John Forbes Nash, Jr.
1928-2015

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- 2015: Abel prize in mathematics (shared with Louis Nirenberg)

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Princeton after WWII

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- the young students around: John Milnor, Lloyd Shapley, Gary Becker, Harold Kuhn, David Gale

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- This is nowadays the standard tool in economics. No one believes in cooperation any more.
- Individual rationality (Nash equilibrium) may lead to outcomes which are bad for everyone.

Global warming

Why this civilisation is doomed

- We are on track for an increase in mean temperatures $> 5^\circ C$ by the end of the century. As a matter of comparison, this is exactly what separates us from the last ice age, when most of Europe was under glaciers. Temperatures around the Mediterranean are set to increase by $> 10^\circ C$ in summertime

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- Let us consider N nations undertaking a climate policy. The cost of such a policy is c, the benefit is nB, where n is the number of nations participating.
- If everyone participates, benefit is $NB >> c$ for everyone. Evidently, it is to everyone's benefit. Will everyone participate ?

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- Everyone does the same calculation, everyone tries to free ride on the others, so no one participates. This is exactly what has been going on for twenty years, and which will happen again in Paris this December (COP 21).
- Not participating is a Nash equilibrium, and it is the only one. We are firmly on track for a $5^\circ C$ increase in mean temperatures (15 in the Arctic)

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- This result belongs to normative economics: a solution is sought satisfying certain assumptions (fairness), whereas his earlier one belongs to positive economics (what people actually do)

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  - If $\dim M < \frac{n-1}{2}$, then M can be C^∞ approximated by a nonsingular real algebraic set
  - any compact smooth manifold is diffeomorphic to a smooth connected component of a algebraic variety

- Consider the classical problem in the calculus of variations

$$\min_{u \in H} \int_{\Omega} F\left(x, u\left(x\right), Du\left(x\right)\right) dx$$

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- To prove that u is in fact smooth, there was a two-step procedure: (a) show that u satisfies the Euler equation, and (b) show that this implies that u is smooth

Consider the problem for $u : R^N \longrightarrow R^K$

$$\int_\Omega F\left(Du\right)dx$$

where $F \in C^\infty\left(R^{KN}\right)$, $\left|F\left(p\right)\right| \leq c\left|p\right|^2$ and the derivatives $A_k^n\left(p\right) := \partial F / \partial p_n^k$ satisfy the growth and ellipticity conditions:

$$\left|A_k^n\left(p\right)\right| \leq c\left|p\right|, \quad \left|\frac{\partial A_k^n}{\partial p_m^j}\left(p\right)\right| \leq c, \quad \frac{\partial A_k^n}{\partial p_m^j}\left(p\right)\xi_n^k \xi_m^j \geq c\left|\xi\right|^2$$

If u is a minimizer of F, the any derivative $v_i := D_i u$ satisfies the elliptic system:

$$\int_\Omega \frac{\partial A_k^n}{\partial p_m^j}\left(Du\right) D_m\left(v^j\right) D_n \varphi^k dx = 0 \quad \forall \varphi \in W^{1,2}\left(\Omega; R^K\right)$$

By bootstrapping this equation, it can be shown that if $u \in C^1$, then $v \in C^1$, and then $u \in C^\infty$ The problem is to start !

- In the scalar case ($K=1$), the system reduces to a single equation for the derivative $v_n := \partial u/\partial x^n$:

$$\frac{\partial}{\partial x^j}\left(\frac{\partial^2 F}{\partial p_i \partial p_j}(Du)\frac{\partial v}{\partial x^i}\right) = 0$$

If u is a weak solution, the coefficients $\frac{\partial A_k^n}{\partial p_m^j}(Du(x))$ are at best L^∞ (not continuous). So we have an elliptic linear equation with L^∞ coefficients. Nash, and independently de Giorgi, showed that the solution is Hölder continuous.

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- In the vector case ($K > 1$), de Giorgi gave an example of a function $F(x, Du)$ with all imaginable blessings, with solution $u(x) = x/|x|^\gamma$. This was extended by Giaquinta and Giusti to a function $F(u, Du)$ with minimizer $u(x) = x/|x|$

In his thesis *Über die Hypothesen, die der Geometrie zugrunde legen* (1854), Bernhard Riemann defined an intrinsic geometry on manifolds by a quadratic form$\sum g_{ij}(x)\,\xi^i\xi^j$ on the tangent space at x. The question immediately arose: does that bring anything new ? Can every such Riemannian manifold be realized as a submanifold of Euclidian space ? Note that, in 1827, Carl Friedrich Gauss had found an obstruction: the curvature is preserved by any isometry. It is the famous theorema egregium. As a consequence, for instance, a sphere of radius R cannot be compressed isometrically into a ball of radius $r < R$ (look at the curvature of extreme points)

Locally, the problem was solved by Janet. The global problem was solved by Nirenberg, in the particular case of a two-dimensional sphere (which can be imbedded as a convex hypersurface in R^3), and in the general case by Nash: any compact Riemannian manifold can be imbedded isometrically into an Euclidian space of sufficiently high dimension.
An isometry is understood as a one-to-one map $\varphi : M \rightarrow R^N$ which preserves the given quadratic form $g(x)$ on $T_x M$:

$$\sum_{j,k=1}^{K} \left(\frac{\partial \varphi^n}{\partial x^j}(x)\, \xi^j, \frac{\partial \varphi^n}{\partial x^k}(x)\, \xi^k \right)_{R^N} = \sum g_{jk}(x)\, \xi^j \xi^k$$

$$\sum_{n=1}^{N} \frac{\partial \varphi^n}{\partial x^j}(x) \frac{\partial \varphi^n}{\partial x^k}(x) = g_{jk}(x)$$

Nash's proof goes by showing that the set of Riemannian structures on M (ie the set of fields $g(x)$) which can be isometrically embedded is both open and closed. The latter is relatively easy, the former is quite difficult. Consider the map $\Phi(u) = F(x, u, Du)$, with $u : R^N \to R^K$. Let us try to apply the IVT to show the image is open. Differentiating at u_0, we get

$$\Phi'(u_0)\, v = F_x + F_u v + F_p Dv$$

So $\Phi'(u)$ maps C^k into C^{k-1}. This derivative is not recovered by inversion: $\Phi'(u)\, v = w$ with $w \in C^{k-1}$ does not imply that $v \in C^k$ except in very special cases. There is a global loss of derivatives, and the usual IVT does not apply. Nash constructed a "hard" IVT to solve the embedding problem. Simultaneously, Kolmogorov in the USSR constructed such an IVT to solve the resonance problem in celestial mechanics. (see the talk by Séré for the state of the art)

We will now consider C^1 embeddings $\varphi : M \to R^N$. Since φ is C^1 only, the Riemannian structure (including curvature) no longer makes sense, only the metric is left. Such an embedding will be called isometric if the length of any path $c(t)$, $0 \leq t \leq 1$, on M, coïncides with the length of its image $\varphi(c(t))$ in R^N.

Theorem (Nash-Kuiper)

Let $f : M \to R^N$ be any map which is contracting:

$$\|f(x) - f(y)\| \leq \|x - y\|$$

Then, for any $\varepsilon > 0$, there exists a C^1 embedding $\varphi : M \to R^N$ such that $\|f(x) - \varphi(x)\| \leq \varepsilon$ on M

The NK theorem has remarkably counterintuitive consequences: a sphere of radius R can be sent isometrically into a ball of radius $r < R$ (which is not possible for C^2), and a piece of paper of format A4 can be put into one's pocket without folding.

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- Gromov's h-principle (if there are no topological obstructions, there are no holonomy obstructions) and convex integration
- the existence of non-energy preserving solutions of the Euler equations in fluid mechanics

The nature of genius