ground before reaching the ore. The walls of the pit must satisfy some mechanical constraints, in order not to collapse. The question then arises how to mine the ore optimally, that is, how to find the optimal pit. We set up the problem in a continuous (as opposed to discrete) framework, and we show, under weak assumptions, the existence of an optimum pit. For this, we formulate an optimal transportation problem, where the criterion is lower semi-continuous and is allowed to take the value $+\infty$. We show that this transportation problem is a strong dual to the optimum pit problem, and also yields optimality (complementarity slackness) conditions.

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1 Introduction

In open pit mining, one tries to extract profitable ore by excavating the (unprofitable) layers of soil above it. One thereby digs a hole (the pit) deep enough to reach the underground ore. The wall cannot be too steep, otherwise the hole will cave in, so there are geomechanical constraints on the slope, which depend on the physics of the soil, which may in turn vary across layers. In any case, the deeper the ore, the wider the pit, and the costlier it is to reach it. Certain parts will simply be too costly to reach, so the question arises: how to determine a most profitable pit? Note that this question encompasses several others: which part of the ore can be most profitably exploited? (i.e., the "cut-off grade" decision); what is the shape of the hole to be dug, taking into account the slope constraints?

The standard approach is to discretize the problem and to solve it by mixed integer programming. However, it is by nature a continuous problem: the ore density is distributed underground, and the shape of the pit may be described as the graph of a function $\varphi : A \to R$, where A is the "claim", i.e., the surface area under which the pit will be dug. The geomechanical requirements on the pit translate into constraints on the derivatives of φ, and one is led to an optimization problem. To our knowledge, the first one to use this approach was G. Matheron, in unpublished technical notes [5] and [6]. After a forty-year interval, this approach was revived by and [1] and [2], who seek the optimal shape in a class of Lipschitz functions. Further references on discrete and continuous approaches to optimum open pit design may be found in [1] and in the references therein.

In this paper, we develop yet another approach, which is to seek the optimal pit as the solution of an optimal transportation problem. Let us recall that optimal transportation is concerned with transporting a given mass from one location to another so as to minimize cost (or maximize profit, which is the same thing), so its relevance to a mining problem (to bring ore out of the ground) should come as no surprise. However the relationship is not obvious. The mathematics will be found is Section 3. The underlying intuition is that in any profitable pit, there unprofitable parts, with no commercial interest, but which must be cleared out to reach the profitable ones. Since the pit is profitable, the profit made from the ore must pay for the excavation, and more. Our idea is to match every unprofitable part of the pit with a profitable part which pays for its extraction; this will leave some profitable parts unmatched, since otherwise there would be no overall profit, and these we match with a sink ω. The amount put into ω is the total profit. There will be unprofitable parts which are unmatched, meaning that they are unexcavated. To account for these, we introduce a sink α, and we match them with the sink.

We model the mechanical constraints by an ordering relation: $x_2 \succsim_\Gamma x_1$ means that if you want to extract x_1, you must first extract x_2. It is obviously transitive, and we assume it to be closed. We also assume that the distribution of profit (or cost) in the ground is known and continuous. Under these weak assumptions, weaker than any other one we have seen in the literature, we prove that there is an optimal pit (Theorem 13). The proof relies on the mathematical theory of optimal transportation, as described in [9], with the added twist that we allow the cost function to take the value $+\infty$. In that case, provided that the transportation cost function is lower semi-continuous, the optimal transportation problem from (X, μ) to (Y, ν) has a solution in the class of probabilities on $X \times Y$ with marginals μ and ν, but not necessarily in the class of maps $X \to Y$ transporting μ to ν. Special arguments have to be used, and this is what we do in this paper. We show that the Kantorovitch dual has a solution (p, q), with $p : X \to \mathbb{R}$ and $q : Y \to \mathbb{R}$ taking only the values 0 and 1, and that the optimal pit is $F = \{p = 1\} \cup \{q = 1\}$. Thus the transportation problem is a strong dual to the optimum pit problem, for which we also present optimality (complementarity slackness) conditions.

2 The model

We are given a compact subset $E \subset \mathbb{R}^3$, representing the domain to be mined. To support intuition, we may think of E as having the special form $E = A \times [h_1,\ h_2]$, where $A \subset \mathbb{R}^2$ is the claim, $h_1 < 0 < h_2$ is the elevation range and $-h_1$ is the maximum depth allowed; in this case we are also given a map $\varphi : A \to [h_1,\ h_2]$ representing ground level before excavation: any $x = (x_1, x_2, x_3)$ with $x_3 < \varphi(x_1, x_2)$ is underground.

In open pit mining, to reach an underground spot $x \in E$, one must first excavate the material above it, thereby creating a pit. There are physical constraints on the shape of these pits, so that their walls do not collapse and fill it up. These constraints depend on the slope and on the material. We represent them by a map $\Gamma : E \twoheadrightarrow E$ with closed graph such that:

$$\text{(reflexivity)} \quad x \in \Gamma(x)$$
$$\text{(transitivity)} \quad [x_2 \in \Gamma(x_1) \text{ and } x_3 \in \Gamma(x_2)] \Longrightarrow x_3 \in \Gamma(x_1)$$

Write $x_2 \succsim_\Gamma x_1$ for $x_2 \in \Gamma(x_1)$. It is a partial ordering of E. The interpretation is that, to mine x, one must first excavate all of $\Gamma(x)$. Reflexivity is then obvious, and transitivity as well. If a pit has been dug, and x has been reached, then all the $x' \in \Gamma(x)$ must have been excavated, i.e. they must also belong to the pit. This leads us to the following:

Definition 2 *A* pit *is a subset $\Omega \subset E$ which is Lebesgue measurable, closed and stable for the ordering:*

$$[z \in \Omega \textit{ and } z' \succsim_\Gamma z] \Longrightarrow z' \in \Omega$$

To support intuition, one may think of $\Gamma(x)$ as being a cone with vertex x and vertical axis, directed upwards. To reach an underground region $\Omega \subset E$, one must excavate the whole pit, up to ground level:

$$\Gamma(\Omega) := \cup_{z \in \Omega} \Gamma(z)$$

Thus Ω is a pit if and only if $\Gamma(\Omega) = \Omega$.

Finally, we are given a continuous function $g : E \to \mathbb{R}$ which satisfies $\int_E \max\{0, g(x)\}\, dx > 0$ (for otherwise there is no hope of making a profit). This represent profit or cost. More precisely, $g(x)\, dx$ is the profit (net of extraction cost) obtained (or, if $g(x) < 0$, the negative of the cost incurred) by extracting the volume $dx = dx_1 dx_2 dx_3$ at x, once it has become accessible, i.e. when all $x' \succsim_\Gamma x$ have been extracted. The region where $g > 0$ is the profitable ore (the higher g the richer the ore). If we want to extract the ore from $F \subset \{g > 0\}$, one has to excavate all the ground above it, that is, the whole pit $\Gamma(F)$, and the corresponding profit is

$$\int_{\Gamma(F)} g(z)\, dz$$

To summarize, the data are E, Γ and g. The family of all pits will be denoted by $\mathcal{S}(E)$:

$$F \in \mathcal{S}(E) \Longleftrightarrow F = \Gamma(F)$$

We are looking for a pit that maximizes profit, that is, we are aiming to solve the optimization problem:

$$\begin{array}{c} \max \int_F g(z)\, dz \\ F \in \mathcal{S}(E) \end{array} \tag{P}$$

3 An optimal transportation problem

Introduce the following subsets of E:

$$E^+ := \overline{\{g(x) > 0\}}$$
$$E^- := \overline{\{g(x) < 0\}}$$

Both E^+ and E^- are compact sets. Introduce two points α (a source) and ω (a sink), and set:

$$X = E^+ \cup \{\alpha\}, \quad Y = E^- \cup \{\omega\}$$

Both X and Y are compact sets. We endow them with the non-negative measures μ and ν defined by:

$$\begin{array}{c} \mu(\{\alpha\}) = \int_{E^-} |g(z)|\, dz, \quad \mu|_{E^+} = g(z)\, dz \\ \nu(\{\omega\}) = \int_{E^+} g(z)\, dz, \quad \nu|_{E^-} = |g(z)|\, dz \end{array}$$

Note that $\mu(E^+) > 0$, $\mu(E^+ \cap E^-) = \nu(E^+ \cap E^-) = 0$, and $\mu(X) = \nu(Y)$. Define the "transportation" cost $c : X \times Y \longrightarrow R$ as follows:

X	Y	$c(x,y)$
$x \in E^+$	$y \in \Gamma(x)$	0
$x \in E^+$	$y \notin \Gamma(x),\ y \in E^-$	$+\infty$
$x \in E^+$	$y = \omega$	1
$x = \alpha$	$y \in Y$	0

Lemma 3 *c is lower semi-continuous (l.s.c.)*

Proof. Let $(x_n, y_n) \to (\bar{x}, \bar{y})$. If $\liminf c(x_n, y_n) = +\infty$, there is nothing to prove. If $\liminf c(x_n, y_n) < +\infty$, there is a subsequence $n(k)$ such that either

$$\liminf c(x_n, y_n) = c(x_{n(k)}, y_{n(k)}) = 0 \text{ for all } n$$

or

$$\liminf c(x_n, y_n) = c(x_{n(k)}, y_{n(k)}) = 1 \text{ for all } n$$

The first case divides into two subcases. Either $x_{n(k)} = \alpha$ for infinitely many k, or $x_{n(k)} \in E^+$ and $y \in \Gamma(x)$ for all $k \geq k_0$. In the first subcase, $\bar{x} = \alpha$ and $c(\bar{x}, \bar{y}) = 0 = \liminf c(x_n, y_n)$. In the second subcase, since E^+ is compact, $\bar{x} \in E^{+}$, and since Γ has closed graph, $\bar{y} \in \Gamma(\bar{x})$, so that $c(\bar{x},\bar{y}) = 0 = \liminf c(x_n, y_n)$ again.

In the second case, $\bar{x} \in E^{+}$ and $\bar{y} = \omega$, so that $c(\bar{x},\bar{y}) = 1 = \liminf c(x_n, y_n)$
■

Let $\Pi(\mu,\nu)$ denote the set of all positive Radon measures π with marginals $\pi_X = \mu$ and $\pi_Y = \nu$. Now consider the optimal transportation problem in Kantorovich form:

$$\begin{array}{c} \min \int_{X \times Y} c(x,y)\, d\pi \\ \pi \in \Pi(\mu,\nu) \end{array} \tag{K}$$

Problem (K) may, as outlined in the introduction, be interpreted as the problem of optimally transferring profits from the profitable parts of the mine to cover the costs of those parts that must be removed in order to reach them. It may also be viewed as a continuous version of a minimum cost, bipartite version of the maximum flow problem which is dual to a minimum cut formulation (e.g., [7]) of the discrete optimal pit problem.

Proposition 4 *Problem (K) has a solution*

Proof. The set of positive Radon measures on the compact space $X \times Y$ is weak-* compact, and the map $\pi \to \mathrm{E}^{\pi}[c]$ is weak-* l.s.c, so the result follows.
■

4 The Kantorovich dual

Introduce an admissible set $\mathcal{A}$ and a criterion J:

$$\mathcal{A} := \left\{ (p,q) \ \mid p \in L^1(X,\mu),\ q \in L^1(Y,\nu),\ p(x) - q(y) \le c(x,y) \quad (\mu,\nu)\text{-a.s.} \right\} \tag{1}$$

$$J(p,q) := \int_X p d\mu - \int_Y q d\nu = \int_{E^+} (p(z) - q(\omega))\, d\mu - \int_{E^-} (q(z) - p(\alpha))\, d\nu \tag{2}$$

In particular, with any pit F we can associate an admissible pair (p_F, q_F):

Lemma 5 *Let $F \in \mathcal{S}(E)$ be a pit. Set $F^{+} := F \cap E^{+}$ and $F^{-} := F \cap E^{-}$. Define $p_F : X \to R$ and $q_F : Y \to R$ by:*

$$p_F(\alpha) = 0,\ p_F(x) = \begin{cases} 1 \text{ if } x \in F^{+} \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

$$q_F(\omega) = 0,\ q_F(y) = \begin{cases} 1 \text{ if } y \in F^{-} \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

Then $(p_F, q_F) \in \mathcal{A}$ and:

$$J(p_F, q_F) = \int_F g(z)\, dz \tag{5}$$

which is just the profit associated with the pit F.

Proof. Since $\mu\left(E^{+}\cap E^{-}\right)=\nu\left(E^{+}\cap E^{-}\right)=0$, the definition of p_F and q_F makes sense. Since F is Lebesgue measurable and X and Y are compact we have $p_F\in L^1(X,\mu)$ and $q_F\in L^1(Y,\nu)$, and we only need to check that $p_F\left(x\right)-q_F\left(y\right)\leq c\left(x,y\right)$ for all $x\in X$ and $y\in Y$. If $x=\alpha$, this becomes $q_F\geq 0$, which is true. Similarly, if $y=\omega$, we get $p_F\leq 1$, which is true as well.

Suppose first $x\in F^{+}\subset E^{+}$, so that $p_F\left(x\right)=1$. If $y\notin\{\omega\}\cup\Gamma\left(x\right)$, we have $c\left(x,y\right)=+\infty$, so the relation holds. If $y\in\Gamma\left(x\right)$, we must have $y\in F$ because F is a pit, hence stable, so that $y\in F\cap E^{-}=F^{-}$ and $q_F\left(y\right)=1$. On the other hand, we have $c\left(x,y\right)=0$, so the relation becomes $q_F\left(y\right)\geq 1$, which is satisfied.

Suppose then $x\notin F^{+}$, so that $p_F\left(x\right)=0$. If $y\notin\Gamma\left(x\right)$ then $c\left(x,y\right)=+\infty$, and the relation holds. If $y\in\Gamma\left(x\right)$, then $c\left(x,y\right)=0$, and the relation becomes $q_F\left(y\right)\geq 0$, which is always true.

As for the last equality, we simply substitute into (2), getting:

$$
\begin{aligned}
\int_X p_F d\mu-\int_Y q_F d\nu &= \int_X\left(p\left(z\right)-q\left(\omega\right)\right)d\mu-\int_Y\left(q\left(z\right)-p\left(\alpha\right)\right)d\nu \\
&= \int_{F^{+}} g\left(z\right)dz-\int_{F^{-}}\left|g\left(z\right)\right|dz=\int_F g dz
\end{aligned}
$$

■

Consider the optimisation problem:

$$
\begin{array}{c}
\sup J\left(p,q\right) \\
\left(p,q\right)\in\mathcal{A}
\end{array}
\tag{D}
$$

Problem (D) is a dual of problem (K), with the weak duality property:

$$
\int_{X\times Y} c\left(x,y\right)d\pi\geq J\left(p,q\right)\quad\forall\pi\in\Pi\left(\mu,\nu\right),\ \forall\left(p,q\right)\in\mathcal{A}
\tag{6}
$$

This implies that problem (K) is a also a dual to our optimum pit problem:

Proposition 6 $\sup\left(P\right)\leq\inf\left(K\right)$

Proof. Combining inequality (6) with Lemma 5, we get:

$$
\forall F\in\mathcal{S}\left(E\right),\quad\int_F g dz\leq\inf\left(\mathrm{K}\right)
$$

■

In fact, by a fundamental result of Kantorovich (see [9], Theorem 1.3), there is no duality gap between (K) and (D):

$$
\inf\left(\mathrm{K}\right)=\sup\left(\mathrm{D}\right)
\tag{7}
$$

We will show that there is also no duality gap between (K) and (P), i.e., that problem (K) is a strong dual to our optimum pit problem (P).

Before we proceed, let us recall some facts from c-convex analysis. For proofs, we refer to [3], or [4]. Given $p : X \to R$ and $q : Y \to R$, we define $q^{\flat} : X \to R$ and $p^{\sharp} : Y \to R$ by:

$$p^{\sharp}(y) := \sup_{x \in X} \{p(x) - c(x,y)\}$$
$$q^{\flat}(x) := \inf_{y \in Y} \{q(y) + c(x,y)\}$$

Computing the right-hand side gives:

$$p^{\sharp}(y) := \max \left\{ p(\alpha), \sup_{y \in \Gamma(x)} p(x) \right\} \text{ for } y \in E^{-} \tag{8}$$
$$p^{\sharp}(\omega) := \max \left\{ p(\alpha), \sup_{x \in E^{+}} \{p(x)\} - 1 \right\}$$
$$q^{\flat}(x) := \min \left\{ 1 + q(\omega), \inf_{y \in \Gamma(x)} q(y) \right\} \text{ for } x \in E^{-} \tag{9}$$
$$q^{\flat}(\alpha) := \min \left\{ q(\omega), \inf \left\{ q(y) \mid y \in E^{-} \right\} \right\}$$

with the understanding that:

$$\sup_{x \in \varnothing} p(x) = -\infty, \quad \inf_{y \in \varnothing} = +\infty$$

It follows from the definition that:

$$p(x) - p^{\sharp}(y) \leq c(x,y)$$
$$q^{\flat}(x) - q(y) \leq c(x,y)$$
$$p(x) \leq c(x,y) + p^{\sharp}(y), \text{ hence } p(x) \leq p^{\sharp\flat}(x)$$
$$q(y) \geq q^{\flat}(x) - c(x,y), \text{ hence } q(y) \geq q^{\flat\sharp}(y)$$

We have the fundamental duality result:

$$p^{\sharp\flat\sharp} = p^{\sharp} \text{ and } q^{\flat\sharp\flat} = q^{\flat}$$

and the monotonicity properties:

$$p_1 \leq p_2 \Longrightarrow p_1^{\sharp} \leq p_2^{\sharp}$$
$$q_1 \leq q_2 \Longrightarrow q_1^{\flat} \leq q_2^{\flat}$$

It also follows from (8), (9) and the transitivity of Γ that

Lemma 7 *$p^{\sharp}$ and $q^{\flat}$ are increasing with respect to $\succsim_{\Gamma}$:*

$$x' \succsim_{\Gamma} x \Longrightarrow q^{\flat}(x') \geq q^{\flat}(x)$$
$$y' \succsim_{\Gamma} y \Longrightarrow p^{\sharp}(y') \geq p^{\sharp}(y)$$

Note that, given a pit F the associated pair (p_F, q_F) defined by (3) and (4) satisfies:

$$p_F = q_F^{\flat} \text{ and } q_F = p_F^{\sharp}$$

5 Solving the dual problem.

Back to problem (D). Note that there is a built-in translation-invariance:

Lemma 8 *Take any pair $(p,q) \in \mathcal{A}$ and any constants p_0, p_1, q_0, q_1 satisfying:*

$$\mu\left(E^{+}\right)\left(q_0 - p_1\right) - \nu\left(E^{-}\right)\left(p_0 - q_1\right) = 0$$

Define $(\bar{p}, \bar{q})$ by:

$$\begin{aligned}
\bar{p}(\alpha) &= p(\alpha) - p_0 \\
\bar{p}(x) &= p(x) - p_1 \quad for \quad x \in E^{+} \\
\bar{q}(\omega) &= q(\omega) - q_0 \\
\bar{q}(y) &= q(y) - q_1 text\ for\ y \in E^{-}
\end{aligned}$$

Then:

$$J(\bar{p}, \bar{q}) = J(p,q)$$

Proof. Substituting, we get:

$$\begin{aligned}
J(\bar{p}, \bar{q}) &= \int_{E^{+}} (\bar{p}(x) - \bar{q}(\omega))\, d\mu - \int_{E^{-}} (\bar{q}(y) - \bar{p}(\alpha))\, d\nu \\
&= J(p,q) + \mu\left(E^{+}\right)\left(q_0 - p_1\right) - \nu\left(E^{-}\right)\left(p_0 - q_1\right)
\end{aligned}$$

■

Lemma 9 *If $(p,q) \in \mathcal{A}$ then $\left(p, p^{\sharp}\right) \in \mathcal{A}$, $\left(q^{\flat}, q\right) \in \mathcal{A}$ and:*

$$\begin{aligned}
J\left(p, p^{\sharp}\right) &\geq J(p,q) \\
J\left(q^{\flat}, q\right) &\geq J(p,q)
\end{aligned}$$

Proof. Since $(p,q) \in \mathcal{A}$, we have $p(x) - q(y) \leq c(x,y)$ for all (x,y) so that:

$$\begin{aligned}
p(x) &\leq \inf_y \{c(x,y) + q(y)\} = q^{\flat}(x) \\
q(y) &\geq \sup_x \{p(x) - c(x,y)\} = p^{\sharp}(y)
\end{aligned}$$

Substituting in J, we get the result. ■

It follows from the Lemma that:

$$J(p,q) \leq J\left(p, p^{\sharp}\right) \leq J\left(p^{\sharp\flat}, p^{\sharp}\right)$$

Setting $\bar{p} := p^{\sharp\flat}$ and $\bar{q} := p^{\sharp}$, we find that:

$$\begin{aligned}
J(p,q) &\leq J(\bar{p}, \bar{q}) \\
\bar{p} = \bar{q}^{\flat} &\text{ and } \bar{q} = \bar{p}^{\sharp}
\end{aligned}$$

Proposition 10 *Problem (D) has a solution* $(\bar{p}, \bar{q})$ *with*

$$\begin{array}{rcl} \bar{p} = \bar{q}^{\flat} & \text{and} & \bar{q} = \bar{p}^{\sharp} \\ 0 \leq \bar{p} \leq 1 & \text{and} & 0 \leq \bar{q} \leq 1 \\ \bar{p}(\alpha) = 0 & \text{and} & \bar{q}(\omega) = 0 \end{array}$$

Proof. Take a maximizing sequence $(p_n, q_n) \in \mathcal{A}$:

$$J(p_n, q_n) \to \sup \{ J(p, q) \mid (p, q) \in \mathcal{A} \}$$

By Lemma 9 and the following observations, we may assume that:

$$p_n = q_n^{\flat}, \quad q_n = p_n^{\sharp} \tag{10}$$

By Lemma 8, we may assume in addition that:

$$p_n(\alpha) = 0, \quad q_n(\omega) = 0, \quad \inf_{y \in E^-} q(y) = 0 \tag{11}$$

If follows from (10) and (9) that:

$$\forall x \in E^+, \quad p_n(x) = \min \left\{ 1, \inf_{y \in \Gamma(x)} q_n(y) \right\}$$

Taking (11) into account, we find that $0 \leq p_n(x) \leq 1$. Similarly, it follows from (10) and (8) that:

$$\forall y \in E^-, \quad q_n(y) = \max \left\{ 0, \sup_{y \in \Gamma(x)} p_n(x) \right\}$$

and, since all $p_n(x) \leq 1$, we find that $0 \leq q_n(x) \leq 1$ as well.

So the family (p_n, q_n) is equi-integrable in $L^1(\mu) \times L^1(\nu)$. By the Dunford-Pettis theorem, we can extract a subsequence which converges weakly to some (p, q). Since the admissible set $\mathcal{A}$ is convex and closed in $L^1(\mu) \times L^1(\nu)$, it is weakly closed, and $(p, q) \in \mathcal{A}$. Since J is linear and continuous on $L^1(\mu) \times L^1(\nu)$, we get:

$$J(\bar{p}, \bar{q}) = \lim_n J(p_n, q_n) = \sup_{\mathcal{A}} J$$

so that $(p, q) \in \mathcal{A}$ is an optimal solution. ■

6 Solving the original problem

We now derive the complementarity conditions arising from the strong duality equation (7). If π is optimal in problem (K) and (p, q) is optimal in problem

(D), we have:

$$\begin{aligned}0 &= J(p,q) - \int_{X\times Y} c(x,y)\,d\pi \\ &= \int_X p d\mu - \int_Y q d\nu - \int_{X\times Y} c(x,y)\,d\pi \\ &= \int_{X\times Y} (p(x) - q(y) - c(x,y))\,d\pi\end{aligned}$$

Since the integrand is non-positive and the integral is zero, the integrand must vanish almost everywhere and we obtain the *complementary slackness* conditions:

$$p(x) - q(y) - c(x,y) = 0 \quad \pi\text{-a.e.} \tag{12}$$

Lemma 11 *If (p,q) is an optimal solution to problem (D) satisfying the properties in Proposition 10, then we have:*

$$y'' \succsim_\Gamma y' \succsim_\Gamma x'' \succsim_\Gamma x' \Longrightarrow q(y'') \geq q(y') \geq p(x'') \geq p(x')$$

Proof. The first and the last inequality come from Lemma 7, and the middle one from (8):

$$q(y') = p^\sharp(y') = \max\left\{0, \max_{x: y' \in \Gamma(x)} p(x)\right\} \geq p(x'')$$

■

Proposition 12 *Let (p,q) be an optimal solution to problem (D) satisfying the properties in Proposition 10. Define the set $F \subset E$ by:*

$$F = \{x \mid p(x) = 1\} \cup \{y \mid q(y) = 1\} \tag{13}$$

Then F is stable, and it is an optimal pit, that is, an optimal solution to problem (P)

Proof. By Lemma 11, F is stable, so it is a pit. Define F^+ and F^- as in Lemma 5. The profit from pit F satisfies:

$$\int_F g(z)\,dz = \int_{F^+} d\mu - \int_{F^-} d\nu \leq \sup(\mathrm{P}) \tag{14}$$

Set $G^+ := E^+ \backslash F^+$ and $G^- := E^- \backslash F^-$. We have, taking into account the fact that $p = 1$ on F^+ and $q = 1$ on F^-, together with $p(\alpha) = q(\omega) = 0$:

$$J(p,q) = \int_{F^+} d\mu - \int_{F^-} d\nu + \int_{G^+} p\,d\mu - \int_{G^-} q\,d\nu \tag{15}$$

Since ν is the marginal of π:

$$\int_{G^-} q(y)\,d\nu(y) = \int_{E^+\times G^-} q(y)\ \,d\pi(x,y)$$

Now observe that, since $c(x,y)=0$ or $+\infty$ for $(x,y)\in E^{+}\times E^{-}$, property (12) and the fact that p and q are bounded (viz., $0\leq p\leq 1$ and $0\leq q\leq 1$) imply that $p(x)=q(y)$ π-a.e. on $E^{+}\times E^{-}$. Therefore

$$\pi(F^{+}\times G^{-})=0=\pi(G^{+}\times F^{-})$$

and thus

$$\begin{aligned}\int_{E^{+}\times G^{-}}q\left(y\right)d\pi(x,y)&=\int_{G^{+}\times G^{-}}q\left(y\right)d\pi(x,y)=\int_{G^{+}\times G^{-}}p\left(y\right)d\pi(x,y)\\&=\int_{G^{+}\times E^{-}}p\left(y\right)d\pi(x,y)=\int_{G^{+}}p\left(x\right)d\mu\left(x\right)\end{aligned}$$

This implies:

$$J\left(p,q\right)=\int_{F^{+}}d\mu-\int_{F^{-}}d\nu=\int_{F}g\left(z\right)dz$$

Since (p,q) is optimal, $J\left(p,q\right)=\sup\left(\mathrm{D}\right)=\inf\left(\mathrm{K}\right)$. By Proposition 6 $\sup\left(\mathrm{P}\right)\leq\inf\left(\mathrm{K}\right)$. So:

$$\int_{F}g\left(z\right)dz=\inf\left(\mathrm{K}\right)\geq\sup\left(\mathrm{P}\right)$$

Comparing with (14) we see that F is an optimal pit for problem (P), as claimed. ■

The pit F consists of two regions, $A:=\{p=1\}$ and $B:=\{q=1\}$. We have $g\geq 0$ on A, so A is the profitable part of the pit, while $g\leq 0$ on B, so B is the costly part, which must be excavated in order to reach A. Note that A need not be equal to the whole E^{+}: there are regions underground which are potentially profitable, but which are too costly to reach.

Summarizing our results:

Theorem 13 *If E is compact, $\succsim_{\Gamma}$ is an order relation with closed graph, and $g\left(x\right)$ is continuous with $\int_{E}\max\{0,g(x)\}\,dx>0$ then:*

1. *Problem (P) has an optimum solution, i.e., there exists an optimal pit F.*

2. *The corresponding pair $(p,q):=(p_F,q_F)$ defined by (3)–(4) is an optimum solution to problem (D).*

3. *Problem (K) has an optimum solution and is a strong dual to problem (P), i.e, $\min(K)=\max(P)$.*

4. *A pit F is optimal if and only if there exists a feasible solution π to problem (K) such that the pair $(p,q):=(p_F,q_F)$ satisfies the complementary slackness conditions (12).*

Proof. By Proposition 10, there is an optimal solution $(\bar{p}, \bar{q})$ to problem (D), and by Proposition 12 we have $J(\bar{p}, \bar{q}) = \int_F g(z)\, dz = \sup(\mathrm{D})$, so that the pit F defined by (13) is optimal. On the other hand, by Lemma 5, we have $J(p_F, q_F) = \int_F g(z)\, dz$, so the pair $(p_F, q_F) \in \mathcal{A}$ is optimal as well. The other statements follow from preceding observations. ■

The optimum pit need not be unique. In fact it is known (Topkis [8]; see also Matheron [5] Th. 2) that the set of optimal pits is closed under (arbitrary) intersections and unions. Therefore, taking the intersection and the union, respectively, of all optimum pits, we have:

Corollary 14 *There exist a unique smallest optimum pit and a unique largest optimum pit.*

The smallest optimum pit may be of particular interest when seeking to minimize the environmental impact of the pit without sacrificing its total profit.

Of course, our solution of the problem is purely static: all the excavation and extraction is done at once. In practice, these processes take time, and it it is of interest to plan the whole mining process so as to optimize discounted revenue over time. This leads to a variant of the optimal transportation problem, where transportation is costly, not only in money, but in time. We hope to investigate it in the not-too-distant future

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