

Optimal Bond Portfolios

Ivar Ekeland Erik Taffin

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1 Motivation

The literature on portfolio management starts with the Markowitz portfolio and the CAPM ([19], [20], [33]). It is a one-period model, where the information on assets is minimal. Every asset is characterized by two numbers, its expected return and its covariance with respect to the market portfolio. With such poor information, one cannot hope to distinguish between stocks and bonds, and indeed part of the beauty of the CAPM lies in its generality: it applies to any type of financial assets.

On the other hand, as soon as one tries to make use of all the information available on assets, important differences appear between stocks and bonds. Bonds mature, that is they are eventually converted into cash, whereas stocks do not. The price of bonds depends on interest rates, and the price of stocks, at least in the academic literature, does not. The bond market is notoriously incomplete, much more so than the stock market, as is observed in practice. As a result, the classical results on portfolio management, such as Merton's ([21], [22]), concern stock portfolios. This paper and the papers [10] and [34] were born from a desire to extend them to bond portfolios.

More generally, we aim to construct a general framework for portfolio management in continuous time, encompassing both stocks and bonds.

The first difficulty to overcome (and, in our opinion, the main financial one) is the fact that such a theory should encompass two very different kinds of financial assets: bonds, which have a finite life, and stocks, which are permanent. We do it by introducing a new type of financial asset, the *rollovers*. A rollover of time to maturity x is a bank deposit and which can be cashed at any time, with accrued interest, provided notice be given time x in advance. Roll-overs have constant time to maturity (as opposed to zero-coupon bonds, for instance), and are similar to stocks, in the sense that their main characteristics do not change with time. By decomposing bonds into rollovers, instead of decomposing them into zero-coupons, we can hope to incorporate

bonds and stocks into a unified theory of portfolio management. Rollovers were considered in [32] under the name “rolling-horizon bond”.

This implies that the time to maturity x , rather than the maturity date T , becomes the relevant characteristic of bonds. Thus, we shall describe bonds using a moving maturity-time frame, where at time t , the origin is the time to maturity $x = 0$, corresponding to the maturity date $T = t$. As we shall see very soon, there will be a mathematical price to pay for that.

At any time t , denote by $p_t(x)$ the price of a unit zero-coupon with time to maturity x . The function $x \mapsto p_t(x)$ will be called the zero-coupon (price) curve at time t ; note that the actual time when that zero-coupon matures is $T = t + x$, and that T , is fixed while x changes with t . The zero-coupon curve p_t will be understood to move randomly, and the second difficulty we face is to describe its motion in some reasonable way. One solution is to decide that p_t belongs to a fixed family of curves, depending on finitely many parameters, so that

$$p_t(x) = f(t, x; r_1, \dots, r_d)$$

and the random motion of p_t is the image of a random motion of the r_i , which could, as in spot-rates models, be modelled, for instance by diffusions. This is the *parametric* approach, which exhibits the classical difficulty of all parametric approaches, namely that there is no theoretical reason why the p_t should be written in that way, so that the choice of the function f has to be dictated by observational fit. One then has to strike the right balance between two evils: if the number of parameters is too small, the model will be unrealistic, and if it is higher, it becomes very difficult to calibrate.

We will operate in a *non-parametric* framework: we will make no assumption on p_t , beyond some very rough ones, regarding smoothness and behavior at infinity, nothing that would much constrain their shape. Mathematically speaking, we will let the curve p_t move freely in a linear space E , which will typically be an infinite-dimensional Banach space, of functions from $[0, \infty[$ to \mathbb{R} .

In order to reflect adequately known financial facts, the correct definition of E must incorporate some basic constraints:

1. At any time t , the zero-coupon prices $p_t(x)$ must depend continuously on the time to maturity x . In order for forward interest rates to be well-defined, they must also have some degree of differentiability with respect to x . So E must consist of continuous curves with some degree of differentiability.
2. The degree of differentiability of functions in E will determine which basic interest rates derivatives can be modelled. If p_t is continuous,

for instance, then we can introduce bonds. The price of a unit zero-coupon bond with time to maturity x is $p_t(x)$; the bond itself, i.e. the value of a portfolio including exactly one bond is represented by the linear form $p_t \mapsto p_t(x)$. Mathematically speaking, this is just the Dirac mass δ_x at x . Now other derivatives such as Call's and Put's on zero-coupon bonds can be introduced, since the pay-off for each of them is a continuous function of the zero-coupon bond price $p_T(x)$, with a given time to maturity x . If p_t is continuously differentiable, then the forward interest rate with time to maturity x , $-\frac{\partial}{\partial x}p_t(x)/p_t(x)$ is well-defined, and further contingent claims can be defined, such as caps, floors and swaps.

3. The curve p_t will be understood to move randomly in E , the randomness being driven by a Brownian motion. We will therefore need to define Brownian motions in the infinite-dimensional space E , which for all practical purposes will require E to be a Hilbert space.
4. The accepted standard in mathematical modelling of zero-coupon prices (the Heath-Jarrow-Morton model, henceforth HJM) is to decide that the real-valued process $t \mapsto p_t(T-t)$, the price at time t of a unit zero-coupon maturing at a given time T , is a diffusion process satisfying an stochastic ordinary differential equation (SODE). As is well-known, for fixed x , the real-valued process $t \mapsto p_t(x)$, which is also a diffusion process, then no longer satisfies an SODE. Indeed, if $f(t, T) \equiv p_t(T-t)$, then we have $p_t(x) = f(t, t+x)$ so that for fixed x :

$$d_t p_t(x) = [d_t f(t, T) + \frac{\partial f(t, T)}{\partial T} dT]_{T=t+x} = [d_t f(t, T)]_{T=t+x} + \frac{\partial p_t(x)}{\partial x} dt. \quad (1)$$

Here the right-hand side (r.h.s) depends, not only on $p_t(x)$, but also on its partial derivative with respect to x . So, equation (1) for p is a SPDE, stochastic partial differential equation, where the first term on the r.h.s depends only on the un-known $p_t(x)$, since $f(\cdot, T)$ satisfies an SODE. This is the well-known difficulty of the Musiela parametrization (see [25]), and the space E shall permit a simple mathematical formulation of the SPDE (1).

5. At any time t , the zero-coupon prices $p_t(x)$ should go to zero as the time to maturity x goes to ∞ . To include also the trivial case, where all interest rates vanish, and also cases where the forward rates converges rapidly to zero as $x \rightarrow \infty$, we only require that $\lim p_t(x)$ exists as $x \rightarrow \infty$. N.B. We will chose E such that the elements $f \in E$ satisfying

$\lim_{x \rightarrow \infty} f(x) = 0$ form a closed sub-space of E , in order to cover easily the case where $p_t(x) \rightarrow 0$.

Formula (1) is really an infinite family of coupled equations, one for each $x \geq 0$, describing the motion of the random variable $p_t(x)$, which we write

$$dp_t(x) = p_t(x)m_t(x)dt + p_t(x)\sigma_t(x)dW_t + \frac{\partial p_t(x)}{\partial x}dt, \quad (2)$$

where for the moment W is thought of as being a high dimensional Brownian motion. Let us rewrite it as a single stochastic evolution equation for the motion of the random curve p_t in E , i.e. as a SODE in E :

$$dp_t = p_t m_t dt + p_t \sigma_t dW_t + (\partial p_t) dt \quad (3)$$

where ∂ is the differentiation operator with respect to *time to maturity*, i.e. it is defined by $(\partial u)(x) = \frac{du(x)}{dx}$, for differentiable $u \in E$. Since the left-hand side “belongs” to E , so must the right-hand side, and then ∂p_t must belong to E . There are ways to achieve that. One is to choose a framework where the operator ∂ is continuous over all of E . Then so is its n -th iterate ∂^n , so that the space E must consist of functions which have infinitely many derivatives. Unfortunately, the natural topology of such spaces cannot be defined by a single norm, except for very particular cases, and the mathematics become more demanding. A second more standard way to proceed is to consider ∂ as an unbounded operator in a Hilbert space E , so that ∂ is defined only on a subspace $\mathcal{D}(\partial) \subset E$, called the domain of the operator. One would then hope to define the solution of equation (3) in such a way that, if the initial condition p_0 lies in $\mathcal{D}(\partial)$, then p_t remains in $\mathcal{D}(\partial)$ for every t , so that $t \mapsto p_t$ is a trajectory in $\mathcal{D}(\partial)$. In other words, if $p_0(x)$ is differentiable with respect to x , so should the functions $x \mapsto p_t(x)$ be for all $t > 0$.

To summarize, the introduction of rollovers and a moving frame forces us to complicate the equations for price dynamics, by incorporating an additional term, ∂p_t . To be able to solve the relevant equations, we have to treat ∂ as an unbounded operator in Hilbert space. The definition of the relevant Hilbert space has to incorporate basic properties which we expect of zero-coupon curves.

This suits our purpose well, for it enables us to work in a non-parametric framework, where no particular shape is assigned to the the zero-coupon curves. On the other hand, we then have to use the theory of Brownian motion in infinite-dimensional Hilbert spaces and the corresponding stochastic integrals, which creates some additional difficulties. We do not limit the number of sources of noise, indeed in our paper there can be infinitely many. This

is natural, since the already mentioned experimental fact, that even using a large number of bonds, not all interest rate derivatives can be hedged. The third difficulty to overcome, is the mathematically significant fact that such a market can not be complete in the usual sense, i.e. every (sufficiently integrable) contingent claim being hedgeable. This has important implications for the solution of the portfolio optimization problem. The now classical two-step solution, so successfully applied to the case of a finite number of stocks (cf. [17], [28]), consisting of first determining the optimal final wealth by duality methods and then determining a hedging portfolio, does not (yet at least) apply to the general infinite dimensional bond markets. In this paper (see [10] and [34]) we give, within the considered general diffusion model, the optimal final wealth for every case it exist (Proposition 43). The existence of an optimal portfolio, is then established by the construction of a hedging portfolio for two cases : The first is for deterministic E -valued drift m and volatility operator σ , where we give a necessary and sufficient condition for the existence of an optimal portfolio. Here there can exist several equivalent martingale measures (e.m.m.), so the market can clearly be incomplete in every sense of the word. The second is for certain stochastic m and σ , for which there is a unique market price of risk process γ . There is then a unique e.m.m. Q . Now, certain integrability conditions on the ℓ^2 -valued Malliavin derivative of the Radon-Nikodym density dQ/dP leads to the construction of a hedging portfolio.

We have tempted to make these notes self-contained, with exception of the general hedging result in Theorem 38. The notes first recall some basic facts concerning linear operators and semi-groups in Hilbert spaces, Sobolev spaces and stochastic integration in Hilbert spaces. The theory of bond portfolios and hedging of interest rate derivatives are then introduced. Once this theory is explained, the paper proceeds to a short solution of the optimization problem, leading to the results of [10] and [34]. In particular, under the assumption that the market prices of risk are deterministic, some explicit formulas are given, very similar in spirit to those who are known in the case of stock portfolios, and a mutual fund theorem is formulated. We conclude by stating an alternative formulation, of the optimization problem, within a Hamilton-Jacobi-Bellman approach.

2 Mathematical preliminaries

2.1 Hilbert spaces and bounded maps

We shall be working with separable infinite-dimensional real Hilbert spaces. Let E be a Hilbert space with scalar product $(\cdot, \cdot)_E$ and norm $\|\cdot\|_E$, simply denoted (\cdot, \cdot) and $\|\cdot\|$ if no risk for confusion. The topology and convergence in E is w.r.t. this norm, if not otherwise stated, i.e. the strong topology and convergence. By definition E is, *separable* if it has a countable dense subset. One shows easily that E is separable iff it has a countable orthonormal basis $e_n, n \in \mathbb{N}$, i.e. $(e_i, e_j) = 0$ for $i \neq j$ and $\|e_i\| = 1$, so that every $x \in E$ can be written:

$$x = \sum_{n=0}^{\infty} (x, e_n) e_n,$$

where the right-hand side converges in E . Since the e_n are orthonormal, we have Parseval's equality:

$$\|x\|^2 = \sum_{n=0}^{\infty} |(x, e_n)|^2.$$

A typical separable Hilbert space is ℓ^2 , which is the space of all real sequences $a_n, n \in \mathbb{N}$, such that $\sum |a_n|^2 < \infty$. The scalar product in ℓ^2 is given by $(a, b) = \sum a_n b_n$. In fact, every infinite dimensional separable Hilbert space E is isomorphic to ℓ^2 . The map

$$x \mapsto a_n = (x, e_n)_E, n \in \mathbb{N}, \tag{4}$$

of E into ℓ^2 is a linear bijection and it preserves norms on both sides.

A linear map $L : E_1 \rightarrow E_2$ is continuous if and only if it is *bounded*, that is if there exists a constant c such that $\|Lx\|_{E_2} \leq c \|x\|_{E_1}$ for every $x \in E_1$. The (operator) norm of L is then defined to be the infimum of all such c :

$$\|L\| = \inf \{ c \mid \|Lx\|_{E_2} \leq c \|x\|_{E_1} \ \forall x \}.$$

For example, the linear map in (4) of E onto ℓ^2 as well as its inverse has norm 1. The linear space of all continuous linear maps from E_1 to E_2 , $L(E_1, E_2)$, is a Banach space when given this norm. One writes $L(E)$ as a shorthand for $L(E, E)$. Linear maps are also called linear operators or just operators. A *bounded operator* on E is a bounded linear map from E into itself. The *dual space* E' of E , i.e. the space of all linear continuous functionals on E , is given by $E' = L(E, \mathbb{R})$. By the F. Riesz representation theorem,

$$F \in E' \text{ iff } \exists f \in E \text{ such that } F(x) = (f, x) \ \forall x \in E. \tag{5}$$

Also $\|F\|_{E'} = \|f\|_E$, so E' and E are isomorphic. In this paper we will often use, in the context of Sobolev spaces, other representations of the dual E' .

By duality, every operator in $L(E_1, E_2)$ corresponds to an operator in $L(E_2', E_1')$. Using the representation of the dual space given by (5), the adjoint operator A^* of $A \in L(E_1, E_2)$ is defined by $A^*y = y^*$, where for $y \in E_2$ the element $y^* \in E_1$ is defined by

$$(y^*, x)_{E_1} = (y, Ax)_{E_2} \quad \forall x \in E_1. \quad (6)$$

This defines an operator $A^* \in L(E_2, E_1)$. One easily checks that $(A^*)^* = A$ and $\|A^*\| = \|A\|$. Let us consider a simple example, which will be relevant in the sequel of this paper:

Example 1 (Left-translation in L^2)

i) Let $E = L^2(\mathbb{R})$ and let a be a given real number. Define the operator A on E by $(Af)(x) = f(x + a)$. Then $\|A\| = 1$ and $(A^*f)(x) = f(x - a)$. We note that A has a bounded inverse A^{-1} given by $(A^{-1}f)(x) = f(x - a)$, so $AA^* = A^*A = I$.

ii) Let $E = L^2([0, \infty[)$ and let $a > 0$ be a given real number. Define the operator A on E by $(Af)(x) = f(x + a)$. Here we find that $\|A\| = 1$, that a.e. $(A^*f)(x) = 0$ if $0 \leq x < a$ and that $(A^*f)(x) = f(x - a)$ if $a \leq x$. In this case A^* is one-to-one and $AA^* = I$. But A^*A is the orthogonal projection on the (non-trivial) closed subspace of E of functions with support in $[a, \infty[$. So $A^*A \neq I$.

An operator $S \in L(E_1, E_2)$ is called unitary if $SS^* = S^*S = I$. This is the case of A in (i) of Example 1. An operator $S \in L(E_1, E_2)$ is called isometric if $S^*S = I$. This is the case of A^* in (ii) of Example 1.

We will be interested in a particular class of bounded operators on E . We begin with an easy result

Lemma 2 *Suppose $L \in L(E_1, E_2)$ and that we have:*

$$\sum_{n=0}^{\infty} \|Le_n\|^2 < \infty$$

for an orthonormal basis $e_n, n \in \mathbb{N}$ in E_1 . Let $f_n, n \in \mathbb{N}$ be another orthonormal basis. Then:

$$\sum_{n=0}^{\infty} \|Le_n\|^2 = \sum_{n=0}^{\infty} \|Lf_n\|^2$$

Definition 3 An operator L on E_1 into E_2 is Hilbert-Schmidt if $\sum_{n=0}^{\infty} \|Le_n\|^2 < \infty$ for some orthonormal basis $e_n, n \in \mathbb{N}$, in E_1 . Its Hilbert-Schmidt norm is defined to be:

$$\|L\|_{\mathcal{HS}} = \left(\sum_{n=0}^{\infty} \|Le_n\|^2 \right)^{1/2}.$$

It does not depend on the choice of the orthonormal basis $e_n, n \in \mathbb{N}$, in E . The linear space of Hilbert-Schmidt operators from E_1 into E_2 is denoted $\mathcal{HS}(E_1, E_2)$.

Hilbert-Schmidt operators are bounded (in fact, $\|L\| \leq \|L\|_{\mathcal{HS}}$) and even compact: they map bounded subsets of E_1 into relatively compact subsets of E_2 . In other words, if L is Hilbert-Schmidt and $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence, then one can extract from $(Lx_n)_{n \in \mathbb{N}}$ a norm-convergent subsequence. This property of a Hilbert-Schmidt operator L follows from the fact that L is the limit in the operator norm of finite rank operators. The space $\mathcal{HS}(E_1, E_2)$ endowed with the Hilbert-Schmidt norm defines a Hilbert space.

Some general references for this subsection are: [16], [18], [30], [31].

2.2 Linear semi-groups and unbounded operators.

Let L be a bounded linear operator on E . For every $t \in \mathbb{R}$, define:

$$\Phi(t) = e^{tL} = \sum_{i=0}^{\infty} \frac{1}{i!} t^i L^i,$$

which converges in the operator norm. Then $\Phi(t)$ is a bounded linear operator for every t , and we have the relation:

$$\Phi(t+s) = \Phi(t)\Phi(s) \quad \forall s, t \in \mathbb{R} \quad \text{and} \quad \Phi(0) = I, \quad (7)$$

where I is the identity operator on E , from which it follows that $\Phi(t)$ and $\Phi(s)$ commute and that $\Phi(t)$ is invertible for every t . Relation (7) states that the map $t \mapsto \Phi(t)$ is a group homomorphism. Note that it is continuous in the norm topology for operators:

$$\|\Phi(t) - I\| \rightarrow 0 \quad \text{when} \quad t \rightarrow 0. \quad (8)$$

The solution of the Cauchy problem:

$$\frac{dx}{dt} = Lx, \quad (9)$$

$$x(0) = x_0 \quad (10)$$

is given by $x(t) = \Phi(t)x(0)$. In other words, $\Phi(t)$ is the flow associated with the ordinary differential equation (9). We can recover L from $\Phi(t)$ by writing:

$$Lx = \lim_{h \rightarrow 0} \frac{1}{h} [\Phi(h)x - x] \quad (11)$$

The norm continuity of the mapping $t \mapsto \Phi(t)$ is exceptional and has to be replaced by a more useful weaker property (cf. Definition 1, Sect. 1, Chap. IX of [35]):

Definition 4 A family $\Phi(t), t \geq 0$, of bounded operators on E is called a one parameter semi-group if $\Phi(0) = I$, and for all $t \geq 0$ and $s \geq 0$ we have:

$$\Phi(t+s) = \Phi(t)\Phi(s) = \Phi(s)\Phi(t). \quad (12)$$

It is said to be strongly continuous or to be of class (C_0) if, for every $x \in E$, we have:

$$\lim_{t \rightarrow 0} \Phi(t)x = x. \quad (13)$$

It is said to be a contraction semi-group if $\|\Phi(t)\| \leq 1$ for all $t \geq 0$.

Note that, since equality (12) is supposed to hold only for positive s and t , the operators $\Phi(t)$ are no longer necessarily invertible, as in the case of a group. It can be proved easily that, if the semi-group $\Phi(t)$ is strongly continuous, then $\lim_{s \rightarrow t} \Phi(s)x = \Phi(t)x$ and there are constants c and C such that $\|\Phi(t)\| \leq C \exp(ct)$. We also note that if $[0, \infty[\ni t \mapsto \Phi(t)$ is a one parameter semi-group, so is the family of adjoint operators $[0, \infty[\ni t \mapsto \Phi^*(t)$, where we define $\Phi^*(t) = (\Phi(t))^*$.

Example 5

In the situation of (i) (resp. of (ii)) of Example 1, for given a , let $\Phi_1(a) = A$ (resp. $\Phi_2(a) = A$). Then $\mathbb{R} \ni t \mapsto \Phi_1(t)$ is a strongly continuous contraction group. However $[0, \infty[\ni t \mapsto \Phi_2(t)$ is only a strongly continuous contraction semi group, which can not be extended to a group. In fact, then $\Phi_2(t)$ is not invertible for $t > 0$.

We now try to extend formula (11). It turns out that when Φ is no longer norm-continuous, but only strongly continuous, the right-hand side does not converge for every x , and if the limit exists, it does not depend continuously on x . The set of x for which the limit exists is obviously a linear subspace of E and on this subspace the limit is a linear function, let's say G of x . More formally, let $\mathcal{D}(G)$ be the subset of E of all elements $x \in E$ for which the strong limit

$$Gx = \lim_{h \rightarrow 0} \frac{1}{h} [\Phi(h)x - x] \quad (14)$$

exists.

Theorem 6 Assume Φ is a strongly continuous semi-group. The set $\mathcal{D}(G)$ is then a dense linear subspace of E and G given by (14) defines is a linear map $G : \mathcal{D}(G) \rightarrow E$. This map is closed, i.e. if x_n is a sequence in $\mathcal{D}(G)$ such that $x_n \rightarrow \bar{x} \in E$ and $Gx_n \rightarrow \bar{y} \in E$ then $\bar{y} = G\bar{x}$.

For every $x \in \mathcal{D}(G)$ and $t \geq 0$ we have $\Phi(t)x \in \mathcal{D}(G)$,

$$G\Phi(t)x = \Phi(t)Gx \quad (15)$$

and

$$\frac{d}{dt}\Phi(t)x = G\Phi(t)x. \quad (16)$$

Proof. By definition $\mathcal{D}(G)$ is the set of x where the limit in formula (14) exists (note that this is a strong limit, meaning that we should have norm-convergence), and Gx then is the value of that limit. Clearly $G : \mathcal{D}(G) \rightarrow E$ is a linear map.

Given any $x \in E$ and $t > 0$, consider the integral:

$$X(t) = \int_0^t \Phi(s)x ds.$$

It is well-defined since the integrand is a continuous function from $[0, t]$ into E . Using the semi-group property, we have:

$$\begin{aligned} \frac{1}{h} [\Phi(h)X(t) - X(t)] &= \frac{1}{h} \left[\Phi(h) \int_0^t \Phi(s)x ds - \int_0^t \Phi(s)x ds \right] \\ &= \frac{1}{h} \left[\int_0^t \Phi(s+h)x ds - \int_0^t \Phi(s)x ds \right] \\ &= \frac{1}{h} \int_t^{t+h} \Phi(s+h)x ds - \frac{1}{h} \int_0^h \Phi(s)x ds \\ &\rightarrow \Phi(t)x - x. \end{aligned}$$

This proves that $X(t)$ belongs to $\mathcal{D}(G)$. Then so does $\frac{1}{t}X(t)$, and when $t \rightarrow 0$, we have $\frac{1}{t}X(t) \rightarrow x$, so $\mathcal{D}(G)$ is dense in H , as announced.

Now write:

$$\frac{1}{h} [\Phi(t+h) - \Phi(t)]x = \Phi(t) \frac{\Phi(h) - I}{h} x = \frac{\Phi(h) - I}{h} \Phi(t)x.$$

If $x \in \mathcal{D}(G)$, the second term converges to $\Phi(t)Gx$ and the third one to $G\Phi(t)x$. So these two terms must be equal. Formulas (15) and (16) follow.

To prove the last condition, note that:

$$\forall x \in \mathcal{D}(G), \quad \Phi(t)x - x = \int_0^t \Phi(s)Gx ds. \quad (17)$$

Indeed, we have two functions of t , with values in E , which are zero for $t = 0$ and which have the same derivative, namely $\Phi(t)Gx$, for every $t > 0$. So they must be equal. Now take a sequence $x_n \rightarrow \bar{x}$, and assume that $Gx_n = y_n \rightarrow \bar{y}$ in E . Writing $x = x_n$ in formula (17), we get:

$$\Phi(t)\bar{x} - \bar{x} = \int_0^t \Phi(s)\bar{y}ds.$$

Dividing by t and letting $t \rightarrow 0$, we find that $\bar{x} \in \mathcal{D}(G)$ and that $\bar{y} = G\bar{x}$. ■

Definition 7 *In the situation of Theorem 6, G is called the infinitesimal generator of the semi-group Φ .*

A linear map $L : \mathcal{D}(L) \rightarrow E_2$, where $\mathcal{D}(L)$ is a subspace of E_1 , is called an operator from E_1 to E_2 with domain $\mathcal{D}(L)$. That two operators are equal, $L_1 = L_2$, means that they have the same domain $\mathcal{D}(L_1) = \mathcal{D}(L_2)$ and that $L_1x = L_2x$ for all x in the domain. The operator L is densely defined if $\mathcal{D}(L)$ is dense in E_1 . It is called a bounded operator if there exists a finite constant $C \geq 0$ such that for all $x \in \mathcal{D}(L)$ one has $\|Lx\| \leq C\|x\|$ and it is called an *unbounded operator* if such C does not exist. It is *closed* if its graph $\{(x, Lx) \mid x \in \mathcal{D}(L)\}$ is a closed subset of $E_1 \times E_2$, which extends the definition in the preceding theorem. With these definitions, we can rephrase part of the preceding theorem by saying that every strongly continuous semi-group in E has a unique infinitesimal generator, which is a densely defined closed operator in E . The problem to determine if a given densely defined closed operator L in E is the infinitesimal generator of a strongly continuous semi-group is more difficult and we refer the interested reader to the references mentioned in the end of this subsection.

The definition of the *adjoint* of an operator can be extended to unbounded operators. Let L be a densely defined operator from E_1 to E_2 . We introduce the adjoint operator L^* to L . The domain of $\mathcal{D}(L^*)$ consists of all $y \in E_2$ for which the linear functional

$$x \mapsto (y, Lx) \tag{18}$$

is continuous on $\mathcal{D}(L)$, endowed with the strong topology of E_1 . For $y \in \mathcal{D}(L^*)$ we define L^*y by

$$(L^*y, x) = (y, Lx) \quad \forall x \in \mathcal{D}(L). \tag{19}$$

This defines L^*y uniquely, since $\mathcal{D}(L)$ is dense in E_1 . One proves that $\mathcal{D}(L^*)$ is dense in E_2 if L is also closed.

An operator L in E is called *selfadjoint* if $L^* = L$ and *skew-adjoint* if $L^* = -L$. We have the following clear-cut result (Stone's theorem): L is the infinitesimal generator of a group of unitary operators iff L is skew-adjoint.

Example 8

In the situation of Example 5, let L_1 and L_2 be the infinitesimal generators of Φ_1 and Φ_2 respectively. L_1 is given by

$$\mathcal{D}(L_1) = \{f \in L^2(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\},$$

and $(L_1 f)(x) = f'(x)$, where f' is the derivative of f . L_2 is given by

$$\mathcal{D}(L_2) = \{f \in L^2([0, \infty[) \mid f' \in L^2([0, \infty[)\},$$

and $(L_2 f)(x) = f'(x)$. Since Φ_1 is a group of unitary operators, we have that $L_1^* = -L_1$. Φ_2 is not a semi-group of unitary operators, so $L_2^* \neq -L_2$. A simple calculation shows that

$$\mathcal{D}(L_2^*) = \{f \in L^2([0, \infty[) \mid f(0) = 0 \text{ and } f' \in L^2([0, \infty[)\}$$

and $(L_2^* f)(x) = -f'(x)$. So here $\mathcal{D}(L_2^*) \subset \mathcal{D}(L_2)$, with strict inclusion. One checks that Φ_2^* is a strongly continuous semi-group in $L^2([0, \infty[)$. It represents right translations of functions. Its infinitesimal generator is L_2^* .

Some general references for this subsection are: [16], [18], [31], [35].

2.3 Sobolev spaces

For any integer $n \geq 0$, the Sobolev space $H^n(\mathbb{R})$ is defined to be the set of functions f which are square-integrable together with all their derivatives of order up to n :

$$f \in H^n(\mathbb{R}) \iff \int_{-\infty}^{\infty} \left[f^2 + \sum_{k=1}^n \left(\frac{d^k f}{dx^k} \right)^2 \right] dx < \infty.$$

This is a linear space, and in fact a Hilbert space with norm given by:

$$\|f\|_{H^n}^2 = \int_{-\infty}^{\infty} \left[f^2 + \sum_{k=1}^n \left(\frac{d^k f}{dx^k} \right)^2 \right] dx.$$

It is a standard fact that this norm of f can be expressed in terms of the Fourier transform \hat{f} (appropriately normalized) of f by:

$$\|f\|_{H^n}^2 = \int_{-\infty}^{\infty} (1 + y^2)^n |\hat{f}(y)|^2 dy.$$

The advantage of that new definition is that it can be extended to non-integral and non-positive values. For any real number s , not necessarily an

integer nor positive, we define the Sobolev space $H^s(\mathbb{R})$ to be the Hilbert space of functions associated with the following norm:

$$\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1+y^2)^s \left| \hat{f}(y) \right|^2 dy. \quad (20)$$

Clearly, $H^0(\mathbb{R}) = L^2(\mathbb{R})$ and $H^s(\mathbb{R}) \subset H^{s'}(\mathbb{R})$ for $s \geq s'$ and in particular $H^s(\mathbb{R}) \subset L^2(\mathbb{R}) \subset H^{-s}(\mathbb{R})$, for $s \geq 0$. $H^s(\mathbb{R})$ is, for general $s \in \mathbb{R}$, a space of (tempered) distributions. For example $\delta^{(k)}$, the k -th derivative of a delta Dirac distribution, is in $H^{-k-1/2-\epsilon}(\mathbb{R})$ for $\epsilon > 0$.

In the case when $s > 1/2$, there are two classical results.

Theorem 9 (Continuity of multiplication) *If $s > 1/2$, if f and g belong to $H^s(\mathbb{R})$, then fg belongs to $H^s(\mathbb{R})$, and the map $(f, g) \rightarrow fg$ from $H^s \times H^s$ to H^s is continuous.*

Denote by $C_b^n(\mathbb{R})$ the space of n times continuously differentiable real-valued functions which are bounded together with all their n first derivatives. Let $C_{b0}^n(\mathbb{R})$ the closed subspace of $C_b^n(\mathbb{R})$ of functions which converges to 0 at $\pm\infty$ together with all their n first derivatives. These are Banach spaces for the norm:

$$\|f\|_{C_b^n} = \max_{0 \leq k \leq n} \sup_x |f^{(k)}(x)| = \max_{0 \leq k \leq n} \|f^{(k)}\|.$$

Theorem 10 (Sobolev embedding) *If $s > n + 1/2$ and if $f \in H^s(\mathbb{R})$, then there is a function g in $C_{b0}^n(\mathbb{R})$ which is equal to f almost everywhere. In addition, there is a constant c_s , depending only on s , such that:*

$$\|g\|_{C_b^n} \leq c_s \|f\|_{H^s}.$$

From now on we shall no longer distinguish between f and g , that is, we shall always take the continuous representative of any function in $H^s(\mathbb{R})$. As a consequence of the Sobolev embedding theorem, if $s > 1/2$, then any function f in $H^s(\mathbb{R})$ is continuous and bounded on the real line and converges to zero at $\pm\infty$, so that its value is defined everywhere.

We define, for $s \in \mathbb{R}$, a continuous bilinear form on $H^{-s}(\mathbb{R}) \times H^s(\mathbb{R})$ by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \overline{\hat{f}(y)} \hat{g}(y) dy, \quad (21)$$

where \bar{z} is the complex conjugate of z . Schwarz inequality and (20) give that

$$|\langle f, g \rangle| \leq \|f\|_{H^{-s}} \|g\|_{H^s}, \quad (22)$$

which indeed shows that the bilinear form in (21) is continuous. We note that formally the bilinear form (21) can be written

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx,$$

where, if $s \geq 0$, f is in a space of distributions $H^{-s}(\mathbb{R})$ and g is in a space of “test functions” $H^s(\mathbb{R})$.

Any continuous linear form $g \rightarrow u(g)$ on $H^s(\mathbb{R})$ is, due to (20), of the form $u(g) = \langle f, g \rangle$ for some $f \in H^{-s}(\mathbb{R})$, with $\|f\|_{H^{-s}} = \|u\|_{(H^s)'}$, so that henceforth we can identify the dual $(H^s(\mathbb{R}))'$ of $H^s(\mathbb{R})$ with $H^{-s}(\mathbb{R})$. In particular, if $s > 1/2$ then $H^s(\mathbb{R}) \subset C_{b0}^0(\mathbb{R})$, so $H^{-s}(\mathbb{R})$ contains all bounded Radon measures.

In the sequel, we will also be interested in functions defined only on the half-line $[0, \infty[$. Let $s \geq 0$. We define the space $H^s([0, \infty[)$ to be the set of restrictions to $[0, \infty[$ of functions in $H^s(\mathbb{R})$. This is clearly a linear space. To turn it into a Hilbert space, we have to use the following norm:

$$\|f\|_{H^s([0, \infty))} = \inf \{ \|g\|_{H^s(\mathbb{R})} \mid g(x) = f(x) \text{ a.e. on } [0, \infty) \}. \quad (23)$$

This is a Hilbert space norm on $H^s([0, \infty[)$, which is the natural restriction of the norm on $H^s(\mathbb{R})$. For instance, if f is a function in $H^s(\mathbb{R})$ such that $f(x) = 0$ for $x \leq 0$, then its restriction f_0 to $[0, \infty[$ belongs to $H^s([0, \infty[)$, and we have:

$$\|f_0\|_{H^s([0, \infty[)} = \|f\|_{H^s(\mathbb{R})}$$

If $s = n$ is an integer, the norm on $H^s([0, \infty[)$ turns out to be equivalent to the following one:

$$\|f\|_{H^s}^2 = \int_0^{\infty} \left[f^2 + \sum_{k=1}^n \left(\frac{d^k f}{dx^k} \right)^2 \right] dx.$$

To establish properties of translations in $H^s([0, \infty[)$, we need to know if there is a continuous linear embedding of $H^s([0, \infty[)$ into $H^s(\mathbb{R})$, i.e. to know if the restriction operator has a continuous right-inverse. Fortunately, as we are in a Hilbert space setting, this problem is easy to solve. Let $s \geq 0$ and let H_-^s be the subset of functions in $H^s(\mathbb{R})$ with support in $] - \infty, 0]$, so that $f \in H_-^s$ if and only if $f \in H^s(\mathbb{R})$ and $f(x) = 0$ for all $x > 0$. H_-^s is a closed subspace of $H^s(\mathbb{R})$. Two functions $f_1, f_2 \in H^s(\mathbb{R})$ have the same restriction to $[0, \infty[$ iff $f_1 - f_2 \in H_-^s$. This means exactly that $H^s([0, \infty[)$ is a quotient space: $H^s([0, \infty[) = H^s(\mathbb{R})/H_-^s$. Introducing the notation \oplus for the Hilbert space direct sum, we have the following result, which proof we omit since its trivial:

Proposition 11 For $s \geq 0$ we have:

i) $H^s(\mathbb{R}) = H^s([0, \infty[) \oplus H^s_-$.

ii) Let M be the orthogonal complement of H^s_- in $H^s(\mathbb{R})$ w.r.t. the scalar product in $H^s(\mathbb{R})$, let κ be the canonical projection of $H^s(\mathbb{R})$ on $H^s([0, \infty[)$ and let ι be the canonical bijection of $H^s([0, \infty[)$ onto M . Then κ is continuous, ι is a Hilbert space isomorphism, $\kappa\iota$ is the identity map on $H^s([0, \infty[)$ and $\iota\kappa$ is the orthogonal projection map in $H^s(\mathbb{R})$ on M .

We note that ι is a continuous operator extending functions on $[0, \infty[$ to functions on \mathbb{R} and that $\|f\|_{H^s([0, \infty[)} = \|\iota f\|_{H^s(\mathbb{R})}$.

The dual space of $H^s([0, \infty[)$ can easily be characterized in terms of distributions. For $s \geq 0$, $H^s([0, \infty[) = H^s(\mathbb{R})/H^s_-$, so

$$(H^s([0, \infty[))' = \{f \in H^{-s}(\mathbb{R}) \mid \langle f, g \rangle = 0 \ \forall g \in H^s_-\}. \quad (24)$$

For $s \geq 0$, we define $H^{-s}([0, \infty[)$ to be the closed subspace of all distributions in $H^{-s}(\mathbb{R})$ with support in $[0, \infty[$. It then follows that $(H^s([0, \infty[))'$ can be identified with $H^{-s}([0, \infty[)$. Since $(H^s([0, \infty[))'' = H^s([0, \infty[)$, it then follows that

$$(H^s([0, \infty[))' = H^{-s}([0, \infty[) \quad s \in \mathbb{R}. \quad (25)$$

If $s \in \mathbb{R}$, then the constant function taking the value 1 is not in $H^s([0, \infty[)$. If $s > 1/2$, then even every function in $H^s([0, \infty[)$ converges to zero at ∞ . For this reason, we will need a larger class of distributions containing the constant functions. Let $s \in \mathbb{R}$ and let f be a distribution with support in $[0, \infty[$ such that it admits the decomposition $f = g + a$, where $g \in H^s([0, \infty[)$ and $a \in \mathbb{R}$. This decomposition of f is then unique and the set of all such distributions is naturally given the Hilbert space structure $H^s([0, \infty[) \oplus \mathbb{R}$. The norm of $f = g + a$ is then given by

$$\|f\|^2 = \|g\|_{H^s([0, \infty[)}^2 + a^2.$$

This unique decomposition property leads us to the following

Definition 12 For $s \in \mathbb{R}$, set $E^s([0, \infty[) = H^s([0, \infty[) \oplus \mathbb{R}$ with the corresponding Hilbert space norm. If $f \in E^s([0, \infty[)$ and if $g \in H^s([0, \infty[)$ and $a \in \mathbb{R}$ are related by the unique decomposition $f = g + a$, then the norm of f is given by

$$\|f\|_{E^s}^2 = \|g\|_{H^s}^2 + a^2.$$

The dual $(E^s([0, \infty[))'$, of $E^s([0, \infty[)$ is identified with $(H^s([0, \infty[))' \oplus \mathbb{R} \approx E^{-s}([0, \infty[)$ by extending the bi-linear form, defined in (21), to $E^{-s}([0, \infty[) \times E^s([0, \infty[)$:

$$\langle F, G \rangle = ab + \langle f, g \rangle, \quad (26)$$

where $F = a + f \in E^{-s}([0, \infty[)$, $G = b + g \in E^s([0, \infty[)$, $a, b \in \mathbb{R}$, $f \in H^{-s}([0, \infty[)$ and $g \in H^s([0, \infty[)$.

For all the Sobolev spaces H^s we have introduced, and also for the spaces E^s , there are two natural realizations of the dual space. Let us consider only the case of $E^s([0, \infty[)$, the other being similar. One possibility, the canonical one, is to identify $(E^s([0, \infty[))'$ with $E^s([0, \infty[)$ by the scalar product in $E^s([0, \infty[)$. This gives the Riesz representation in (5). Another possibility, is, as we have seen, to identify $(E^s([0, \infty[))'$ with $E^{-s}([0, \infty[)$, by the bi-linear form defined in (26). There is a linear continuous map $\mathcal{S} : E^s([0, \infty[) \rightarrow (E^s([0, \infty[))'$ with continuous inverse, relating the two realizations. It is defined by:

$$(f, g)_{E^s([0, \infty[)} = \langle \mathcal{S}f, g \rangle, \forall f, g \in E^s([0, \infty[). \quad (27)$$

Now, different realizations of the dual space leads to different realizations of adjoint operators. Let A be a closed and densely defined operator from a Hilbert-space H to $E^s([0, \infty[)$. We have already defined in (18) its adjoint operator A^* from $E^s([0, \infty[)$ to H w.r.t. the duality defined by the scalar product. Let the dual H' of H be realized by H_1 and the continuous bi-linear form $\langle \cdot, \cdot \rangle_1 : H_1 \times H \rightarrow \mathbb{R}$. The adjoint A' , w.r.t. the duality realized by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle$ is the operator from $E^{-s}([0, \infty[)$ to H_1 , defined by: The domain of $\mathcal{D}(A')$ consists of all $y \in E^{-s}([0, \infty[)$ for which the linear functional

$$x \mapsto \langle y, Ax \rangle \quad (28)$$

is continuous on $\mathcal{D}(A)$. For $y \in \mathcal{D}(A')$ we define $A'y$ by

$$\langle A'y, x \rangle_1 = \langle y, Ax \rangle \quad \forall x \in \mathcal{D}(A). \quad (29)$$

This defines $A'y$ uniquely, since $\mathcal{D}(A)$ is dense in H .

We now study translation semi-groups in the different spaces we have introduced. It follows directly from the definition (20) of the norm in $H^s(\mathbb{R})$ and by dominated convergence that that left-translations defines a strongly continuous group of unitary operators $\tilde{\mathcal{L}}$ in $H^s(\mathbb{R})$ for $s \in \mathbb{R}$ (similarly as to the case of Φ_1 in Example 5):

$$(\tilde{\mathcal{L}}_t f)(x) = f(x + t), \quad \forall f \in H^s(\mathbb{R}) \text{ and } t \in \mathbb{R}. \quad (30)$$

Since, for $s \geq 0$, the closed subspace H^s_- of $H^s(\mathbb{R})$ is invariant under the semi-group $\tilde{\mathcal{L}}_t$, $t \geq 0$, it defines a semi-group \mathcal{L} in $H^s([0, \infty[)$. Defining \mathcal{L} also on constants $a \in \mathbb{R}$ by $\mathcal{L}_t a = a$ we extend the semi-group \mathcal{L} to $E^s([0, \infty[)$, $s \geq 0$:

$$(\mathcal{L}_t f)(x) = f(x + t), \quad \forall f \in E^s([0, \infty[) \text{ and } t \geq 0. \quad (31)$$

Proposition 13 *If $s \geq 0$, then \mathcal{L} is a strongly continuous contraction semi-group on $E^s([0, \infty[)$. Its infinitesimal generator, denoted ∂ , has domain $\mathcal{D}(\partial) = E^{s+1}([0, \infty[)$. If $f \in E^{s+1}([0, \infty[)$ then $\partial f = f'$, where f' is the derivative of f .*

Proof. We first observe that, in the canonical decomposition $E^s([0, \infty[) = H^s([0, \infty[) \oplus \mathbb{R}$ in Definition 12, \mathcal{L}_a leaves the subspace $H^s([0, \infty[)$ invariant and acts trivially on \mathbb{R} . It is therefore sufficient to prove the statement with $E^s([0, \infty[)$ replaced by $H^s([0, \infty[)$.

We use the notations of Proposition 11 and let $P = \iota\kappa$ be the orthogonal projection on M . Since $\tilde{\mathcal{L}}_t H_-^s \subset H_-^s$, for $t \geq 0$, it follows that $P\tilde{\mathcal{L}}_t(I-P) = 0$. The group composition law $\tilde{\mathcal{L}}_t\tilde{\mathcal{L}}_u = \tilde{\mathcal{L}}_{t+u}$, then gives for $t, u \geq 0$:

$$(P\tilde{\mathcal{L}}_tP)(P\tilde{\mathcal{L}}_uP) = P\tilde{\mathcal{L}}_{t+u}P$$

So, $[0, \infty[\ni t \mapsto P\tilde{\mathcal{L}}_tP$ is a semi-group of bounded operators on M . It is a strongly continuous contraction semi-group since this is the case for $\tilde{\mathcal{L}}$ and $\|P\| = 1$.

We have that $\mathcal{L}_t = \kappa\tilde{\mathcal{L}}_t\iota$, for $t \geq 0$. Using that $\mathcal{L}_t = \kappa P\tilde{\mathcal{L}}_tP\iota$ it easily follows from the semi-group property of $P\tilde{\mathcal{L}}_tP$ that \mathcal{L} is a semi-group on $H^s([0, \infty[)$. It is a strongly continuous contraction semi-group, since this is the case for $P\tilde{\mathcal{L}}P$ and since $\|\kappa\| = \|\iota\| = 1$. Let ∂ be the infinitesimal generator of \mathcal{L} . By the definition of \mathcal{L} it follows that $\mathcal{D}(\partial) = \{f \in H^s([0, \infty[) \mid f' \in H^s([0, \infty[)\}$ and $\partial f = f'$ for $f \in \mathcal{D}(\partial)$. But $H^{s+1}([0, \infty[) = \{f \in H^s([0, \infty[) \mid f' \in H^s([0, \infty[)\}$, which proves the proposition. ■

Example 14

Let $\mathcal{L}'_t : E^{-s}([0, \infty[) \rightarrow E^{-s}([0, \infty[)$ be the adjoint of \mathcal{L}_t , $t \geq 0$ in Proposition 13, w.r.t. duality defined by the bilinear form $\langle \cdot, \cdot \rangle$. \mathcal{L}' is then a semi-group of right-translations on the space of distributions $E^{-s}([0, \infty[)$. Loosely speaking $(\mathcal{L}'_t f)(x) = f(x - t)$. Let $s \geq 1$. Then the generator ∂' has domain $E^{-s+1}([0, \infty[)$ and $-\partial'$ is the derivative of distributions, so $(\partial' f)(x) = -df(x)/dx$ if f is a differentiable function. One is easily convinced that the expressions for \mathcal{L}'_t and ∂'^* are more complicated.

Some general references for this subsection are: [1], [5], [13].

2.4 Infinite-dimensional Brownian motion

In this sub-section we consider a separable Hilbert space E and an index-set \mathbb{I} with the cardinality equal to the dimension of E . The space E can be

infinite-dimensional or finite-dimensional. There is given a family $W^i, i \in \mathbb{I}$ of standard independent Brownian motions on a complete filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{A})$. The filtration $\mathcal{A} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, is generated by the W^i , and $\mathcal{F} = \mathcal{F}_T$.

Definition 15 *A standard cylindrical Brownian motion $W_t, 0 \leq t \leq T$, on E is a sequence $e_i W_t^i, i \in \mathbb{I}$ of E -valued processes, where the e_i are the elements of an orthonormal basis of E and the $W_t^i, i \in \mathbb{I}$, are independent real-valued standard Brownian motions on a filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{A})$.*

From now on, given a standard cylindrical Brownian motion W , we shall write informally $W_t = \sum_{i \in \mathbb{I}} W_t^i e_i$. If \mathbb{I} is finite, we have:

$$\|W_t\|^2 = \sum_{i \in \mathbb{I}} \|W_t^i\|^2 < \infty \text{ a.s.}$$

and W_t is a stochastic process with values in E . If \mathbb{I} is infinite, then for every t the right-hand side is the sum of infinitely many i.i.d. positive random variables, which does not converge in any reasonable way. In that case, the formula $W_t = \sum_{i \in \mathbb{I}} W_t^i e_i$ cannot be understood as an equality in E , and must be given another meaning.

Proposition 16 *If $W_t = \sum_{i \in \mathbb{I}} W_t^i e_i$ is a standard cylindrical Brownian motion, then, for every $f \in E$ with $\|f\| = 1$, the real-valued stochastic process W_t^f defined by*

$$W_t^f = \sum_{i \in \mathbb{I}} (e_i, f) W_t^i \tag{32}$$

is a standard Brownian motion on the real line.

Proof. If \mathbb{I} is finite, the result is obvious. Let us then consider the case when $\mathbb{I} = \mathbb{N}$. We first have to check if the right-hand side is well-defined. By Doob's inequality for martingales:

$$\begin{aligned} E \left[\sup_{0 \leq t \leq T} \left| \sum_{i=n}^{n+p} (e_i, f) W_t^i \right|^2 \right] &\leq 4E \left[\left| \sum_{i=n}^{n+p} (e_i, f) W_T^i \right|^2 \right] \\ &\leq 4T \sum_{i=n}^{n+p} (e_i, f)^2 \rightarrow 0. \end{aligned}$$

This implies that the right-hand side of (32) converges in probability to a continuous process. Since each finite sum is Gaussian, so is the limit, and

the result follows. ■

So, in the case when \mathbb{I} is infinite, the r.h.s. of $W_t = \sum_{i \in \mathbb{I}} W_t^i e_i$ makes no sense in E , but every projection does. Equation (32) can be rewritten as:

$$\forall f \in E, \quad (W_t, f) = \sum_{i \in \mathbb{I}} (e_i, f) W_t^i.$$

We will now show that the stochastic integrals with respect to cylindrical Brownian motion make sense, provided the integrand satisfies a strong integrability condition. Consider the space $\mathcal{HS}(E, F)$ of all Hilbert-Schmidt operators from E into a Hilbert space F . Let the space $\mathcal{L}^2(\mathcal{HS}(E, F))$ consist of all progressively measurable processes A with values in the Hilbert space $\mathcal{HS}(E, F)$, such that:

$$E \left[\int_0^T \|A_t\|_{\mathcal{HS}}^2 dt \right] < \infty.$$

Recall that we have, according to Definition 3:

$$\|A_t\|_{\mathcal{HS}}^2 = \sum_{n=0}^{\infty} \|A_t e_n\|^2,$$

where $(e_n)_{n \in \mathbb{N}}$ is any orthonormal basis of E .

Theorem 17 *The stochastic integral:*

$$\int_0^T A_t dW_t$$

is well-defined for every process $A \in \mathcal{L}^2(\mathcal{HS}(E, F))$. It is a continuous martingale with values in E , and we have the usual isometry:

$$\left\| \int_0^T A_t dW_t \right\|_{L^2}^2 = \int_0^T E [\|A_t\|_{\mathcal{HS}}^2] dt.$$

In other words, the random variable $\int_0^T A_t dW_t$ has mean 0 and its variance is $\sum_{n=0}^{\infty} \int_0^T E [\|A_t e_n\|^2] dt$, the sum of the variances of the independent sources of Gaussian noise.

As usual, by localization the stochastic integral can be extended to a wider class of processes. Denote by $\mathcal{L}_{loc}^2(\mathcal{HS})$ the set of all progressively measurable processes with values in \mathcal{HS} , such that:

$$P \left[\int_0^T \|\Phi\|_{\mathcal{HS}}^2 dt < \infty \right] = 1.$$

Then the stochastic integral defines a continuous local martingale.

Some general references for this subsection are: [7], [14], [23], [24], [26].

3 The dynamics of bond prices

3.1 The non-parametric framework

From now on, and for the rest of the paper, we are given a finite time interval of possible trading times $\mathbb{T} = [0, \bar{T}]$ and we are given a family W^i , $i \in \mathbb{I}$ of standard independent Brownian motions on a complete filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{A})$, the filtration $\mathcal{A} = \{\mathcal{F}_t\}_{0 \leq t \leq \bar{T}}$, is generated by the W^i , and $\mathcal{F} = \mathcal{F}_{\bar{T}}$. The family \mathbb{I} itself can be finite or infinite, in which case we take $\mathbb{I} = \mathbb{N}$. Let $\ell^2(\mathbb{I})$, be the Hilbert space of all real sequences $x = (x_i)_{i \in \mathbb{I}}$, such that $\|x\|_{\ell^2(\mathbb{I})} = (\sum_{i \in \mathbb{I}} (a_n)^2)^{1/2} < \infty$. So, when \mathbb{I} has a finite number \bar{m} of elements, then $\ell^2(\mathbb{I}) = \mathbb{R}^{\bar{m}}$. Often we write just ℓ^2 for $\ell^2(\mathbb{I})$.

Heath, Jarrow and Morton (henceforth HJM) were the first to study the term structure of interest rates in a non-parametric framework. Their basic idea (see [12]) consists of writing one equation for the price of every zero-coupon at time t . Denoting by $\hat{B}_t(T)$ the price at time t of a zero-coupon bond maturing at time $T \geq t$, the HJB equation has the following form:

$$\hat{B}_t(T) = \hat{B}_0(T) + \int_0^t \hat{B}_s(T) a_s(T) ds + \int_0^t \sum_{i \in \mathbb{I}} \hat{B}_s(T) v_s^i(T) dW_s^i, \quad 0 \leq t \leq T \quad (33)$$

There are infinitely many such equations, one for each maturity $T \geq t$.

The trend $a_t(T)$ and the volatilities $v_t^i(T)$ are supposed to be progressively measurable processes, which means, for instance, that they could be functions of all the $\hat{B}_s(S)$, $S \geq s$ and $s \leq t$. In due course, we will make further assumptions so as to ensure that equations such as (33) make mathematical sense.

Let us discount all prices to $t = 0$, by the spot interest rate r_t , which in terms of the zero-coupon bond price is given by

$$r_t = - \frac{\partial \hat{B}_t(T)}{\partial T} \Big|_{T=t}. \quad (34)$$

The discounted prices of zero-coupons are now:

$$B_t(T) = \hat{B}_t(T) \exp\left(- \int_0^t r_s ds\right) \quad (35)$$

and the equations (33) become:

$$B_t(T) = B_0(T) + \int_0^t B_s(T) (a_s(T) - r_s) ds + \int_0^t \sum_{i \in \mathbb{I}} B_s(T) v_s^i(T) dW_s^i, \quad 0 \leq t \leq T \quad (36)$$

and, again, there is one such equation for every maturity $T \geq t$. Note the boundary condition $\hat{B}_T(T) = 1$, and hence, from (35):

$$B_t(t) = \exp\left(-\int_0^t r_s ds\right). \quad (37)$$

3.2 The bond dynamics in the moving frame

For every $x \geq 0$, we denote by $\hat{p}_t(x)$ the price and by $p_t(x)$ the discounted price at time t of a zero-coupon maturing at time $t + x$. The stochastic processes $B_t(T)$ and $p_t(x)$ are related by:

$$p_t(x) = B_t(t + x).$$

In other words, as explained in the introduction, instead of dating events by their distance from a fixed origin, defined to be $t = 0$, we are dating them by their distance from today: we are using a time frame which moves with the observer. The equation for p_t in the moving frame, is easily obtained from (36). For every $x \geq 0$, we have:

$$p_t(x) = p_0(t+x) + \int_0^t p_s(t-s+x)m_s(t-s+x)ds + \int_0^t \sum_{i \in \mathbb{I}} p_s(t-s+x)\sigma_s^i(t-s+x)dW_s^i, \quad (38)$$

where

$$m_t(x) = a(t, t+x) - r_t \quad \text{and} \quad \sigma_t^i(x) = v^i(t, t+x), \quad (39)$$

for all $0 \leq t \leq \bar{T}$ and $x \geq 0$. Here, again, the trends $t \mapsto m_t(x)$ and the volatilities $t \mapsto \sigma_t^i(x)$ are progressively measurable processes.

Instead of looking at (38) as an infinite family of coupled equations, one for each $x \geq 0$, we shall interpret it as a single equation describing the dynamics of an infinite-dimensional object, the curve $x \mapsto p_t(x)$, which will be seen as a vector p_t in the Hilbert space $E^s([0, \infty[)$, for some fixed $s > 1/2$, chosen so that the functions m_t and σ_t^i belong to $E^s([0, \infty[)$.

Let \mathcal{L} be the semi-group left translations on $E^s([0, \infty[)$ (see formula (31) and Proposition 13). From now on we shall just write E^s instead of $E^s([0, \infty[)$, when there is no risk of confusion. The equations in (38) can be rewritten as one equation in E^s :

$$p_t = \mathcal{L}_t p_0 + \int_0^t (\mathcal{L}_{t-s}(p_s m_s))ds + \int_0^t \sum_{i \in \mathbb{I}} (\mathcal{L}_{t-s}(p_s \sigma_s^i))dW_s^i. \quad (40)$$

Theorem 18 *Let $s > 1/2$. Assume that $p_0 \in E^s$ and assume that m_t and the σ_t^i , $i \in \mathbb{I}$, are progressively measurable processes in E^s satisfying:*

$$\int_0^{\bar{T}} (\|m_t\|_{E^s} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^s}^2) dt < \infty \quad a.s. \quad (41)$$

Then equation (40) defines a unique process p in E^s satisfying:

$$\int_0^{\bar{T}} (\|p_t\|_{E^s} + \|p_t m_t\|_{E^s} + \sum_{i \in \mathbb{I}} \|p_t \sigma_t^i\|_{E^s}^2) dt < \infty \quad a.s. \quad (42)$$

The process p has continuous trajectories in E^s ,

$$p_t = \exp \left\{ \int_0^t \mathcal{L}_{t-s} \left(m_s - \frac{1}{2} \sum_{i \in \mathbb{I}} (\sigma_s^i)^2 \right) ds + \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i \right\} \mathcal{L}_t p_0. \quad (43)$$

and if $p_0 \in H^s$ then the process p takes its values in H^s . If $p_0 \in E^s$ satisfies $p_0 \geq 0$ (resp. $p_0 > 0$), i.e. $p_0(x) \geq 0$ (resp. $p_0(x) > 0$) for all $x \geq 0$, then so does p_t .

For a proof of this theorem see Lemma A.1 of [10], which is reproduced in the appendix of this article (Lemma 48). Note that equation (40) implies that p_0 is the value of p_t for $t = 0$.

A word here about the choice of function spaces. Assuming that p_t belongs to H^s for some $s > 1/2$ is minimal: it is basically saying that the zero-coupon prices depend continuously on time to maturity and go to zero at infinity. This, however, is too strong a requirement for m_t and the σ_t^i : we cannot expect the trend and the volatilities to go to zero when the time to maturity increases to infinity. This is why we are assuming that m_t and the σ_t^i belong to E^s . To simplify the mathematical formalism and also to include interest rate models, with vanishing long term rates, we have permitted that $p_t \in E^s$. Now according to Theorem 18, p_t is in-fact in H^s if $p_0 \in H^s$.

Condition (41) implies that $\sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^s}^2$ is finite for almost every $(t, \omega) \in \mathbb{T} \times \Omega$. This means, when $\mathbb{I} = \mathbb{N}$, that the operator σ_t from $\ell^2(\mathbb{I})$ to E^s defined by:

$$\sigma_t e_i = \sigma_t^i, \quad i \in \mathbb{I}, \quad (44)$$

where e_i are the elements of the standard basis of $\ell^2(\mathbb{I})$, is Hilbert-Schmidt a.e. (t, ω) . We have

$$\|\sigma_t\|_{\mathcal{HS}(\ell^2, E^s)}^2 = \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^s}^2.$$

We shall refer to σ as the *volatility operator* process. It takes its values in $\mathcal{HS}(\ell^2, E^s)$ and when we say that it is progressively measurable, it is meant that all the σ^i are progressively measurable.

We can now, using the stochastic integral introduced in Theorem 17, rewrite equation (40) on a more compact form in E^s , where $s > 1/2$:

$$p_t = \mathcal{L}_t p_0 + \int_0^t \mathcal{L}_{t-s}(p_s m_s) ds + \int_0^t \mathcal{L}_{t-s}(p_s \sigma_s) dW_s. \quad (45)$$

This makes sense in E^s . Indeed, the only difference with equation (40) is the last term on the r.h.s. When condition (41) is satisfied then the volatility operator σ_u , defined by (44), from ℓ^2 to E^s , is Hilbert-Schmidt a.e. (u, ω) . Since pointwise multiplication of functions in E^s is a continuous operation for $s > 1/2$ it follows that the linear operator $x \mapsto p_u \sigma_u x$, from ℓ^2 to E^s , is Hilbert-Schmidt a.e. (u, ω) . \mathcal{L}_v is bounded for every $v \geq 0$, so the integrand is a progressively measurable $\mathcal{HS}(\ell^2, E^s)$ -valued process satisfying the conditions of Theorem 17.

A process p with values in E^s satisfying (45) (or equivalently (40)) and (42) will be called a *mild solution* of the bonds dynamics.

Note that we are not worrying about the boundary condition (37) at this time, because it does not make mathematical sense: how do we define r_t ? This will be taken care of in the next section.

3.3 Smoothness of the zero-coupon curve.

Another way to proceed is to write (38) in differentiated form. For fixed $x \geq 0$, a formal calculation using Itô's lemma and which can be rigorously justified gives:

$$\begin{aligned} dp_t(x) - p_t(x) m_t(x) dt - \sum_{i \in \mathbb{I}} p_t(x) \sigma_t^i(x) dW_t^i \\ = \frac{\partial}{\partial t} p_0(t+x) + \int_0^t \frac{\partial}{\partial t} (p_s(t-s+x) m_s(t-s+x)) ds \\ + \int_0^t \frac{\partial}{\partial t} \sum_{i \in \mathbb{I}} p_s(t-s+x) \sigma_s^i(t-s+x) dW_s^i. \end{aligned}$$

In the expression on r.h.s. we can replace $\partial/\partial t$ by $\partial/\partial x$, since p_0 and the integrands on the r.h.s. are functions of $t+x$. Derivation w.r.t. x under the

integral then gives:

$$\begin{aligned} & dp_t(x) - p_t(x) m_t(x) dt - \sum_{i \in \mathbb{I}} p_t(x) \sigma_t^i(x) dW_t^i \\ &= \frac{\partial}{\partial x} \left(p_0(t+x) + \int_0^t p_s(t-s+x) m_s(t-s+x) ds + \int_0^t \sum_{i \in \mathbb{I}} p_s(t-s+x) \sigma_s^i(t-s+x) dW_s^i \right). \end{aligned}$$

The l.h.s. is equal to $(\partial/\partial x)p_t(x)$, according to (38), so

$$dp_t(x) - p_t(x) m_t(x) dt - \sum_{i \in \mathbb{I}} p_t(x) \sigma_t^i(x) dW_t^i = \frac{\partial}{\partial x} p_t(x), \quad (46)$$

for all $x \geq 0$ and $t \in \mathbb{T}$.

Introducing the infinitesimal generator ∂ of the semi-group \mathcal{L} (see Proposition 13), this can be understood as an equation in E^s :

$$dp_t = \partial p_t + p_t(m_t dt + \sum_{i \in \mathbb{I}} \sigma_t^i dW_t^i) \quad (47)$$

or equivalently:

$$p_t = p_0 + \int_0^t (\partial p_s + p_s m_s) ds + \int_0^t \sum_{i \in \mathbb{I}} p_s \sigma_s^i dW_s^i. \quad (48)$$

Equation (40) is the integrated version of (48), w.r.t. the semi-group \mathcal{L} . The connection between formulas (48) and (40) is similar to the *variations of constants* formula for ODE's in finite dimension.

We now have to give some mathematical meaning to equation (48). This will require beefing up the existence conditions given in Theorem 18. The following corollary follows from applying Theorem 18 with $s+1$ instead of s :

Corollary 19 *Let $s > 1/2$. Assume that $p_0 \in \mathcal{D}(\partial) = E^{s+1}$ and assume that m_t and the σ_t^i , $i \in \mathbb{I}$ are progressively measurable processes with values in E^{s+1} satisfying*

$$\int_0^{\bar{T}} (\|m_t\|_{E^{s+1}} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^{s+1}}^2) dt < \infty \quad a.s. \quad (49)$$

Then the mild solution p , in Theorem 18, of the bonds dynamics satisfies the following condition:

$$p_t \in E^{s+1} \text{ and } \int_0^{\bar{T}} (\|p_t\|_{E^{s+1}} + \|p_t m_t\|_{E^s} + \sum_{i \in \mathbb{I}} \|p_t \sigma_t^i\|_{E^s}^2) dt < \infty \quad a.s. \quad (50)$$

Equation (48) holds for every t . In addition p has continuous paths in E^{s+1} and $p_t \in H^{s+1}$ if $p_0 \in H^{s+1}$.

By definition a solution of equation (40) is called a strong solution of the equation (48), when condition (50) is satisfied. Here we shall say that p is a *strong solution* of the bonds dynamics.

As a consequence, in the situation of Corollary 19, the term structure $x \mapsto p_t(x)$ is C^1 for every t , and interest rates are well defined. The instantaneous forward rate $R_t(x)$ contracted at $t \in \mathbb{T}$ for time to maturity x and the spot rate r_t at time t , for instance, are defined by:

$$R_t(x) = -\frac{\partial \log p_t(x)}{\partial x} = -\frac{(\partial p_t)(x)}{p_t(x)} \quad \text{and} \quad r_t = R_t(0) = -\frac{(\partial p_t)(0)}{p_t(0)}. \quad (51)$$

By Corollary 19, p is a strong solution and the maps $t \mapsto p_t$ and $t \mapsto \partial p_t$ are continuous from \mathbb{T} into E^s , and hence into $C^0([0, \infty[)$ endowed with the topology of uniform convergence. So $p_s(0)$ and $(\partial p_s)(0)$ converge to $p_t(0)$ and $(\partial p_t)(0)$, when $s \rightarrow t$. In other words, r_t is a continuous function of t , when $p_t(0) > 0$ for all $t \in \mathbb{R}$.

We are now able to make sense of the boundary condition (37), which we rewrite in terms of p :

$$p_t(0) = \exp\left(\int_0^t \frac{(\partial p_s)(0)}{p_s(0)} ds\right), \quad (52)$$

for every $t \in \mathbb{T}$.

Proposition 20 *Let $s > 1/2$. Assume that m_t and the σ_t^i are progressively measurable processes with values in E^{s+1} satisfying (49) and*

$$m_t(0) = 0, \quad \sigma_t^i(0) = 0 \quad \forall i \in \mathbb{I} \quad (53)$$

and assume that p_0 satisfies

$$p_0 \in E^{s+1}, \quad p_0(0) = 1, \quad p_0(x) > 0 \quad \forall x \geq 0. \quad (54)$$

Then the solution of the bond dynamics, given by Corollary 19, satisfies the boundary condition (52).

Proof. Since m_t and the σ_t^i take values in E^{s+1} , they are continuous function on $[0, \infty[$, and condition (53) makes sense. As $p_0 > 0$ it follows from Proposition 18 that $p_t > 0$. We have shown that, if p_t is a strong and strictly

positive solution of the bond dynamics, then r_t given by (51) is a continuous function of t . Writing conditions (53) into equation (48), we get:

$$\begin{aligned} p_t(0) &= p_0(0) + \int_0^t ((\partial p_s)(0) + p_s(0)m_s(0))ds + \int_0^t \sum_{i \in \mathbb{I}} p_s(0)\sigma_s^i(0)dW_s^i \\ &= 1 + \int_0^t (\partial p_s)(0)ds = 1 - \int_0^t r_s p_s(0)ds. \end{aligned}$$

In other words, $\varphi(t) = p_t(0)$ must satisfy the differential equation $\varphi'(t) = -r_t\varphi(t)$, with the initial condition $\varphi(0) = 1$. The result follows. ■

When we get to optimizing portfolios, we will need L^p estimates on the solutions of the bond dynamics. They are provided by the following result:

Theorem 21 *Let $q(t) = p_t/\mathcal{L}_t p_0$ and $\hat{q}(t) = \hat{p}_t/\mathcal{L}_t \hat{p}_0$. If p_0, σ and m in Proposition 20 also satisfy the following additional conditions:*

$$E\left(\left(\int_0^{\bar{T}} \|\sigma_t\|_{\mathcal{H}\mathcal{S}(\ell^2, E^{s+1})}^2 dt\right)^a + \exp\left(a \int_0^{\bar{T}} \|\sigma_t\|_{\mathcal{H}\mathcal{S}(\ell^2, E^s)}^2 dt\right)\right) < \infty, \forall a \in [1, \infty[\quad (55)$$

and

$$E\left(\left(\int_0^{\bar{T}} \|m_t\|_{E^{s+1}} dt\right)^a + \exp\left(a \int_0^{\bar{T}} \|m_t\|_{E^s} dt\right)\right) < \infty, \forall a \in [1, \infty[, \quad (56)$$

then the solution p in Proposition 20 has the following property:

$$p, \hat{p}, q, \hat{q}, 1/q, 1/\hat{q} \in L^u(\Omega, P, L^\infty(\mathbb{T}, E^{s+1})), \forall u \in [1, \infty[. \quad (57)$$

Proof. We use the notation

$$\tilde{\mathcal{E}}_t(L) = \exp\left(\int_0^t \mathcal{L}_{t-s}\left((m_s - \frac{1}{2} \sum_{i \in \mathbb{I}} (\sigma_s^i)^2)ds + \sigma_s dW_s\right)\right), \quad (58)$$

for

$$L_t = \int_0^t (m_s ds + \sigma_s dW_s), \quad \text{if } 0 \leq t \leq \bar{T}. \quad (59)$$

Conditions (i)–(iv) of Lemma 49 are satisfied for p . Estimate (142) of Lemma 49 then shows that $p \in L^u(\Omega, P, L^\infty(\mathbb{T}, E^{s+1})) \forall u \in [1, \infty[$. By the explicit expression (43), $q = \tilde{\mathcal{E}}(L)$, so it follows from Lemma 49 that the conclusion holds true also for q .

Let $N_t = \int_0^t ((-m_s + \sum_{i \in \mathbb{I}} (\sigma_s^i)^2)ds - \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i)$. Then $1/q = \tilde{\mathcal{E}}(N)$. According to conditions (55), (56), the conditions (i) – (iv) of Lemma 49

(with N instead of L) are satisfied. We now apply estimate (142) to $1/q$, which proves that $1/q \in L^u(\Omega, P, L^\infty(\mathbb{T}, E^{s+1}))$, for all $u \geq 1$.

To prove the cases of \hat{q}^α , $\alpha = 1$ or $\alpha = -1$, we note that $q(t) = \hat{q}(t)p_t(0)$. Using that the case of q^α is already proved and Hölders inequality, it is enough to prove that $g \in L^u(\Omega, P, L^\infty(\mathbb{T}, \mathbb{R}))$, where $g(t) = (p_t(0))^{-\alpha}$. Since $p_t(0) = (\mathcal{L}_t p_0)(0)(q(t))(0) = p_0(t)(q(t))(0)$, it follows that $0 \leq g(t) = (p_0(t))^{-\alpha}((q(t))(0))^{-\alpha}$. By Sobolev embedding, p_0 is a continuous real valued function on $[0, \infty[$ and it is also strictly positive, so the function $t \mapsto (p_0(t))^{-\alpha}$ is bounded on \mathbb{T} . Once more by Sobolev embedding, $((q(t))(0))^{-\alpha} \leq C\|(q(t))^{-\alpha}\|_{E^s}$. The result now follows, since we have already proved the case of q^α . The case of \hat{p} is so similar to the previous cases that we omit it. ■

Under the hypotheses of Proposition 20, $p_t(0)$ satisfies (52), so it is the discount factor (37). It has nice properties, as follows from the second part of the proof of Theorem 21

Corollary 22 *Under the hypotheses of Theorem 21, if $\alpha \in \mathbb{R}$, then the discount factor $p_t(0)$ satisfies*

$$E(\sup_{t \in \mathbb{T}} (p_t(0))^\alpha) < \infty.$$

Remark 23 *It follows from Theorem 21 that for all $t \in \mathbb{T}$, p_t and p_0 have similar asymptotic behavior. In fact for some r.v. $A > 0$, $A^{-1}p_0(t+x) \leq p_t(x) \leq Ap_0(t+x)$, for all $t \in \mathbb{T}$ and $x \geq 0$, where A is independent of x and t and $A \in L^u(\Omega, P)$ for all $u \geq 1$.*

In a different context, Hilbert spaces of forward rate curves were considered in [4] and [11]. The space E^s , with $s > 1/2$ sufficiently small, contains the image of these spaces, under the nonlinear map of forward rates to zero-coupons prices. Or more precisely, it contains the image of subsets of forward rate curves f with positive long term interest rate, i.e. $f(x) \geq 0$ for all x sufficiently big.

4 Portfolio theory

In this section $s > 1/2$, $E^s = E^s[0, \infty)[\cdot]$ and $\mathbb{T} = [0, \bar{T}]$, where \bar{T} is the time horizon of the model. We also write E for $E^s = E^s[0, \infty)[\cdot]$ and E' for $E^{-s}[0, \infty)[\cdot]$.

4.1 Basic definitions.

We recall that, by the bilinear form $\langle \cdot, \cdot \rangle$, the space E^{-s} is identified with the dual of E^s , that is, the space of continuous linear functionals on E^s . It is important to note that, since $s > 1/2$, the space E^s is contained in $C_b^0([0, \infty[)$, the space of bounded continuous functions on $[0, \infty[$, so that H^{-s} contains the dual of $C_b^0([0, \infty[)$, which is the space of bounded Radon measure on $[0, \infty)$. In particular, all Dirac masses δ_x , for $x \geq 0$, belong to H^{-s} .

Definition 24 *A portfolio is progressively measurable process on the time interval \mathbb{T} , with values in E^{-s} . If θ is a portfolio, then its discounted value at time $t \in \mathbb{T}$ is*

$$V_t(\theta) = \langle \theta_t, p_t \rangle. \quad (60)$$

The basic example is a portfolio of one zero-coupon:

Example 25

Consider a portfolio containing exactly one zero-coupon bond with maturity date T , i.e. *time of maturity* T :

1) Let $T \geq \bar{T}$ and let T be fixed. The portfolio θ is then defined by

$$\theta_t = \delta_{T-t}, \quad \forall t \leq \bar{T}. \quad (61)$$

Since $T \geq \bar{T}$, we have indeed that the support of the distribution θ_t is contained in $[0, \infty[$, so $\theta_t \in E^{-s}$. With this definition, the value of the zero-coupon is:

$$\langle \delta_{T-t}, p_t \rangle = p_t(T - t)$$

which is precisely what we had in mind.

2) Let $T < \bar{T}$ and let T fixed. In this case we note that the process in (61) does not continue after time T : the zero-coupon is converted into cash. So the buy-and-hold strategy is not possible for zero-coupon bonds, unless the horizon \bar{T} is less than the maturity T .

3) Let $T = t + x$ and $x \geq 0$ a fixed *time to maturity*. Then the portfolio is defined by

$$\theta_t = \delta_x, \quad \text{for } t \leq \bar{T}. \quad (62)$$

We note that the higher we choose s , the more portfolios can be incorporated into the model. For instance, if $s > 3/2$, all curves in E^s are C^1 , so that the derivative δ'_x of the Dirac mass belongs to E^{-s} . The value of δ'_{T-t} is:

$$\langle \delta'_{T-t}, p_t \rangle = p'_t(T - t) = -R_t(T - t)p_t(T - t), \quad (63)$$

where $p'_t(x) = \partial p_t(x)/\partial x$ and where $R_t(x)$, defined in (51), is the instantaneous forward rate with time to maturity x , contracted at time t . This also implies that the higher we choose s , the more interest rates derivatives can be incorporated into the model. If $s > 1/2$, then we can contract directly on the values of zero-coupon bond prices, and if $s > 3/2$, then we can contract directly on the values of interest rates.

We next introduce the notion of *self-financing* portfolio. We state a definition such that it will makes sense for mild solutions of the bonds dynamics:

Definition 26 *A portfolio is called self-financing if, for every $t \in \mathbb{T}$*

$$V_t(\theta) = V_0(\theta) + \int_0^t \langle \theta_s, p_s m_s \rangle ds + \sum_{i \in \mathbb{I}} p_s \sigma_s^i dW_s^i. \quad (64)$$

Given a strong solution p of the bonds dynamics, we have for a self-financing portfolio:

$$dV_t(\theta) = \langle \theta_t, dp_t - \partial p_t dt \rangle. \quad (65)$$

Note that this is not the standard definition: this is because we are in the moving frame. Changes in portfolio value are due to two causes: changes in prices, as in the fixed frame, and also to changes in time to maturity.

For the right-hand side of (64) to make mathematical sense and to introduce later arbitrage free markets, we need a further definition.

Definition 27 *A portfolio θ is an admissible portfolio if $\|\theta\|_{\mathbb{P}} < \infty$, where*

$$\|\theta\|_{\mathbb{P}}^2 = E \left[\left(\int_0^{\bar{T}} |\langle \theta_t, p_t m_t \rangle| dt \right)^2 + \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} (\langle \theta_t, p_t \sigma_t^i \rangle)^2 dt \right].$$

\mathbb{P} is the linear space of all admissible portfolios and \mathbb{P}_{sf} the subspace of self-financing portfolios.

The discounted gains process G , defined by

$$G(t, \theta) = \int_0^t (\langle \theta_s, p_s m_s \rangle ds + \langle \theta_s, p_s \sigma_s dW_s \rangle), \quad (66)$$

is well-defined for admissible portfolios:

Proposition 28 *Assume that p_0 , m and σ are as in Proposition 20. If $\theta \in \mathbb{P}$, then $G(\cdot, \theta)$ is continuous a.s. and $E(\sup_{t \in \mathbb{T}} (G(t, \theta))^2) < \infty$.*

Proof. Let $\theta \in \mathbf{P}$ and introduce $X = \sup_{t \in \mathbb{T}} |G(t, \theta)|$, $Y(t) = \int_0^t \langle \theta_s, p_s m_s \rangle ds$ and $Z(t) = \int_0^t \langle \theta_s, p_s \sigma_s dW_s \rangle$. Then $G(t, \theta) = Y(t) + Z(t)$, according to formula (66). Let p be given by Proposition 20, of which the hypotheses are satisfied.

We shall give estimates for Y and Z . By the definition of \mathbf{P} :

$$E((\sup_{t \in \mathbb{T}} Y(t))^2) \leq E((\int_0^{\bar{T}} |\langle \theta_s, p_s m_s \rangle| ds)^2) \leq \|\theta\|_{\mathbf{P}}^2. \quad (67)$$

By isometry we obtain

$$\begin{aligned} E(Z(t)^2) &= E(\int_0^t \langle \theta_s, p_s \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i \rangle)^2 \\ &= E(\int_0^t \sum_{i \in \mathbb{I}} (\langle \theta_s, p_s \sigma_s^i \rangle)^2 ds) \leq \|\theta\|_{\mathbf{P}}^2. \end{aligned} \quad (68)$$

Doob's L^2 inequality and inequality (68) give $E(\sup_{t \in \mathbb{T}} Z(t)^2) \leq 4\|\theta\|_{\mathbf{P}}^2$. Inequality (67) then gives $E(X^2) \leq 10\|\theta\|_{\mathbf{P}}^2$, which proves the proposition. ■

Example 29

- 1) The portfolio in 1) of Example 29 is self-financing and the portfolios in 2) and 3) of Example 29 are not self-financing.
- 2) The interest rate portfolio in formula (63) is self-financing.

4.2 Rollovers

Definition 30 Let $S \geq 0$. A S -rollover is a self-financing portfolio θ of a number of zero-coupon bonds with constant time to maturity S and with initial price $V_0(\theta) = p_0(S)$.

It follows directly from the definition that a S -rollover have the same initial price as a zero-coupon with maturity date S . It also follows that, if x_t is the number of zero-coupon bonds in the portfolio at t , then we must have:

$$\theta_t = x_t \delta_S,$$

where the real-valued process x makes the portfolio self-financing.

Proposition 31 If θ_t is a S -rollover, then:

$$x_t = p_0(S) \exp\left(\int_0^t R_s(S) ds\right). \quad (69)$$

Proof. The portfolio θ_t only contains zero-coupons with time to maturity S , so that $V_t(\theta) = x_t p_t(S)$. Assuming the process x to be of bounded variation it follows that:

$$dV_t(\theta) = p_t(S)dx_t + x_t dp_t(S).$$

Substituting the expression for $dp_t(S)$ this becomes:

$$\begin{aligned} dV_t(\theta) &= p_t(S) \frac{dx_t}{dt} dt + x_t \partial_x p_t(S) dt + x_t p_t(S) (m_t(S) dt + \sum_{i \in \mathbb{I}} \sigma_t^i(S) dW_t^i) \\ &= (p_t(S) \frac{dx_t}{dt} + x_t \partial_x p_t(S) + x_t p_t(S) m_t(S)) dt + x_t p_t(S) \sum_{i \in \mathbb{I}} \sigma_t^i(S) dW_t^i. \end{aligned}$$

According to (64) the portfolio is then self-financing if and only if:

$$p_t(S) \frac{dx_t}{dt} + x_t \partial_x p_t(S) = 0.$$

This means that:

$$\frac{1}{x_t} \frac{dx_t}{dt} = - \frac{1}{p_t(S)} \frac{\partial p_t(S)}{\partial S} = R_s(S).$$

and the formula (69) follows by integration. This proves the proposition since x then is of bounded variation. ■

In particular, if $S = 0$, then we get the usual bank account with spot rate r_t .

Henceforth, we will denote by $q_t(S)$ the value (discounted to $t = 0$) at time t of a S -rollover. In the preceding notation, $q_t(S) = V_t(\theta)$.

Introducing the price curve of the roll-over at time t , $q_t : [0, \infty[\rightarrow \mathbb{R}$, we find that the price dynamics of roll-overs is given by:

$$q_t = p_0 + \int_0^t q_s m_s ds + \int_0^t q_s \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i, \quad (70)$$

Note that, compared to the same formula for bond prices, the term in ∂ has disappeared from the right-hand side.

A S -rollover is a bank account which needs advance notice to be cashed: if notice is given at time t , the rollover will then pay x_t units of account at time $t + S$. In other words, at time t , when notice is given, the rollover is exchanged for $q_t(S)/p_t(S) = x_t$ units of a unit zero-coupon with time of maturity $t + S$.

As we noted earlier, zero-coupons do not in general allow buy-and-hold strategies. However rollovers do: a constant portfolio of rollovers is always self-financing. A general bond portfolio θ_t can be expressed in terms of a portfolio of rollovers η_t and vice versa.

4.3 Absence of arbitrage opportunities.

Let p be a mild solution of the price dynamics. Suppose that θ_t is a self-financing portfolio such that, for almost every $(t, \omega) \in \mathbb{T} \times \Omega$, we have:

$$\forall i \in \mathbb{I}, \quad \langle \theta_t(\omega), p_t(\omega)\sigma_t^i(\omega) \rangle = 0. \quad (71)$$

(We note that $p_t(\omega) \in E^s$ is a function of time to maturity, $x \mapsto p_t(\omega, x)$, and similarly for θ_t etc.) Then (64) gives $dV_t(\theta) = \langle \theta_t, m_t p_t \rangle dt$, so that θ_t is risk-free. Since the spot rate is zero (after discounting values to $t = 0$), in an arbitrage free market it must follow that for almost every (t, ω) :

$$\langle \theta_t(\omega), m_t(\omega)p_t(\omega) \rangle = 0. \quad (72)$$

Comparing (71) and (72), we find that $p_t(\omega)m_t(\omega)$ must belong to the closure of the linear span of $\{p_t(\omega)\sigma_t^i(\omega) \mid i \in \mathbb{I}\}$. In fact this follows rigorously using Lemma 34, proved independently of this subsection. There are now two cases:

- \mathbb{I} is finite. Then the linear span is finite-dimensional, and it coincides with its closure. So there are numbers $\gamma_t^i(\omega), i \in \mathbb{I}$ such that $p_t(\omega)m_t(\omega) = p_t(\omega) \sum_{i \in \mathbb{I}} \gamma_t^i(\omega)\sigma_t^i(\omega)$ (finite sum). Since $p_t(\omega) > 0$ for almost every (t, ω) , this leads to:

$$m_t(\omega) = \sum_{i \in \mathbb{I}} \gamma_t^i(\omega)\sigma_t^i(\omega)$$

and since the processes m and σ^i are progressively measurable, so can one choose the processes γ^i . Note that the preceding equation holds in E^s , and that it translates into a family of equations in $[0, \infty[$:

$$m_t(\omega, x) = \sum_{i \in \mathbb{I}} \gamma_t^i(\omega)\sigma_t^i(\omega, x) \quad \forall x \geq 0$$

or, as usual, omitting to mention the ω variable:

$$m_t(x) = \sum_{i \in \mathbb{I}} \gamma_t^i \sigma_t^i(x) \quad \forall x \geq 0.$$

The γ_t^i are the components of a market price of risk, and they do not depend on the time to maturity x . Using the volatility operator process σ the last equality reads

$$m_t = \sigma_t \gamma_t \quad \forall t \in \mathbb{T} \quad (73)$$

and any γ , progressively measurable with values in $\ell^2(\mathbb{I})$, satisfying this equation is called a market price of risk process.

- $\mathbb{I} = \mathbb{N}$. Then the linear span is not closed in general; in fact, it is closed if and only if it is finite-dimensional. In that case, we shall impose a stronger condition. To prove that the market is arbitrage-free, we shall use that $m_t(\omega)$ is in the range of the volatility operator $\sigma_t(\omega)$ which is a subset of the above closed linear span. So, once more we impose that the condition (73) should be satisfied, but for γ with values in $\ell^2(\mathbb{I})$. If the range of $\sigma_t(\omega)$ is infinite dimensional, then this condition is indeed stronger, since $\sigma_t(\omega)$ is a.e. a compact operator.

In both cases, we also need that γ satisfy some integrability condition in (ω, t) . This leads us to the following

Definition 32 *We shall say that the market is strongly arbitrage-free if there exists a progressively measurable process γ with values in $\ell^2(\mathbb{I})$, such that*

$$m_t = \sigma_t \gamma_t, \quad \forall t \in \mathbb{T} \quad (74)$$

and

$$E \left[\exp(a \int_0^{\bar{T}} \|\gamma_t\|_{\ell^2}^2 dt) \right] < \infty, \quad \forall a \geq 0. \quad (75)$$

If the market is strongly arbitrage-free then, by the Girsanov theorem, a martingale measure is given by $dQ = \xi_{\bar{T}} dP$, with:

$$\xi_t = \exp \left(-\frac{1}{2} \int_0^t \|\gamma_s\|_{\ell^2}^2 ds - \sum_{i \in \mathbb{I}} \gamma_s^i dW_s^i \right). \quad (76)$$

The \tilde{W}^i , $i \in \mathbb{I}$, where

$$\tilde{W}_t^i = W_t^i + \int_0^t \gamma_s^i ds, \quad (77)$$

are independent Wiener process with respect to Q . The expected value of a random variable X with respect to Q is given by:

$$E_Q(X) = E(\xi_{\bar{T}} X).$$

Under a martingale measure, the discounted zero-coupon price process p satisfies the equation

$$p_t = \mathcal{L}_t p_0 + \int_0^t \mathcal{L}_{t-s} (p_s \sigma_s) d\tilde{W}_s \quad (78)$$

and also the equation

$$p_t = p_0 + \int_0^t \partial p_s ds + \int_0^t p_s \sigma_s d\tilde{W}_s. \quad (79)$$

The discounted roll-over price process q_t is given by:

$$q_t = p_0 + \int_0^t q_s \sigma_s d\tilde{W}_s. \quad (80)$$

Lemma 33 *A portfolio θ is self-financing if and only if:*

$$V_t(\theta) = V_0(\theta) + \int_0^t \sum_{i \in \mathbb{I}} \langle \theta_t, p_t \sigma_t^i \rangle d\tilde{W}_s^i. \quad (81)$$

We note that the integrand is in fact the adjoint operator of the operator $b_t(\omega) = p_t(\omega) \sigma_t(\omega)$ from $\ell^2(\mathbb{I})$ to $E^s([0, \infty[)$:

$$(b_t(\omega)' \theta_t)^i = \langle \theta_t, p_t \sigma_t^i \rangle, \quad \forall i \in \mathbb{I}. \quad (82)$$

To see this, with $x_t^i(\omega) = \langle \theta_t, p_t \sigma_t^i \rangle$, rewrite it as follows: for all (t, ω) and all $z \in \ell^2(\mathbb{I})$

$$\begin{aligned} (z, x_t(\omega))_{\ell^2} &= \sum_{i \in \mathbb{I}} z^i \langle \theta_t(\omega), p_t(\omega) \sigma_t^i(\omega) \rangle \\ &= \langle \theta_t(\omega), p_t(\omega) \sum_{i \in \mathbb{I}} \sigma_t^i(\omega) z^i \rangle \\ &= \langle \theta_t(\omega), b_t(\omega) z \rangle = \langle b_t(\omega)' \theta_t(\omega), z \rangle. \end{aligned}$$

If the market is strongly arbitrage-free and if condition (55) of Theorem 21 is satisfied, then also condition (56) is satisfied and the Theorem 21 applies.

5 Hedging of interest derivatives

From now on, it will be a standing assumption that p_0 satisfies condition (54), that σ satisfy conditions (53) and (55) and that the market is strongly arbitrage-free according to Definition 32.

Before we solve the optimal portfolio problem, we shall study the problem of hedging a European interest rates derivative with payoff X at maturity \bar{T} . X is said to be an attainable contingent claim or derivative if $V_{\bar{T}}(\theta) = X$ for some admissible self-financing portfolio θ . Here we are only interested in of payoffs, relevant for the optimal portfolio problem considered in these notes, i.e. $X \in L^p(\Omega, \mathcal{F}, P)$ for every $p \geq 1$ (see Lemma 41). We first introduce the hedging equation, the Malliavin derivative and the Clark-Ocone representation formula, which then permits the reader, if he wish, to proceed

directly to the study of the optimization problem in the case of deterministic σ and γ in §6.2.1

Assume that $X \in L^2(\Omega, \mathcal{F}, Q)$, where Q is one equivalent martingale measure given by (76). Then, by the martingale representation theorem, X can be written as a stochastic integral:

$$X = E_Q[X] + \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} x_t^i d\tilde{W}_t^i, \quad (83)$$

with:

$$E_Q \left[\int_0^{\bar{T}} \|x_t\|_{\ell^2}^2 dt \right] < \infty. \quad (84)$$

Comparing with equations (81) and (82) for a self-financing portfolio, we obtain the hedging equation

$$b_t(\omega)' \theta_t(\omega) = x_t(\omega), \text{ a.e. } (t, \omega), \quad (85)$$

where the operator $b_t(\omega) = p_t(\omega) \sigma_t(\omega)$ from $\ell^2(\mathbb{I})$ to $E^s([0, \infty[)$ was introduced in (82). Equivalently: for almost every (t, ω) ,

$$x_t^i(\omega) = \langle \theta_t(\omega), p_t(\omega) \sigma_t^i(\omega) \rangle, \forall i \in \mathbb{I}.$$

We next introduce the Malliavin derivative (c.f. [26]), $D_t X$, with respect to \tilde{W} , at time $t \in \mathbb{T}$ of certain $\mathcal{F} = \mathcal{F}_{\bar{T}}$ measurable real random variables X by:

- D1) $D_t X = 0$, if X is a constant,
- D2) $D_t X = h_t$, if $h \in L^2(\mathbb{T}, \ell^2(\mathbb{I}))$ and $X = \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} h_t^i d\tilde{W}_t^i$,
- D3) $D_t(XY) = X D_t Y + Y D_t X$.

The algebra of such random variables is dense in $L^2(\Omega, \mathcal{F}, Q)$, which can be used to extend the definition to larger sets. $D_t X$ takes its values in $\ell^2(\mathbb{I})$. The partial derivative, with respect to \tilde{W}^i , $D_{i,t} X$, is the i -th component of $D_t X$.

We will use the following expression for the Malliavin derivative of an Itô stochastic integral:

$$D_t \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} x_s^i d\tilde{W}_s^i = x_t + \int_t^{\bar{T}} \sum_{i \in \mathbb{I}} (D_t x_s^i) d\tilde{W}_s^i, \quad (86)$$

when almost all the x_s^i are Malliavin differentiable and sufficiently integrable.

In the case when X is Malliavin differentiable, the Clark-Ocone representation formula states that the integrand x_t in (83) is given by

$$x_t = E_Q [D_t X | \mathcal{F}_t]. \quad (87)$$

We now come back to the hedging equation (85). The fact that $\theta_t = \delta_0$ is a solution to the homogeneous equation (85) permits us to construct self-financed solutions of the in-homogeneous equation (85), from solutions, which are not self-financed:

Lemma 34 *If $\bar{\theta}$ is an admissible portfolio (not necessarily self-financed) which satisfies (85), then there is a unique self-financing admissible portfolio θ_t such that the difference $\theta_t - \bar{\theta}_t$ is risk-free. It is given by:*

$$\theta_t = a_t \delta_0 + \bar{\theta}_t, \quad (88)$$

$$a_t = \frac{1}{p_t(0)} [E_Q[X | \mathcal{F}_t] - V_t(\bar{\theta})]. \quad (89)$$

Proof. We here omit the argument ω . Since the portfolio $\theta_t - \bar{\theta}_t$ is risk-free, it must have time to maturity 0, and the first of the formulas in (88) is true by definition. Substituting into equation (85), and bearing in mind that $\sigma_t^i(0) = 0$:

$$\begin{aligned} ((p_t \sigma_t)' \theta_t)^i &= \langle \theta_t, p_t \sigma_t^i \rangle = \langle a_t \delta_0 + \bar{\theta}_t, p_t \sigma_t^i \rangle \\ &= a_t p_t(0) \sigma_t^i(0) + \langle \bar{\theta}_t, p_t \sigma_t^i \rangle \\ &= x_t^i \quad \forall i \in \mathbb{I}. \end{aligned}$$

So θ_t satisfies (85). It is then a hedging portfolio of X if $V_t(\theta) = E_Q[X | \mathcal{F}_t]$. Substituting again (88) and then (89), we get:

$$V_t(\theta) = a_t V_t(\delta_0) + V_t(\bar{\theta}) = a_t p_t(0) + V_t(\bar{\theta}) = E_Q[X | \mathcal{F}_t].$$

If $\bar{\theta}$ is an admissible portfolio, then θ is also admissible, since $\|\theta\|_{\mathbf{P}} = \|\bar{\theta}\|_{\mathbf{P}}$. ■

By the lemma, the construction of a hedging portfolio for X is reduced to solve equation (85) in $\theta_t(\omega)$ for every (t, ω) , in such a way that $\theta \in \mathbf{P}$, i.e. θ is admissible. Any such solution θ of this equation contains the risky part of the portfolio.

To solve equation (85), for given (t, ω) , we have to know if $x_t(\omega)$ is in the range of the operator $b_t(\omega)'$. The closure of the range of $b_t(\omega)'$ is equal to the orthogonal complement $(\mathcal{K}(b_t(\omega)))^\perp$ of the kernel $\mathcal{K}(b_t(\omega))$ of $b_t(\omega)$.

Consider the cases of \mathbb{I} finite: Then the range $\mathcal{R}((b_t(\omega))')$ is closed. The kernel is trivial, so $(b_t(\omega))'$ is surjective and there is a (non-unique) solution, for every $x_t(\omega)$, iff the $p_t(\omega)$ $\sigma_t^i(\omega)$ are linearly independent.

Consider the cases of \mathbb{I} infinite: The map $(b_t(\omega))'$ from $E^{-s}([0, \infty[)$ to $\ell^2(\mathbb{I})$, is then never surjective. In fact, $b_t(\omega)$ is a Hilbert-Schmidt operator, so it is compact. The adjoint is then also compact and since $\ell^2(\mathbb{I})$ is infinite dimensional, its range must be a proper subspace of $\ell^2(\mathbb{I})$. This is the basic reason why there are always non-attainable contingent claims, when \mathbb{I} is infinite.

We have the following result (see Th.4.1 and Th.4.2 of [34] for the case $\mathbb{I} = \mathbb{N}$):

Theorem 35 *Let $D_0 = \cap_{p \geq 1} L^p(\Omega, P, \mathcal{F})$.*

i) If $\mathbb{I} = \mathbb{N}$, then there exists $X \in D_0$ such that $V_{\bar{T}}(\theta) \neq X$ for all $\theta \in P_{sf}$.

ii) D_0 has a dense subspace of attainable contingent claims if and only if the operator $\sigma_t(\omega)$ has a trivial kernel a.e. $(t, \omega) \in \mathbb{T} \times \Omega$.

Statement *ii)* says by definition that the bond market is approximately complete (notion introduced in [2] and [3]) if and only if $\sigma_t(\omega)$ has a trivial kernel a.e.

In the sequel of this section, we are interested in the hedging problem for approximately complete markets, so we only consider the solution of the hedging equation (85) in the case when $\sigma_t(\omega)$ has a trivial kernel a.e. $(t, \omega) \in \mathbb{T} \times \Omega$.

Consider the case when $\mathbb{I} = \mathbb{N}$ is an infinite and let $\ell^2 = \ell^2(\mathbb{I})$. To derive a condition under which (85) has a solution and to derive a closed formula for one of the solutions, we rewrite the l.h.s. of (85) using the notations

$$l_t = \mathcal{L}_t p_0, \quad B_t(\omega) = l_t \sigma_t(\omega) \quad \text{and} \quad \eta_t(\omega) = \mathcal{S}^{-1}(p_t(\omega)/l_t) \theta_t(\omega). \quad (90)$$

Then

$$\begin{aligned} (\sigma_t(\omega))' p_t(\omega) \theta_t(\omega) &= (\sigma_t(\omega))' l_t (p_t(\omega)/l_t) \theta_t(\omega) = (l_t \sigma_t(\omega))' (p_t(\omega)/l_t) \theta_t(\omega) \\ &= (l_t \sigma_t(\omega))^* \mathcal{S}^{-1}(p_t(\omega)/l_t) \theta_t(\omega) = (B_t(\omega))^* \eta_t(\omega). \end{aligned}$$

The linear operator $B_t(\omega)$ is given, since p_0 and $\sigma_t(\omega)$ are supposed given. Applying Theorem 21 to the factor p/l , it follows that equation (85) is equivalent to find a progressive E^s -valued process η satisfying the equation

$$(B_t(\omega))^* \eta_t(\omega) = x_t(\omega), \quad \text{a.e. } (t, \omega) \in \mathbb{T} \times \Omega. \quad (91)$$

We define the self-adjoint operator $A_t(\omega)$ in ℓ^2 by

$$A_t(\omega) = (B_t(\omega))^* B_t(\omega). \quad (92)$$

It is a fact of basic Hilbert space operator theory (cf. [16]) that the range $\mathcal{R}((B_t(\omega))^*) = \mathcal{R}((A_t(\omega))^{1/2})$. The solvability of each one of equations (85) and (91) is therefore equivalent to the existence of a progressive ℓ^2 -valued process z satisfying

$$(A_t(\omega))^{1/2}z_t(\omega) = x_t(\omega), \text{ a.e. } (t, \omega) \in \mathbb{T} \times \Omega. \quad (93)$$

The kernel $\mathcal{K}((A_t(\omega))^{1/2})$ is trivial since $\mathcal{K}((A_t(\omega))^{1/2}) = \mathcal{K}(A_t(\omega)) = \mathcal{K}(B_t(\omega)) = \{0\}$. Now, if $x_t(\omega) \in \mathcal{R}((B_t(\omega))^*)$ then the unique solution of (93) is $z_t(\omega) = ((A_t(\omega))^{1/2})^{-1}x_t(\omega)$ and a solution of (91) is given by

$$\eta_t(\omega) = S_t(\omega)(A_t(\omega))^{-1/2}x_t(\omega), \quad (94)$$

where $S_t(\omega)$, the closure of the operator $B_t(\omega)(A_t(\omega))^{-1/2}$, is isometric (cf. [16]) from ℓ^2 to E^s . Let a be as in (89) and

$$\theta = a\delta_0 + \bar{\theta} \text{ and } \bar{\theta}_t = (l_t/p_t)\mathcal{S}\eta_t. \quad (95)$$

Then θ is a hedging portfolio according to Lemma 34.

In order to ensure that $x_t(\omega)$ of (85) is in the range of $(\sigma'_t p_t)(\omega)$, we introduce spaces $\ell^{s,2}$, of vectors decreasing faster (for $s > 0$) than those of ℓ^2 . For $s \in \mathbb{R}$, let $\ell^{s,2}$ be the Hilbert space of real sequences endowed with the norm

$$\|x\|_{\ell^{s,2}} = \left(\sum_{i \in \mathbb{N}} (1+i^2)^s |x^i|^2 \right)^{1/2}. \quad (96)$$

Obviously $\ell^2 = \ell^{0,2}$ and $\ell^{s',2} \subset \ell^{s,2}$, if $s' \geq s$. Although $(A_t(\omega))^{-1/2}$ is an unbounded operator in ℓ^2 its restriction to $\ell^{s,2}$ can be a bounded operator for some sufficient large $s > 0$, i.e. $(A_t(\omega))^{-1/2}\ell^{s,2} \subset \ell^2$. This is the idea of our assumption, which will ensure hedgeability. However a precise formulation of this assumption must, as in the case of a finite of Bm., take care of integrability properties in (t, ω) .

To consider also the case of a finite \mathbb{I} , we define after obvious modifications the operator $A_t(\omega)$ in $\ell^2(\mathbb{I})$ by formula (92). In this case $A_t(\omega)$ has obviously a bounded inverse.

Condition 36 *i) If $\text{Card}(\mathbb{I}) < \infty$, then there exists $k \in \mathbf{D}_0$, such that for all $x \in \ell^2(\mathbb{I})$:*

$$\|x\|_{\ell^2} \leq k(\omega) \|(A_t(\omega))^{1/2}x\|_{\ell^2} \text{ a.e. } (t, \omega) \in \mathbb{T} \times \Omega. \quad (97)$$

ii) If $\mathbb{I} = \mathbb{N}$, then there exists $s > 0$ and $k \in \mathbf{D}_0$, such that for all $x \in \ell^2(\mathbb{I})$:

$$\|x\|_{\ell^2} \leq k(\omega) \|(A_t(\omega))^{1/2}x\|_{\ell^{s,2}} \text{ a.e. } (t, \omega) \in \mathbb{T} \times \Omega. \quad (98)$$

In the case of a finite number of Bm. Condition 36 *i*) leads to a complete market and one can choose a hedging portfolio such that it is continuous in the asset to hedge. To state the result let us introduce the notation $D_0(F) = \cap_{p \geq 1} L^p(\Omega, P, \mathcal{F}, F)$, where F is a Banach space. $D_0 = D_0(\mathbb{R})$.

Theorem 37 (Finite number of random-sources, $Card(\mathbb{I}) < \infty$)

If (i) of Condition 36 is satisfied and if $X \in D_0$, then the portfolio given by equation (95) satisfies $\theta \in P_{sf}$ and $V_{\bar{T}}(\theta) = X$. Moreover the linear mapping $D_0 \ni X \mapsto \theta \in P \cap D_0(L^2(\mathbb{T}, E'))$, is continuous.

Proof. We only outline the proof of the theorem. Here $\ell^2 = \ell^2(\mathbb{I}) = \mathbb{R}^{\bar{m}}$ is finite dimensional.

Let $X \in D_0$ and let x be given by (83). First one proves (see Lemma 3.1 of [34]) that

$$D_0(F) = \cap_{p \geq 1} L^p(\Omega, Q, \mathcal{F}, F). \quad (99)$$

Applying the BDG inequalities to equation (83) it follows that

$$x \in D_0(L^2(\mathbb{T}, \ell^2)), \quad (100)$$

where x is progressively measurable. The definition of η in (94) and the condition (97) give

$$\|\eta_t(\omega)\|_{\ell^2} \leq k_t(\omega) \|x_t(\omega)\|_{\ell^2}.$$

Inequality (100) then leads to $\eta \in D_0(L^2(\mathbb{T}, E))$. Using the definition (95) of $\bar{\theta}$ we then obtain

$$\bar{\theta} \in D_0(L^2(\mathbb{T}, E')). \quad (101)$$

Since $\bar{\theta}$ satisfies equation (85) by construction and since formulas (100) and (101) shows that $\bar{\theta}$ is admissible, the hypotheses of Lemma 34 are satisfied, so $\theta \in P_{sf}$. This shows that θ is a hedging portfolio of X .

All the linear maps $X \mapsto x \mapsto \eta \mapsto \theta$ are continuous in the above spaces, which also proves the claimed continuity of the map $X \mapsto \theta$. ■

The solution of the hedging problem, given by Theorem 37, is highly non-unique, since when $Card(\mathbb{I}) = \bar{m} < \infty$ then the kernel $\mathcal{K}((\sigma'_t p_t)(\omega))$ has infinite dimension. For instance there is a hedging portfolio ϑ consisting of $\bar{m} + 1$ rollovers at any time.

To state the result in the case of a infinite number of Bm., we first introduce spaces of contingent claims D_s , smaller than D_0 if $s > 0$ and corresponding to that the integrand x in (83) takes values in $\ell^{s,2}$. More precisely, for $s > 0$ let

$$D_s = \{X \in D_0 \mid x \in D_0(L^2(\mathbb{T}, \ell^{s,2})) \text{ where } x \text{ is given by (83)}\}. \quad (102)$$

Condition 36 *ii*) leads to a D_s -complete market, i.e. D_s is a space of attainable contingent claims, D_s is a dense subspace of D_0 and D_s is itself a complete topological vectorspace. This concept gives a natural framework to study existence and continuity of hedging portfolios. We have (see Theorem 4.3 of [34]):

Theorem 38 (Infinite number of random-sources $\mathbb{I} = \mathbb{N}$)

If (ii) of Condition 36 is satisfied and if $X \in D_s$, where $s > 0$ is given by Condition 36, then the portfolio given by equation (95) satisfies $\theta \in P_{sf}$ and $V_{\bar{T}}(\theta) = X$. Moreover the linear map $D_s \ni X \mapsto \theta \in P \cap D_0(L^2(\mathbb{T}, E^I))$, is continuous.

For the proof, which only uses elementary spectral properties of self-adjoint operators and compact operators, the reader is referred to [34].

A Malliavin-Clark-Ocone formalism was adapted recently in reference [6], for the construction of hedging portfolios in a Markovian context, with a Lipschitz continuous (in the bond price) volatility operator. This guaranties that the Malliavin derivative of the bond price is proportional to the volatility operator (formula (30) of [6]). Hedging is then achieved for a restricted class of claims, namely European claims being a Lipschitz continuous function in the price of the bond at maturity.

References [8] and [27] studies the hedging problem in a weaker sense of approximate hedging, which in our context simply boils down to the well-known existence of the integrand x in the decomposition (83).

6 Optimal portfolio management

We now consider an investor, characterized by a von-Neumann-Morgenstern utility function U , an initial wealth v , and a horizon \bar{T} . The money is invested in a market portfolio, and the investor seeks to maximize the terminal (discounted) value $V_{\bar{T}}(\theta)$ of the portfolio. Transaction costs and taxes are neglected. The optimal portfolio problem is then to find a admissible self-financing portfolio $\hat{\theta}$ with $V_{\bar{T}}(\hat{\theta}) = v$, such that:

$$(P_0) \begin{cases} \sup E_P [U (V_{\bar{T}}(\theta))] = U(V_{\bar{T}}(\hat{\theta})) \\ V_{\bar{T}}(\theta) = v \\ \theta \in P_{sf}. \end{cases}$$

We will follow the now classical two-step approach (cf. [17], [28]) towards solving that problem. If the portfolio is self-financing and is worth v at time 0, then, by the martingale property:

$$E_P [\xi_{\bar{T}} V_{\bar{T}}] = v$$

where the random variable $\xi_{\bar{T}}$, arising from Girsanov's theorem, was introduced earlier in (76). In general there can be several possible $\xi_{\bar{T}}$, one for each γ satisfying the conditions of Definition 32. The first step (optimization) consists of finding, among $\mathcal{F}_{\bar{T}}$ -measurable random variables X such that $E_P[\xi_{\bar{T}}X] = v$, the one(s) that maximize expected utility $E_P[U(X)]$. This problem has in our setting a general solution \hat{X} , given by Proposition 43. The second one (accessibility) consists in hedging one of the contingent claims \hat{X} by a self-financing portfolio $\hat{\theta}$. This portfolio is then a solution of the optimal portfolio problem (P_0).

6.1 Optimization

Consider the optimization problem:

$$\left\{ \begin{array}{l} \sup E_P[U(X)] \\ E_P[\xi_{\bar{T}}X] = v \\ X \in L^2(\Omega, \mathcal{F}_{\bar{T}}, P) \end{array} \right.$$

which we can rewrite in a more geometric way, involving the scalar product in $L^2(\Omega, \mathcal{F}_{\bar{T}}, P)$:

$$(P) \left\{ \begin{array}{l} \sup \int_{\Omega} U(X) dP \\ \int_{\Omega} \xi_{\bar{T}} X dP = (\xi_{\bar{T}}, X)_{L^2} = v \\ X \in L^2(\Omega, \mathcal{F}_{\bar{T}}, P) \end{array} \right.$$

Problem (P) consists of maximizing a concave function on a closed linear subspace of L^2 . Assume there is a maximizer \hat{X} . If the usual theory of Lagrange multipliers applies, there will be some $\lambda \in \mathbb{R}$ such that \hat{X} actually optimizes the functional

$$\int_{\Omega} [U(X) - \lambda \xi_{\bar{T}} X] dP$$

over all of L^2 . Maximizing pointwise under the integral, and bearing in mind that U is concave, we are led to the equation:

$$U'(\hat{X}(\omega)) = \lambda \xi_{\bar{T}}(\omega) \quad P\text{-a.e.}, \quad (103)$$

which fully characterizes the solution \hat{X} . Unfortunately this program cannot be carried through, for the function $E_P[U(X)]$ has no point of continuity in L^2 unless U is bounded, so the constraint qualification conditions do not hold for problem (P), cf. [9]. We will therefore proceed by a roundabout way: use (103) to define \hat{X} , and then prove that \hat{X} is optimal for a suitable choice of λ . For this, we need some conditions on U .

Definition 39 *The utility function U will be called admissible if it satisfies the following properties:*

1. $U : \mathbb{R} \rightarrow \{-\infty\} \cup \mathbb{R}$ is concave and upper semi-continuous
2. there is some $a \in \{-\infty\} \cup]-\infty, 0]$, such that $U(x) = -\infty$ if $x < a$ and $U(x) > -\infty$ if $x > a$
3. U is twice differentiable on the interval $A =]a, \infty[$; set $B = U'(A)$
4. $\sup B = +\infty$; $\inf B = 0$ or $\inf B = -\infty$.
5. $U' : A \rightarrow B$ is one-to-one, and there are some positive constants r, c_1, c_2 and c_3 such that its inverse $I = [U']^{-1}$ satisfies the estimate $|I(y)| + |yI'(y)| \leq c_1 + c_2 |y|^r + c_3 |y|^{-r}$ for $y \in B$.

It follows from these assumptions that I is continuous and strictly decreasing, with:

$$\begin{aligned} I(\lambda) &\rightarrow +\infty \text{ when } \lambda \rightarrow \inf B \\ I(\lambda) &\rightarrow a \text{ when } \lambda \rightarrow +\infty. \end{aligned}$$

We note that the estimate, in point 5) of Definition 39, is satisfied iff there exist $C \geq 0$ such that

$$|I(y)| + |yI'(y)| \leq C (|y|^r + |y|^{-r}),$$

for all $y \in B$. All usual utility functions are admissible:

Example 40

i) *Quadratic utility; Set $U(x) = \mu x - \frac{1}{2}x^2$, $\mu \in \mathbb{R}$. Then $a = -\infty$, and $U'(x) = \mu - x$, so that $B = \mathbb{R}$ and $I(y) = \mu - y$. The estimate is satisfied with $r = 1$.*

ii) *Exponential utility; Set $U(x) = 1 - \frac{1}{\mu} \exp(-\mu x)$, $\mu > 0$. Then $a = -\infty$, and $U'(x) = \exp(-\mu x)$, so that $B =]0, \infty[$ and $I(y) = -\frac{1}{\mu} \ln(y)$. The estimate is satisfied for any $r > 0$.*

iii) *Power utility; Set $U(x) = \frac{1}{\mu} x^\mu$ for some $\mu < 1$ and $\mu \neq 0$ (note that μ may be negative). Then $a = 0$, and $U'(x) = x^{\mu-1}$, so that $B =]0, \infty[$ and $I(y) = y^{1/(\mu-1)}$. The estimate is satisfied with $r = \frac{1}{1-\mu}$.*

iv) *Logarithmic utility; Set $U(x) = \ln x$. Then $a = 0$ and $U'(x) = \frac{1}{x}$, so that $B =]0, \infty[$ and $I(y) = \frac{1}{y}$. The estimate is satisfied with $r = 1$.*

Take some $\lambda \in B$, and define a random variable X_λ by:

$$X_\lambda(\omega) = I(\lambda \xi_{\bar{T}}(\omega)).$$

X_λ is $\mathcal{F}_{\bar{T}}$ -measurable. In addition, we have:

Lemma 41 $X_\lambda \in L^p(\Omega, \mathcal{F}_{\bar{T}}, P)$ for every $p \geq 1$.

Proof. Since U is admissible, we know from condition 4 that, for some $r > 0$ we have:

$$\begin{aligned} |I(\lambda \xi_{\bar{T}})|^p &\leq (c_1 + c_2 |\lambda \xi_{\bar{T}}|^r + c_3 |\lambda \xi_{\bar{T}}|^{-r})^p \\ &\leq k_1 + k_2 |\lambda|^{pr} |\xi_{\bar{T}}|^{pr} + k_3 |\lambda|^{-pr} |\xi_{\bar{T}}|^{-pr} \end{aligned}$$

and the right-hand side is integrable, for we know that $\xi_{\bar{T}}^s \in L^1(\Omega, \mathcal{F}_{\bar{T}}, P)$ for every $s \in \mathbb{R}$. ■

Lemma 42 Let $v \in A$. There is a unique $\hat{\lambda} \in \mathbb{R}$ such that $E_P[X_{\hat{\lambda}} \xi_{\bar{T}}] = v$

Proof. Consider the map $\varphi : B \rightarrow \mathbb{R}$ defined by $\varphi(\lambda) = E_P[X_\lambda \xi_{\bar{T}}] = E_P[I(\lambda \xi_{\bar{T}}) \xi_{\bar{T}}]$. Since $\xi_{\bar{T}} > 0$ P -a.e., and f is strictly decreasing, φ is strictly decreasing. Using the Lebesgue dominated convergence theorem, we find that it is continuous. Using Fatou's lemma, we find that:

- $\varphi(\lambda) \rightarrow +\infty$ when $\lambda \rightarrow \inf B$
- $\liminf \varphi(\lambda) \leq 0$ when $\lambda \rightarrow +\infty$

Since $v \in A$, it follows that there is a unique $\hat{\lambda}$ such that $\varphi(\hat{\lambda}) = v$. ■

Denote $X_{\hat{\lambda}}$ by \hat{X} . We now conclude:

Proposition 43 \hat{X} is the unique solution of problem (P).

Proof. Let us show that \hat{X} is indeed a solution of problem (P). Uniqueness follows from the strict concavity of U .

We have shown that \hat{X} is in L^2 , and $E_P[\hat{X} \xi_{\bar{T}}] = v$, so \hat{X} satisfies the constraints. Take another $X \in L^2$ such that $E_P[X \xi_{\bar{T}}] = v$. Since U is concave, we have:

$$U(X(\omega)) \leq U(\hat{X}(\omega)) + (X(\omega) - \hat{X}(\omega))U'(\hat{X}(\omega)) \quad P\text{-a.e.}$$

By definition, $U'(\hat{X}(\omega)) = \lambda \xi_{\bar{T}}(\omega)$. Substituting into the inequality and integrating, we get:

$$\int_{\Omega} U(X) dP \leq \int_{\Omega} U(\hat{X}) dP + \lambda \int_{\Omega} (X - \hat{X}) \xi_{\bar{T}} dP$$

and the last term vanishes because it is just $\lambda(v - v)$. So \hat{X} is indeed a optimizer, and the result follows. ■

6.2 Hedging

6.2.1 Deterministic case

In this paragraph we suppose that m and σ are deterministic processes.

Once the solution \hat{X} of the optimization problem is found, for a given γ , the question is whether it can be hedged by a self-financing portfolio θ_t , so that $V_{\hat{T}}(\theta) = \hat{X}$. We shall now use the general hedging results of §5 to solve this problem, in the case when the m and σ , are deterministic (i.e. they do not depend on ω).

Under these conditions, there can be several γ that satisfy the conditions of Definition 32 and some γ can even be non-deterministic. However, as we have supposed that the market is strongly arbitrage free, so equation (74) has a solution, we can choose γ to be the unique solution with the property of being orthogonal in ℓ^2 to the kernel of the volatility operator. More precisely, we choose the unique γ such that

$$(\gamma_t, x)_{\ell^2} = 0, \quad \forall x \in \ell^2(\mathbb{I}) \text{ s.t. } \sigma_t x = 0. \quad (104)$$

The γ defined by this condition is deterministic. In the sequel of this paragraph γ is given by (104). In that case, it follows from formula (76) that $\xi_{\hat{T}}$ is Malliavin differentiable. It follows from formula (86) that the partial derivative with respect to \tilde{W}^i is given by:

$$D_{i,t}\xi_{\hat{T}} = -\gamma_t^i \xi_{\hat{T}}$$

and $\hat{X} = I(\hat{\lambda}\xi_{\hat{T}})$ is Malliavin differentiable as well, with:

$$D_{i,t}\hat{X} = -\hat{\lambda}\gamma_t^i \xi_{\hat{T}} I'(\hat{\lambda}\xi_{\hat{T}}).$$

The Clarke-Ocone formula now reads:

$$X = E_Q[X | \mathcal{F}_0] + \sum_{i \in \mathbb{I}} \int_0^{\hat{T}} E_Q[D_{i,t}X | \mathcal{F}_t] d\tilde{W}_t^i \quad (105)$$

$$= v - \hat{\lambda} \sum_{i \in \mathbb{I}} \int_t^{\hat{T}} \gamma_t^i E_Q[\xi_{\hat{T}} I'(\hat{\lambda}\xi_{\hat{T}}) | \mathcal{F}_t] d\tilde{W}_t^i \quad (106)$$

We then write the equation (85) for the hedging portfolio $\hat{\theta}$, and we substitute the Clark-Ocone formula for $x_t^i(\omega)$:

$$b_t(\omega)' \theta_t(\omega) = -\hat{\lambda} E_Q[\xi_{\hat{T}} I'(\hat{\lambda}\xi_{\hat{T}}) | \mathcal{F}_t] \gamma_t. \quad (107)$$

This equation has a solution iff γ_t is in the range of $b_t(\omega)'$. Since σ is deterministic, this condition simplifies. In fact, let l_t and B_t be given by (90), which here both are deterministic, and let $q(t, \omega) = p_t(\omega)/l_t$. Then the expression (82) of $b_t(\omega)'$ give:

$$(b_t(\omega)' \theta_t(\omega))^i = \langle \theta_t(\omega), p_t(\omega) \sigma_t^i \rangle = \langle \theta_t(\omega) q(t, \omega), l_t \sigma_t^i \rangle = (B_t' f_t(\omega))^i,$$

where $f_t(\omega) \in E^{-s}$ is given by $f_t(\omega) = q(t, \omega) \theta_t(\omega)$. So, equation (107) has a solution iff γ_t is in the range of B_t' . This is always true when \mathbb{I} is finite, since then the range of B_t' is equal to the orthogonal complement of the kernel of σ_t (we remember that $p_t(\omega, x) > 0$ for $x \geq 0$). When $\mathbb{I} = \mathbb{N}$, then the range is only a strictly smaller dense subset.

We are lead to following condition

Definition 44 *We shall say that the market satisfies condition (C) if there exists a deterministic portfolio θ_t^0 which is admissible and satisfies:*

$$\langle \theta_t^0, (\mathcal{L}_t p_0) \sigma_t^i \rangle = \gamma_t^i, \quad (108)$$

for each $i \in \mathbb{I}$ and t .

Condition C is then equivalent to $\gamma_t \in \mathcal{R}(B_t')$, the range of B_t' . In the case when \mathbb{I} is finite, there is never uniqueness in the choice of θ_t^0 .

In the case when \mathbb{I} is finite, we know that condition (C) is satisfied and it can easily be verified, with n elements say, by picking n maturities $0 < S_1 < \dots < S_n$ and by seeking θ_t^0 as a linear combination of rollovers: $\theta_t^0 = \sum x_t^i \delta_{S_i}$. Condition (108) then reduces to a system of n linear equations with n unknowns which determines the x_t^i .

In the case when $\mathbb{I} = \mathbb{N}$, condition (C) may not be satisfied. We will be content with reminding that the left-hand side of equation (108) is meaningful, since $(\mathcal{L}_t p_0) \sigma_t^i$ belongs to the space E^s .

If condition (C) is satisfied, equation (107) becomes:

$$\begin{aligned} \langle \theta_t, p_t \sigma_t^i \rangle &= -\hat{\lambda} E_Q[\xi_{\bar{T}} I'(\hat{\lambda} \xi_{\bar{T}}) \mid \mathcal{F}_t] \langle \theta_t^0, \frac{\mathcal{L}_t p_0}{p_t} p_t \sigma_t^i \rangle \\ &= \langle -\hat{\lambda} E_Q[\xi_{\bar{T}} I'(\hat{\lambda} \xi_{\bar{T}}) \mid \mathcal{F}_t] \frac{\mathcal{L}_t p_0}{p_t} \theta_t^0, p_t \sigma_t^i \rangle \end{aligned}$$

and an obvious solution $\theta_t = \bar{\theta}_t$ (the risky part of the optimal portfolio) is given by:

$$\bar{\theta}_t = -\hat{\lambda} E_Q[\xi_{\bar{T}} I'(\hat{\lambda} \xi_{\bar{T}}) \mid \mathcal{F}_t] \frac{\mathcal{L}_t p_0}{p_t} \theta_t^0.$$

Applying Lemma 34, with x defined by (89), we obtain a hedging portfolio $\hat{\theta} = x\delta_0 + \bar{\theta}$ of \hat{X} , where $\bar{\theta}$ is as above, and:

$$x_t = \frac{1}{p_t(0)} \left(E_Q \left[I \left(\hat{\lambda}_{\xi_{\bar{T}}} \right) \mid \mathcal{F}_t \right] - \langle \bar{\theta}_t, p_t \rangle \right).$$

To sum up, in the case when the m_s and the $\sigma_s^i, i \in \mathbb{I}$, are deterministic, with $\sigma_t^i(0) = 0$, with condition (C) and equation (74) satisfied, an optimal admissible and self-financing portfolio is given by

$$\hat{\theta}_t = x_t \delta_0 + \bar{\theta}_t, \quad \text{where } \bar{\theta}_t = y_t \frac{(\mathcal{L}_t p_0)}{p_t} \theta_t^0 \quad (109)$$

and where the coefficients x_t and y_t are real-valued progressively measurable processes given by

$$y_t = -E_Q(\hat{\lambda}_{\xi_{\bar{T}}} I'(\hat{\lambda}_{\xi_{\bar{T}}}) \mid \mathcal{F}_t) \quad (110)$$

$$x_t = (p_t(0))^{-1} \left(E_Q[I(\hat{\lambda}_{\xi_{\bar{T}}}) \mid \mathcal{F}_t] - y_t \langle \theta_t^0, \mathcal{L}_t p_0 \rangle \right). \quad (111)$$

This leads immediately to a mutual fund theorem: whatever the utility function and the initial wealth, the optimal portfolio at time t is a linear combination of the current account δ_0 and the portfolio $f \mapsto \langle \theta_t^0, \frac{\mathcal{L}_t p_0}{p_t} f \rangle$, i.e. the portfolio $\frac{\mathcal{L}_t p_0}{p_t} \theta_t^0$. This portfolio is in general not self-financed, so it can not be given the status of a *market portfolio*. However we can easily reformulate the result with a self-financed portfolio. In fact, chose an admissible utility function, with $a = 0$, according to Definition 39. For this utility function, let Θ be the optimal portfolio given by (109), with unit initial wealth. Obviously $\frac{\mathcal{L}_t p_0}{p_t} \theta_t^0$ is a linear combination of δ_0 and Θ_t . This gives us:

Theorem 45 (Mutual fund theorem) *The optimal portfolio Θ has the following properties:*

i) Θ is an admissible self-financing portfolio, with unit initial value, i.e. $\langle \Theta_0, p_0 \rangle = 1$, and the value at each time $t \in \mathbb{T}$ is strictly positive, i.e. $\langle \Theta_t, p_t \rangle > 0$.

ii) For each utility function U , admissible according to Definition 39 and each initial wealth $v \in]a, \infty[$, there exist two real valued processes c and d such that if $\hat{\theta}_t = c_t \delta_0 + d_t \Theta_t$, then $\hat{\theta}$ is an optimal self financing portfolio for U , i.e. a solution of problem (P_0) .

6.2.2 Stochastic m and σ

We shall here concentrate on the case of an approximately complete market, which is equivalent to that the volatility operator is non-degenerated. In fact, according to *iii*) of Theorem 35, the market is approximately complete if and only if $\sigma_t(\omega)$ has a trivial kernel a.e. $(t, \omega) \in \mathbb{T} \times \Omega$.

In the case of a finite number of Bm. we obtain easily from Lemma 41 and Theorem 37 the following result (see Theorem 3.6 of [10]):

Theorem 46 *Let \mathbb{I} be a finite set, let U be admissible in the sens of Definition 39 and let *i*) of Condition 36 be satisfied. The problem (P_0) then has a solution $\hat{\theta}$. One solution $\hat{\theta} = a\delta_0 + \bar{\theta} \in \mathbb{P}_{sf}$ is given by (95).*

In the case of an infinite number of Bm. we shall impose Malliavin differentiability properties on the market price of risk γ . To this end we introduce the space \mathbb{D}_s^1 , for $s > 0$ by

$$\mathbb{D}_s^1 = \{X \in \mathbb{D}_0 \mid DX \in \mathbb{D}_0(L^2(\mathbb{T}, \ell^{s,2}))\}. \quad (112)$$

We can now state a result in the case of an infinite number of Bm., quite analog to the case of a finite number of Bm. (see Theorem 4.5 of [34]):

Theorem 47 *Let $\mathbb{I} = \mathbb{N}$, let U be admissible in the sens of Definition 39, let *ii*) of Condition 36 be satisfied and let $\ln(\xi_{\bar{T}}) \in \mathbb{D}_s^1$, where $s > 0$ is given by *ii*) of Condition 36. The problem (P_0) then has a solution $\hat{\theta}$. One solution $\hat{\theta} = a\delta_0 + \theta \in \mathbb{P}_{sf}$ is given by (95).*

Proof. We only consider the case of $U' > 0$, since the case of $U'(x) = 0$ for some x is so similar. Let the hypotheses of the theorem be satisfied. The portfolio $\hat{\theta}$ is a solution of equation (P_0) , if $\hat{\theta} \in \mathbb{P}_{sf}$ and if it hedges \hat{X} given by Proposition 43. (See Corollary 3.4 of [10]). It is enough to verify that Theorem 38 applies to $\hat{X} = I(\hat{\lambda}\xi_{\bar{T}})$ for a certain given $\hat{\lambda} > 0$.

I is C^1 , so $D_t \hat{X} = \lambda \xi_{\bar{T}} \varphi'(\lambda \xi_{\bar{T}}) D_t \ln(\xi_{\bar{T}})$. Since $\ln(\xi_{\bar{T}}) \in \mathbb{D}_s^1$, this gives $\|D\hat{X}\|_{L^2(\mathbb{T}, \ell^{s,2})} = |\lambda \xi_{\bar{T}} \varphi'(\lambda \xi_{\bar{T}})| \|D \ln(\xi_{\bar{T}})\|_{L^2(\mathbb{T}, \ell^{s,2})}$. The inequality in 5) of Definition 39 gives $\|D\hat{X}\|_{L^2(\mathbb{T}, \ell^{s,2})} \leq C((\lambda \xi_{\bar{T}})^p + (\lambda \xi_{\bar{T}})^{-p}) \|D \ln(\xi_{\bar{T}})\|_{L^2(\mathbb{T}, \ell^{s,2})}$, for some $p \geq 1$. Condition (75) of Definition 32 shows that $(\lambda \xi_{\bar{T}})^p + (\lambda \xi_{\bar{T}})^{-p} \in L^q(\Omega, P)$, for all $q \geq 1$. By hypothesis $\|D \ln(\xi_{\bar{T}})\|_{L^2(\mathbb{T}, \ell^{s,2})} \in \mathbb{D}_0$, so Hölder's inequality now gives that $\|D\hat{X}\|_{L^2(\mathbb{T}, \ell^{s,2})} \in \mathbb{D}_0$, i.e. $D\hat{X} \in \mathbb{D}_0(L^2(\mathbb{T}, \ell^{s,2}))$. By Lemma 41, $\hat{X} \in \mathbb{D}_0$. It follows that $\hat{X} \in \mathbb{D}_s^1$. We can now apply Theorem 38, which proves the existence of $\hat{\theta}$. ■

6.2.3 Examples.

We now give examples of optimal bond portfolios for certain utility functions U . We assume the drift function m_t and the volatility operator σ_t to be deterministic. We shall therefore suppose that the market satisfy condition (C), of Definition 44, so the market prices of risk γ is deterministic and satisfy condition (108).

The final optimal discounted wealth is $\hat{X} = I(\hat{\lambda}\xi_{\bar{T}})$. The corresponding optimal discounted wealth process Y is given by $Y_t = E_Q[I(\hat{\lambda}\xi_{\bar{T}}) | \mathcal{F}_t]$. The initial wealth $Y_0 = v$ determines $\hat{\lambda}$ by the equation

$$v = Y_0 = E_Q[I(\hat{\lambda}\xi_{\bar{T}})]. \quad (113)$$

We recall that $(p_t)^{-1}\mathcal{L}_t p_0 \in E^s$ a.s and that $p_t(0) > 0$ a.s. We shall derive the optimal portfolio directly, going through the steps leading to the general solution (109).

Logarithmic utility Let

$$U(x) = \ln(x). \quad (114)$$

We have $I(x) = 1/x$, and $\hat{X} = (\hat{\lambda}\xi_{\bar{T}})^{-1}$, so that equation (113) gives:

$$v = E_Q[1/(\hat{\lambda}\xi_{\bar{T}})] = E_P[\xi_{\bar{T}}/(\hat{\lambda}\xi_{\bar{T}})] = 1/\hat{\lambda}.$$

Then using the expression (76) for ξ_t and \tilde{W}_t^i we have:

$$\frac{1}{\xi_t} = \exp\left(-\frac{1}{2}\int_0^t \sum_{i \in \mathbb{I}} (\gamma_s^i)^2 ds + \int_0^t \sum_{i \in \mathbb{I}} \gamma_s^i d\tilde{W}_s^i\right). \quad (115)$$

The right-hand side is a Q -martingale, then so is $1/\xi_t$. It follows that the optimal discounted wealth at t is

$$Y_t = E_Q[I(\hat{\lambda}\xi_{\bar{T}}) | \mathcal{F}_t] = \frac{1}{\hat{\lambda}\xi_t} = \frac{v}{\xi_t}.$$

Since $d(1/\xi_t) = \sum_{i \in \mathbb{I}} (\gamma_t^i/\xi_t) d\tilde{W}_t^i$ and $\hat{X} = Y_{\bar{T}}$, it then follows that:

$$\hat{X} = v \left(1 + \sum_{i \in \mathbb{I}} \int_0^{\bar{T}} \gamma_t^i \frac{1}{\xi_t} d\tilde{W}_t^i\right). \quad (116)$$

The hedging equation (85) and the above formula give:

$$\forall i \in \mathbb{I}, \quad \langle \theta_t(\omega), p_t(\omega) \sigma_t^i(\omega) \rangle = \frac{v}{\xi_t(\omega)} \gamma_t^i \quad (117)$$

By condition (C) we find a portfolio θ^0 satisfying $\gamma_t^i = \langle \theta_t^0, (\mathcal{L}_t p_0) \sigma_t^i \rangle$, so

$$\gamma_t^i = \langle (\mathcal{L}_t p_0) \theta_t^0, \sigma_t^i \rangle. \quad (118)$$

Substituting this expression of γ into (117) we obtain:

$$\forall i \in \mathbb{I}, \quad \langle p_t(\omega) \theta_t(\omega) - \frac{v}{\xi_t(\omega)} (\mathcal{L}_t p_0) \theta_t^0, \sigma_t^i(\omega) \rangle = 0. \quad (119)$$

One solution of this equation is obviously given by $\theta = \bar{\theta}$, where

$$\bar{\theta}_t(\omega) = y_t(\omega) \frac{(\mathcal{L}_t p_0)}{p_t(\omega)} \theta_t^0, \quad y_t(\omega) = \frac{v}{\xi_t(\omega)}. \quad (120)$$

The discounted value of $\bar{\theta}$ at time t in state ω is then

$$(V_t(\bar{\theta}))(\omega) = \langle \bar{\theta}_t, p_t \rangle = \frac{v}{\xi_t(\omega)} \langle \theta_t^0, \mathcal{L}_t p_0 \rangle. \quad (121)$$

The optimal portfolio $\hat{\theta}$ is now obtained by using Lemma 34: $\hat{\theta}_t = x_t \delta_0 + \bar{\theta}_t$, where

$$x_t = \frac{1}{p_t(0)} \frac{v}{\xi_t} (1 - \langle \theta_t^0, \hat{\theta}_t(\omega) \mathcal{L}_t p_0 \rangle). \quad (122)$$

As it should, the discounted value of $\hat{\theta}$ is then $V_t(\hat{\theta}) = Y_t = v/\xi_t$.

We note the following useful property: the ratio of the investment in bonds with time to maturity $S > 0$ to the total investment is deterministic. In fact this ratio is simply price at $t = 0$, of a zero-coupon bond with time to maturity $S + t$:

$$\frac{\bar{\theta}_t(S, \omega) p_t(S, \omega)}{(V_t(\bar{\theta}))(\omega)} = p_0(S + t). \quad (123)$$

Quadratic utility Let the utility function be:

$$U(x) = \mu x - \frac{1}{2} x^2$$

As in *i*) of Example 40, we find that

$$I(y) = \mu - y.$$

The final discounted optimal wealth is $\hat{X} = I(\hat{\lambda} \xi_{\bar{T}})$, so

$$\hat{X} = \mu - \hat{\lambda} \xi_{\bar{T}}.$$

We determine $\hat{\lambda}$ by the condition:

$$v = E_Q [\hat{X}] = E_Q [\mu - \hat{\lambda} \xi_{\bar{T}}] = \mu - \hat{\lambda} E_Q [\xi_{\bar{T}}]. \quad (124)$$

Set

$$Z_t = \exp \left(-\frac{1}{2} \int_0^t \sum_{i \in \mathbb{I}} (\gamma_s^i)^2 ds - \int_0^t \sum_{i \in \mathbb{I}} \gamma_s^i d\tilde{W}_s^i \right).$$

Then Z is a martingale with respect to Q and formula (77) gives

$$\xi_t = Z_t \exp \left(\int_0^t \sum_{i \in \mathbb{I}} (\gamma_s^i)^2 ds \right). \quad (125)$$

We have, by substitution into (124):

$$v = \mu - \hat{\lambda} E_Q [\xi_{\bar{T}}] = \mu - \hat{\lambda} \exp \left(\int_0^{\bar{T}} \sum_{i \in \mathbb{I}} (\gamma_s^i)^2 ds \right).$$

This gives

$$\hat{\lambda} = (\mu - v) \exp \left(- \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} (\gamma_s^i)^2 ds \right). \quad (126)$$

It now follows from (125) that

$$\hat{X} = \mu - \hat{\lambda} \xi_{\bar{T}} = \mu + (v - \mu) Z_{\bar{T}} \quad (127)$$

and the optimal discounted wealth at t is

$$Y_t = E_Q [I(\hat{\lambda} \xi_{\bar{T}}) | \mathcal{F}_t] = \mu + (v - \mu) Z_t.$$

Since $dZ_t = -Z_t \sum_{i \in \mathbb{I}} \gamma_t^i d\tilde{W}_t^i$, we have that

$$\hat{X} = \mu - (v - \mu) \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} Z_t \gamma_t^i d\tilde{W}_t^i = \mu + \int_0^{\bar{T}} \sum_{i \in \mathbb{I}} (\mu - Y_t) \gamma_t^i d\tilde{W}_t^i,$$

so the hedging equation reads (see (85)):

$$\forall i \in \mathbb{I}, \quad \langle \theta_t(\omega), p_t(\omega) \sigma_t^i(\omega) \rangle = -(\mu - Y_t(\omega)) \gamma_t^i. \quad (128)$$

As usually, condition (C) gives a portfolio θ^0 satisfying $\gamma_t^i = \langle \theta_t^0, (\mathcal{L}_t p_0) \sigma_t^i \rangle$, which together with (128) gives:

$$\forall i \in \mathbb{I}, \quad \langle p_t(\omega) \theta_t(\omega) + (Y_t(\omega) - \mu) (\mathcal{L}_t p_0) \theta_t^0, \sigma_t^i(\omega) \rangle = 0.$$

One solution of this equation is $\theta = \bar{\theta}$, where

$$\bar{\theta}_t(\omega) = y_t(\omega) \frac{(\mathcal{L}_t p_0)}{p_t(\omega)} \theta_t^0, \quad y_t(\omega) = \mu - Y_t(\omega).$$

$\bar{\theta}$ gives the risky part of the optimal portfolio.

Applying Lemma 34 we obtain the optimal portfolio $\hat{\theta}_t = x_t \delta_0 + \bar{\theta}_t$, where

$$x_t = (p_t(0))^{-1} (Y(t) - (\mu - Y(t)) \langle \theta_t^0, \mathcal{L}_t \hat{p}_0 \rangle). \quad (129)$$

6.3 The H-J-B approach

When m_t and σ_t^i are given functions of p_t , for every t , then the optimal portfolio problem (P_0) can be considered within a Hamilton-Jacobi-Bellman approach. In this subsection we illustrate this approach, without being rigorous and we suppose that the utility function U satisfies the conditions of Definition 39.

The optimal value function, here denoted by F , then only depends of time t , of the value of the discounted wealth w and the discounted price function $f \in E^s$ of Zero-Coupons at time t :

$$F(t, w, f) = \sup \{ E(u(V_T(\theta))) \mid V_t(\theta) = w, p_t = f \mid \theta \in \mathbf{P}_{sf} \}.$$

The derivative $DG(f; g)$ of a function $E^s \ni f \mapsto G(f)$ in the direction $g \in E^s$, is as usually defined by

$$DG(f; g) = \lim_{\epsilon \rightarrow 0} \frac{G(f + \epsilon g) - G(f)}{\epsilon}.$$

Suppose that G is C^2 . Writing $DG(f)$ for the map $g \mapsto DG(f; g)$ and $D^2G(f)$ for the map $g_1 \times g_2 \mapsto DG(f; g_1, g_2)$, we have that $DG(f)$ is a linear continuous form on E^s and $D^2G(f)$ is a bi-linear continuous form.

Define the Hamiltonian $H(t, w, f, x)$ by:

$$\begin{aligned} H(t, w, f, x) &= \sum_{i \in \mathbb{I}} x^i(t, w, f) \gamma_t^i \frac{\partial F}{\partial w}(t, w, f) + DF(t, w, f; \partial f + \sum_{i \in \mathbb{I}} \gamma_t^i \sigma_t^i f) \\ &+ \sum_{i \in \mathbb{I}} \left(\frac{1}{2} (x^i(t, w, f))^2 \frac{\partial^2 F}{\partial w^2}(t, w, f) + x^i(t, w, f) \frac{\partial}{\partial w} DF(t, w, f; \sigma_t^i f) \right. \\ &\left. + \frac{1}{2} D^2 F(t, w, f; \sigma_t^i f, \sigma_t^i f) \right). \end{aligned} \quad (130)$$

In that formula, $x = (x^i)_{i \in \mathbb{I}} \in \ell^2$ is the control, which is related to the optimal terminal wealth by formula (83). A control x is called *admissible* if

$$x^i(t, V_t(\theta), p_t) = \langle \theta_t, p_t \sigma_t^i \rangle$$

for all $\theta \in \mathbf{P}_{sf}$. In other words, x^i can be interpreted as the value invested in the i -th source of noise. Using the Ito formula, one derives the (formal) HJB equation:

$$\frac{\partial F}{\partial t}(t, w, f) + \sup_x H(t, w, f, x) = 0, \quad (131)$$

with the boundary condition

$$F(\bar{T}, w) = U(w). \quad (132)$$

The optimal control \hat{x} , solution of the optimization problem

$$\sup_x H(t, w, f, x),$$

is given by

$$\hat{x}^i(t, w, f) = - \left(\frac{\partial^2 F}{\partial w^2} \right)^{-1} \left(\gamma_t^i \frac{\partial F}{\partial w} + (D \frac{\partial F}{\partial w})(t, w, f; \sigma_t^i f) \right), \quad i \in \mathbb{I}. \quad (133)$$

Now, substitution of $H(t, w, f, \hat{x}(t, w, f))$ into equation (131) gives:

$$\begin{aligned} & \frac{\partial^2 F}{\partial w^2}(t, w, f) \left(\frac{\partial F}{\partial t}(t, w, f) + DF(t, w, f; \partial f + m_t f) + \frac{1}{2} \sum_{i \in \mathbb{I}} D^2 F(t, w, f; \sigma_t^i f, \sigma_t^i f) \right) \\ &= \frac{1}{2} \sum_{i \in \mathbb{I}} \left(\gamma_t^i \frac{\partial F}{\partial w} + (D \frac{\partial F}{\partial w})(t, w, f; \sigma_t^i f) \right)^2. \end{aligned} \quad (134)$$

Once the solution F of (134), with boundary condition (132), is found, the optimal control \hat{x} is given by (133). Any optimal portfolio $\hat{\theta}$ is then a solution of the equation:

$$\hat{x}^i(t, V_t(\hat{\theta}), p_t) = \langle \hat{\theta}_t, p_t \sigma_t^i \rangle, \quad \forall i \in \mathbb{I}, \quad t \in \mathbb{T}.$$

In the case when m_t and σ_t^i are independent of p_t , then the \hat{x}^i are independent of f and the above equations simplify:

$$\frac{\partial F}{\partial t} \frac{\partial^2 F}{\partial w^2} = \frac{1}{2} \left(\sum_{i \in \mathbb{I}} \|\gamma_t^i\|^2 \right) \left(\frac{\partial F}{\partial w} \right)^2,$$

with the boundary condition

$$F(\bar{T}, w) = U(w), \quad w \in \mathbb{R}.$$

Each self financing portfolio $\hat{\theta} \in \mathbf{P}_{sf}$, such that

$$\langle \hat{\theta}_t, p_t \sigma_t^i \rangle = -\gamma_t^i \left(\frac{\partial F}{\partial w} \right) \left(\frac{\partial^2 F}{\partial w^2} \right)^{-1}, \quad \forall i \in \mathbb{I}, t \in \mathbb{T},$$

where $w = V_t(\hat{\theta})$, is then a solution of problem (P_0) . The solutions in the examples in §6.2.3, as well as the general solution (109) for deterministic m and σ , are easily obtained by solving these equations.

A Appendix

In this appendix, we reproduce results (proved in the appendix of [10]), used in this article, concerning existence of solutions of some SDE's and L^p estimates of these solutions. The notations $\mathbb{T} = [0, \bar{T}]$, W^i , \mathbb{I} and $(\Omega, P, \mathcal{F}, \mathcal{A})$ are defined in §3.1. Through the appendix m and σ^i , $i \in \mathbb{I}$, are \mathcal{A} -progressively measurable E^s -valued processes satisfying

$$\int_0^{\bar{T}} (\|m_t\|_{E^s} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^s}^2) dt < \infty, \text{ a.s.} \quad (135)$$

The E^s -valued semi-martingale L is given by

$$L(t) = \int_0^t (m_s ds + \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i), \quad \text{if } 0 \leq t \leq \bar{T} \quad (136)$$

and by $L(t) = L(\bar{T})$, if $t > \bar{T}$. We introduce, for $t \geq 0$, the random variable

$$\mu(t) = t + \int_0^t (\|m_s\|_{E^s} + \sum_{i \in \mathbb{I}} \|\sigma_s^i\|_{E^s}^2) ds, \quad \text{if } 0 \leq t \leq \bar{T} \quad (137)$$

and $\mu(t) = t - \bar{T} + \mu(\bar{T})$ if $t > \bar{T}$. μ is a.s. strictly increasing, absolutely continuous and on-to $[0, \infty[$. The inverse τ of μ also have these properties and $\tau(t) \leq t$. For a continuous E^s -valued processes Y on $[0, \bar{T}]$ we introduce

$$\rho_t(Y) = (E(\sup_{s \in [0, t]} \|Y(\tau(s))\|_{E^s}^2))^{1/2}, \quad (138)$$

for $t \in [0, \infty[$, where we have defined $Y(t)$ for $t > \bar{T}$ by $Y(t) = Y(\bar{T})$. We note that $\rho_t(Y) \leq (E(\sup_{s \in [0, t]} \|Y(s)\|_{E^s}^2))^{1/2}$, since $\tau(t) \leq t$.

Lemma 48 *If condition (135) is satisfied and if Y is an \mathcal{A} -progressively measurable E^s -valued continuous process on $[0, \bar{T}]$, satisfying $\rho_t(Y) < \infty$, for all $t \geq 0$, then the equation*

$$X(t) = Y(t) + \int_0^t \mathcal{L}_{t-s} X(s) (m_s ds + \sum_{i \in \mathbb{I}} \sigma_s^i dW_s^i), \quad (139)$$

in $t \in [0, \bar{T}]$, has a unique solution X , in the set of \mathcal{A} -progressively measurable E^s -valued continuous process satisfying:

$$\int_0^{\bar{T}} (\|X(t)\|_{E^s} + \|X(t)m_t\|_{E^s} + \sum_{i \in \mathbb{I}} \|X(t)\sigma_t^i\|_{E^s}^2) dt < \infty \text{ a.s.} \quad (140)$$

Moreover this solution satisfies:

i) If $\int_0^{\bar{T}} (\|m_t\|_{E^{s+1}} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^{s+1}}^2) dt < \infty$ and Y is a continuous E^{s+1} -valued process with $\rho_t(\partial Y) < \infty$, for all $t \geq 0$, then X is a continuous E^{s+1} -valued process.

ii) If (i) is satisfied and if Y is a semi-martingale, then X is a semi-martingale.

iii) If Y is H^s -valued, then X is H^s -valued.

The next lemma establish conditions under which the solution of equation (139) is in L^p , $p \in [0, \infty[$. The notation $\tilde{\mathcal{E}}$ was introduced in (58).

Lemma 49 *Let condition (135) be satisfied and let (i) $E(\exp(p \int_0^{\bar{T}} (\|m_t\|_{E^s} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^s}^2) dt)) < \infty$, for each $p \in [1, \infty[$. Suppose that Y in Lemma 48 satisfies (ii) $E(\sup_{t \in \mathbb{T}} \|Y(t)\|_{E^s}^p) < \infty$, for each $p \in [1, \infty[$. Then the unique solution X of equation (139) in Lemma 48 satisfies*

$$E(\sup_{t \in \mathbb{T}} \|X(t)\|_{E^s}^p) < \infty, \quad \forall p \in [1, \infty[. \quad (141)$$

Moreover if (iii) $E((\int_0^{\bar{T}} (\|m_t\|_{E^{s+1}} + \sum_{i \in \mathbb{I}} \|\sigma_t^i\|_{E^{s+1}}^2) dt)^p) < \infty$ and (iv) $E(\sup_{t \in \mathbb{T}} \|Y(t)\|_{E^{s+1}}^p) < \infty$, for each $p \in [1, \infty[$, then also

$$E(\sup_{t \in \mathbb{T}} \|X(t)\|_{E^{s+1}}^p) < \infty, \quad \forall p \in [1, \infty[. \quad (142)$$

In particular, estimates (141) and (142) applies to $X = \tilde{\mathcal{E}}(L)$.

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