Testable implications of general equilibrium theory: a differentiable approach

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Abstract

Is general equilibrium theory empirically testable? Our perspective on this question differs from the standard, Sonnenschein-Debreu-Mantel (SDM) viewpoint. While SDM tradition considers aggregate (excess) demand as a function of prices, we assume that what is observable is the equilibrium price vector as a function of the fundamentals of the economy. We apply this perspective to an exchange economy where equilibrium prices and individual endowments are observable. We derive necessary and sufficient conditions that characterize the equilibrium prices, as functions of initial endowments. Furthermore, we show that, if these conditions are satisfied, then the economy can generically be identified. Finally, we show that when only aggregate data are available, observable restrictions vanish. We conclude that the availability of individual data is essential for the derivation of testable consequences of the general equilibrium construct.

Key words: aggregation, excess demand, equilibrium manifold, identification.

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1 Introduction

Is general equilibrium theory empirically testable? This question has attracted considerable attention for at least thirty years; that is, at least since the statement of the "Sonnenschein problems". In two seminal papers, Sonnenschein [20], [21] posed the question whether the individualistic foundations of general equilibrium theory could generate non-trivial testable restrictions on the aggregate excess demand or market demand functions of an exchange economy. The case of excess demand was solved by Mantel [17] and Debreu [9]; the market demand problem was solved by Andreu [1] for finite sets of data, and, recently, by Chiappori and Ekeland [7] for analytic demand functions. In all cases, the answer is negative, provided there are enough individuals in the economy — a conclusion that confirmed Sonnenschein's intuition and initial arguments.

These (by now classical) results have widely been interpreted as pointing out a severe weakness of general equilibrium theory, namely its inability to generate empirically falsifiable predictions. A prominent illustration of this stand is provided for instance by Kenneth Arrow, who, in a recent survey, listed among the main developments of utility theory the result that "*in the aggregate, the hypothesis of rational behavior has in general no implications*", and drew the conclusion that "*if agents are different in unspecifiable ways, then [...] very few, if any, inferences can be made*" ([2], p. 201).

The main claim of the present paper is that this view is overly pessimistic, and that general equilibrium theory can actually generate strong testable predictions, even for large economies. The main idea is in the line of recent contributions by Brown and Matzkin [4] and Brown and Shannon [5], and can be summarized as follows. The approach by Sonnenschein, Debreu and Mantel concentrates on the properties of excess (or market) demand as a function of *prices* only. There are, of course, deep theoretical reasons for the investigation of the structure of aggregate demand as a function of prices; for instance, the SDM result has strong implications for the convergence of tâtonnement processes.However, this viewpoint is not the only possible one, and actually not the most adequate for assessing the testability of general equilibrium theory. As far as testable predictions are concerned, the structure of aggregate excess demand is not the relevant issue, if only because excess demand is, in principle, not observable, except at equilibrium prices — where, by definition, it vanishes. However, prices are not the only variables that can be observed to vary. Price movements reflect fluctuations of fundamentals, and the relationship between these fundamentals and the resulting equilibrium prices is the natural object for empirical observation. One of the goals of general equilibrium theory is precisely to characterize the properties of this relationship. As it turns out, this characterization generates strong testable restrictions.

We develop our claim in the simple but natural context of an exchange economy, where excess demand depends on both prices *and* initial endowments. The equilibrium equations then relate prices to endowments; the *equilibrium manifold* is defined as the set of prices and endowments for which excess demand is zero. We are interested in the local structure of that manifold; that is, we study equilibrium prices, *locally,* as a smooth function of initial endowments. We derive two main results. First, there exist strong *restrictions* on the local structure of the equilibrium manifold. Some of these restrictions come from the individualism assumption (the aggregate demand arises as the sum of individual demands each of which is a function *solely* of prices and individual income), and others stem from the rationality assumption (each individual is a utility maximizer). In other words, although none of these assumptions constrains the shape of excess demand as a function of prices (the SDM conclusion), they do restrict the form of the equilibrium manifold, which is of empirical relevance.

Second, and perhaps more surprisingly, we prove that, if income effects do not vanish, observing equilibrium prices as a function of initial endowment generically identifies the underlying economy, in the sense that individual preferences can be recovered without ambiguity. In a way, this result is the exact opposite of the SDM conclusion. In the SDM perspective, *all* the structure due to individual utility maximization is *lost* by aggregation. Adopting the equilibrium manifold perspective, we reach the opposite conclusion that *all* the relevant structure is generically *preserved*, in the sense that the initial economy can be recovered from the local structure of the equilibrium manifold.

These results indicate that the two lines contrasted above — the 'manifold' point of view versus the SDM excess demand approach — generate different (and in a sense opposite) conclusions. How can this striking discrepancy be explained? Our interpretation emphasizes a crucial difference: in the manifold approach, individual data (initial endowments) are available, whereas only aggregate variables can be observed in the SDM setting. In other words, we understand our results as suggesting the important conclusion that *whenever data are available at the individual level, then utility* *maximization generates very stringent restrictions upon observed behavior, even if the observed variables (equilibrium prices in our case) are aggregate.* From this perspective, whether individual transactions can be observed is irrelevant. Individual determinants of individual choices (such as initial endowments or individual incomes) may do just as well.

A natural question, then, is whether the converse claim also holds: is it the case that, when aggregate variables only are observed, no testable restriction can be generated, at least if the number of individuals is "large enough"? Specifically, assume that only *aggregate* endowments Ω can be recorded. These aggregate endowments are redistributed among individuals in the economy according to some rule that is not observed. In particular, fluctuations in Ω generate changes in individual endowments that are not recorded. What is observed, however, are the corresponding movements of equilibrium prices. In this new context, the equilibrium manifold is observed as a function of aggregate endowments only. Is there any restriction on the form of this relationship?

We show that, under an analyticity condition, when the number of individuals is at least equal to the number of commodities, any (sufficiently smooth) manifold can be (locally) rationalized as the equilibrium manifold of an exchange economy with utility maximizing individuals, for some 'well chosen' redistribution rule. This result closes the argument by confirming that the Walrasian framework cannot generate restrictions on the local structure of the equilibrium manifold when only aggregate data are observable. In this sense, although our results emphasize a new aspect of aggregation theory, they remain fully consistent with the conventional wisdom of the field.

Our work is in the line of a former contribution by Brown and Matzkin [4], who study the restrictions on the structure of the equilibrium manifold from a "non-parametric", revealed preferences perspective. In their paper, Brown and Matzkin derive a set of necessary and sufficient conditions under the form of linear equalities and inequalities that have to be satisfied by any finite data set, and they show that these relationships are indeed restrictive. This approach has been recently extended by Kübler [14], Snyder [19] and Brown and Shannon [5].. Our work complements these results in three ways. First, we adopt a differentiable viewpoint, so that our necessary and sufficient conditions take the somewhat more familiar form of a system of partial differential equations, reminiscent of Slutsky conditions. In particular, our conditions can readily be imposed on a parametric estimation of the equilibrium manifold; hence they can be tested using the standard econometric tools of consumer analysis. We provide an example of such a parametric analysis in Section 3. Secondly, the result that these restrictions, if fulfilled, are sufficient to generically recover the underlying economy is original. Thirdly, we extend the analysis to the case where only aggregate endowments are observable, and provide a formal non testability result.

2 The framework

2.1 The model

We consider an exchange economy with K commodities and N individuals. Initial endowments of individual n we denote by $\omega_n = (\omega_n^1, \ldots, \omega_n^K) \in \mathbf{R}_+^K$, and his wealth by $y_n = p'\omega_n = \sum_k p_k \omega_n^k$.Here and throughout the paper, x' denotes the transpose of the vector x, and $E^\perp$ denotes the orthogonal of a subspace E.

Individual n is characterized by a demand function, $x_n(p, y_n)$, which we assume to be smooth and homogeneous of degree 0, and to satisfy the Walras law. As a consequence, differentiating the relation $p'x\,(p, p'\omega_n) = p'\omega_n$ with respect to ω_n, we get the identity:

$$p' D_y x\,(p, p'\omega_n) = 1 \tag{1}$$

We shall say that this demand function is *rationalizable* if it is derived from the maximization of smooth, strongly quasi-concave utilities. It is well-known that $x_n(p, y_n)$ is rationalizable if and only if it satisfies the Slutsky conditions on symmetry and negative definiteness.

A smooth map Z, defined on $\mathbf{R}_+^K \times \mathbf{R}_+^{KN}$, is an *excess demand function* if there exist N individual demand functions $x_1, ..., x_N$, such that

$$Z(p, \omega) = \sum_{n=1}^{N} (x_n(p, p'\omega_n) - \omega_n). \tag{2}$$

If Z is an excess demand function, then it is is homogeneous of degree zero with respect to p, and, by Walras' law,

$$p'Z(p, \omega) = 0.$$

We use, henceforth, the normalization

$$p'p = 1.$$

We denote by $\mathbf{S}^{K-1}$ the unit sphere in $\mathbf{R}^K$, and by $\mathbf{S}_+^{K-1}$ its intersection with $\mathbf{R}_+^K$. With the normalization of prices,

$$Z: \ \mathbf{S}_+^{K-1} \times \mathbf{R}_+^{KN} \longrightarrow \ \mathbf{R}^K,$$

and $Z(\bullet, \omega)$ is a map from the unit sphere into $\mathbf{R}^K$. Note that the tangent space to $\mathbf{S}_+^{K-1}$ at the point p is the orthogonal subspace $[p]^\perp$ so that

$$DZ\,(p, \omega) : [p]^\perp \times \mathbf{R}^{KN} \longrightarrow \mathbf{R}^K$$

There are other restrictions on the derivative DZ. Differentiating the Walras law, we get:

$$p' D_p Z = Z \tag{3}$$
$$p' D_{\omega_n} Z = 0 \tag{4}$$

Finally, the equilibrium manifold is defined as

$$\mathcal{E} = \left\{ (p, \omega) \in \mathbf{S}_+^{K-1} \times \mathbf{R}_+^{KN} \ \mid \ Z(p, \omega) = 0 \right\}.$$

In particular, at any point (p, ω) belonging to $\mathcal{E}$,

$$p' D_p Z = 0. \tag{5}$$

This means that, at every point, (p, ω) on $\mathcal{E}$, the Jacobian $D_p Z$ maps the tangent space $[p]^\perp$ into itself. Its rank is at most $(K-1)$, and if it is exactly $(K-1)$, then $D_p Z$ has a pseudo-inverse Δ, that is, there is a map

$$\Delta\,(p, \omega) : [p]^\perp \longrightarrow [p]^\perp$$

such that $D_p Z\,(p, \omega) \circ \Delta\,(p, \omega)$ is the identity in $[p]^\perp$.

2.2 The problem

Under standard assumptions, see [3], the graph of the competitive equilibrium correspondence has the structure of a continuously differentiable manifold. Locally, in a neighborhood, $\mathcal{N}\,(\overline{p}, \bar{\omega})$, of some arbitrary, non-singular point $(\overline{p}, \bar{\omega})$, the equilibrium price can be defined as a function of individual endowments:

$$(p, \omega) \in \mathcal{E} \cap \mathcal{N}\,(\overline{p}, \bar{\omega}) \Rightarrow p = \pi\,(\omega).$$

We denote by $\mathcal{N}\,(\bar{\omega})$ the projection of $\mathcal{N}\,(\overline{p}, \bar{\omega})$ over $\mathbf{R}_+^{KN}$.

Definition 2 *A smooth map, $\pi : \mathcal{N}(\bar{\omega}) \mapsto \mathbf{R}_+^K$, with $\pi'\pi = 1$, is a* regular equilibrium map *over $\mathcal{N}(\bar{\omega})$ if there exists a smooth aggregate excess demand function $Z(p, \omega)$, defined on $\mathcal{N}(\overline{p}, \bar{\omega})$, such that $Z(\pi(\omega), \omega) = 0$ and $D_p Z(\pi(\omega), \omega)$ has rank $(K-1)$ for all $\omega \in \mathcal{N}(\bar{\omega})$.*

Our basic question is thus: *What are the conditions for a smooth map, π, to be a regular equilibrium map* over $N(\bar{\omega})$?

An immediate remark is that the *local* nature of the problem is crucial. Indeed, assume that the equilibrium manifold is known *globally*, including at the boundaries of $\mathbf{R}_+^{KN}$. Then, one can set $\omega_2, ..., \omega_n$ to zero, so that aggregate excess demand coincides with the excess demand of individual 1; the corresponding section of the manifold gives the inverse demand function of individual 1, and the same trick can be used for all individuals. The interesting, and more difficult, question we consider refer to the neighborhood of an interior point $(\overline{p}, \bar{\omega})$, where the non-negativity constraints are not binding: $x_n^k(\bar{p}, \bar{p}'\bar{\omega}_n) > 0$, for all commodities k and individuals n.

3 Characterization of the equilibrium manifold

3.1 Necessary conditions

There are two sets of necessary conditions. The first one derive from individualism, that is, the fact that aggregate demand is the sum of individual demands, the second one derives from the fact that individuals are maximizers.

Proposition 3 *If π is a regular equilibrium map over $\mathcal{N}(\bar{\omega})$, then there is an invertible linear map $\Theta(\omega)$ from $[\pi(\omega)]^\perp$ into itself, and vectors $\theta_n(\omega), 1 \leq n \leq N$, depending smoothly on ω, such that, for every n, we have:*

$$\Theta(\omega) D_{\omega_n} \pi(\omega) = I - \theta_n(\omega) \pi(\omega)', \quad \omega \in \mathcal{N}(\bar{\omega}) \tag{6}$$

The linear map $\Theta(\omega)$ and the vectors $\theta_n(\omega)$ are determined uniquely by the map π.

Proof. Differentiating formula (2) for the excess demand function Z, we obtain:

$$\frac{\partial Z^k}{\partial \omega_n^j} = \frac{\partial x_n^k}{\partial y_n} p_j - \delta_k^j$$

where δ_k^j is equal to 1 if $j = k$, and to 0 otherwise. In matrix notation, this is written:

$$D_{\omega_n} Z = \frac{\partial x_n^k}{\partial y_n}(p, p'\omega_n)p' - I.$$

Differentiating the equation $Z(\pi(\omega), \omega)) = 0$ with respect to ω_n yields:

$$D_p Z \; D_{\omega_n} \pi = -D_{\omega_n} Z$$

Note that, by equations (4) and (5), both sides are contained in $[p]^{\perp}$.

Substituting the last equation into the preceding one, we get:

$$D_p Z \; D_{\omega_n} \pi = I - \theta_n \pi', \tag{7}$$

where

$$\theta_n = \frac{\partial x_n}{\partial y}\left(\pi, \pi'\omega_n\right)$$

Note that, because of relations (4) and (1), both sides of relation (7) map $\mathbf{R}^K$ into $[\pi(\omega)]^{\perp}$.Recall that $D_{\omega_n}\pi$ maps $[\pi(\omega)]^{\perp}$, the tangent space to $\mathbf{S}^{K-1}$ at $\pi(\omega)$ into $\mathbf{R}^K$. Applying $\Delta(\pi(\omega), \omega)$, the pseudo-inverse of $DZ(\pi(\omega), \omega)$, to both sides of (7), we get:

$$D_{\omega_n}\pi = \Delta(\pi(\omega), \omega)\left(I - \theta_n(\omega)\pi(\omega)'\right) \tag{8}$$

Setting $\Theta(\omega) = \Delta(\pi(\omega), \omega)^{-1}$ we get the desired decomposition.

We now prove that the linear map $\Theta(\omega)$ and the vectors $\theta_n(\omega)$ above are uniquely defined from π.Note first that it follows from equation (6) and the fact that $\Theta(\omega) : [\pi(\omega)]^{\perp} \longrightarrow [\pi(\omega)]^{\perp}$ is invertible that $D_{\omega_n}\pi$ has rank $(K-1)$ Now suppose we have:

$$D_{\omega_n}\pi = \Gamma\left(I - \gamma_n \pi'\right)$$

for some other operator $\Gamma : [\pi]^{\perp} \longrightarrow [\pi]^{\perp}$ and vector γ_n. Since $D_{\omega_n}\pi$ has rank $(K-1)$, so must Γ. We have:

$$\Gamma\left(I - \gamma_n \pi'\right) = \Theta^{-1}\left(I - \theta_n \pi'\right)$$

This yields $\Gamma\xi = \Theta^{-1}\xi$ for all $\xi \in [\pi]^{\perp}$, so $\Gamma = \Theta^{-1}$, and then $\gamma_n = \theta_n$ follows. ∎

Proposition 3 describes the testable properties of equilibrium prices, as function of the initial allocation, stemming from the individualism assumption. underlying, general equilibrium framework. It states that there exists a linear map Θ such that, for any n, the matrix $(\Theta(\omega)D_{\omega_n}\pi(\omega) - I)$ is of rank one and vanishes over the subspace $[\pi(\omega)]^{\perp}$.Note that the operator $\Theta(\omega) : [\pi(\omega)]^{\perp} \longrightarrow [\pi(\omega)]^{\perp}$ is independent of n. As a consequence, for any i and j, the rank of the operator $(D_{\omega_j}\pi - D_{\omega_i}\pi)$ is at most one. Indeed, from (6) we get:

$$(D_{\omega_j}\pi - D_{\omega_i}\pi) = \Theta(\theta_i - \theta_j)\pi'$$

Proposition 4 *Assume that*

$$\sum_{i,j=1}^{K} \omega^j \frac{\partial \pi_j}{\partial \omega_n^i} \pi_i \neq -1$$

at a certain $\omega = \overline{\omega}$. *Then, in some neighbourhood of* $\overline{\omega}$, *knowledge of* $\Theta(\omega)$ *and* $\theta_n(\omega)$ *uniquely identifies the marginal propensity to consume,* $a_n(p, y) = D_y x_n(p, y)$, *of individual* n*:*

$$a_n\left[\pi(\omega), \pi(\omega)' \omega_n\right] = \theta_n(\omega)$$

Proof. Consider the map $\Phi_n : \omega \longmapsto \left(\pi(\omega)' \omega_n, \pi(\omega)\right)$, which sends $\mathbf{R}_+^{KN}$ into $\mathbf{R} \times \mathbf{S}_+^K$. The derivative $D_{\omega_n}\Phi_n(\omega)$ maps $\mathbf{R}^K$ into $\mathbf{R}\times [\pi(\omega)]^{\perp}$ .We have:

$$D_{\omega_n}\Phi_n(\omega) = \left(\omega_n' D_{\omega_n}\pi(\omega) + \pi(\omega)', D_{\omega_n}\pi(\omega)\right)$$

Splitting $\mathbf{R}_+^K$ into the orthogonal sum of $\pi(\omega)$ and $[\pi(\omega)]^{\perp}$, we get:

$$D_{\omega_n}\Phi_n(\omega) = \begin{matrix} \omega_n'(D_{\omega_n}\pi)\pi + 1 & 0...0 \\ \xi_n & \Theta(\omega)^{-1} \end{matrix}$$

where ξ_n is some $(K-1)$ vector.

It follows from the assumption that $D_{\omega_n}\Phi_n(\overline{\omega})$ is invertible.This will allow us, by the implicit functions theorem, to use the following change of variable:

$$\omega = (\omega_1, ...\omega_N) \longrightarrow \left(\omega_1, ..., \omega_{n-1}, \pi(\omega)' \omega_n, \pi(\omega), \omega_{n+1}, ..., \omega_N\right).$$

in some neigbourhood of $\overline{\omega}$.

By definition, we have $\theta_n(\omega) = \frac{\partial x_n}{\partial y}\left(\pi(\omega), \pi(\omega)' \omega_n\right)$. This means that, in teh nex coordinates, θ_n is a function of $\pi(\omega)' \omega_n$ and $\pi(\omega)$ only, and this gives $\frac{\partial x_n}{\partial y}$, as announced. ■

Proposition 5 *Assume in addition that agent n is a utility-maximizer. Then the function $a_n(p,y)$ satisfies the equations*

$$\left(\frac{\partial a_n^k}{\partial p_j}-\frac{\partial a_n^j}{\partial p_k}\right)\frac{\partial a_n^i}{\partial y}+\left(\frac{\partial a_n^j}{\partial p_i}-\frac{\partial a_n^i}{\partial p_j}\right)\frac{\partial a_n^k}{\partial y}+\left(\frac{\partial a_n^i}{\partial p_k}-\frac{\partial a_n^k}{\partial p_i}\right)\frac{\partial a_n^j}{\partial y}=0,\ \forall i,j,k \tag{9}$$

Proof. To ease notations, let us drop the index n in the following proof, and write a for a_n and x for x_n. Since $x(p,y)$ is a demand function, it satisfies Slutsky symmetry:

$$\frac{\partial x^k}{\partial p_j}-\frac{\partial x^j}{\partial p_k}=x^k\frac{\partial x^j}{\partial y}-x^j\frac{\partial x^k}{\partial y}.$$

Differentiating with respect to y,

$$\frac{\partial^2 x^k}{\partial y\partial p_j}-\frac{\partial^2 x^j}{\partial y\partial p_k}=x^k\frac{\partial^2 x^j}{\partial y^2}-x^j\frac{\partial^2 x^k}{\partial y^2},$$

which can be written as

$$\frac{\partial a^k}{\partial p_j}-\frac{\partial a^j}{\partial p_k}=x^k\frac{\partial a^j}{\partial y}-x^j\frac{\partial a^k}{\partial y} \tag{10}$$

This provides a system of equations in the x^i where all the coefficients are known. It can readily be checked that this system cannot be of full rank. In fact, the equations are not compatible unless condition (9) is fulfilled ■

Condition (9) is just one of the testable properties of the price function which derive from the assumption that aggregate demand is rationalizable. As we shall see later on, there are many more testable properties stemming from this assumption: in general, knowledge of the $a_n(p,y)$ determines the $x_n(p,y)$ and the latter must then satisfy the Slutsky relations.

3.2 An example

To see how restrictive condition (6) is in general, assume that $K=2$ (then π_2 can be normalized to one) and consider the following functional form for π_1 as a function of the ω :

$$\pi_1(\omega)=\frac{\sum_n\left(A_n^1\omega_n^1+A_n^2\omega_n^2\right)}{\sum_n\left(B_n^1\omega_n^1+B_n^2\omega_n^2\right)}, \tag{11}$$

with the normalization $A_1^2 + B_1^1 = 1$.

We now derive the restrictions implied by (6) on the coefficients A_n^i, B_n^i. First,

$$\frac{\partial \pi_1}{\partial \omega_s^k} - \frac{\partial \pi_1}{\partial \omega_t^k} = \frac{\sum_{j=1}^{2} \sum_n \left[\left(A_s^k B_n^j - B_s^k A_n^j \right) - \left(A_t^k B_n^j - B_t^k A_n^j \right) \right] \omega_n^j}{\left[\sum_n \left(B_n^1 \omega_n^1 + B_n^2 \omega_n^2 \right) . \right]^2}$$

The decomposition property (6) implies that

$$\frac{\frac{\partial \pi_1}{\partial \omega_s^1} - \frac{\partial \pi_1}{\partial \omega_t^1}}{\frac{\partial \pi_1}{\partial \omega_s^2} - \frac{\partial \pi_1}{\partial \omega_t^2}} = \pi_1 \quad \text{for all } s, t,$$

which gives

$$\frac{\sum_k \sum_n \left[\left(A_s^1 B_n^k - B_s^1 A_n^k \right) - \left(A_t^1 B_n^k - B_t^1 A_n^k \right) \right] \omega_n^k}{\sum_k \sum_n \left[\left(A_s^2 B_n^k - B_s^2 A_n^k \right) - \left(A_t^2 B_n^k - B_t^2 A_n^k \right) \right] \omega_n^k} = \frac{\sum_n \left(A_n^1 \omega_n^1 + A_n^2 \omega_n^2 \right)}{\sum_n \left(B_n^1 \omega_n^1 + B_n^2 \omega_n^2 \right)}.$$

This equation must be satisfied for all $\left(\omega_1^1, \omega_N^K \right)$. Simple (although tedious) algebra shows that the only form possible for π is then:

$$\pi_1 (\omega) = \frac{A^1 \Omega^1 + \sum_n A_n^2 \omega_n^2}{\sum_n \left(1 - A_n^2 \right) \omega_n^1 + B^2 \Omega^2}.$$

The important message of the example is that should an econometric test be based on the relatively flexible functional form (11), then the decomposition condition (6) implies that $A_s^1 = A_t^1, B_s^2 = B_t^2$ and $A_s^2 + B_s^1 = 1$ for all s, t — that is, a set of strong parametric restrictions.

3.3 Recovering individual demands

We now consider the identification problem: to what extent is it possible to recover preferences from the observation of the *local structure* of the equilibrium manifold? A first remark is that from Proposition 4, the local structure of the equilibrium manifold fully allows to identify individual income effects. We are thus left with a problem in consumer theory, namely: *is it possible to recover a demand function $x(p, y)$ from the sole knowledge of its partial derivatives with respect to income, $a(p, y) = D_y x(p, y)$?* We proceed to show that the answer is positive in general. We start with the following restriction:

Assumption 1 : *The demand function $x(p, y)$ is such that*

1. *the income effect for every commodity* i, $\partial x^i/\partial y$, *is a twice differentiable function of income, and*
$$\frac{\partial^2 x^i}{\partial y^2} \neq 0;$$
2. *there exist at least two commodities* j *and* k *such that*
$$\frac{\partial}{\partial y}(\ln \frac{\partial^2 x^j}{\partial y^2}) \neq \frac{\partial}{\partial y}(\ln \frac{\partial^2 x^k}{\partial y^2}).$$

Assumption 1 requires that income effects do not vanish for any commodity, while there are two commodities for which the partial elasticities of the income effects with respect to revenue do not vanish. A few remarks are in order here. First, Assumption 1 implies that there are at least three commodities: $L \geq 3$; a different argument is required for economies with two commodities, $L = 2$. Secondly, Assumption 1 rules out specific preferences, such as homothetic or quasi-linear utility functions. Indeed, it can readily be checked that identification is not possible for homothetic utility functions. Intuitively, this is due to the fact that homothetic utilities permit aggregation. In general, however, if demand is non-linear in income, and if income effect do not vanish, Assumption 1 is satisfied for an open and dense set of prices and incomes; which suffices, since continuity then allows for identification. More precisely, using the concept of the "generalized rank" of a demand system, introduced in Lewbel [16], we find that Assumption 1 is generically satisfied for systems of rank at least 2.

Finally, it is important to note that Assumption 1 involves only derivatives of the income effects. As such, it can be directly expressed in terms of $a(p, y)$. In particular, from the general perspective of the paper, Assumption 1 can be tested from the sole knowledge of the equilibrium manifold, since the latter identifies $a(p, y)$.

The main result is then the following:

Proposition 6 *If the demand function,* $x(p, y)$, *satisfies Assumption 1, then it is uniquely identified by its partial derivatives with respect to income* $a(p, y) = D_y x(p, y)$: *for any demand function,* $\xi(p, y)$, *if* $D_y \xi(p, y) = D_y x(p, y)$ *for all* (p, y), *then* $\xi(p, y) = x(p, y)$, *for all* (p, y).

Proof. If (9) holds, the system is indeterminate, and one further derivation in y is needed. Specifically,

$$\frac{\partial^2 a^k}{\partial y \partial p_j} - \frac{\partial^2 a^j}{\partial y \partial p_k} - \left(a^k \frac{\partial a^j}{\partial y} - a^j \frac{\partial a^k}{\partial y} \right) = x^k \frac{\partial^2 a^j}{\partial y^2} - x^j \frac{\partial^2 a^k}{\partial y^2} \tag{12}$$

From Assumption 1, there exist commodities j and k such that the system consisting of the two equations $(k,j)_1$ and $(k,j)_2$ in x^j and x^k is of full rank. This identifies x^j and x^k. Then $(i,k)_1$ written for x^k and x^i, allows to identify x^i. ∎

Note that this implies a further restriction on a : the identification gives the same result using j and i instead of k and i. This gives

$$\begin{array}{l}
\frac{\partial^2 a^k}{\partial y^2} \frac{\partial a^j}{\partial y} \left(\left(\frac{\partial a^i}{\partial p_k} - \frac{\partial a^k}{\partial p_i} \right) \frac{\partial a^j}{\partial y} - \frac{\partial a^i}{\partial y} \left(\frac{\partial a^j}{\partial p_k} - \frac{\partial a^k}{\partial p_j} \right) \right) = \\
\frac{\partial^2 a^j}{\partial y^2} \frac{\partial a^k}{\partial y} \left(\left(\frac{\partial a^j}{\partial p_i} - \frac{\partial a^i}{\partial p_j} \right) \frac{\partial a^k}{\partial y} - \frac{\partial a^i}{\partial y} \left(\frac{\partial a^j}{\partial p_k} - \frac{\partial a^k}{\partial p_j} \right) \right) + \\
\frac{\partial a^j}{\partial y} \frac{\partial a^k}{\partial y} \left(\left(\frac{\partial a^i}{\partial p_k} - \frac{\partial a^k}{\partial p_i} \right) \frac{\partial^2 a^j}{\partial y^2} - \left(\frac{\partial a^j}{\partial p_i} - \frac{\partial a^i}{\partial p_j} \right) \frac{\partial^2 a^k}{\partial y^2} \right).
\end{array}$$

Note also that if $x(p,y)$ satisfies Assumption 1 and ξ is such that $D_y \xi(p,y) = D_y x(p,y)$ for all (p,y), then $\xi(p,y)$ also satisfies Assumption 1.

We can thus summarize our findings:

Theorem 7 *A given smooth map π on $\mathcal{N}(\bar{\omega})$ cannot be a regular equilibrium manifold unless it satisfies the testable restrictions given in Propositions 3 and 4. Conversely, if π is a regular equilibrium map over $\mathcal{N}(\bar{\omega})$, and if Assumption 1 is satisfied, then the underlying economy, if it exists, is uniquely identified.*

4 The case of aggregate endowments

4.1 The problem

The previous restrictions obtain under a specific hypotheses, namely, that *individual* endowments are observable. This fact is quite interesting; it suggests, indeed, that testable restrictions require that some data are available *at the individual level.* In this section, we substantiate this claim by considering the case when aggregate endowments only can be observed. Do restrictions still exist?

Quite obviously, the answer depends on the number of individuals. Take the extreme case of a one individual economy. Then the equilibrium condition boils down to $Z = z_1 = 0$, which means that $\Omega = \omega_1$ must be the agent's equilibrium consumption at prices $\pi(\Omega)$. Then $\pi(\Omega)$ is an inverse demand function; as such, it has to satisfy the Slutsky relations, that is, $D_{\omega_1}\pi$ must be symmetric on $\omega_1{}^{\perp}$.

This fact is by no means unexpected. With one individual, utility maximization is known to generate restrictive conditions on behavior. What the previous literature suggests, however, is that these conditions might become less and less restrictive as the number of individuals is increased. This intuition turns out to be true, as we now proceed to demonstrate.

4.2 A formal statement

Suppose that we can no longer observe the individual endowments ω_n, but only the aggregate endowment $\Omega = \sum_n \omega_n$. Suppose furthermore that, for each value of Ω, this total endowment is distributed across individuals in a way which is not observed, and that we only observe some set of equilibrium prices, p. What can we predict on the local structure of the mapping $\pi : \Omega \mapsto p$? More precisely, is it possible to find utility functions $U^1, \ldots, U^N$ and some distribution of endowment $(\omega_1(\Omega), \ldots, \omega_N(\Omega))$, with $\sum \omega_i(\Omega) = \Omega$, such that the price vector $\pi(\Omega)$ is an equilibrium price for an economy with N individuals, the preference of the n-th individual being U^n and his endowment $\omega_n(\Omega)$?

We now answer positively a local version of this problem. Assume that $N \geq K$, and suppose we are given a mapping $\pi : \mathbf{R}^K \mapsto S_+^{K-1}$. Chose an $\bar{\Omega}$ that satisfies the following, smoothness restriction:

Assumption 2 : *There exists an open neighborhood $V\left(\bar{\Omega}\right)$ of $\bar{\Omega}$ in which the mapping $\Omega \rightarrow \left(\pi\left(\Omega\right), \pi\left(\Omega\right)'\Omega\right)$ is (locally) invertible, and the inverse mapping $A : (p, Y) \rightarrow \Omega$ is analytic in a neighborhood of $\left(\pi\left(\bar{\Omega}\right), \pi\left(\bar{\Omega}\right)'\bar{\Omega}\right)$.*

Here, $Y = \pi\left(\Omega\right)'\Omega$ denotes the economy's total wealth. This assumption deserves a few comments. First, local invertibility does not raise specific problems. It is a standard regularity assumption, that can be expected to hold for almost every Ω. Its main use, here, is to allow to consider the mapping A as a change of variables; that is, any function of Ω can alternatively be expressed as a function of prices and aggregate income. This technique will be helpful in what follows. Analyticity is more demanding; it can be viewed as an extreme case of smoothness. However, it is by now known that it is a very useful assumption for this kind of problem (see [7])

4.3 The main result

If Assumption 2 is satisfied, then the mapping A has two obvious properties: it is homogenous, and it satisfies $p\prime A(p, Y) = Y$.

Now, let us just assume that resources are shared equally; the distribution of endowment $(\omega_1(\Omega), \ldots, \omega_N(\Omega))$ is thus defined by

$$\omega_n(\Omega) = \frac{1}{N}\Omega.$$

This implies, in particular, that

$$\pi(\Omega)' \omega_n(\Omega) = \frac{Y}{N} = \frac{\pi(\Omega)\prime\Omega}{N}.$$

The problem can be stated as follows: can one find N individual demand functions $x_1(p, y_1), ..., x_N(p, y_N)$, such that

$$\sum_{n=1}^{N} x_n\left(\pi(\Omega), \frac{\pi(\Omega)'\Omega}{N}\right) = \Omega, \quad \forall \Omega \in \mathcal{V}(\bar{\Omega}).$$

We can use the change in variables defined by Assumption 2; the previous equation becomes, with obvious notation,

$$\sum_{n=1}^{N} x_n\left(p, \frac{Y}{N}\right) = A(p, Y)$$

or, using homogeneity

$$\sum_{n=1}^{N} x_n\left(\frac{p}{Y}, \frac{1}{N}\right) = A\left(\frac{p}{Y}, 1\right).$$

In words, we are now looking for an economy, the aggregate demand of which is locally equal to some given, analytic function $B\left(\frac{p}{Y}\right) = A\left(\frac{p}{Y}, 1\right)$. The answer is given by a recent result ([7]), which states that this is always possible. Formally,

Proposition 8 *Under Assumption 2, and assuming* $N \geq K$*, there is an open neighborhood* $\mathcal{V}$ *of* $\overline{\Omega}$*, and* N *functions* $U_1, ..., U_N$*, concave and analytic on* $\mathbf{R}^K$*, such that, for all* Ω *in* $\mathcal{V}$*,* $\pi(\Omega)$ *is a system of equilibrium prices for the economy where individual* n *is characterized by the utility function* U_n *and the endowment*

$$\omega_n = \frac{\Omega}{N}.$$

One the one hand, this result confirms the intuition, stated in introduction, that the observation of individual data is necessary to generate testable restrictions. These restrictions reflect both the decentralized nature of the problem *and* the maximization assumptions made at the individual level; furthermore, they allow, generically, to recover the entire economy. If, on the contrary, only fluctuations in aggregate income can be observed, then no structure is preserved, at least if the number of individuals is large enough.

One the other hand, Proposition 8 seems at variance with other resuls in aggregation theory, obtained within the same framework of collinear endowments with collinear perturbations. The basic intuition of Hildenbrand [11] is that, provided there is sufficient dispersion in preferences across agents, the resulting aggregate demand will satisfy the law of demand. This result has by now been shown to hold in wider conditions of hererogeneity, see for instance Chiappori [6], Hildenbrand [12], Grandmont [10],Quah [18]. In contrast with these situations, proposition 8 makes no restriction at all on agents' preferences, except for the fact that they are convex. The intuition here is that no collective demand function can be so weird as not to arise from some well-chosen individual preferences, but then, of course, it can no longer be assumed that these preferences are uncorrelated across agents. So these two classes of result in aggregation theory are distinct. An earlier result which is in the spirit of Proposition 8 is due to Kirman and Koch [13]

5 Concluding remarks

A first and obvious conclusion of our work is that the "equilibrium manifold" approach leads to conclusions that differ deeply from the Sonnenschein-Debreu-Mantel excess demand perspective. The main conclusion of the latter literature is that *all* the structure due to individual utility maximization is *lost* by aggregation. Adopting the equilibrium manifold perspective, we reach the opposite conclusion that *all* the relevant structure is generically *preserved*, in the sense that the initial economy can be recovered from the structure of the equilibrium manifold. In that sense, our results both generalize Brown and Matzkin's findings and shed a new light on their scope and status. We refer to [15] for an extension to the case of uncertainty and incomplete markets. Also, out interpretation of these results is simple. Rephrasing Arrow's statement quoted in the introduction, we believe that in the aggregate, the hypothesis of rational behavior and market equilibrium has in general strong implications even if individuals are different in unspecifiable ways; however, the latter can be tested only insofar as data are available at the *individual* level. In short, rationality may be testable, but not without individual data.

Finally, what is the empirical relevance of the restrictions derived in the paper? An obvious qualification is that they rely on the impact of changes in individual endowments on aggregate prices. Obviously, the larger the economy, the smaller such effects, and the more difficult it will be to produce empirical work on them. It should be stressed, however, that general equilibrium does not apply only to 'large' economies. On the contrary, the tools of general equilibrium theory have been recently applied, in a very successful way, to the analysis of the behavior of 'small' groups. For instance, standard demand theory uses data on households or families, most of which gather several individuals. Models aimed at taking into account the 'non unitary' nature of the interactions at stake usually rely on a 'collective' approach, that postulates only efficiency. With private consumptions - a framework that has been used in most empirical applications - efficient allocations and market equilibria coincide, and general equilibrium theory is a relevant tool (see for instance Chiappori and Ekeland [8]). The same approach has also been adopted to the analysis of such groups as committees, clubs, villages and other local organizations, which have also attracted much interest. For instance, many micro studies in development, starting with Townsend's seminal investigation of risk sharing within an Indian village [22], are based on data collected at the local level; it is not uncommon to observe endowments (say, individual crops) and prices within the village, a context to which our framework directly applies. Even in large economies, our result may still apply directly when individuals belong to a finite (and "small") number of homogeneous "classes". Finally, an interesting question is how our results can be extended to production economies. The idea is that, in a production context, changes in factor endowments will have an observable impact on factor prices, and that the corresponding equilibrium manifold can in principle be studied in a similar way. All this shall be the subject of further research

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