Inverse function theorems: soft and hard

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THE INVERSE FUNCTION THEOREM
IN BANACH SPACES

A variational principle

Theorem (Ekeland, 1972)

Let (X, d) *be a complete metric space, and* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semi-continuous map, bounded from below:*

$$\left\{(x, a) \mid a \geq f(x)\right\} \text{ is closed in } X \times \mathbb{R}$$
$$f(x) \geq 0, \qquad \forall x$$

Suppose $f(0) < \infty$*. Then for every* $A > 0$*, there exists some* $\bar{x}$ *such that:*

$$f(\bar{x}) \leq f(0)$$
$$d(\bar{x}, 0) \leq A$$
$$f(x) \geq f(\bar{x}) - \frac{f(0)}{A} d(x, \bar{x}) \quad \forall x$$

This is a Baire-type result: relies on completeness, no compactnes needed

Gâteaux-differentiability

Definition

Let X and Y be Banach spaces. We shall say that $F : X \to Y$ is *Gâteaux-differentiable* at x if there exists a continous linear map $DF(x) : X \to Y$ such that

$$\forall \xi \in X, \quad \lim_{t} \frac{1}{t} \left[F(x + t\xi) - F(x) \right] = DF(x)\,\xi \ \text{ in } \ Y$$

Example

Let $\Omega \subset R^n$ be a bounded domain. Define $F : L^1(\Omega) \to L^1(\Omega)$ by $F(u) := \int_\Omega f(u(x))\,dx$, where f is C^1 and f' is bounded: $|f'(u)| \leq a$. Then F is G-differentiable, with:

$$[DF(u)\,v](x) = f'(u(x))\,v(x)$$

but it is NOT C^1 (unless $f(u) = au + b$). If $f'(u) \geq c > 0$, then $DF(u)$ is invertible, and the inverse $L(u)$ is uniformly bounded

First-order version

Suppose X is a Banach space, and $d(x_1, x_2) = \|x_1 - x_2\|$. Apply EVP to $x = \bar{x} + tu$ and let $u \to 0$. We get:

$$f(\bar{x} + tu) \geq f(\bar{x}) - \frac{f(0)}{A} t \|u\| \quad \forall (t, u)$$

$$\lim_{t \to +0} \frac{1}{t} \left(f(\bar{x} + tu) - f(\bar{x}) \right) \geq -\frac{f(0)}{A} \|u\| \quad \forall u$$

$$\langle Df(x), u \rangle \geq -\frac{f(0)}{A} \|u\| \quad \forall u, \text{ or } \|Df(x)\|^* \leq \frac{f(0)}{A}$$

Corollary

Suppose F is everywhere finite and Gâteaux-differentiable. Then there is a sequence x_n such that:

$$f(x_n) \to \inf f$$

$$\|Df(x_n)\|^* \to 0$$

A non-smooth inverse function theorem

Theorem

Let X and Y be Banach spaces. Let $F : X \to Y$ be continuous and Gâteaux-differentiable, with $F(0) = 0$. Assume that the derivative $DF(x)$ has a right-inverse $L(x)$, uniformly bounded in a neighbourhood of 0:

$$\forall v \in Y, \quad DF(x) L(x) v = v$$
$$\sup \{ \|L(x)\| \mid \|x\| \leq R \} < m$$

Then, for every $\bar{y}$ such that

$$\|\bar{y}\| \leq \frac{R}{m}$$

there is some $\bar{x}$ such that:

$$\|\bar{x}\| \leq m \|\bar{y}\|$$
$$F(\bar{x}) = \bar{y}$$

Proof

Consider the function $f : X \to R$ defined by:

$$f(x) = \|F(x) - \bar{y}\|$$

It is continuous and bounded from below, so that we can apply EVP with $A = m\|\bar{y}\|$. We can find $\bar{x}$ with:

$$f(\bar{x}) \leq f(0) = \|\bar{y}\|$$
$$\|\bar{x}\| \leq m\|\bar{y}\| \leq R$$
$$\forall x, \quad f(x) \geq f(\bar{x}) - \frac{f(0)}{m\|\bar{y}\|}\|x - \bar{x}\| = f(\bar{x}) - \frac{1}{m}\|x - \bar{x}\|$$

I claim $F(\bar{x}) = \bar{y}$.

Proof (ct'd)

Assume $F(\bar{x}) \neq \bar{y}$. The last equation can be rewritten:

$$\forall t \geq 0,\ \forall u \in X, \qquad \frac{f(\bar{x}+tu) - f(\bar{x})}{t} \geq -\frac{1}{m}\|u\|$$

Simplify matters by assuming X is Hilbert. Then:

$$\left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}\|}, DF(\bar{x})\, u\right) = \langle Df(\bar{x}), u\rangle \geq -\frac{1}{m}\|u\|$$

We now take $u = -L(\bar{x})(F(\bar{x}) - \bar{y})$, so that $DF(\bar{x})\, u = -(F(\bar{x}) - \bar{y})$. We get a contradiction:

$$\|F(\bar{x}) - \bar{y}\| \leq \frac{\|L(\bar{x})\|}{m}\|F(\bar{x}) - \bar{y}\| < \|F(\bar{x}) - \bar{y}\|$$

THE INVERSE FUNCTION THEOREM
IN FRÉCHET SPACES

Fréchet spaces.

A Fréchet space X is *graded* if its topology is defined by an increasing sequence of norms:

$$\forall x \in X, \quad \|x\|_k \leq \|x\|_{k+1}, \quad k \geq 0$$

A point $x \in X$ is *controlled* if there is a constant $c_0(x)$ such that:

$$\|x\|_k \leq c_0(x)^k$$

Definitions

A graded Fréchet space is *standard* if, for every $x \in X$, there is a constant $c_3(x)$ and a sequence x_n of controlled vectors such that:

$$\forall k \quad \lim_{n \to \infty} \|x_n - x\|_k = 0$$

$$\forall n, \quad \|x_n\|_k \leq c_3(x) \|x\|_k$$

The graded Fréchet spaces $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \cap C^k(\bar{\Omega}, \mathbb{R}^d)$ and $C^\infty(\bar{\Omega}, \mathbb{R}^d) = \cap H^k(\Omega, \mathbb{R}^d)$ are both standard.

Normal maps

We are given two Fréchet spaces X and Y, and a neighbourhood of zero $B = \{x \mid \|x\|_{k_0} \leq R\}$ in X

Definition

A map $F : X \to Y$ is *normal* over B if there are two integers $d_1, \ d_2$ and two non-decreasing sequences $m_k > 0, \ m'_k > 0$ such that:

1. $F(0) = 0$ and F is continuous on B
2. F is Gâteaux-differentiable on B and for all $x \in B$

$$\forall k \in \mathbb{N}, \ \|DF(x)\, u\|_k \leq m_k \|u\|_{k+d_1}$$

There exists a linear map $L(x) : Y \longrightarrow X$ such that:

$$\forall v \in Y, \ DF(x)\, L(x)\, v = v$$
$$\forall k \in \mathbb{N}, \quad \sup_{x \in B} \|L(x)\, v\|_k < m'_k \|v\|_{k+d_2}$$

An inverse function theorem

Theorem

Suppose Y is standard, and $F : X \to Y$ is normal over $B = \left\{x \mid \|x\|_{k_0} \leq R\right\}$. Then, for every y with

$$\|y\|_{k_0+d_2} \leq \frac{R}{m'_{k_0}}$$

there is some $x \in B$ such that:

$$\|x\|_{k_0} \leq m'_{k_0} \|y\|_{k_0+d_2} \text{ and } F(x) = y$$

Corollary (Lipschitz inverse)

For every y_1, y_2 *with* $\|y_i\|_{k_0+d_2} \leq m_{k_0}'^{-1} R$ *and every* $x_1 \in B$ *with* $F(x_1) = y_1$, *there is some* x_2 *with:*

$$\|x_2 - x_1\|_{k_0} \leq m'_{k_0} \|y_2 - y_1\|_{k_0+d_2} \text{ and } F(x_2) = y_2$$

Corollary (Finite regularity)

Suppose F *extends to a continuous map* $\bar{F} : X_{k_0} \to Y_{k_0-d_1}$. *Take some* $y \in Y_{k_0+d_2}$ *with* $\|y\|_{k_0+d_2} < R m_{k_0}'^{-1}$. *Then there is some* $x \in X_{k_0}$ *such that* $\|x\|_{k_0} < R$ *and* $\bar{F}(x) = y$.

Proof: step 1

Let $\bar{y}$ be given, with $\|\bar{y}\|_{k_0+d_2} \leq \frac{R}{m'_{k_0}}$. Let $\beta_k \geq 0$ be a sequence with unbounded support satisfying:

$$\sum_{k=0}^{\infty} \beta_k m_k m'_{k+d_1} n^k < \infty, \quad \forall n \in \mathbb{N},$$

$$\frac{1}{\beta_{k_0+d_2}} \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k \leq \frac{R}{m'_{k_0}}$$

Set $\alpha_k := m'^{-1}_{k_0} \beta_{k+d_2}$ and define:

$$\|x\|_\alpha := \sum_{k=0}^{\infty} \alpha_k \|x\|_k, \quad X_\alpha = \{x \in X \mid \|x\|_\alpha < \infty\}$$

Then $X_\alpha \subsetneq X$ is a linear subspace, X_α is a Banach space and the identity map $X_\alpha \to X$ is continuous: So the restriction $F : X_\alpha \to Y$ is continuous.

Step 1 (ct'd)

Now consider the function $f : X_\alpha \longrightarrow \mathbb{R} \cup \{+\infty\}$ (the value $+\infty$ is allowed) defined by:

$$f(x) = \sum_{k=0}^{\infty} \beta_k \|F(x) - \bar{y}\|_k$$

f is lower semi-continuous, and $0 \leq \inf f \leq f(0) = \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \infty$. By the EVP there is a point $\bar{x} \in X_\alpha$ such that:

$$f(\bar{x}) \leq f(0) = \sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k, \quad \|\bar{x}\|_\alpha \leq \alpha_{k_0} R$$

$$f(x) \geq f(\bar{x}) - \frac{f(0)}{\alpha_{k_0} R} \|x - \bar{x}\|_\alpha, \quad \forall x \in X_\alpha$$

It follows that:

$$\sum_{k=0}^{\infty} \alpha_k \|\bar{x}\|_k \leq \alpha_{k_0} R, \text{so } \|\bar{x}\|_{k_0} \leq R$$

$$\sum_{k=0}^{\infty} \beta_k \|\bar{y}\|_k < \infty, \ \sum_{k=0}^{\infty} \beta_k \|F(\bar{x}) - \bar{y}\|_k < \infty, \ \sum_{k=0}^{\infty} \beta_k \|F(\bar{x})\|_k < \infty$$

Proof: step 2

Assume then $F(\bar{x}) \neq \bar{y}$. If $u \in X_\alpha$, we can set $x = \bar{x} + tu$, replace f by its value and divide by t.

$$-\lim_{\substack{t \longrightarrow 0 \\ t>0}} \frac{1}{t} \left[\sum_{k=0}^{\infty} \beta_k \left\| \bar{y} - F(\bar{x} + tu) \right\|_k - \sum_{k=0}^{\infty} \beta_k \left\| \bar{y} - F(\bar{x}) \right\|_k \right] \leq A \sum_{k \geq 0} \alpha_k \left\| u \right\|$$

with $A = \sum \beta_k \left\| \bar{y} \right\|_k (\alpha_{k_0} R)^{-1} < 1$. We would like to go one step further:

$$-\sum_{k=0}^{\infty} \beta_k \left(\frac{F(\bar{x}) - \bar{y}}{\left\| F(\bar{x}) - \bar{y}_k \right\|_k}, DF(\bar{x})\, u \right)_k \leq A \sum_{k \geq 0} \alpha_k \left\| u \right\|_k$$

This program can be carried through (by repeated use of Lebesgue's dominated convergence theorem) if we take $u = u_n$, where $u_n = L(\bar{x})\, v_n$ and:

$$v_n \longrightarrow F(\bar{x}) - \bar{y}, \quad v_n \text{ controlled,}$$
$$\left\| v_n \right\|_k \leq c_3 \left(F(\bar{x}) - \bar{y} \right) \left\| F(\bar{x}) - \bar{y} \right\|_k$$

Proof: step 3

Plugging in $u_n = L(\bar{x})\, v_n$, we get:

$$-\sum_{k=0}^{\infty} \beta_k \left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}_k\|_k}, DF(\bar{x})\, L(\bar{x})\, v_n \right)_k \leq A \sum_{k \geq 0} \alpha_k \|L(\bar{x})\, v_n\|_k$$

$$-\sum_{k=0}^{\infty} \beta_k \left(\frac{F(\bar{x}) - \bar{y}}{\|F(\bar{x}) - \bar{y}_k\|_k}, v_n \right)_k \leq A \sum_{k \geq 0} \alpha_k \|L(\bar{x})\, v_n\|_k$$

Letting $n \to \infty$, so $v_n \to F(\bar{x}) - \bar{y}$, this becomes:

$$-\sum_{k=0}^{\infty} \beta_k \|F(\bar{x}) - \bar{y}_k\|_k \leq A \sum_{k \geq 0} \alpha_k m_k \|F(\bar{x}) - \bar{y}_k\|_{k+d_2}$$

$$= A \sum_{k \geq d_2} \beta_k \|F(\bar{x}) - \bar{y}_k\|_k$$

which is a contradiction since $A < 1$

THE HARD INVERSE FUNCTION THEOREM:
NASH-MOSER FOR NON-SMOOTH FUNCTION

Let $(X_s, \|\cdot\|_s)$, $0 \leq s \leq S$, be a scale of Banach spaces:

$$0 \leq s_1 \leq s_2 \leq S \Longrightarrow \left(X_{s_2} \subset X_{s_1} \quad \text{and} \quad \|\cdot\|_{s_1} \leq \|\cdot\|_{s_2} \right)$$

We shall assume that there exists a sequence of projectors $\Pi_N : X_0 \to E_N$ (smoothing operators) where $E_N \subset \bigcap_{s \geq 0} X_s$ is the range of Π_N, with $\Pi_0 = 0$, $E_N \subset E_{N+1}$ and $\bigcup_{N \geq 1} E_N$ is dense in each space X_s for the norm $\|\cdot\|_s$. We assume that:

$$\|\Pi_N u\|_{s+d} \leq C N^d \|u\|_s$$

$$\|(1 - \Pi_N) u\|_s \leq C N^{-d} \|u\|_{s+d}$$

Note that these properties imply some interpolation inequalities, for $0 \leq t \leq 1$ and $0 \leq s_1\,,\, s_2 \leq A$

$$\|x\|_{ts_1 + (1-t)s_2} \leq C_2^A \|x\|_{s_1}^t \|x\|_{s_2}^{1-t} \ .$$

Let $(Y_s, \|\cdot\|'_s)_{s \geq 0}$ be another regular scale of Banach spaces, with smoothing operators $\Pi'_N : Y_0 \to E'_N \subset \bigcap_{s \geq 0} Y_s$

In the following, $R > 0$ and $S > 0$ are prescribed, with possibly $S = \infty$

Definition

We shall say that $F : B_0(R) \to Y_0$ is *roughly tame* with loss of regularity μ if:

(a) F is continuous and Gâteaux-differentiable from $B_0(R) \cap X_s$ to Y_s for any $s \in [0, S)$.

(b) There is a constant K such that, for all $s \leq S$ and $x \in B_0(R)$:

$$\forall h \in X, \quad \|DF(x)h\|'_s \leq K(\|h\|_s + \|x\|_s \|h\|_0)$$

(c) For $x \in B_0(R) \cap E_N$, the linear maps $\Pi'_N DF(x)|_{E_N} : E_N \to E'_N$ have a right-inverse, denoted by $L_N(x)$. There are constants $\mu > 0$ and $\gamma > 0$, such that, for all $s \leq S$ and $x \in B_0(R)$ we have:

$$\forall k \in E'_N, \quad \|L_N(x)k\|_s \leq \frac{1}{\gamma} N^{\mu}(\|k\|'_s + \|x\|_s \|k\|'_0)$$

A hard inverse function theorem.

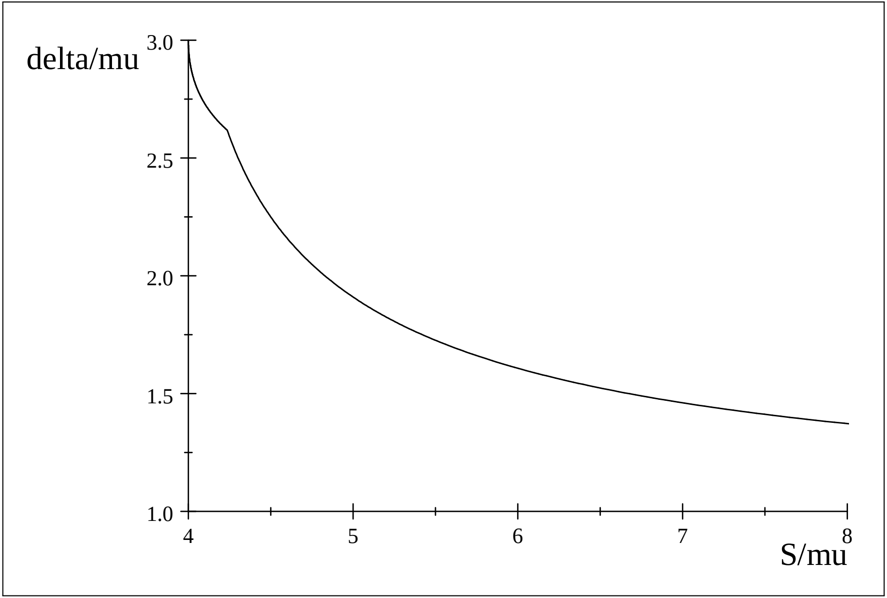
- Note the nonlinear estimates in H^s: if $\|u\|_\infty$ and $\|v\|_\infty$ are finite, then:

$$\|f(u,v)\|_{H^s} \leq C\left(\|u\|_{H^0}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{H^0}\right)$$

- The loss of derivatives for F and DF is normalized to 0.
- The loss of derivatives for DF^{-1} is μ, and we cannot expect any better for F^{-1}

Define a real function φ on $[4,\ \infty[$ by:

$$\varphi(x) = \begin{cases} \frac{1}{2}x\left(1-\sqrt{1-\frac{4}{x}}\right)+1 & \text{if} \quad 4 < x \leq \frac{3+\sqrt{5}}{\sqrt{5}-1} \\ \frac{x^2}{4}\left(1-\sqrt{1-\frac{4}{x}}\right)^2 & \text{if} \quad x \geq \frac{3+\sqrt{5}}{\sqrt{5}-1} \end{cases}$$



The authorized region is above the graph

Theorem (IE, Eric Séré)

Assume $F(0) = 0$ *and* $F : B_0(R) \cap X_s \to Y_s$, $0 \leq s < S$, *is roughly tame with loss of regularity* μ. *If* δ/μ *is in the authorized region, then, for any* α *with*

$$\frac{\alpha}{\mu} < \min\left\{\frac{\delta}{\mu} - \varphi\left(\frac{S}{\mu}\right), \frac{S}{\mu} - \varphi^{-1}\left(\frac{\delta}{\mu}\right)\right\}$$

one can solve $F(x) = y$ *with* $y \in Y_\delta$ *and* $x \in X_\alpha$. *More precisely, we can find* $\rho > 0$ *and* $C > 0$ *such that, whenever* $\|y\|'_\delta \leq \rho$, *there is some* $x \in X_\alpha$ *with:*

$$F(x) = y$$
$$\|x\|_0 \leq 1$$
$$\|x\|_\alpha \leq C \|y\|_\delta$$

Comments

- $S > 4\mu$ (you need some room upstairs)
- $3\mu > \delta - \alpha > \mu$ (the loss of derivatives for F is larger than the one for DF)
- if $S/\mu \to \infty$ (lots of room upstairs), $\delta - \alpha \to \mu$ (lowest possible value)
- If $S/\mu \to 4$ (little room upstairs), $\delta - \alpha \to 3\mu$ (large loss of regularity)

Corollary

Assume $F(x)$ sends X_s into Y_s and is roughly tame at $\bar{x}$ with loss of regularity μ. Suppose $\bar{x} \in X_S$, $\bar{y} \in Y_S$ and $F(\bar{x}) = \bar{y}$. Then one can solve $F(x) = y$, with $\|x - \bar{x}\|_\alpha \leq C\|y - \bar{y}\|'_\delta$.

Proof.

Consider the map $\Phi(x, y) := F(x) - y + \bar{y}$ from $X_s \times Y_s$ into Y_s. It is roughly tame, with $F(0,0) = 0$, and we can apply the preceding Theorem

The proof: constructing approximate solutions

Theorem

Consider an integer $N_0 \geq 2$ and define $N_n \simeq (N_0)^{\kappa^n}$, where $\kappa = \kappa(\delta, \mu) > 1$ is appropriately chosen. Then one can find $\rho > 0$ and $c > 0$ such that, for any y with $\|y\|_\delta < \rho$, there is a sequence $(x_n)_{n\geq 1}$ with $\|x_n\|_0 \leq 1$ satisfying:

$$\text{(case 1) } \Pi'_{N_n} F(x_n) = \Pi'_{N_{n-1}} y \text{ and } x_n \in E_{N_n}$$
$$\text{(case 2) } \Pi'_{N_n} F(x_n) = \Pi'_{N_n} y \text{ and } x_n \in E_{N_n}$$

and in both cases, for appropriate σ and β with $\kappa\beta < \sigma < S$:

$$\|x_1\|_\sigma \leq c N_1^\beta \|y\|'_\delta \text{ and } \|x_{n+1} - x_n\|_\sigma \leq c\, N_n^{\kappa\beta} \|y\|'_\delta$$
$$\|x_1\|_0 \leq c N_1^\mu \|y\|'_\delta \text{ and } \|x_{n+1} - x_n\|_0 \leq c\, N_n^{\kappa\beta-\sigma} \|y\|'_\delta$$

Auxiliary constants

(S, δ, μ) are given.
We first choose κ :

$$1 < \kappa < 2 \text{ and } \frac{\kappa^2}{\kappa - 1} < \frac{S}{\mu} \text{ and } \min\{\kappa^2, \kappa + 1\} < \mu\frac{\delta}{\mu}$$

This gives two possibilities: $1 < \kappa \leq \frac{1+\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2} \leq \kappa < 2$. Then we choose σ and β :

$$\frac{\kappa^2}{\kappa - 1} < \frac{\kappa}{\mu}\beta < \frac{1}{\mu}\sigma < \frac{S}{\mu}$$

$$\kappa\beta > \sigma + \kappa\mu - \frac{\delta}{\kappa} \text{ for } 1 < \kappa \leq \frac{1+\sqrt{5}}{2}$$

$$\beta > \mu + \sigma - \delta \text{ for } \frac{1+\sqrt{5}}{2} \leq \kappa < 2$$

Passing to the limit

The sequence (x_n) has a limit x in X_0, with $\|x\|_0 \leq C\|y\|'_\delta$, for $C := c(N_1^\mu + \sum_{n\geq 1} N_n^{\kappa\beta-\sigma})$. Then $F(x_n)$ converges to $F(x)$ in Y_0, by the continuity of $F : X_0 \to Y_0$. On the other hand:

$$F(x_n) = (1 - \Pi'_{N_n})F(x_n) + \Pi'_{N_{n-1}}y\,.$$

One proves by induction an estimate of the form

$$\|(1 - \Pi'_N)F(x_n)\|'_0 \leq C\,N_n^{\beta-\sigma}\|y\|'_\delta \to 0$$

because the exponent $(\beta - \sigma)$ is negative. So $(1 - \Pi'_N)F(x_n)$ converges to zero in Y_0. On the other hand, by the definition of a smoothing operator,

$$\|(1 - \Pi'_{N_{n-1}})y\|'_0 \leq CN_{n-1}^{-\delta}\|y\|'_\delta \to 0$$

Passing to the limit we get $F(x) = y$.

Checking the induction.

Here we assume that we have chosen ρ and c, and found $x_1, \cdots, x_n$. We are going to construct $x_{n+1} = x_n + u$. Using the induction hypothesis $\Pi'_{N_n} F(x_n) = \Pi'_{N_{n-1}} y$, the equation to be solved by Δx_n may be written in the following form:

$$f_n(u) = e_n + \Delta y_{n-1}$$

$$f_n(u) := \Pi_{N_{n+1}} \left(F(x_n + u) - F(x_n) \right) \in E'_{N_{n+1}}$$
$$e_k := \Pi_{N_{k+1}} (\Pi_{N_k} - 1) F(x_k)$$
$$\Delta y_k := \Pi_{N_{k+1}} (1 - \Pi_{N_k}) y$$

The function f_n is continuous and Gâteaux-differentiable with $f(0) = 0$. So this is an inverse problem for u

Choosing the right norms

We choose the norm

$$\mathcal{N}_n(u) = \|u\|_0 + N_n^{-\sigma}\|u\|_\sigma \quad \text{on} \quad E_{N_{n+1}}$$
$$\mathcal{N}_n'(v) = \|v\|_0' + N_n^{-\sigma}\|v\|_\sigma' \quad \text{on} \quad E_{N_{n+1}}'$$

Set $R_n := cN_n^{\kappa\beta-\sigma}\|y\|_\delta$. For $\mathcal{N}_n(u) \leq R_n$ we have:

$$\left\| [Df_n(u)]^{-1} k \right\|_0 \leq \frac{2N_{n+1}^{\mu}}{\gamma}\|k\|_0$$
$$\left\| [Df_n(u)]^{-1} k \right\|_\sigma \leq \frac{N_{n+1}^{\mu}}{\gamma}\left(\|k\|_\sigma + B^{(2)}\, R_n\, N_n^{\sigma}\|k\|_0\right)$$

Hence:

$$\mathcal{N}_n([Df(u)]^{-1}k) \leq \frac{B^{(2)}+2}{\gamma} N_{n+1}^{\mu}\mathcal{N}_n'(k)$$

Applying the soft IVT

The IVT gives the existence of $\bar{u}_n \in \mathcal{B}_n(R_n)$ such that $f_n(\bar{u}_n) = e_n + \Delta y_n$ provided:

$$\mathcal{N}_n'(e_n + \Delta y_{n-1}) < \frac{R_n}{\left(B^{(2)} + 2\right)\gamma^{-1} N_{n+1}^{\mu}}$$

. The condition on $e_n + \Delta y_{n-1}$ is fulfilled provided

$$\mathcal{N}_n'(e_n) + \mathcal{N}_n'(\Delta y_{n-1}) < \frac{2c}{B^{(2)} + 2} N_{n+1}^{-\mu} N_n^{\kappa\beta - \sigma} \|y\|_\delta \,.$$

This is satisfied if:

$$2\,B^{(3)}\,c\,N_n^{\beta} + C^{(1)} \left(g^{\frac{\delta}{\kappa}} N_n^{\sigma - \frac{\delta}{\kappa}} + \left(\frac{g}{N_n}\right)^{\frac{(\delta-\sigma)_+}{\kappa}} N_n^{(\sigma-\delta)_+} \right) < \frac{2c\,g^{-\mu}}{B^{(2)} + 2} N_n^{\kappa(\beta-\mu)}$$

holds. We find that the exponents of N_n in the left-hand side of are strictly smaller than the one in the right-hand side. So, for N_0 chosen large enough, is satisfied for all n

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I. Ekeland and E. Séré, work in progress

All papers can be found on my website:

http://www.ceremade.dauphine.fr/~ekeland/