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# Periodic Solutions of Hamiltonian Equations and a Theorem of P. Rabinowitz

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The existence of nontrivial orbits with prescribed period is proved by a direct variational method.

## I. STATEMENT

Let $H$ be a $C^2$ function on $\mathbb{R}^n \times \mathbb{R}^n$, the Hamiltonian, and consider Hamilton's equations on the time interval $[0, T]$

$$\dot{x} = \frac{\partial H}{\partial p}(x, p), \qquad \dot{p} = -\frac{\partial H}{\partial x}(x, p).$$

It is well known (Maupertuis principle of least action) that the solutions $(x, p)$ of these equations are the extremals of the functional

$$\int [p\dot{x} - H(x, p)]\, dt.$$

However, from a calculus of variations viewpoint, this functional is quite untractable: it is unbounded, and is linear with respect to derivatives. It is of very little use in existence theory (finding solutions satisfying given boundary conditions, $x(0) = x_0$ and $x(T) = x_1$ for instance, or $x(0) = x(T)$ and $p(0) = p(T)$).

The main point of this paper is that the solutions of Hamilton's equations can be related to a much more tractable functional, involving the Legendre transform $G$ of $H$. From now on, we will assume $H$ to be convex, which will ensure that $G$ is defined globally by Fenchel's formula

$$G(y, q) = \operatorname*{Sup}_{(x,p)} \{xy + pq - H(x, p)\}$$



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and is a lower semicontinuous convex function, which may assume the value $+\infty$. If $H$ is minimum at (0, 0) with value zero, we also have

$$G(y, q) \geqslant G(0, 0) = 0 \qquad \text{for all } (y, q).$$

These functions need not be differentiable. However, at every continuity point they have a well-defined subgradient (see any textbook on convex analysis, for instance, [7] or [4]). The following three statements are known to be equivalent:

(i) $(y, q) \in \partial H(x, p)$,

(ii) $(x, p) \in \partial G(y, q)$,

(iii) $xy + pq = H(x, p) + G(y, q)$.

Consider Hamilton's equations

$$(-\dot{p}(t), \dot{x}(t)) \in \partial H(x(t), p(t)) \qquad \text{a.e.} \tag{$\mathscr{E}$}$$

If $H$ is assumed to be differentiable, this relation becomes $\dot{x} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial x$, which may be more readily recognized. We are interested in finding periodic solutions to $(\mathscr{E})$—apart from the obvious one, $x(t) = 0$ and $p(t) = 0$ for all $t$. Specifically, we want to prove that for any $T > 0$, Eq. $(\mathscr{E})$ has a nontrivial solution of period $T$.

This was done first by Rabinowitz [6], under assumptions on $H$ which imply quick growth at infinity and slow growth near the origin; the orbits of small periods then are to be found near infinity. Recently [3] adapting an idea of Clarke [2], we have been able to prove the same results under different assumptions, that $H$ be convex, grow slowly at infinity and quickly near the origin; in that case, the orbits of smaller periods are to be found near the origin, and it can actually be proved that $T$ is the minimal period.

This paper borrows something from both approaches. The growth assumptions are Rabinowitz'. We have to assume that $H$ is convex, but we prove an a priori estimate on the solution. The method is a critical point argument a la Rabinowitz, but applied to the same variational problem as in [3], the result being considerably simpler than [6].

THEOREM 1. *Assume that $H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is convex, minimum at $(0, 0)$, with minimum value zero, and that there is a constant $\theta \in [0, \frac{1}{2})$ such that, for all $\lambda > 1$ and all $(x, p) \neq (0, 0)$,*

$$H(\lambda x, \lambda p) \geqslant \lambda^{1/\theta} H(x, p) > 0. \tag{0}$$

*Then, for every $T > 0$, there is a nonconstant $T$-periodic solution of $(\mathscr{E})$ such that*

$$\begin{aligned} 0 < H(x(t), p(t)) = h \qquad & a.e., \\ h \leqslant C/T^{1/(1-2\theta)}, \end{aligned} \tag{2}$$

*where $C$ is a constant depending only on $\theta$ and the minimum value of $H$ on the unit sphere.*

Inequality (2) implies that the energy level $h$ goes to zero as $T$ goes to infinity: The orbits of large period are to be found near the origin. Note that $T$ may not actually be the minimal period of such orbits. This is in contrast with the Clarke–Ekeland setting [3], where the orbits of large period are to be found at infinity, and where the period $T$ can be proved to be minimal.

Inequality (0) implies that the Hamiltonian $H$ has more than quadratic growth at infinity, and that its Hessian at the origin $U''(0, 0)$ is zero.

Indeed, calling $M$ the maximum value of $H$ on the sphere $x^2 + p^2 = 1$, and $m$ its minimum value, we get

$$\begin{aligned} H(x, p) &\geqslant (x^2 + p^2)^{1/2\theta} m \qquad \text{when } x^2 + p^2 \to \infty, \\ H(x, p) &\leqslant (x^2 + p^2)^{1/2\theta} M \qquad \text{when } x^2 + p^2 \to 0 \end{aligned}$$

(please note that the condition is $\lambda \geqslant 1$, and not $\lambda > 0$). It will now be put in another form, which is the one Rabinowitz stated and which is equivalent in the convex case. The following lemma was pointed out to me by G. Haddad:

LEMMA. *Assume $H$ is continuously differentiable. Then inequality* (0) *is equivalent to the following*:

$$H(x, p) \leqslant \theta(xH'_x(x, p) + pH'_p(x, p)), \qquad \textit{all } (x, p). \tag{1}$$

*Proof.* First, we prove that (0) implies (1). Consider the two functions $\Phi$ and $\Psi$ of the real variable $t$

$$\begin{aligned} \Phi(t) &= H(tx, tp), \\ \Psi(t) &= t^{1/\theta} H(x, p). \end{aligned}$$

They are both defined for $t \geqslant 0$, and satisfy

$$\begin{aligned} \Phi(1) &= \Psi(1), \\ \Phi(t) &\geqslant \Psi(t) \qquad \text{for} \quad t \geqslant 1. \end{aligned}$$

It follows that $\Phi'(1) \geqslant \Psi'(1)$, which yields

$$xH'_x(x, p) + pH'_p(x, p) \geqslant \frac{1}{\theta} H(x, p).$$

We now prove that (1) implies (0). Define as above the function $\Phi$ by $\Phi(t) = H(tx, (p))$. It is assumed that

$$\begin{aligned}\Phi'(t) &= xH_x(tx, tp) + pH'_p(tx, tp)\\ &= \frac{1}{t}[txH_x(tx, tp) + tpH'_p(tx, tp)]\\ &\geqslant \frac{1}{t\theta} H(tx, tp) = \frac{1}{t\theta}\Phi(t).\end{aligned}$$

Using Gronwall's inequality for $\Phi(t)$, we get

$$\begin{aligned}\Phi(t) &\geqslant t^{1/\theta}\Phi(1),\\ H(tx, tp) &\geqslant t^{1/\theta} H(x, p). \quad \blacksquare\end{aligned}$$

## II. Proof

We shall prove Theorem 1 under the added assumption that $H$ is differentiable and strictly convex and that there is some constant $a > 0$ such that

$$H(x, p) \leqslant \frac{a}{2}(x^2 + p^2)^{1/2\theta} \qquad \text{for all } (x, p). \tag{3}$$

The general case will be derived later.

We first spell out some consequences for $H$ and $G$ of assumptions (1) and (2).

Lemma 1. *Set $b = 2\,\mathrm{Min}\{H(x, p) \mid x^2 + p^2 = 1\}$. It is strictly positive, and we have for all $(x, p)$ in $\mathbb{R}^n \times \mathbb{R}^n$*

$$H(x, p) \geqslant \frac{b}{2}(x^2 + p^2)^{1/2\theta}, \tag{4}$$

$$H'(x, p) \leqslant \tfrac{1}{2}(a2^{1/\theta} - b)(x^2 + p^2)^{(1-\theta)/2\theta}. \tag{5}$$

*Proof.* Since $H$ is convex, we have the inequality

$$H(u) + H'(u)(v - u) \leqslant H(v) \qquad \text{for all} \quad v \in \mathbb{R}^n \times \mathbb{R}^n.$$

We take the supremum of both sides over all $v$ such that $|v - u| = |u|$. Using assumption (3), we get

$$|H'(u)| \leqslant \frac{1}{|u|}\left(\frac{a}{2}|2u|^{1/\theta} - \frac{b}{2}|u|^{1/\theta}\right).$$

Setting $u = (x, p)$ and $|u| = (x^2 + p^2)^{1/2}$, we get the desired result. $\blacksquare$

Lemma 2. *$G$ is everywhere finite and $C^1$.*

*Proof.* By Lemma 1, the supremum is always achieved in the right-hand side of Fenchel's formula, and $G(y, q)$ has to be finite. Moreover, since $H$ is strictly convex, this supremum is achieved at a single point $(x, p)$, which is the one element in $\partial G(y, q)$, by the equivalence (ii) $\Leftrightarrow$ (iii). It is a standard fact that if $\partial G(y, q)$ is a singleton, over an open set, $G$ is continuously differentiable on that set, with $\partial G(y, q) = \{G'(y, q)\}$.

LEMMA 3. *We have, for all* $(y, q) \in \mathbb{R}^n \times \mathbb{R}^n$

$$G(y, q) \geqslant \frac{1}{2a}(y^2 + q^2)^{1/2(1-\theta)}, \tag{6}$$

$$G(y, q) \geqslant (1-\theta)(yG_1'(y, q) + qG_2'(y, q)), \tag{7}$$

$$G(y, q) \leqslant \frac{1}{2b}(y^2 + q^2)^{1/2(1-\theta)}, \tag{8}$$

$$|G'(y, q)| \leqslant \left[\frac{2\theta}{b}\right]^{\theta/(1-\theta)} (y^2 + q^2)^{\theta/2(1-\theta)}. \tag{9}$$

*Proof.* Condition (6)–(9) follow from the corresponding conditions on $H$, by using Fenchel's formula and the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii); we derive (6) from (3), (8) from (4), and (9) from (5). As for (7), simply write condition (4) as follows:

$$H(x, p) \leqslant \theta(xy + pq),$$
$$H(x, p) = xy + pq - G(y, q).$$

Subtracting, we get the desired result. ∎

We now proceed to the proof of the theorem, which is another extension of the direct method introduced by Clarke in [2]. Let $T > 0$ be given. Setting $\alpha = 1/(1-\theta)$, with $1 < \alpha < 2$, we consider the following classical problem in the calculus of variations:

$$\text{extremize} \int_0^1 \{G(-\dot{q}, \dot{y}) - T\dot{y}q\}\, dt;$$
$$\dot{y} \in L^\alpha(0, 1; \mathbb{R}^n), \qquad y(0) = 0 = y(1), \tag{$\mathscr{P}$}$$
$$\dot{q} \in L^\alpha(0, 1; \mathbb{R}^n), \qquad q(0) = 0 = q(1).$$

In other words, consider in $L^\alpha(0, 1; \mathbb{R}^n)$ the closed linear subspace $E$ of all functions whose integral vanishes. Set $q(t) = \int_0^t \dot{q}(s)\, ds$, and consider the functional $J$ on $E \times E$ defined by

$$J(\dot{y}, \dot{q}) = \int_0^1 \{G(-\dot{q}, \dot{y}) - T\dot{y}q\}\, dt.$$

LEMMA 4. *Suppose $(y, q)$ is an extremal of problem $(\mathscr{P})$, i.e., $(\dot{y}, \dot{q})$ is a critical point of $J$ on $E \times E$. Then, $(x, p)$ defined by*

$$x(t) = G_1'\left(-\dot{q}\left(\frac{t}{T}\right), \dot{y}\left(\frac{t}{T}\right)\right),$$
$$p(t) = G_2'\left(-\dot{q}\left(\frac{t}{T}\right), \dot{y}\left(\frac{t}{T}\right)\right) \tag{10}$$

*is a nonconstant $T$-periodic solution of Eq. $(\mathscr{E})$.*

*We also have constants $\mu$ and $\mu' \in \mathbb{R}^n$ such that*

$$x(T) = Ty\left(\frac{t}{T}\right) + \mu,$$
$$p(T) = Tq\left(\frac{t}{T}\right) - \mu'. \tag{11}$$

*Proof.* Inequality (8) implies that $J$ is finite everywhere; by Fatou's lemma, it is lower semicontinuous, and since it is convex, it is continuous as well (see [4, pp. 13, 239]). Inequality (9), with $\theta/(1 - \theta) = \alpha - 1$, implies that $J$ is Gateaux differentiable, and since it is convex, it is $C^1$ as well (see [4, p. 347]). The two components of its gradient in $E^* \times E^*$ are

$$G_2'(-\dot{q}, \dot{y}) - Tq + \mu = f \in L^{\alpha'},$$
$$-G_1'(-\dot{q}, \dot{y}) + Ty + \mu' = g \in L^{\alpha'},$$

where $\alpha' = \alpha/(\alpha - 1) = 1/\theta$ is the conjugate exponent of $\alpha$. Here $\mu$ and $\mu'$ are constants such that

$$\int_0^1 f(t)\, dt = \int_0^1 g(t)\, dt = 0.$$

If $(\dot{y}, \dot{q})$ is a critical point, then $f = g = 0$. Setting $x(t) = Ty(t/T) + \mu$ and $p(t) = Tq(t/T) - \mu'$, which are obviously $T$-periodic, we get

$$G_2'(-\dot{p}(tT), \dot{x}(tT)) - p(tT) = 0, \qquad \text{a.e.,}$$
$$-G_1'(-\dot{p}(tT), \dot{x}(tT)) + x(tT) = 0, \qquad \text{a.e.}$$

In other words, $(x, p) \in \partial G(-\dot{p}, \dot{x})$ for almost every $t$. Using the equivalence (i) $\Leftrightarrow$ (iii), we get $(-\dot{p}, \dot{x}) \in \partial H(x, p)$, the desired result. ∎

It is clear that the constant (0, 0) is a critical point of $J$ (we will see presently that it is a local minimum). The problem is reduced to showing that this functional has at least one more. This will be done by applying a theorem of Ambrosetti and Rabinowitz: if a functional $J$ on some Banach space satisfies condition (C) of Palais and Smale, is zero at the origin, is bounded away from zero on the boundary of some open ball $B$, and is zero again at some point outside $B$, then it has a critical point outside $B$ (see [1, 2]). We first set the stage.

LEMMA 5. *Let $B$ be the unit ball in $E \times E$. Constants $\gamma > 0$ and $\rho > 0$ can be found such that*

$$(0,0) \neq (\dot{y}, \dot{q}) \in \rho B \Rightarrow J(\dot{y}, \dot{q}) > J(0,0) = 0, \tag{12}$$

$$J(\dot{y}, \dot{q}) \geqslant \gamma \qquad \text{on the boundary of } \rho B. \tag{13}$$

*Proof.* Clearly, $J(0, 0) = 0$. By condition (6), we have

$$J(\dot{y}, \dot{q}) \geqslant \int_0^1 \left\{ \frac{1}{2a} (\dot{q}^2 + \dot{y}^2)^{\alpha/2} - T\dot{y}q \right\} dt.$$

Since $q(0) = 0$, and using the Cauchy–Schwarz inequality, we get

$$\int_0^1 \dot{y}q \, dt \leqslant \| \dot{y} \|_\alpha \| q \|_{\alpha'} \leqslant \| \dot{y} \|_\alpha \| \dot{q} \|_\alpha / 2^{1/\alpha'}$$

On the other hand, we have

$$(\dot{q}^2 + \dot{y}^2)^{\alpha/2} \geqslant | \dot{q} |^\alpha + | \dot{y} |^\alpha.$$

Combining these inequalities, we get

$$J(\dot{y}, \dot{q}) \geqslant \frac{1}{2a} (\| \dot{q} \|_\alpha^\alpha + \| \dot{y} \|_\alpha^\alpha) - T \| \dot{y} \|_\alpha \| \dot{q} \|_\alpha / 2^{1/\alpha'}.$$

Since $\alpha < 2$, the desired conclusions follow. ∎

LEMMA 6. *There is a nonzero $(e, f) \in E \times E$ where the functional is zero:*

$$(e, f) \neq (0, 0) \qquad \text{and} \qquad J(e, f) = 0.$$

*Proof.* Choose $y_0 \in \mathbb{R}^n$, and set $y(t) = y_0 \sin 2\pi t$, $q(t) = y_0 \cos 2\pi t$, so that

$$\int_0^1 \dot{y}(t) \, q(t) \, dt = \pi {y_0}^2.$$

On the other hand, by inequality (8), we have

$$\int_0^1 G(-\dot{q}(t), \dot{y}(t)) \, dt \leqslant \frac{1}{2b} | y_0 |^\alpha.$$

This gives us

$$J(\dot{y}, \dot{q}) \leqslant \frac{1}{2b} | y_0 |^\alpha - T\pi {y_0}^2.$$

Since $\alpha < 2$, we can choose $y_0$ so large that

$$J(\dot{y}, \dot{q}) < 0.$$

Now consider the path $s \to (s\dot{y}, s\dot{q})$ in $E \times E$, joining $(\dot{y}, \dot{q})$ (for $s = 1$) to the origin (for $s = 0$). The function $s \to J(s\dot{y}, s\dot{q})$ is strictly negative for $s = 1$, and because of Lemma 5 it is strictly positive for small value $s_0 \in (0, 1)$. By the intermediate value theorem, it has to vanish somewhere on $(s_0, 1)$. ∎

LEMMA 7. *The functional $J$ satisfies condition* (C) *of Palais and Smale: any sequence $(\dot{y}_n, \dot{q}_n)$ in $E \times E$, along which $J(\dot{y}_n, \dot{q}_n)$ is bounded and $J'(\dot{y}_n, \dot{q}_n)$ converges to zero, possesses a convergent subsequence.*

*Proof.* Let $(\dot{y}_n, \dot{q}_n)$ be such a sequence. In other words:

$$c_1 \leqslant \int_0^1 \{G(-\dot{q}_n, \dot{y}_n) - Tq_n\dot{y}_n\}\, dt \leqslant c_2, \tag{14}$$

$$G_2'(-\dot{q}_n, \dot{y}_n) - Tq_n + \mu_n = f_n \to 0 \quad \text{in } L^{\alpha'}, \tag{15}$$

$$-G_1'(-\dot{q}_n, \dot{y}_n) + Ty_n + \mu_n' = g_n \to 0 \quad \text{in } L^{\alpha'}. \tag{16}$$

It follows from (14) that

$$\int_0^1 Tq_n\dot{y}_n\, dt = -\int_0^1 Ty_n\dot{q}_n\, dt \geqslant -c_2 + \int_0^1 G(-\dot{q}_n, \dot{y}_n)\, dt,$$

$$T\int_0^1 \{q_n\dot{y}_n - y_n\dot{q}_n\}\, dt \geqslant -2c_2 + 2\int_0^1 G(-\dot{q}_n, \dot{y}_n)\, dt.$$

The left-hand side can be written differently, using (15) and (16):

$$T\int_0^1 \{q_n\dot{y}_n - y_n\dot{q}_n\}\, dt = \int_0^1 \{\dot{y}_n G_2'(-\dot{q}_n, \dot{y}_n) - \dot{q}_n G_1'(-\dot{q}_n, \dot{y}_n)\}\, dt$$
$$- \int_0^1 \{f_n\dot{y}_n + g_n\dot{q}_n\}\, dt.$$

By condition (7), the first term on the right-hand side can be estimated as follows:

$$\int_0^1 \{\dot{y}_n G_2'(-\dot{q}_n, \dot{y}_n) - \dot{q}_n G_1'(-\dot{q}_n, \dot{y}_n)\}\, dt \leqslant \alpha \int_0^1 G(-\dot{q}_n, \dot{y}_n)\, dt.$$

Adding all these inequalities, we get

$$-2c_2 + 2\int_0^1 G(-\dot{q}_n, \dot{y}_n)\, dt \leqslant \alpha \int_0^1 G(-\dot{q}_n, \dot{y}_n)\, dt - \int_0^1 \{f_n \dot{y}_n + g_n \dot{q}_n\}\, dt,$$

$$(2-\alpha)\int_0^1 G(-\dot{q}_n, \dot{y}_n)\, dt \leqslant 2c_2 - \int_0^1 \{f_n \dot{y}_n + g_n \dot{q}_n\}\, dt.$$

Using inequality (6), this yields

$$\frac{2-\alpha}{2a}(\| \dot{q}_n \|_\alpha^\alpha + \| \dot{y}_n \|_\alpha^\alpha) \leqslant 2c_2 + \| f_n \|_{\alpha'} \| \dot{y}_n \|_\alpha + \| g_n \|_{\alpha'} \| \dot{q}_n \|_\alpha .$$

Since $\alpha > 1$ and the $\| f_n \|_{\alpha'}$ are bounded, as well as the $\| g_n \|_{\alpha'}$, we find a constant $c_3$ such that

$$\| \dot{q}_n \|_\alpha \leqslant c_3 \qquad \text{and} \qquad \| \dot{y}_n \|_\alpha \leqslant c_3 . \tag{17}$$

It follows that there is a subsequence, still denoted by $(\dot{y}_n, \dot{q}_n)$, and some $(\dot{y}, \dot{q})$ in $E \times E$ such that

$$\begin{aligned} \dot{y}_n &\to \dot{y} && \text{weakly in } L^\alpha, \\ \dot{q}_n &\to \dot{q} && \text{weakly in } L^\alpha, \\ y_n &\to y && \text{uniformly on } [0, 1], \\ q_n &\to q && \text{uniformly on } [0, 1]. \end{aligned}$$

Because of inequalities (9) and (17), the functions $G_1'(-\dot{q}_n, \dot{y}_n)$ and $G_2'(-\dot{q}_n, \dot{y}_n)$ are bounded in $L^{\alpha'}$. It then follows from (15) and (16) that the sequences $\mu_n$ and $\mu_n'$ are bounded, so that, after extracting a subsequence:

$$\mu_n \to \mu \qquad \text{and} \qquad \mu_n' \to \mu'.$$

We now write (15) and (16) somewhat differently, using the equivalence (i) $\Leftrightarrow$ (ii); this yields

$$\begin{aligned} \dot{y}_n &= H_2'(Ty_n + \mu_n' - g_n, Tq_n - \mu_n + f_n) && \text{a.e.,} \\ \dot{q}_n &= -H_1'(Ty_n + \mu_n' - g_n, Tq_n - \mu_n + f_n) && \text{a.e.} \end{aligned} \tag{18}$$

with

$$\begin{aligned} Ty_n + \mu_n' - g_n &\to Ty + \mu' && \text{in } L^{\alpha'}, \\ Tq_n - \mu_n + f_n &\to Tq - \mu && \text{in } L^{\alpha'}. \end{aligned} \tag{19}$$

Since the derivative $H'$ is continuous and satisfies estimate (5), the mapping $(\phi, \psi) \to H'(\phi, \psi)$ is continuous from $L^{\alpha'} \times L^{\alpha'}$ into $L^\alpha \times L^\alpha$ (this is a theorem of Krasnoselskii; see [4, p. 77]). It follows that the right-hand sides of formulas

(18) converge in $L^\alpha$; then so do the left-hand sides: $\dot{y}_n \to \dot{y}$ and $\dot{q}_n \to \dot{q}$ strongly, as desired. ∎

All the assumptions of the Ambrosetti–Rabinowitz theorem have been checked; it follows that the functional $J$ does have a nontrivial critical point in $E \times E$. Hence Theorem 1.

To prove estimate (2), we first relate the critical value $c$ to the energy level $h$. Denoting by $(\dot{y}, \dot{q})$ the critical point we have just found, we have

$$c = \int_0^1 \{G(-\dot{q}, \dot{y}) - Tq\dot{y}\}\, dt.$$

Taking into account the relation $\int_0^1 (q\dot{y} + y\dot{q})\, dt = 0$, this becomes

$$\begin{aligned} c &= \int_0^1 \{G(-\dot{q}, \dot{y}) - \tfrac{1}{2}(Tq\dot{y} - Ty\dot{q})\}\, dt \\ &= \int_0^1 \{G(-\dot{q}, \dot{y}) - \tfrac{1}{2}(\dot{y}G_2'(-\dot{q}, \dot{y}) - \dot{q}G_1'(-\dot{q}, \dot{y}))\}\, dt. \end{aligned}$$

We now use the equivalence (ii) $\Leftrightarrow$ (iii), taking into account Lemma 4. We get

$$c = \frac{1}{T} \int_0^T \{\tfrac{1}{2}(xH_1'(x, p) + pH_2'(x, p)) - H(x, p)\}\, dt.$$

By assumption (1), this yields

$$c \geqslant \frac{1}{T} \int_0^T \left(\frac{1}{2\theta} - 1\right) H(x, p)\, dt.$$

By the constancy of $H$ along the trajectory, we get

$$c \geqslant h \left(\frac{1}{2\theta} - 1\right). \tag{20}$$

We now try to relate the critical value $c$ to the period $T$. This will require some more information about $c$. The Ambrosetti–Rabinowitz theorem defines it as follows:

$$c = \operatorname*{Inf}_{\gamma \in \Gamma} \operatorname*{Max}_{0 \leqslant t \leqslant 1} J(\gamma(t)),$$

where $\Gamma$ is the set of all continuous paths $\gamma$: $[0, 1] \to E \times E$, such that $\gamma(0) = (0, 0)$ and $\gamma(1) = (e, f)$. Taking the same path $\gamma$ as in Lemma 6,

$$\gamma(s) = (s\dot{y}, s\dot{q}),$$

with $y(t) = y_0 \sin 2\pi t$ and $q(t) = y_0 \cos 2\pi t$, we get

$$c \leqslant \operatorname*{Max}_{0 \leqslant t \leqslant 1} J(\bar{y}(t)).$$

The right-hand side is readily estimated; we have seen that

$$J(s\dot{y}, s\dot{q}) \leqslant \frac{1}{2b} s^\alpha \mid y_0 \mid^\alpha - T\pi s^2 y_0{}^2.$$

The right-hand side is maximum for

$$s = \frac{1}{\mid y_0 \mid} \left( \frac{\alpha}{4b\pi T} \right)^{1/(2-\alpha)}.$$

Hence the desired estimate:

$$c \leqslant \frac{1}{T^{\alpha/(2-\alpha)}} \left( \frac{\alpha}{4\pi b} \right)^{\alpha/(2-\alpha)} \frac{2-\alpha}{4b}. \tag{21}$$

The conclusion follows by putting together (20) and (21).

Now for the general case, where $H$ is no longer differentiable nor strictly convex, and where inequality (3) does not hold. We leave it to the reader to check that there is a sequence $H_n$ of convex functions, differentiable, strictly convex, satisfying

$$H_n(x, p) \leqslant \theta_n(xH_1'(x, p) + yH_2'(x, p)), \tag{22}$$

$$H_n(x, p) \leqslant \frac{a_n}{2} (x^2 + p^2)^{1/2\theta}, \tag{23}$$

for constants $\theta_n \to \theta$ and $a_n \to a$, and converging to $H$ uniformly on the ball of radius $\operatorname{Max}(1, C/T^{1/(1-2\theta)})$. Here $a$ is the supremum of $2H(x, p)/(x^2 + p^2)^{1/2\theta}$ on $B$. By the preceding result, there is for each $n$ a $T$-periodic function $(x_n, p_n)$ such that

$$(-\dot{p}_n(t), \dot{x}_n(t)) = H_n'(x_n(t), p_n(t)) \qquad \text{a.e.}, \tag{24}$$

$$H_n(x_n(t), p_n(t)) \leqslant C_n/T^{1/(1-2\theta)} \qquad \text{for all } t. \tag{25}$$

The constants $C_n$, given by formula (21), are uniformly bounded because $H_n$ converges to $H$ uniformly on the unit sphere. It then follows from inequality (25) that the $(x_n, p_n)$ are uniformly bounded in $L^\infty$, and by Eq. (24) so are the $(\dot{x}_n, \dot{p}_n)$. We can extract a subsequence $(\dot{x}_n, \dot{p}_n)$ which converges in $L^\infty$ weak-star, so that $(x_n, p_n)$ converges everywhere on $[0, T]$. Passing to the limit in (24) (using Mazur's lemma and the fact that $\partial H$ is convex compact-valued; see [4, p. 258] and (25), we get the desired result:

$$(-\dot{p}(t), \dot{x}(t)) \in \partial H(x(t), p(t)) \qquad \text{a.e.}, \tag{26}$$

$$H(x(t), p(t)) \leqslant C/T^{1/(1-2\theta)} \qquad \text{for all } t. \tag{27}$$

The solution $(x(t), p(t))$ is clearly $T$-periodic. To see that it is nontrivial, we note that Lemma 5 gives us for each $n$ an inequality $c_n \geqslant \gamma_n$, with $\gamma_n > 0$ depending only on $a_n$. Since $a_n \to a$, we have $\gamma_n \geqslant \gamma > 0$. Taking limits in the equation

$$c_n = \frac{1}{T} \int_0^T \left\{ \frac{1}{2} (x_n H'_{nx}(x_n, p_n) + p_n H'_{np}(x_n, p_n)) - H_n(x_n, p_n) \right\} dt$$

we see that the right-hand side cannot go to zero.

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## References

1. A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14** (1973), 349–381.
2. F. Clarke, Periodic solutions to Hamiltonian inclusions, to appear.
3. F. Clarke and I. Ekeland, Hamiltonian trajectories having prescribed minimal period, *Comm. Pure Appl. Math.*, in press.
4. I. Ekeland and R. Temam, "Convex Analysis and Variational Problems," North-Holland, Amsterdam, 1976.
5. P. Rabinowitz, "Variational Methods for Nonlinear Eigenvalue Problems," C.I.M.E., 1974.
6. P. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **31** (1978), 157–184.
7. R. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N. J., 1970.