Myopia and Inconsistency in Dynamic Utility Maximization [1]

" but you must bind me hard and fast, so that I cannot stir from the spot where you will stand me . . . and if I beg you to release me, you must tighten and add to my bonds."—*The Odyssey.*

I. INTRODUCTION

This paper presents a problem which I believe has not heretofore been analysed[2] and provides a theory to explain, under different circumstances, three related phenomena : (1) spendthriftiness ; (2) the deliberate regimenting of one's future economic behaviour—even at a cost; and (3) thrift. The senses in which we deal with these topics can probably not be very well understood, however, until after the paper has been read ; but a few sentences at this point may shed some light on what we are up to.

An individual is imagined to choose a plan of consumption for a future period of time so as to maximize the utility of the plan as evaluated at the present moment. His choice is, of course, subject to a budget constraint. Our problem arises when we ask: If he is free to reconsider his plan at later dates, will he abide by it or disobey it—*even though his original expectations of future desires and means of consumption are verified* ? Our answer is that the optimal plan of the present moment is generally one which will *not* be obeyed, or that the individual's future behaviour will be inconsistent with his optimal plan. If this inconsistency is not recognized, our subject will typically be a " spendthrift," a term which has had no meaning in existing utility theory but which becomes explicated in the theory presented here. If the inconsistency is recognized, the rational individual will do one of two things. He may " precommit " his future behaviour by precluding future options so that it will conform to his present desire as to what it should be. Or, alternatively, he may modify his chosen plan to take account of future disobedience, realizing that the possibility of disobedience imposes a further constraint—beyond the budget constraint—on the set of plans which are attainable. It is in this way that the individual becomes " thrifty "—a term which also acquires meaning in the context of the analysis. What is crucial to all this is that the discount applied to a future utility should depend on the time-distance from the present date and not upon the calendar date at which it occurs. In a final section, we shall have some things to say about the meaning of " consumer sovereignty " in the framework of dynamic choice.

[1] I am indebted to many colleagues at Northwestern University and elsewhere who have commented helpfully on this manuscript, and I am especially indebted to Mr. Fred Westfield and Dr. Alvin Marty for many sustained and fruitful discussions of the subject as well as to M. Jacques Drèze, Prof. Harry Johnson, Prof. R. Solow, Mr. S. A. Thore, Professor Gerhard Tintner, Professor H. Wold, and members of the seminar of the Department of Applied Economics, Cambridge University, for criticisms rendered. The usual caveat protecting these courteous people from further responsibility is, of course, in order.

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[2] But it has been alluded to by Paul A. Samuelson, " A Note on Measurement of Utility," *Review of Economic Studies*, IV, 2 (Feb., 1937), esp. p. 160. See also Friedrich A. Hayek, *The Pure Theory of Capital* (1941), p. 218, and M. Allais, *L'Economie et Intérêt,* Annexe 3.

II. THE UTILITY OF THE CONSUMPTION PLAN

The general problem of intertemporal utility maximization is that of an economic decision-maker who must choose among various possible functions relating his economic activities to time. The decision-maker may be a firm or an individual and the object of choice may be the variation through time of one or more economic magnitudes such as corporate sales, profits, the supply of a labor service, the consumption of a given commodity, or the expenditure of income. It suits our purpose, however, to deal with a quite simple version of this problem and later to show the considerable extent to which our results can be generalized. It is only as a standard illustration, then, that we have chosen to think of an individual who must decide how a single economic magnitude, his consumption, is to vary over some period of time subject to a simple budget constraint. As a further simplification we abstract from all considerations of risk and uncertainty. This may disturb those who feel that the essence of dynamical problems is thereby ignored ; but I think it will become clear as we go along that to introduce risk and uncertainty would only clutter up the analysis and prevent our getting a clear view of the particular issues to be considered here. Risk and uncertainty do have some bearing on the topic, but that may best be examined much later on.

To begin, assume that an individual must choose at time $t = \tau$ among a (possibly infinite) number of alternative time paths of consumption each of which is certain. For example, in Figure 1 the curves $C_1, \ldots, C_5$ represent functions relating consumption to time. These curves, which may be called " consumption-time " curves, are the objects of choice and are defined over a specified time period, $0 \leqslant t \leqslant T$. The case in which the period is infinite need not be excluded, but, to fix ideas, we suppose that it is finite. This is merely

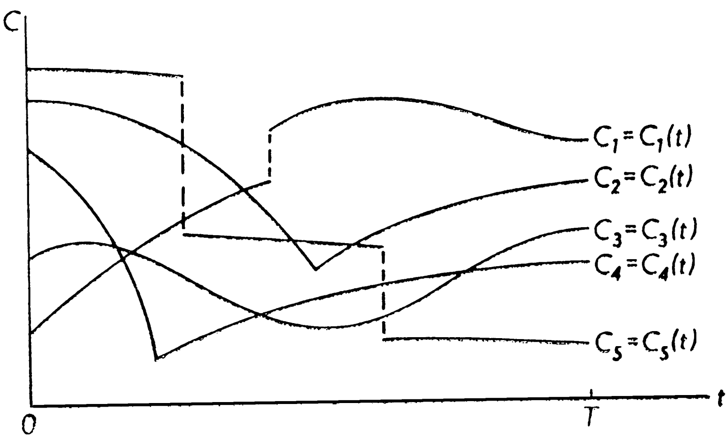

Figure 1

to assume that the individual has only idle curiosity in what happens after several millenia. Each function $C(t)$ is, of course, single-valued and bounded from both above and below, and time is treated as a continuous variable.[1]

The next step is to assume that an individual who faces the alternatives $C_1(t)$, $C_2(t)$, . . .

[1] There is no difficulty in regarding time as discrete and one can then employ the method of Lagrange multipliers rather than the Calculus of Variations used here. I find that some issues can be treated more smoothly by dealing with the continuous case, however, and prefer the approach taken here.

can order them transitively and that his ordering can be represented by a *utility functional*[1]

$$\Phi_\tau = \Phi_\tau \begin{Bmatrix} T \\ C(t) \\ 0 \end{Bmatrix} \tag{1}$$

or any monotonic increasing function thereof. The subscript τ appears on the functional to indicate that the preferences expressed are those at time τ although choices are among consumption paths for the entire period.

To derive any interesting theorems from the analysis, however, one must assume more than just the transitivity of preferences among the consumption-time curves. Accordingly, I suppose that the ordering of these curves is such that the utility functional can be written as :

$$\Phi_\tau = \int_0^T \lambda(t - \tau)\ u\,[C(t), t]\,dt, \tag{2}$$

where $u[C(t),\ t]$ is an " instantaneous utility function "[2] assigning at each time t a value $u(t)$ to $C(t)$, and where $\lambda(t - \tau)$ is a weight or discount function whose value depends notably on the time-distance between a future (or past) date t and the present date τ. Expression (2) may be normalized and this we do by setting $\lambda(0) = 1$.

In our later effort at generalization (Section VII) we find it crucial that the utility functional should be representable as an integral with respect to time and that the integrand function can, as in this illustration, be factored into two functions, one depending only on the time distance of future (or past) consumption from the present date, and the other being independent of the present date. What changes then as time (τ) marches on is only the discount function $\lambda(t - \tau)$, and it undergoes only a linear shift.

If $\tau > 0$, one may question the relevance for choice making of the range of integration from 0 to τ. For the especially simple functional given by (2), the value of $\int_0^\tau \lambda(t - \tau) \cdot u[C(t),\ t]\,dt$ is historically given. In the more general case, however, this is not technically necessary, because the u function might contain as one of its arguments a lead value of C, *e.g.*, $C(t + \theta)$. We prefer, therefore, to integrate from 0, noting that the budget constraint to be introduced later requires that $\begin{Bmatrix} \tau \\ C(t) \\ 0 \end{Bmatrix}$ be taken as given, anyway. This also allows (trivially) for the possibility that a person is not indifferent to his consumption history but enjoys his memories of it.

The relative weight which a person may assign to the satisfaction of a future act of consumption (the manner of discounting) may depend on either or both of two things : (1) the *time distance* of the future date from the present moment, or (2) the *calendar date*

[1] A " functional " is simply a function of the form of a function, or, more exactly, a function of an infinite number of variables. In (1), for example, Φ_τ depends not on any particular value assigned $C(t)$, but on the form of the function $C(t)$ as t goes from 0 to T. Change the form of the function and you change Φ_τ. The economist will find a good introduction to the topic and to the related subject of the " calculus of variations " in R. G. D. Allen, *Mathematical Analysis for Economists*, ch. XX. For early applications of the calculus of variations to utility problems, see G. Tintner, " Distribution of Income Over Time," *Econometrica*, 1936, and " Maximization of Utility Over Time," *Econometrica*, 1938.

[2] This is doubtless a misnomer, although the term may facilitate one's understanding in the first instance of its use. It would be more precise to say that u is a real number assigned to $[C(t),t]$ such that the individual may be said to maximize a weighted sum (integral) of numbers assigned in this way.

Otherwise, it is awkward to think of u as a utility " experienced " at a point of time when in a later section we allow the possibility of its depending on the consumption of a *later* date.

of the future act of consumption. The weight I assign to my pleasure (u function) in drinking champagne next September 26 may depend either on the fact that that date is a certain length of time away from the present or on the fact that that is my birthday. To the extent that time-distance is important, I may assign a different (and probably higher) weight to September 26 as it draws nigh ; if only the calendar date is important, the weight will not change as that date approaches.[1] Both bases for discounting a future date are included in the functional (2). The importance of the calendar date enters through the appearance of t in the instantaneous utility function, whereas the importance of time distance is given by $\lambda(t - \tau)$. The distinction between these two causes of discounting is commonly overlooked, because it has no consequence in those theories which regard the present date as fixed.[2] A truly dynamic theory of utility maximization must, however, assume that the present date changes, and, as we shall see, the distinction is then an important one.

The reader may wonder whether the instantaneous utility function could be replaced in (2) with any monotonically increasing function thereof; the answer is that it cannot. It is determined up to a linear transformation.[3] Just as the von Neumann-Morgenstern utility function is determined up to a linear transformation by assuming that the individual behaves so as to maximize the probability-weighted sum of the utilities resulting from the various outcomes of a gamble, so too is this function specified by assuming that an individual acts as if to maximize a weighted sum of instantaneous utilities arising at different points of time.[4]

III. THE OPTIMAL PLAN AS SEEN TO-DAY

In this section we explore the following preliminary problem. A consumer at time $\tau = 0$ wishes to maximize his utility functional :

$$\Phi_0 = \int_0^T \lambda(t - 0) \cdot u\,[C(t),\, t]\; dt \tag{3}$$

[1] Of course, it could drop to zero once the date is passed. This possibility is excluded in the functional (2), as is any interaction between the time-distance and calendar-date bases of discounting ; otherwise either λ or u or both would have both $t - \tau$ and t as arguments.

[2] I rather assume that in speaking of economic myopia (*The Economics of Welfare*) Professor Pigou had the time-distance concept in mind. Certainly Jevons did when he wrote: " people of good sense will not discount the future except for uncertainty—but people do discount the future in accordance with its remoteness." (*The Theory of Political Economy*, pp. 77-80), and Böhm-Bawerk : " It is one of the most pregnant facts of experience that we attach a less importance to future pleasures and pains simply because they are future, and in the measure that they are future." (*The Positive Theory of Capital*, tr. by W. Smart, p. 253). For other examples see also Alfred Marshall, *Principles of Economics*, 8th ed., p. 120, Adolphe Landry, *L'Interet du Capital* (1904), ch. X, §150 and E. C. K. Gonner, *Interest and Saving* (1906), p. 36.

[3] See, on this point, Paul A. Samuelson, *op. cit.*

[4] There is no presumption that these two measures of utility are the same. Measurement is arbitrary and for different purposes different measures may be the most convenient. Cf. my statement in " Cardinal Utility," *American Economic Review*, XLIII, 2 (May, 1953), p. 397 : " Furthermore, the acceptance of the von Neumann-Morgenstern measure does not preclude the definition of still other measures. It is true that the von Neumann-Morgenstern measure is convenient and manageable for the class of problems involving risk, but it need not prove convenient for all classes of utility problems that may conceivably arise. Nothing rules out the usefulness of another measure for another purpose."

If we were to introduce risk explicitly into the present analysis, we would have to write:

$$\Phi_\tau = \int_0^T \lambda\,(t - \tau) \cdot u\,\Big[\int_0^\infty p(C,t) \cdot v(C)\, dC,\; t\Big]\, dt,$$

where $p(C,t)$ is the probability density at t of consumption C and where $v(C)$ is the von Neumann-Morgenstern measure of the utility of C. Probabilities are, of course, introduced here in a very special way, being made to depend *only* on C and t.

with respect to $\left\{\begin{matrix} T \\ C(t) \\ 0 \end{matrix}\right\}$ and subject to the constraint :

$$\int_0^T C(t)\ dt = K(0) \tag{4}$$

where $K(0)$ is a constant stock at $t = 0$. The constraint here is a simple one, but it will do for our purpose.[1]

The present problem is one in the calculus of variations and, assuming piecewise differentiability, first-order conditions for a maximum may be found as follows.

Define :

$$y(t) = \int_0^t C(t)\ dt \tag{5}$$

so that:

$$\begin{aligned} y(0) &= 0, \\ y(T) &= K(0) \end{aligned} \tag{6}$$

and :

$$\mathring{y} = C(t). \tag{7}$$

(*The dot above the y indicates a derivative with respect to time*).

The problem then is to maximize :

$$\int_0^T \lambda(t - 0) \cdot u[\mathring{y}(t), t]\ dt = \int_0^T \lambda(t) \cdot u[\mathring{y}(t), t]\ dt \tag{8}$$

subject to the fixed end points given by (6). This is an elementary problem now in standard form, and the solution is provided by Euler's differential equation :

$$\frac{d}{dt}\left[\frac{\partial}{\partial \mathring{y}}\ \lambda(t)u[\mathring{y}(t), t]\right] = \frac{\partial}{\partial y}\lambda(t)u[\mathring{y}(t), t] \tag{9}$$

which, upon substituting $C(t)$ for $\mathring{y}(t)$ and letting $u_C(t) = \partial u[C(t), t]/\partial C$, is :

$$\lambda(t)\mathring{u}_C(t) + u_C(t) \quad \lambda^\circ(t) = 0, \qquad 0 \leqslant t \leqslant T. \tag{10}$$

(10) may be written as :

$$\lambda^\circ(t)/\ \lambda(t) = -\mathring{u}_C(t)/u_C(t), \qquad 0 \leqslant t \leqslant T, \tag{11}$$

or as :

$$d \ln \lambda(t)/dt = -d \ln u_C(t)/dt, \qquad 0 \leqslant t \leqslant T. \tag{12}$$

The solution to this differential equation is simply :

$$\lambda(t) \cdot u_C(t) = \text{constant}, \qquad 0 \leqslant t \leqslant T, \tag{13}$$

with the constant dependent on $K(0)$. This is to say that the " stock of consumption " $K(0)$ must be distributed over the interval 0 to T so that the discounted marginal utility of consumption is the same for all dates. This is, of course, a quite obvious condition, and, in view of the absence of intertemporal complementarity, will assure us of a maximum for Φ_0 provided that the instantaneous utility function displays a diminishing marginal rate of utility. This would be the case for any consumer who does not concentrate his entire consumption for the period at a single point of time.

[1] Often, for a constraint of this sort (as in the case of a man who must ration fresh water to himself during an ocean voyage), there is the additional requirement that $C(t) \geqslant 0$ for all t, but such a condition is not imposed in what follows.

IV. THE QUESTION OF INCONSISTENCY

Equation (11) along with (4) therefore picks out from all possible consumption-time curves that one which is optimal for the consumer at $\tau = 0$. But it would be a mistake to conclude that, *even under conditions of certainty*, the optimal curve is the one which the individual will actually follow. The difficulty arises because all we really know is that this is the curve he will *start* to follow. It is his best consumption plan at $\tau = 0$. At a later date, $\tau > 0$, he may (or must, if he is to maximise Φ_τ, $\tau > 0$) reconsider his plan, and τ cannot then be dropped from (8) because it is no longer zero.[1] The problem then is to maximize :

(14) $$\int_0^T \lambda\,(t - \tau)\, u[C(t), t]\, dt$$

with respect to $\left\{ \begin{matrix} T \\ C(t) \\ 0 \end{matrix} \right\}$ and subject to :

(15) $$\begin{cases} \left\{ \begin{matrix} \tau \\ C(t) \\ 0 \end{matrix} \right\} \text{ given, and} \\ \displaystyle\int_\tau^T C(t)dt = K(\tau), \end{cases}$$

where :

(16) $$K(\tau) = K(0) - \int_0^\tau C(t)dt.$$

Define :

(17) $$y(t) = \int_0^t C(t)dt$$

so that :

(18) $$\begin{aligned} y(0) &= 0 \\ y(\tau) &= K(0) - K(\tau) \\ y(T) &= K(0) \end{aligned}$$

and :

(19) $$\mathring{y}(t) = C(t).$$

The solution is given by :

(20) $$\lambda^\circ(t - \tau)/\,\lambda\,(t - \tau) = -\mathring{u}_C(t)/u_C(t), \qquad \tau \leqslant t \leqslant T,$$

with $K(\tau)$ determining the constant of integration. This solution may be entirely different from that of (11) because the discount function has been shifted. To continue to obey a fixed consumption plan just because it was otpimal when viewed at an earlier date is not

[1] Following a suggestion by P. N. Rosenstein-Rodan ("The Role of Time in Economic Theory," *Economica*, Feb., 1934, p. 84), we may want the economic horizon to move uniformly with an increase in τ. Although T remains constant, any horizon in the utility functional may be considered to change under the following assumptions. Let the horizon at $\tau = 0$ be H, $0 < H < T$, with H defined by the condition that $\lambda(t) = 0$ for $H < t < T$. Then at time τ, the horizon may be taken as $H + \tau$ with $\lambda\,(t - \tau)$ always zero for $t - \tau > H$, provided $H + \tau \leqslant T$. Our analysis is valid then over the period $0 \leqslant \tau \leqslant T - H$. It is to be noted, however, that there is no moving horizon possible in the budget constraint considered here.

rational if that plan is not the optimal one at the present date. The best plan will generally change with a change in τ, and there is nothing patently irrational about the individual who finds that he is in an intertemporal tussle with himself—except that rational behaviour requires he take the prospect of such a tussle into account.

Should the individual re-evaluate his plan periodically, his actual behaviour could be described in terms of the graph below. C_0 may be his best plan at $\tau = 0$; C_1 may be his best plan at $\tau = \tau_1$; C_2 at $\tau = \tau_2$; etc. If he does not reconsider his originally best plan during the period $0 \leqslant \tau < \tau_1$, he abides by it and follows C_0 during that period as is indicated by the heavy part of the curve. At τ_1, however, he reconsiders and, taking into account that the " stock of consumption " available from then until T is the original amount minus what he has already consumed, he chooses C_1 as his best plan and follows it (along the heavy portion) until τ_2 when he reconsiders once more, etc. This means his actual behaviour is represented by the sequence of heavy arcs shown in the figure. The integral of this sequence of arcs must, of course, be $K(0)$.

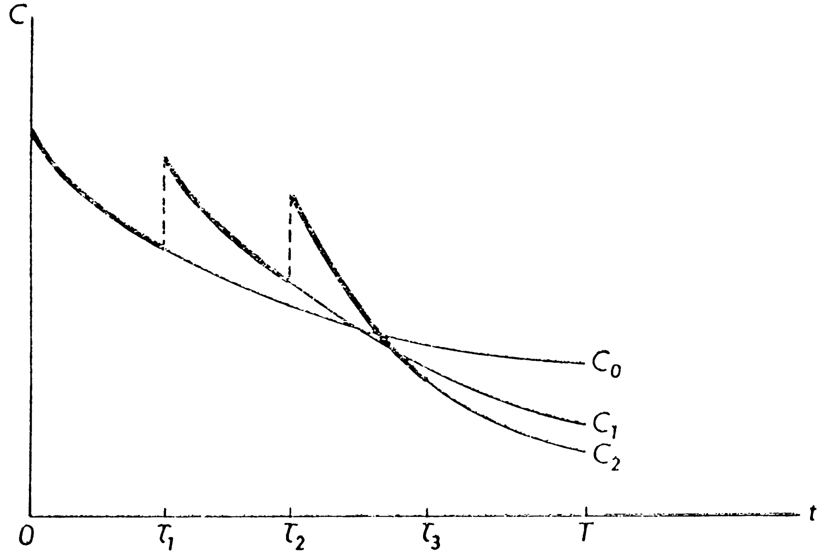

Figure 2

If the plan is re-evaluated continuously, any single plan chosen has validity only at $t = \tau$, and actual behaviour is then given by the locus of the $C(t)$ for $t = \tau$ as determined by (15) and (20) *as τ proceeds from* 0 *to* T.

In this section we have considered the actual *dynamics* of utility maximization, as distinct from the mere plan for the future which is made at a given moment, and have questioned whether the actual path of consumption over the period would be the same as that which is chosen as optimal at the beginning of the period. In the next section we show that it need not be.

V. CONSISTENCY, INCONSISTENCY, AND THE NATURE OF THE DISCOUNT FUNCTION

Under what circumstances will an individual who continuously re-evalutes his planned course of consumption confirm his earlier choices and follow out the consumption plan originally selected ?

This requires that if $\left\{\begin{matrix}\tau' \\ C^*(t) \\ 0\end{matrix}\right\}$ and $\left\{\begin{matrix}T \\ C^*(t) \\ \tau'\end{matrix}\right\}$ mazimixe

(21a) $$\int_0^{\tau'} \lambda(t-0)\cdot u[C(t),\, t]dt + \int_{\tau'}^{T} \lambda(t-0)\cdot u[C(t),\, t]dt$$

subject to $\int_0^{\tau'} C(t)dt + \int_{\tau'}^{T} C(t)dt = K(0)$, then $\left\{\begin{matrix}T \\ C^*(t) \\ \tau'\end{matrix}\right\}$

should maximize

(21b) $$\int_{\tau'}^{T} \lambda(t-\tau')\cdot u[C(t)]dt$$

subject to $\int_{\tau'}^{T} C(t)dt = K(0) - \int_0^{\tau'} C^*(t)dt$, or that $\left\{\begin{matrix}T \\ C^*(t) \\ \tau'\end{matrix}\right\}$

should be the solution both to

(22a) $$\lambda^\circ(t-0)/\lambda(t-0) = -\mathring{u}_c(t)/u_c(t), \qquad \tau' \leqslant t \leqslant T,$$

and

(22b) $$\lambda^\circ(t-\tau')/\lambda(t-\tau') = -\mathring{u}_c(t)/u_c(t), \qquad \tau' \leqslant t \leqslant T.$$

This must hold for all τ'. Equating the left-hand sides of (22a) and (22b), it is clear that as a necessary and sufficient condition the logarithmic rate of change in the discount function must be a constant, so that for $\tau = 0$.

(23) $$d \ln \lambda(t)/dt = \ln k \text{ (a constant)}, \qquad 0 \leqslant t \leqslant T,$$

or :

(24) $$\lambda(t) = k^t, \qquad 0 \leqslant t \leqslant T.$$

This we shall call the harmony case. It requires that the discount function be of a very special form, namely that all future dates should be discounted at a constant rate of interest.[1]

This is what our intuition should lead us to expect. In the language of the discrete case, a discount function of this sort means that the relative importance of 1957 and 1958 is the same in 1957 as in 1956. Consequently, when in 1956 one decides how to apportion consumption between 1957 and 1958, this is the same decision one would make in 1957. Thus, in 1957 the plan laid down in 1956 is confirmed.

But so far we have adduced no reason why an individual should have such a special discount function, *i.e.*, no reason why the defect in the telescope that Professor Pigou spoke of should be logarithmically linear with respect to the distance of the object being viewed. Indeed, if it is believed that this special case is realistic, a rationale is needed. We provide one in the next section.

[1] The reader is cautioned that this is not the same as the familiar proposition that the marginal rate of substitution between present and future consumption is determined in equilibrium by discounting at the market rate of interest. The marginal rate of substitution between consumption at time 0 and at time t is, for the utility functional used here, equal to:

$$\lambda(t)\, \frac{\partial u}{\partial C(t)} \Big/ \lambda(0)\, \frac{\partial u}{\partial C(0)}$$

and not to $\lambda(t) / \lambda(0)$. If one unit of consumption at time 0 can be exchanged for $(1 + r)^t$ units at time t, where r is the rate of interest, utility maximization requires that:

$$\frac{1}{(1+r)^t} = \frac{\lambda(t)}{\lambda(0)}\, \frac{\partial u \,/\, \partial C(t)}{\partial u \,/\, \partial C(0)},$$

which is quite different from the requirement that:

$$\lambda(t) = \left(\frac{1}{1+r}\right)^t \lambda(0) = k^t \lambda(0).$$

VI. TWO STRATEGIES IN THE FACE OF INCONSISTENCY

An individual who because he does not discount all future pleasures at a constant rate of interest finds himself continuously repudiating his past plans may learn to distrust his future behaviour, and may do something about it. Two kinds of action are possible. (1) He may try to precommit his future activities either irrevocably or by contriving a penalty for his future self if he should misbehave. This we call the *strategy of precommitment.* (2) He may resign himself to the fact of intertemporal conflict and decide that his "optimal" plan at any date is a will-o'-the-wisp which cannot be attained, and learn to select the present action which will be best in the light of future disobedience. This we call the *strategy of consistent planning.* These possibilities will be discussed in order.

1. *The Strategy of Precommitment.* To-day it will be rational for a man to jettison his "optimal" plan of yesterday, not because his tastes have changed in any unexpected way nor because his knowledge of the future is different, but because to-day he is a different person with a new discount function—the old one shifted forward in time. Yet it is also rational for the man to-day to try to ensure that he will do tomorrow that which is best from the standpoint of to-day's desires. Unpleasant things which to-day we want to do sometime in the future are continually put off until tomorrow (the "*mañana* effect") unless we can find some way of precommitting ourselves to actually doing the task tomorrow. Consequently, we are often willing even to pay a price to precommit future actions (and to avoid temptation). Evidence of this in economic and other social behaviour is not difficult to find. It varies from the gratuitous promise, from the familiar phrase "Give me a good kick if I don't do such and such" to savings plans such as insurance policies and Christmas Clubs which may often be hard to justify in view of the low rates of return. (I select the option of having my annual salary dispersed to me on a twelve- rather than on a nine-month basis, although I could use the interest!) Personal financial management firms, such as are sometimes employed by high-income professional people (*e.g.*, actors), while having many other and perhaps more important functions, represent the logical conclusion of the desire to precommit one's future economic activity. Joining the army is perhaps the supreme device open to most people, unless it be marriage for the sake of "settling down." And, of course, regretting either course later on (at least for the moment) is to be expected, for otherwise precommitment would have had no purpose. The worker whose income is garnished *chronically* or who is continually harassed by creditors, and who, when one oppressive debt is paid, immediately incurs another is commonly precommitting. There is nothing irrational about such behaviour (quite the contrary) and attempts to default on debts are simply the later consequences which are to be expected. Inability to default is the force of the precommitment.

What needs to be explained is not that people do precommit their future actions, but that the practice is not still more wide-spread. The reason it is not, I believe, is because of the presence of risk and uncertainty, both as to future tastes and future opportunities. Because of risk and uncertainty, people are also willing to pay for options permitting them a greater range of choice at future dates[1] and this is of overwhelming importance, especially as it affects the detailed aspects of future behaviour.

2. *The Strategy of Consistent Planning.* Since precommitment is not always a feasible solution to the problem of intertemporal conflict, the man with insight into his future unreliability may adopt a different strategy and reject any plan which he will not follow through. His problem is then to find the best plan among those that he will actually follow.

[1] See Tjalling C. Koopmans, "Utility Analysis of Decisions Affecting Future Well-Being," (abs.), *Econometrica*, April, 1950, 18 : 174-175, and "La notion d'utilité dans le cas de décisions concernant le bien-être futur", *Cahiers du Séminaire d'Econométrie*, ed. René Roy, 1953.

Returning to the narrow framework of our consumption problem, let the best plan of consumption be given by the function $\left\{\begin{matrix} T \\ \mathring{z}(t) \\ 0 \end{matrix}\right\}$. As a solution, $\mathring{z}(t)$ has the property that at the limit where $\triangle\tau \longrightarrow 0$:

$$(25) \qquad \int_0^{\tau} \lambda(t-\tau)\cdot u\,[\mathring{z}(t), t]\,dt + \int_{\tau}^{\tau+\triangle\tau} \lambda(t-\tau)\cdot u\,[\mathring{z}(t), t]\,dt + \int_{\tau+\triangle\tau}^{T} \lambda(t-\tau)\cdot u\,[\mathring{z}(t), t]\,dt$$

is a maximum of:

$$(26) \qquad \int_0^{\tau} \lambda(t-\tau)\cdot u\,[\mathring{z}(t), t]\,dt + \int_{\tau}^{\tau+\triangle\tau} \lambda(t-\tau)\cdot u\,[\mathring{y}(t), t]\,dt + \int_{\tau+\triangle\tau}^{T} \lambda(t-\tau)\cdot u\,[\mathring{z}(t), t]\,dt$$

with respect to the function $\left\{\begin{matrix} \tau+\triangle\tau \\ \mathring{y}(t) \\ \tau \end{matrix}\right\}$ when the individual at τ can force himself to follow during the period τ to $\tau + \triangle\tau$ any plan which he may select subject to the constraint that $\int_{\tau}^{\tau+\triangle\tau} \mathring{y}(t)dt = K(0) - \int_0^{\tau} \mathring{z}(t)dt - \int_{\tau+\triangle\tau}^{T} \mathring{z}(t)dt$, a constant.

To maximize (26) requires only that the middle term be maximized. This gives :

$$(27) \qquad \lambda^{\circ}(t-\tau)/\,\lambda(t-\tau) = -\,\mathring{u}_C(t)/u_C(t), \qquad \tau \leqslant t < \tau+\triangle\tau,$$

so that as $\triangle\tau \longrightarrow 0$, the condition on $\mathring{z}(t)$ becomes :

$$(28) \qquad \lambda^{\circ}(0)/\,\lambda(0) = -\,\mathring{u}_C(\tau)/u_C(\tau).$$

Since this must hold for every τ it may be written :

$$(29) \qquad \lambda^{\circ}(0)/\,\lambda(0) = -\,\mathring{u}_C(t)/u_C(t), \qquad 0 \leqslant t \leqslant T,$$

which is the solution to the " harmony case," given by :

$$(30) \qquad \int_0^{T} k^{t-\tau}\, u\,[\mathring{y}(t), t]\,dt.$$

Again, this should not be in any way surprising because only those plans that maximize a functional such as (30) are attainable (will be obeyed). The individual must, therefore, first substitute for his true discount function one that is linear in the logarithm and then maximize. The appropriate value for k has already been determined by the analysis which has preceded. At each point of time the individual equates $-\,\mathring{u}_C(t)/u_C(t)$ to $\lambda^{\circ}(0)/\,\lambda(0)$, where $\lambda\,(t-\tau)$ is his true discount function. Since, when acting in this way he also equates $-\,\mathring{u}_C(t)/u_C(t)$ to $\dfrac{dk^{t-\tau}}{dt}\,/k^{t-\tau}$ it follows that :

$$(31) \qquad \mathrm{d}\,\ln\,k^{t-\tau}/dt = d\,\ln\,\lambda(t-\tau)/dt\,\Big|_{t=\tau}$$

Consequently :

(32) $$\ln k = \lambda^{\circ}(0)/\,\lambda(0) = \lambda^{\circ}(0)$$

(the second equality by normalization) or :

(33) $$k = \text{antilog}\ \lambda^{\circ}(0).$$

The individual therefore abides by the consumption-time function which maximizes :

(34) $$\Phi = \int_0^T (\text{antilog}\ \lambda^{\circ}(0)\,)^{t-\tau}\, u\,[C(t),\, t]\, dt.$$

The subscript τ is deleted from Φ because Φ is the same functional for all τ under these circumstances.

A graphic interpretation of this result is given by Figure 3. For the one-parameter family of functions $k^{t-\tau}$, that one is chosen which is tangent at $t = \tau$ to the true discount function $\lambda(t - \tau)$. For the " strategical " man who cannot precommit his future conduct it is now clear that the only relevant characteristic of his true discount function is the rate at which it changes at the *present* moment (at $t = \tau$).[1]

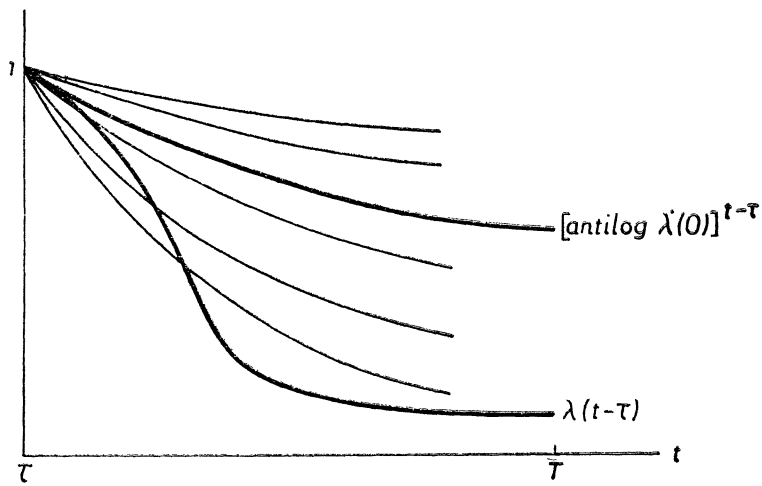

Figure 3

[1] A word on the rationale of this solution is in order. We think of all deliberate action through time as involving precommitments over successive intervals, if the intervals are sufficiently small ; and in the mathematical treatment here we are simply considering the limit form of such behaviour where those small intervals approach zero. Actually, it is impossible for an individual to choose his rate of consumption at *every* time τ independently of his consumption-time path immediately preceding and still have piecewise differentiability, for then we brush up against Taylor's Theorem. (Consider the person who can vary manually the flow of water out of a tap and the deliberation of which he is capable while letting water flow out.) The solution we have obtained in this section is, however, entirely consistent with Taylor's Theorem.

In the discrete case two different interpretations of the problem are possible : (1) Each period the individual selects his consumption for that period and there is no intra-period discounting or allocation problem. (2) At alternative periods the individual selects his consumption for that period and the next and allocates this amount between the two periods. It is only the latter interpretation which is comparable with our continuous model.

A theory based on interpretation (1) appears to be more complex and less fruitful in its implications, although similar to the present theory in that strategies of precommitment and of consistent planning arise. The author's work on this alternative approach is, however, not yet complete.

VII. GENERALIZATION

What becomes of our results if more general utility functionals and budget constraints are considered ?

Suppose that :

$$\Phi_\tau = \int_0^T \lambda(t - \tau) \cdot u[\ldots, t]\, dt \tag{35}$$

is to be maximized with respect to a variety of functions $\left\{ \begin{matrix} T \\ C_1(t) \\ 0 \end{matrix} \right\}$, $\left\{ \begin{matrix} T \\ C_2(t) \\ 0 \end{matrix} \right\}$, . . ., each relating an economic variable to time. Let u depend on these functions in as complicated a way as we please, *provided only that τ does not appear in the u function.* u may, for example have as its arguments various integrals, derivatives, and lead or lagged values of $\left\{ \begin{matrix} T \\ C_1(t) \\ 0 \end{matrix} \right\}$, $\left\{ \begin{matrix} T \\ C_2(t) \\ 0 \end{matrix} \right\}$, . . . Moreover, let the maximization problem be subject to constraints of whatever complexity, provided only that there exists a solution *and that τ does not enter into these constraints*, except that $\left\{ \begin{matrix} \tau \\ C_1(t) \\ 0 \end{matrix} \right\}$, $\left\{ \begin{matrix} \tau \\ C_2(t) \\ 0 \end{matrix} \right\}$, . . . are taken as historically given.

What is the necessary and sufficient condition on $\lambda(t - \tau)$ in order that planning should be consistent ? This requires that whatever maximizes Φ_τ should maximize $\Phi_{\tau+\Delta\tau}$, for any $\Delta\tau$, $0 \leqslant \tau + \Delta\tau \leqslant T$. For this to occur $\Phi_{\tau+\Delta\tau}$ must be a monotonic increasing function of Φ_τ. Since τ does not enter in $u[\ldots, t]$ we need not concern ourselves with that function. The question then is, how must $\lambda(t - \tau - \Delta\tau)$ be related to $\lambda(t - \tau)$? It is not enough that these should be monotonically increasing functions of one another because of the variational nature of the problem. Each must be a positive scalar multiple of the other, so that the scalar can be factored out of the integrand function and placed in front of the integral sign. Thus, if $\lambda(t - \tau - \Delta\tau) = \text{constant} \times \lambda(t - \tau)$, the solution to which is $\lambda(t - \tau) = k^{t-\tau}$, with the constant ratio equal to $k^{-\Delta\tau}$, we have:

$$\begin{aligned} \Phi_{\tau+\Delta\tau} &= \int_0^T \lambda(t - \tau - \Delta\tau) \cdot u[\ldots, t]\, dt \\ &= k^{-\Delta\tau} \int_0^T \lambda(t - \tau) \cdot u[\ldots, t]\, dt = k^{-\Delta\tau}\, \Phi_\tau, \end{aligned} \tag{36}$$

and this is the same solution as was obtained for the illustrative case treated earlier. The important thing is that the relative weights of different dates should be invariant, and no harm is done to the analysis by considering this more general type of functional and constraint.

The qualifications underlined two paragraphs above are the significant ones. Should τ enter the problem in a more general way than by merely causing a linear shift of the discount function, the problem of consistent planning becomes more involved. However, even though the instantaneous utility function or the basic constraints do change with the passage of time, if the individual does not take these future changes into account in deciding what to do *now*, our analysis will still have validity for any fixed *point* of time.

VIII. THE DISCOUNT FUNCTION

Special attention should be given, I feel, to a discount function, such as that shown in Figure 3, which differs from a logarithmically linear one in that it " over-values " the more proximate satisfactions relative to the more distant ones.[1] Such a function suggests that individuals who precommit their future actions or who naively resolve now what they " will do " in the future, commonly do not schedule the beginning of austerity until a later date. How familiar the sentence that begins, " I resolve, starting next . . ." ! It seems very human for a person who decides that he ought to increase his savings to plan to start *next* month, after first satisfying some current desires ; or for one to decide to quit smoking or drinking *after* the week-end, or to say that " the next one is the last one."

It has been customary for the United States Army to offer voluntary enlistees a furlough starting with the date of enlistment. This practice is not needed to enable a man to put his affairs in order—he can do that first and then enlist—but it does serve as an enticement to those who want the paternalism (" security ") of the army, but do not want it right now.[2] The many schemes for instalment buying (notably of used automobiles in the U.S.) which require " no down payment and nothing due for two months " are evidence of the effectiveness of enticements of this same kind. Indeed, all purchases on credit can be viewed as precommitments that often (although not always) exchange future costs for a present pleasure.[3]

My own supposition is that most of us are " born " with discount functions of the sort considered here, that precommitment is only occasionally a feasible strategy (because of risk and uncertainty), and that we are taught to plan consistently by substituting the proper log-linear function for the true one. Children are known to discount the future most precipitously and the " virtue " of frugality is something to be instilled when building " character." True discount functions become sublimated by parental teaching and social pressure, and the inconsistency problem considered in this paper becomes lost from sight. There is a rationale for discounting at a constant rate of interest. In some cases training may be so effective that the individual's original discount function can no longer be said to be his " true " one. His tastes have changed and his discount function has become log-linear or perhaps even constant. His is the " harmony " case. Precommitment, therefore, is never attractive to him—even under certainty. In other cases,[4] however, we may say instead that a person has been taught to plan and behave consistently and not that his tastes have been molded. His is not the harmony case. Such a person will, from time to time, depart from the consistent pattern of behaviour, sometimes because precommitment becomes feasible (and this is always his preferred strategy under conditions because precommitment becomes feasible (and this is always his preferred strategy under conditions of certainty) and sometimes because of lapses that result when the true weight

[1] As an empirical supposition, there is a precedent for this in Böhm-Bawerk, *op. cit.*, pp. 257-258, ". . . the original subjective undervaluations are, in the highest degree, unequal and irregular. In particular, so far as the undervaluation is caused by defects of will, there may be a strong difference between an enjoyment which offers itself at the very moment, and one which does not ; while, on the other hand, there may be a very small difference, or no difference at all, between an enjoyment which is pretty far away, and one which is farther away." In context it is clear that he is referring here to changes in the *rate* of discount.
This same possibility was considered by Marshall, *op. cit.*, Mathematical Appendix, Note V.

[2] Since marriage was mentioned earlier alongside joining the army as a possible precommitment strategy, I cannot avoid remarking facetiously that marriage too is commonly preceded by a period of engagement!

[3] There are, of course, other important reasons for buying on credit.

[4] For the distinction made here I am indebted to Mr. W. B. Reddaway and others in attendance at a seminar on this paper at the Department of Applied Economics, Cambridge University.

function becomes momentarily ascendant. These lapses are the splurges, binges, and extravagances which we all know.[1]

This picture is typical, I suppose for most of us, but there are no doubt some who, either through lack of training or insight, have never learned to behave consistently and for whom the intertemporal tussle remains unsolved. These people we call " spendthrifts."[2] By contrast, those who have taken on log-linear discount functions have learned to be " thrifty."

Spendthriftiness, in the general sense of inconsistent or imprudent planning,[3] is by no means insignificant. It is especially among the lower-income classes, where education and training are commonly blighted, that one would expect to find imprudent behaviour of this sort.[4] In America, lower-income people tend to gorge themselves with food after pay-day ; overheat their homes when they have money for a bucket of coal ; are extravagant, going on sprees on pay-day, not budgeting their money, and engaging in heavy instalment buying ; do not keep their children in school ; and are freer in the expression of their sexual and aggressive impulses.[5] Their high birth rate is well-known. All these behaviour characteristics can be explained as a failure to cope intelligently with the problem of the intertemporal tussle. Obviously, this is not the entire story ; but the observations are consistent with the hypothesis presented, and it would be upsetting if the facts were otherwise.

The character of behaviour under precommitment is more difficult to label. Its results are somewhat ambiguous. Sometimes precommitment causes an individual to sacrifice future pleasures heavily for the sake of present ones, *e.g.*, to go into debt to make possible an expenditure providing mostly present gratification. But at other times precommitment seems more " wholesome,"[6] as when a person contracts to save a certain amount each month or goes into debt to buy a house. The distinction, I feel, is this : in the one case the present is heavily favored at the expense of the future ; in the other an allocation is made among various dates of the future in accordance with weights given by the lower, but more level portion of the $\lambda(t - \tau)$ function, as depicted in Figure 3. Precommitment then has the effect of precluding grossly unequal allocation *within* that future period of time once it moves into the present.

[1] To one who would say that to discount the future for remoteness at all seems to him foolish and irrational, I should reply that he is one who received very strong training as a child which went beyond simply teaching him the strategy of consistent planning and effected such a change in his tastes that he now finds it unnatural to discount the future on this account. Moralizing against discounting the future in this way has, of course, found its way into the prominent literature on this subject. See, *e.g.*, Böhm-Bawerk, *op. cit.*, pp. 253-255.

[2] If the $\lambda(t-\tau)$ function rose to the right of τ, inconsistency between plans and behaviour would lead to " miserliness " with the individual saving for a future planned expenditure which he continuously postpones. True miserliness would, however, appear to be better explained in other terms, money or wealth becoming an object of desire *per se*.

[3] The " spendthrift " might, of course, also be defined as anyone who discounts the future because of its remoteness, but I think this catches somewhat less satisfactorily the essence of the term because such a person may behave quite prudently—empty moralizing aside. Mr. Thore has pointed out, however, that my spendthrift might display more conservative behaviour than *another person* who has a lower k even though he plans consistently.

[4] Perhaps in some underdeveloped economies this problem would take on its most serious dimensions. The notion that the incidence of the sharp discounting of the future is greater among " primitive races, children, and other uninstructive groups in society " was asserted clearly by Fisher in *The Theory of Interest*, ch. IV, § 9, and earlier by Jevons in *The Theory of Political Economy*, ch. II and Böhm-Bawerk, *op. cit.*, p. 244.

[5] See W. Allison Davis, " Child Rearing in the Class Structure of American Society " in *The Family in a Democratic Society: Anniversary Papers of the Community Service Society of New York*, 1949 and Robert F. Winch, *The Modern Family*, 1952, pp. 93-94. These authors ascribe these behaviour patterns to anxiety feelings stemming from the insecurity that results from low and irregular income. It is not clear to me why this anxiety should lead to spendthriftiness rather than to miserliness or to painstaking family budgeting. If the facts were the opposite, would one feel any less comfortable with the explanation given?

[6] The word is in quotes because we do not intend to provide moral judgments at this point.

IX. CONSUMER SOVEREIGNTY

What becomes of the concept of consumer sovereignty for dynamic decision-making problems? To the extent that consumer sovereignty is one of our values, ought we to allow people to behave imprudently? Should we permit them the strategy of precommitment ?—*e.g.*, should a man be allowed to sell himself into bondage for the sake of an immediate gratification of desires ?[1] What ought to be our view of the *irrevocable* trust, especially if the maker tries to revoke it? At which date should sovereignty inhere in the maker? And ought we to instruct people to substitute a log-linear discount function for the true one ? If so, should the constant k be selected so that the derivatives of the log-linear function and the true one are equal at $t = \tau$? Why should this be the appropriate value ?

My view is that these questions are difficult to answer mainly because consumer sovereignty has no meaning in the context of the dynamic decision-making problem. The individual over time is an infinity of individuals, and the familiar problems of interpersonal utility comparisons are there to plague us. The interpersonal aspect of the intertemporal problem becomes clear if we think of a similar problem involving a family of brothers where each has a utility functional depending not only on his own utility but upon a weighted sum of the utilities of all of them. Suppose the oldest brother always has the power to allocate the annual proceeds of an estate, but with it being foreknown that each year one brother will die off, the oldest next. The shifting of the discount function of the family head gives rise to the danger of inconsistent planning ; and the family head of the moment may consider the alternative strategies of (*a*) an irrevocable trust, or (*b*) playing his favorites extra heavily now knowing that they will be out of favor at a later date. What can the detached view of consumer sovereignty be in this context !

Pigou[2] and others have regarded " myopia " as an excuse for state intervention in determining allocations over time (investment). But on what basis ought the state to make these decisions ? Ramsey[3] contended that all weights should be equal ; but at least for those problems involving the allocation of a limited stock of goods over time (*i.e.*, the problem of " conservation "), this proposal becomes meaningless if one contemplates future generations ad infinitum.

More questions have been raised here than I am prepared to answer ; but somewhat out of practical considerations I would suggest the following : The individual can probably do as good a job as the state or any other agency in determining allocations for himself as of future dates, provided the future dates include none which are proximate. That is to say that I would have confidence in the judiciousness of a person to-day, if he is not ignorant of future facts, to decide how much to save and how much to spend for the rest of his life, *starting a couple of years from now*. Mr. Smith is probably in as good a position to evaluate the *relative* importance of the Mr. Smith of 1960 and the Mr. Smith of 1970 as anybody else, and I would therefore permit him, in the absence of risk and uncertainty, to precommit his future allocations provided the period of precommitment did not begin for a couple more years. The real decisions to worry about are those where an immediate or proximate satisfaction is gained at the expense of still-more-future costs. Precommitment may be regarded as either good or evil depending upon whether the period of precommitment begins now or later.

[1] Typically, our common law has held many contracts that partake of this character to be unenforceable. The problem here, ethically, is akin to that of whether a body politic should be permitted to vote itself into a dictatorship.

[2] A. C. Pigou, *The Economics of Welfare*, ch. 2, pp. 23-30.

[3] F. Ramsey, " A Mathematical Theory of Saving," *Economic Journal*, 1928.

The strategy of using a log-linear function also seems to have some appeal from the "social" point of view. It means that the individual always decides what to do now on the assumption that he has no authority over his future self. The individual cannot decide what $C(t)$ shall be for $t > \tau$, except that he can decide now what $K(t)$ shall be for $t = \tau + dt$. If the derivative of $\lambda(t - \tau)$ equals zero at $t = \tau$, then, of course, this case reduces to that of "no myopia," the situation which Ramsay regarded as best. But, these remarks notwithstanding, my own view is that the ethical issues presented by the problem of dynamic choice remain basically unanswered if not unanswerable.

X. SUMMARY

To summarize, we have said that the optimal plan of future behaviour chosen as of a given time

(*A*) may be a plan which will be followed under conditions of certainty (*the harmony case*), or

(*B*) may be inconsistent with the optimizing future behaviour of the individual (*the intertemporal tussle*). In this latter case

- (1) the conflict may not be recognized and the individual will then be *spendthrifty* (or *miserly*), his behavior being inconsistent with his plans, or
- (2) the conflict may be recognized and solved either by
  - (*a*) a *strategy of precommitment*, or
  - (*b*) a *strategy of consistent planning*.

We have, moreover, hypothesized that the typical discount function has the shape of $\lambda(t - \tau)$ as shown in Figure 3, and have argued that this hypothesis is consistent with observed behavior.

Finally, we have challenged the meaning of the concept of consumer sovereignty in this context of dynamic utility maximization.[1]

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[1] We have here treated the problem of the intertemporal tussle only in the context of microeconomics Similar issues may arise, however, in the aggregate case where a group of persons or an economy must decide the distribution of economic activity over time. Political decisions to eliminate a foreign trade deficit or to balance a budget not this year, but next may serve as illustrations.