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FRANK RAMSEY'S A MATHEMATICAL THEORY OF SAVING*

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In 1928, Frank Ramsey, a British mathematician and philosopher, at the time aged only 25, published an article (Ramsey, 1928) whose content was utterly innovative and sowed the seeds of many subsequent developments. Beside the theory of optimal growth, as developed in Cass (1965) and Koopmans (1965), one could argue that the essence of several subsequent influential theories, such as the permanent income/life cycle theory of consumption to model individual saving choices, were already contained in that seminal article.

In this note, I first summarise the content of Ramsey's paper and then relate it to subsequent developments in economics. Before starting, however, I cannot help mentioning the modernity and originality one perceives in Ramsey's writings when going through the original texts. His papers and contributions are astonishingly ahead of their time. This is true for his article on optimal taxation, (Ramsey, 1927), which is reviewed elsewhere in this issue and for his article on 'Truth and probability' (Ramsey, 1931), in which he discusses choices under uncertainty decades ahead of Von Neumann and Morgenstern (1944) and even proposes an experimental way to disentangle preferences and probability assessments from choices. In this note, I discuss how the article on saving and optimal growth anticipated several branches of the literature that followed in subsequent decades. The treatment of individual consumption and saving choices, for instance, is a very modern one. Interestingly, Ramsey's article refers explicitly to conversations with John Maynard Keynes, who was Ramsey's colleague at Cambridge and the ECONOMIC JOURNAL's editor, in formulating part of the main proposition. And yet, in his *General Theory*, Keynes (1936) used a much more simplistic and stylised theory of consumption, which had profound implications for the working of his model of the macroeconomy.

1. A Mathematical Theory of Saving

The article sets out to answer an interesting and important question: 'how much of its income should a nation save?'. The problem is set, therefore, as a normative one. And indeed, in the first Section, in which the basic problem is set and the main result is derived, there is no reference to factor prices or markets. In the third Section, however, Ramsey discusses explicitly the determination of the interest rate, both in the short and the long run and how the equilibrium interest rate would support the optimal solution.

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The article is divided into three Sections. In Section I, Ramsey starts by making a number of stark assumptions to define the problem and to derive the main result. Many of these assumptions can be easily relaxed and, indeed, some are relaxed in subsequent Sections to make a number of specific points. In Section II, Ramsey, considers wages and interest rates to be exogenously fixed constants, both to give a diagrammatic version of his main result and to extend it to consider finite lives and discounting. Finally, in Section III, Ramsey discusses the determination of the equilibrium interest rate, seen as the price that equates the demand and supply of savings. Interestingly, in the last part of this Section, Ramsey even introduces heterogeneity in discount factors to generate inequality in wealth holdings, a topic that has recently received much attention.

Here, I briefly review each of the three Sections. Rather than reproducing the structure of the article exactly, I describe its main components and stress the novelty that they represented.

1.1. *Individual Behaviour*

The first component in the argument is individual behaviour. Ramsey starts the article assuming a representative consumer with an infinite life. Both of these assumptions are relaxed later on in the article. He works within a continuous time framework and assumes a homogeneous consumption good that is produced with capital and labour. The output of the production process can be consumed or can be invested to increase the capital stock. As he considers a closed economy with no government, the first equation Ramsey introduces is the national accounts identity that equates total output to the sum of consumption and investment, with the latter going to increase the existing capital stock:

$$\frac{\mathrm{d}k}{\mathrm{d}t} + c = f(h, k), \qquad (1)$$

where k is the capital stock, $c(t)$ the rate of consumption and $h(t)$ labour supply.[1] The consumer is assumed to maximise an intertemporally additive and separable utility function that is also separable between consumption and labour supply. The instantaneous felicity function is given by:

$$U(c) - V(h),$$

where the two functions $U(.)$ and $V(.)$ are concave and convex respectively and their first derivatives are denoted with $U_c(c)$ and $V_h(h)$. Instantaneous felicity will be a function of the available capital and will not be decreasing in it. Ramsey assumes that such a function is bounded from above, with the upper bound, which he denotes as the bliss level B achievable either with a finite level of capital or asymptotically, as capital increases to infinity.

It is interesting to note how Ramsey recognises the need for an upper bound ('we can see that the community must save enough either to reach Bliss after a finite time,

[1] In the article, Ramsey uses the symbol c for capital, x for consumption and a for labour supply. I follow what has become the standard notation and denote capital with k, consumption with c and labour supply with h.

or at least to approximate to it indefinitely. For in this way alone is it possible to make the amount by which enjoyment falls short of bliss summed throughout time a finite quantity'. p. 545) and the way he justifies it.

Having established a trade-off between the need to approach the bliss level and the sacrifice that it implies in terms of deferred consumption and increased labour supply, he proceeds to characterise such a trade-off and derive his main proposition. In doing so, the first step is to consider the within period choice and the trade-off between work and consumption, which gives the equality between the marginal rate of substitution and the productivity of labour (which will later be given by the wage):

$$\frac{V_h(h)}{U_c(c)} = \frac{\partial f}{\partial h}. \tag{2}$$

The second equation he considers is the Euler equation that guarantees that the saving choice is inter-temporally optimal:

$$\frac{\mathrm{d}}{\mathrm{d}t} U_c[c(t)] = -\frac{\partial f}{\partial k} U_c[c(t)]. \tag{3}$$

As Ramsey puts it: 'This equation means that $U_c(c)$, the marginal utility of consumption, falls at a proportionate rate given by the rate of interest' (p. 546). This equation is at the essence of the life cycle and permanent income model and implies that consumers allocate income between consumption and saving to keep the discounted level of marginal utility constant over time.

Given (1)–(3), Ramsey derives by simple calculus the main result in the article:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = f(h,k) - c = \frac{B - [U(c) - V(h)]}{U_c(c)}, \tag{4}$$

or, in words: '[The] rate of saving multiplied by marginal utility of consumption should always equal bliss minus actual rate of utility enjoyed' (p. 547). It is interesting to note the elegant change of variable that Ramsey uses (using capital as the relevant state variable rather than time) to show that the constant of integration that one has to compute to solve the differential equation given by the Euler equation is equal to the bliss level B. After a description of the heuristic proof suggested by Keynes, Ramsey concludes the Section by discussing technical progress, population growth and exogenous destruction of the capital and suggesting how most of these phenomena could be incorporated in the analysis.

1.2. *Factor Prices and Extensions*

In the second Section, Ramsey starts with the assumption that factor prices, that is the wage rate and the interest rate, are constants, which will be denoted here by w and r:

$$f(h,k) = wh + rk. \tag{5}$$

Given this, the Section does three things:

(*i*) it develops a useful graphical analysis;
(*ii*) finite lives; and
(*iii*) discounting.

I will not reproduce here the graphical analysis Ramsey uses. However, it is interesting how Ramsey defines a new variable as a part of 'non-earned income' which is available for consumption; $y = c - wh$ and the function $\omega(y) = U_c(c) = V_h(h)/w$. Integrating $\omega(y)$ one obtains:

$$\Omega(y) = \int \omega(y)\mathrm{d}y = U(c) - V(h).$$

He then performs the graphical analysis in terms of such variables and functions. Notice that y is the part of income from capital that is not saved and is available to consume. y can be negative if consumption exceeds earned income.

When extending the model to consider the case of finite lives, Ramsey uses the graphic analysis developed in the first part of the Section. He shows that, whilst the formula for optimal savings is similar to that for the case with infinite lives, the optimal level of capital (and therefore saving) is different. In particular, the optimal path converges to a level which is below that allows achieving the bliss level. In the process of deriving the bound for utility, he also considers, as a parameter of the model, the amount of bequest that the representative consumer wants to leave to its heirs. The main result of this Section is that the optimal level of savings is given by:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = f(h,k) - c = \frac{Q - [U(c) - V(h)]}{U_c(c)}, \tag{6}$$

where $Q \leq B$ depends on the parameters of the model, including the importance of the bequest motive.

In the last part of the Section, Ramsey tackles the issue of discounting, recognising that future utility is somewhat discounted when making investment choices. He assumes geometric discounting at a constant rate ρ and, in passing, he mentions how this induces what now would be called time-consistent choices and the fact that the infinite horizon with discounting might be representing the utility of subsequent generations. He then goes on to show how the presence of discounting affects the main results in the article, starting with the Euler equation (3) which now becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} U_c[c(t)] = -\left\{\frac{\partial f}{\partial k} - \rho\right\} U_c[c(t)] = -\{r - \rho\} U_c[c(t)]. \tag{7}$$

From this, he derives the new equilibrium condition:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = rk - y = \frac{Q - \int_b^y \omega(y)^{\frac{r}{r-\rho}}\mathrm{d}y}{\omega(y)^{\frac{r}{r-\rho}}}, \tag{8}$$

where $y = c - wh$ and $\omega(y) = U_c(c) - V_h(h)$.

The result of the previous Section is neatly obtained for $\rho = 0$ (except when $r = 0$).

1.3. *Equilibrium and Inequality*

In the final Section of the article, Ramsey considers the determination of the equilibrium interest rate. He first considers the steady state in which the optima level of the capital stock has been achieved so that, in the absence of technological or population growth, capital and consumption are constant and, consequently, saving is equal to zero. Given that $dc/dt = dk/dt = 0$ the three equations that determine the equilibrium levels of c, h and k are:

$$c = f(h, k),$$
$$V_h(h) = \frac{\partial f}{\partial h} U_c(c),$$
$$\frac{\partial f}{\partial k} = \rho.$$

Clearly the third equation determines the steady-state level of the capital stock and the interest rate equals the discount factor. Ramsey then goes on to discuss the fact that such a long-run equilibrium might never be reached or reached only in the very long run. In the short run, he argues, the interest rate will be determined by the marginal product of the existing capital. That interest rate will then generate a supply of capital, which in turn will determine a new level for the interest rate, through a process that might converge to the long-run equilibrium. Before reaching the steady state, Ramsey stresses, the rate of interest can differ from ρ and, at the low level of capital, can exceed it substantially.

After this first discussion, Ramsey goes on to discuss the determination of equilibrium in the finite lives case where individuals do not care about their heirs. The discussion there is remarkable as it is clear that, on one hand, Ramsey is using a sophisticated version of the life cycle model, and, on the other, describes what many years later, with the contributions of Samuelson (1958) and Diamond (1965) became another work-horse of modern macroeconomics and public finance: the overlapping generation model. It is worth citing the following passage from Ramsey:

> different people discount future utility at different rates, and, quite apart from the time factor, are not so interested in their heirs as in themselves.
>
> Let us suppose that they are not concerned with their heirs at all; that each man is charged with a share of the maintenance of such children as are necessary to maintain the population, but starts his working life without any capital and ends it without any, having spent his savings on an annuity; that within his own lifetime he has a constant utility schedule for consumption and discounts future utility at a constant rate, but that this rate may be supposed different for different people.
>
> When such a community is in equilibrium, the rate of interest must, of course, equal the demand price of capital at $\partial f/\partial k$. And it will also equal the 'supply price', which arises in the following way. Suppose that the rate of interest is constant and equal to r, and that the rate of discount for a given individual is ρ. Then if $r > \rho$, he will save when he is young, not only to provide for loss of earning power in old age, but also because he can get more pounds to spend at a later date for those he forgoes spending now. If we neglect variations in his earning power, his action can be calculated by modifying the equations of lIc to apply to a finite life as in IIb. He will for a time accumulate capital, and then spend it before he dies. Besides this man, we must suppose there to be in our community other men, exactly like him except for being

> born at different times. The total capital possessed by n men of this sort whose birthdays are spread evenly through the period of a lifetime will be n times the average capital possessed by each in the course of his life. The class of men of this sort will, therefore, possess a constant capital depending on the rate of interest, and this will be the amount of capital supplied by them at that price. (If $p > \rho$, it may be negative, as they may borrow when young and pay back when old). We can then obtain the total supply curve of capital by adding together the supplies provided at a given price by each class of individual (pp. 557–8).

The modernity of such an approach is stunning, given the time at which the article was written. I will come back to this in the concluding Section.

Another aspect of Ramsey's article that has not received much attention in the literature is his digression, in the concluding subsection of the article, on heterogeneity about discount factors and the possibility that difference in tastes can generate large inequality in wealth and consumption in the long run. In particular, in Section III (γ) of the article, he goes back to study infinite lives and assumes that the population is divided into two groups, one with a high and one with a low discount rate. He then shows that, in such an economy, there will be increasing inequality in wealth and consumption and with a utility function bounded above and with a subsistence level of consumption, in the long run society will be divided 'into two classes, the thrifty enjoying bliss and the improvident at the subsistence level'. Once again, Ramsey anticipates in his concluding sentence a large literature to which I briefly refer in my concluding Sections.

2. Influences and Anticipations

As should be clear from my short summary in the previous Section, Ramsey's article was extremely innovative and ahead of its time. What is remarkable is that many of the numerous insights and novelties in the article did not emerge in the literature until much later.

The most obvious anticipation in the article is its central theme and result: the optimal growth model, as formulated by Ramsey, is very similar to what has become a basic workhorse of modern macroeconomics. In a 1998 interview Cass (1998) recounts that he read Ramsey's paper after writing the first chapter of his PhD dissertation in 1963, which eventually became the review of economic studies article (Cass, 1965).[2] Talking about his celebrated 1965 article Cass (1998) says: 'In fact I always have been kind of embarrassed because that paper is always cited although now I think of it as an exercise, almost re-creating and going a little beyond the Ramsey model' (p. 534).

A recent paper by Spear and Young (2014) argues that the optimal growth model, which is often cited as the Cass–Koopmans model, evolved through a process of

[2] Spear and Young (2014) cite the Cass interview as follows: 'But didn't some people know about Ramsey? Didn't Uzawa?' to which Cass frankly replied: 'No. I don't think so, because I didn't find out about Ramsey until after I had written the first chapter of my thesis on optimum growth. And then, I was, to be perfectly honest, I was a bit embarrassed about it' (p. 535).

'sequential cross-fertilisation' that involved Cass, Koopmans, Malinvaud and Uzawa (who was Cass's advisor at Stanford). Many of the discussions were about discounting and about the appropriate mathematics to be used.[3] A re-read of Ramsey's article, however, makes it clear, as stated in Cass's quote, that the main ideas were all in the original article.

In addition to the influence the article had on optimal growth, one cannot help noticing, given when the article was written, how it anticipated many other contributions that appeared much later. In particular, three aspects of Ramsey's article are worth mentioning: his approach to the intertemporal allocation of consumption; his discussion of saving in the context of an economy populated by multiple and overlapping generations and its brief discussion of inequality.

When considering the optimal saving problem, Ramsey uses as a first building block an intertemporal consumption problem which essentially defines the permanent income model. Consumptions at different points in time are considered as different commodities and, at the optimum, the consumer keeps the discounted value of marginal utility as constant. This is evident, obviously, in (3), but is also apparent in the language Ramsey uses when describing the heuristic proof apparently suggested by Keynes. And in the passage I cited above, in which he refers to a finite live context, he explicitly mentions life time considerations.[4] Interestingly, in several instances, he discusses bequest motives (or the lack thereof as a simplifying assumption). He also mentions the importance of the ability to borrow, especially in situations where the interest rate is below the discount factor. These intuitions and this way of modelling were written 30 years before the publication of Friedman's (1957) book and Modigliani and Brumberg's (1954) seminal paper on the life cycle model of consumption.

Analogously, the brief description on an economy populated by individuals with 'different birthdays' and how their individual savings aggregates into the supply of capital is essentially a description of the overlapping generation model which was Samuelson (1958) developed 30 years later[5] and subsequently enriched and studied by Diamond (1965). Obviously the welfare implications of such a model (such as the possibility of dynamic inefficiencies related to the over-accumulation of capital that can arise in this model), or even the mathematical complexities that can arise in these models related to the multiplicity of equilibria, are not mentioned in the text. Nonetheless, it is remarkable how he lays down in a single paragraph the essence of that model that, subsequently, became another workhorse of modern macro and public finance. Indeed the discussion of the mechanism that determines the equilibrium of interest rate in that context is similar to that used, for instance, in Auerbach and Kotlikoff (1987). More generally, it is impressive to note how, in the various versions of the model he presents, equilibrium considerations are always present and important. In this respect, his treatment is extremely modern.

[3] In his Nobel lecture, Koopmans (1976) mentions: 'Ramsey used an ingenious mathematical device that gets around the non-convergence of the utility integral for $\rho = 0$, and also leads to a proof simpler than the one for positive ρ' (p. 547).

[4] '...if $r > \rho$, he will save when he is young, not only to provide for loss of earning power in old age, but also because he can get more pounds to spend at a later date for those he forgoes spending now' (p. 558).

[5] Malinvaud (1987) has pointed out that the OLG model was already discussed in Allais (1947).

Finally, in the last page of the article, Ramsey tackles the issue of how inequality can arise in his model. Given that the model is dealing with is essentially the basic optimal growth model,[6] it is not surprising that the basic equations that govern the accumulation of capital are not dissimilar from those used, for instance by Piketty's (2014) recent book on capital and inequality. However, unlike in Piketty (2014) or in Solow's (1956) growth model, saving is endogenously determined in Ramsey's construct. In such a situation, with consumers that are homogeneous in terms of preferences, it is obviously impossible to generate inequality. Ramsey does not consider heterogeneity in terms of type of income: in his model capitalists and workers are not distinguished and the representative consumer holds the capital stock. Instead Ramsey introduces heterogeneity in tastes and points out to the fact that if some individuals are more patient than others, they will end up owning all the capital and inequality will increase, as described in the last sentence of the article.

It is interesting to note that the same approach to generate inequality in a modern macro model was taken 70 years after the publication of Ramsey's article by Krusell and Smith (1998) who try to match the observed wealth inequality in the US and, for such a purpose, populate their economy with three types of individuals, differing in their degree of patience.[7]

In reading Ramsey's (1928) article, one cannot avoid wondering what contributions the author would have produced had he not died at the age of 27. His only two article in economics, Ramsey (1927, 1928) opened up new fields and, as I have argued, contained the seeds of many subsequent developments that were only taken up by the profession decades after Ramsey's death.

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[6] What most textbooks would call the Ramsey–Cass–Koopman model.

[7] Other articles have studied the dynamics of inequality in wealth and income in growth models, such as Stiglitz (1969), Chatterjee (1994) and Caselli and Ventura (2000).

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Appendix A. Ramsey, F.P. (1928). 'A mathematical theory of saving', Economic Journal, vol. 38(152), pp. 543–59.