

**ON “ARNOLD’S THEOREM”
ON THE STABILITY OF THE SOLAR SYSTEM**

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Abstract: Arnold’s theorem on the planetary problem states that, assuming that the masses of n planets are small enough, there exists in the phase space a set of initial conditions of positive Lebesgue measure, leading to quasiperiodic motions with $3n - 1$ frequencies. Arnold’s initial proof is complete only for the plane 2-planet problem. Arnold had missed a resonance later discovered by Herman. The first complete proof, by Herman-Féjoz, relies on the weak non-degeneracy condition of Arnold-Pyartli. A second proof, by Chierchia-Pinzari, is closer to Arnold’s initial idea and shows the strong non-degeneracy of the problem after suitable reduction by (part of) the symmetry of rotation. We review and compare these proofs. In an appendix, we define the Poincaré coordinates and prove their symplectic nature through the shortest possible computation.

Consider $1 + n$ point bodies with masses $m_0, \epsilon m_1, \dots, \epsilon m_n > 0$ ($\epsilon > 0$ and $n \geq 2$) and position vectors $x_0, x_1, \dots, x_n \in \mathbb{R}^3$, undergoing Newton’s universal attraction:

$$\ddot{x}_0 = \epsilon \sum_{k \geq 1} m_k \frac{x_k - x_0}{\|x_k - x_0\|^3},$$

and

$$\ddot{x}_j = m_0 \frac{x_0 - x_j}{\|x_0 - x_j\|^3} + \epsilon \sum_{k \geq 1, k \neq j} m_k \frac{x_k - x_j}{\|x_k - x_j\|^3} \quad (j = 1, \dots, n).$$

We think of the body 0 as the Sun and the other bodies as n planets revolving around the Sun. In our Solar System, the mass of Jupiter is 1/1000 that of the Sun, which justifies that we consider small values of ϵ . The equations have a limit when $\epsilon \rightarrow 0$, for which the Sun is still (in an appropriate frame of reference) and each planet undergoes the only attraction of the Sun. If their energies are negative, planets describe Keplerian ellipses, with some fixed semi major axes and excentricities. As a whole, the system is quasiperiodic with n frequencies. For a generic Hamiltonian system with $3n$ degrees of freedom, one would expect $3n$ frequencies; due to this dynamical degeneracy of the Newtonian potential, we are facing a problem of singular perturbation.

In 1963, V. Arnold [2] published the following remarkable result.

Theorem 1. *For every $m_0, m_1, \dots, m_n > 0$ and for every $0 < a_1 < \dots < a_n$ there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, in the phase space in the neighborhood of circular and coplanar Keplerian motions with semi*

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major axes a_1, \dots, a_n , there is a subset of positive Lebesgue measure of initial conditions leading to quasiperiodic motions with $3n - 1$ frequencies.

Arnold’s theorem is a breakthrough in respect of the oldest question in Dynamical Systems —the stability of the Solar System. Yet it should be noted that it hardly “applies” to the Solar System. There is a first difficulty with the upper value of the small parameter ϵ . Hénon noticed that, without any additional care, the first proofs of Kolmogorov’s theorem show the existence of invariant tori only for a derisory ϵ of the order of 10^{-300} [13]! However, Robutel has shown numerically that some significant parts of the Solar System, in particular of the system consisting of the Sun, Jupiter and Saturn [15, 17, 33], display a quasiperiodic behavior. Also, Celletti–Chierchia [4, 5] and Locatelli–Giorgilli [19] have proved quantitative versions of the KAM theorem, which they have applied to the systems Sun–Jupiter–asteroid Victoria and Sun–Jupiter–Saturn; these applications are assisted by computer symbolic processors, requiring in the second case the manipulation of series of ten million terms. Whether bounded motions form a set of positive Lebesgue measure for all ϵ —and not only for $\epsilon \ll 1$ —remains a completely open problem.

Another matter for discontent when applying KAM theory to astronomy, is that the KAM tori in phase space fills a transversely Cantor set, parametrized by Diophantine frequencies, which is topologically meager. Given the approximation which is made by substituting the Newtonian planetary system to the real Solar System, whether the planet’s mean motions are Diophantine or not, is not a question with any straightforward meaning. Incidentally, Molchanov has speculated on the opposite hypothesis that these mean motions could be totally periodic [22]. Hence the direct conclusion of Arnold’s theorem over an infinite time interval, is illusory in astronomy. Yet KAM theory provides a fundamental conceptual tool in the study of conservative systems, and, to paraphrase Poincaré, quasiperiodic orbits, as much as periodic orbits, are part of the breach through which we can try to enter a place up to now deemed unapproachable.

Nekhoroshev’s theorem [25] on the stability of almost integrable convex Hamiltonian systems, over an exponentially long time interval with respect to the small parameter, is more directly relevant to astronomy. By applying a theorem of this type, Niederman has shown the stability of a Solar System with two planets having small masses, not quite equal but much closer to realistic values [28]. In order to describe the slow evolutions more accurately, Neishtadt has developed the theory of adiabatic invariants [27], and extended related results to non-Hamiltonian perturbations [26].

Over the centuries, mathematicians have spent an inordinate amount of energy proving stronger and stronger stability theorems for dynamical systems more or less closely related to the Solar System. It was a huge surprise when the numerical computations of Laskar showed that over the life span of the Sun, or even over a few hundred million years, collisions and ejections of inner planets are probable (see [16] for a recent account). Our Solar System is now widely believed unstable. Works of Sitnikov and Alekseev [23], Moeckel [21], Simó–Stuchi [34] and Galante-Kaloshin [12], among others (see [3] for other references), show the complexity already of the

simplest non-integrable n -body problem, the restricted three-body problem (*restricted* meaning that the third body has zero mass and thus moves under the influence of, but without influencing, the two primaries); for a planetary problem more closely related to the Solar System, see [11].

The proof of Arnold's theorem is rendered difficult by the multitudinous degeneracies of the planetary system. Arnold's initial proof does not fully describe these degeneracies. Arnold's initial understanding was wrong, as he admitted, due to a resonance he had not foreseen, discovered by Herman in the 1990's. However, this resonance happens not to play such an important role. In 1998, in a series of lectures M. Herman sketched a complete and more conceptual proof of this theorem, showing some weak non-degeneracy property of the planetary problem, and then concluding by calling upon a theorem of Diophantine approximation of Arnold-Pyartli [10]. Later, Chierchia and Pinzari strengthened the result qualitatively and quantitatively with a proof closer to Arnold initial strategy, showing in particular the strong non-degeneracy property (as defined by Poincaré in his proof of existence of Linstedt series [30], and later used also by Kolmogorov [14] or Arnold) of the planetary problem [8, 9]. We now review some ideas of these proofs.

1. **Hamiltonian.** Newton's equations are equivalent to Hamilton's equations (Cauchy, 1831)

$$\begin{cases} \dot{x}_0 = \epsilon \partial_{y_0} H \\ \dot{y}_0 = -\epsilon \partial_{x_0} H \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_j = \partial_{y_j} H \\ \dot{y}_j = -\partial_{x_j} H \end{cases}$$

if the linear momenta are $y_0, \epsilon y_1, \dots, \epsilon y_n$ and the Hamiltonian

$$\epsilon H = \frac{1}{2} \frac{\|y_0\|^2}{m_0} + \epsilon \sum_j \left(\frac{\|y_j\|^2}{2m_j} - \frac{m_0 m_j}{\|x_j - x_0\|} \right) - \epsilon^2 \sum_{j < k} \frac{m_j m_k}{\|x_j - x_k\|}$$

(planets's indices j, k vary from 1 to n).

2. **Reduction by translations.** Switch to the symplectic heliocentric coordinates:

$$\begin{cases} X_0 = x_0 \\ Y_0 = y_0 + \epsilon y_1 + \epsilon y_2 + \dots + \epsilon y_n \end{cases} \quad \text{and} \quad \begin{cases} X_j = x_j - x_0 \\ Y_j = y_j. \end{cases}$$

The conservation of the total linear momentum Y_0 and the invariance by translation of Newton's equations allow us to focus on the subspace $Y_0 = 0$ without loss of generality, and to ignore the variable X_0 . The equations now read

$$\dot{X}_j = \partial_{Y_j} H, \quad \dot{Y}_j = -\partial_{X_j} H,$$

with

$$H = \sum_j \left(\frac{\|Y_j\|^2}{2\mu_j} - \frac{\mu_j M_j}{\|X_j\|} \right) + \epsilon \sum_{j < k} \left(-\frac{m_j m_k}{\|X_j - X_k\|} + \frac{Y_j \cdot Y_k}{m_0} \right);$$

the 'fictitious' masses $\mu_j \sim m_j$ and $M_j \sim m_0$ are functions of the m_j 's and of ϵ , defined by $\frac{1}{\epsilon \mu_j} = \frac{1}{m_0} + \frac{1}{\epsilon m_j}$ and $M_j = m_0 + \epsilon m_j$.

3. The Keplerian part and the perturbing function. The Hamiltonian splits into $H = H_{Kep} + \epsilon H_{per}$, the sum of a *Keplerian Hamiltonian* and an ϵ -small *perturbing function*.

Let $(\lambda_j, \Lambda_j, \xi_j, \eta_j, p_j, q_j) \in \mathbb{T}^1 \times]0, +\infty[\times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be the Poincaré coordinates of the j -th planet (see appendix A). Those coordinates are analytic and symplectic on a neighborhood of the union of circular, direct, horizontal (with respect to some given plane in space) Keplerian ellipses:

- the angle λ_j parameterizes the ellipse proportionally to the area swept by the position vector,

- $\Lambda_j = \mu_j \sqrt{M_j a_j}$,

- ξ_j and η_j determine the eccentricity and orientation of the ellipse within its plane,

- p_j and q_j determine the position of the plane of the ellipse.

Those coordinates straighten the (degenerate) Keplerian dynamics, since H_{Kep} now only depends on Λ_j 's:

$$H_{Kep} = \sum_j -\frac{\mu_j^3 M_j^2}{2\Lambda_j^2}.$$

Let

$$\nu_j = \frac{\partial H_{Kep}}{\partial \Lambda_j} = \frac{\mu_j^3 M_j^2}{\Lambda_j^3} = \frac{\sqrt{M_j}}{a_j^{3/2}}$$

be the so-called *mean motions* (an old name for the Keplerian frequencies); Kepler's third law follows from this expression of ν_j .

4. The averaged Hamiltonian. From now on, restrict to the open set, diffeomorphic to $\mathbb{T}^n \times \mathbb{R}^{5n}$, over which Keplerian ellipses do not meet. Up to renumbering, we can assume that $0 < a_1 < \dots < a_n$. Away from the boundary and in a neighborhood of circular Keplerian ellipses, the perturbing function is uniformly ϵ -small. Thus a change of coordinates ϵ -close to the identity transforms H into the *averaged Hamiltonian*

$$H_{Kep} + \epsilon \langle H_{per} \rangle = H_{Kep} + \epsilon \int_{\mathbb{T}^n} H_{per} \frac{d\lambda_1 \cdots d\lambda_n}{(2\pi)^n},$$

up to terms of order 2 in ϵ , along the Cantor set of Diophantine Keplerian frequencies. (Technically, this means that the new Hamiltonian equals the above expression, plus a term which is C^∞ -flat on the Cantor set, which plays no role and which will be ignored here.)

For the averaged Hamiltonian, the momenta Λ_j are first integrals (this is the first stability theorem of Lagrange and Laplace). It descends to the quotient by the Keplerian action of \mathbb{T}^n and induces a Hamiltonian system on the space, diffeomorphic to $\mathbb{R}^{4n} = \{(\xi_j, \eta_j, p_j, q_j)_{j \in \{1, \dots, n\}}\}$, of Keplerian tori with fixed semi major axes. This Hamiltonian, up to a constant, is $\langle H_{per} \rangle$; it is called the *secular Hamiltonian* and its phase space the *secular space*. This system describes the slow variations of excentricity and orientation in space of the Keplerian ellipses of the planets, under the influence of mutual attraction, at the first order in ϵ , outside resonances in mean motions. Contrary to its analogue in the Lunar problem (see [18]), it seems non-integrable.

5. **The elliptic secular singularity.** By symmetry, the origin $\xi = \eta = p = q = 0$ of the secular space is an equilibrium point. It turns out to be an elliptic critical point of the averaged Hamiltonian (this is the second stability theorem of Laplace), having the following remarkable expansion:

$$\langle H_{per} \rangle = C_0(m, a) + Q_h \cdot \xi^2 + Q_h \cdot \eta^2 + Q_v \cdot p^2 + Q_v \cdot q^2 + O(4),$$

where the “horizontal” and “vertical” quadratic forms Q_h and Q_v are of the form

$$\begin{cases} Q_h \cdot \xi^2 &= \sum_{1 \leq j < k \leq n} m_j m_k \left(C_1(a_j, a_k) \left(\frac{\xi_j^2}{\Lambda_j} + \frac{\xi_k^2}{\Lambda_k} \right) + 2C_2(a_j, a_k) \frac{\xi_j \xi_k}{\sqrt{\Lambda_j \Lambda_k}} \right) \\ Q_v \cdot p^2 &= \sum_{1 \leq j < k \leq n} -m_j m_k C_1(a_j, a_k) \left(\frac{p_j}{\sqrt{\Lambda_j}} - \frac{p_k}{\sqrt{\Lambda_k}} \right)^2; \end{cases}$$

coefficients C_j are real analytic with respect to the semi major axes, and can be expressed in terms of the Laplace coefficients.

6. **The secular frequencies.** Let ρ_h and ρ_v be orthogonal diagonalizing transformations of Q_h and Q_v (depending analytically on the masses and semi major axes):

$$\rho_h^* Q_h = \sum_j \sigma_j d\xi_j^2 \quad \text{and} \quad \rho_v^* Q_v = \sum_j \sigma_{n+j} dp_j^2.$$

In the full phase space, the map

$$\rho : (\xi, \eta, p, q) \mapsto (\rho_h \cdot \xi, \rho_h \cdot \eta, \rho_v \cdot p, \rho_v \cdot q)$$

is symplectic and the problem thus boils down to studying a Hamiltonian of the form

$$\begin{aligned} H_{Kep}(\Lambda) + \epsilon \sum_j (\sigma_j(\Lambda)(\xi_j^2 + \eta_j^2) + \sigma_{n+j}(\Lambda)(p_j^2 + q_j^2)) \\ + \epsilon O_4(\xi, \eta, p, q) + O_2(\epsilon), \end{aligned} \quad (1)$$

where the first remainder does not depend on λ . The Hamiltonian obtained by neglecting the two remainders is integrable and its integral curves (outside the elliptic singularity) are quasiperiodic, with the $3n$ frequencies given by

$$\alpha^o := (\nu_1, \dots, \nu_n, \epsilon\sigma_1, \dots, \epsilon\sigma_n, \epsilon\sigma_{n+1}, \dots, \epsilon\sigma_{2n}).$$

The arithmetic properties of the frequency vector α^o play a deciding role in the existence of higher order normal forms; α^o depends on the masses and the semi major axes, and can also be thought of as a constant function of the actions $\xi_j^2 + \eta_j^2$ and $p_j^2 + q_j^2$ when one moves away from the elliptic secular singularity.

7. **Properties of the frequency vector.** KAM theory asserts that there is a perturbed frequency vector α , not explicitly known but tending to α^o when ϵ , $\xi_j^2 + \eta_j^2$ and $p_j^2 + q_j^2$ tend to zero, such that whenever α is Diophantine there is a corresponding invariant torus.

It is customary in this theory to measure the abundance of invariant tori given by the KAM theorem *with fixed masses*. J. Moser justified this habit with the argument that one cannot change the masses of the planets. We

will let the reader decide from himself whether it is easier to change semi major axes.

— If α is seen as a function of $a = (a_1, \dots, a_n)$ alone (implying that we look for tori very close to circular coplanar Keplerian ellipses), the best we can hope for is the weak non-degeneracy property of Arnold-Pyartli i.e., the local image of α does not lie in a hyperplane. In this case, the theory of Diophantine approximations shows that the set of parameters a such that the frequency vector is Diophantine has positive Lebesgue measure.

— If α is seen as a function of both a and the actions $\xi_j^2 + \eta_j^2$'s and $p_j^2 + q_j^2$'s, one can hope for the strong non-degeneracy property of Kolmogorov i.e., the map α is a local diffeomorphism, thus obviously reaching Diophantine vectors for a set of parameters of positive measure.

There are difficulties in both cases, which are removed when one takes advantage of the invariance of the system by rotations. This symmetry allows us to decrease the number of degrees of freedom by two units and get rid of annoying resonances.

8. Weak non-degeneracy. In the plane ($p = q = 0$), the application of KAM theory is quite straightforward, due to the following fact.

Proposition 2. *Outside an analytic proper subset of values of $a = (a_1, \dots, a_n)$, the vector $(\nu_1, \dots, \nu_n, \epsilon\sigma_1, \dots, \epsilon\sigma_n)$ locally defines an analytic function of a , whose image is not contained in any vector hyperplane.*

This means that no resonance is identically verified and, by Pyartli's theorem [32], that the perturbed frequency vector passes through Diophantine vectors in positive measure. Arnold's theorem in the plane follows. The proof of the proposition goes along the following lines.

For $n = 2$ planets, there is an explicit asymptotics of α in the 'well-spaced regime' where $a_1 \ll a_2$, where the conclusion of the proposition is readily checked. Complexifying the semi major axes and using the analytic continuation of Q_h , one can deduce the proposition for $n = 2$. In the case of $n + 1$ planets, in the well-spaced regime the spectrum of Q_h splits into a very small eigenvalue and the spectrum corresponding to n planets. By induction over the number n of planets the conclusion follows.

In space, the situation is more intricate.

Proposition 3. *Outside an analytic proper subset of values of a , the frequency vector α^o locally defines an analytic function of a , whose image entirely lies in the codimension-2 space of \mathbb{R}^{3n} of equations*

$$\sigma_{2n} = 0, \quad \sum_{1 \leq j \leq 2n} \sigma_j = 0,$$

(up to reordering of the σ_j 's) but is contained in no space of larger codimension.

It is straightforward to check the two resonances. The proof of the proposition then goes along the same lines as for the plane problem.

Unsurprisingly, the first resonance is due to the invariance of the system by rotations about any horizontal axis (hence the linearized secular vector field has two vanishing eigenvalues), so that one of the spatial secular frequencies,

say σ_{2n} , vanishes. Thus this resonance disappears when one restricts to a fixed direction, say vertical, of the angular momentum —a codimension-2, hence $(6n - 2)$ -dimensional, symplectic submanifold.

The second resonance is mysterious and does not seem to be associated with any symmetry. Although it was known to astronomers in particular cases ($n = 2$), it was discovered in its full generality by Herman. It happens to disappear when one completes the reduction by rotations, by fixing the angular momentum vector and quotienting by rotations around the fixed direction of the angular momentum. For $n = 2$ planets, this is checked by carrying out the classical reduction of the node of Jacobi.

The difficulty when $n \geq 3$ is that we do not know of simple coordinates adapted to the reduction by rotations. Arnold thus suggested to merely fix the direction of the angular momentum, not being aware of Herman's resonance. In order to check without much computation that Herman's resonance does not exist at the fully reduced level, one can use a trick (already used by Poincaré in order to find some periodic orbits [30]), consisting in adding to the Hamiltonian a term in $\delta\|C\|$, where δ is small real number, and $\|C\| = C_z$ is the length of the angular momentum. This amounts to switching to a rotating frame of coordinates. The KAM theorem applies to show the existence of Diophantine Lagrangian tori of dimension $3n - 1$, invariant for the modified Hamiltonian, for a positive measure of values of δ (one value of δ would suffice). Then, by an argument of Lagrangian intersection, these tori must be invariant by the flow of the initial Hamiltonian; as Chierchia-Pinzari's results will later prove, these tori are sometimes ergodic, and sometimes foliated in ergodic invariant tori of dimension $3n - 2$. This proves Arnold's theorem.

9. Strong non-degeneracy. Chierchia and Pinzari have strengthened and completed the previous result by investigating the part of the secular Hamiltonian which is quartic with respect to the secular Poincaré coordinates ξ, η, p, q , as in Arnold's original strategy.

Rotations around a horizontal axis generate a degeneracy of all Birkhoff invariants, which disappears by fixing the direction of the angular momentum (but the norm of the angular momentum is not fixed and we do not descend to the quotient by rotations around the angular momentum). Now the phase space has dimension $6n - 2$. This partial reduction by rotations, which already played a key role above for proving the weak non-degeneracy condition, can be understood as follows. Topologically, the group SO_3 is an S^1 -bundle over S^2 . The action of the maximal torus $S^1 = SO(2)$ carries all the non-trivial dynamical information, while all directions of the angular momentum are equivalent to each other.

For the Birkhoff normal form to exist at the order 4, a priori we need the secular frequencies to avoid resonances of order ≤ 4 ; see [7] for the applicability of KAM which follows. According to the proposition above, the only remaining resonance is Herman's, which is order $2n - 1$. This resonance is not an obstruction for $n \geq 3$ planets. For $n = 2$ planets (a case which can be handled anyway by completing the reduction by rotations, by carrying out Jacobi's reduction of the node, as in Robutel's thesis [33]), it happens that the resonant terms vanish, even up to terms of order 10 [20]! So, in

all cases, the secular Hamiltonian (1) has a non-resonant Birkhoff normal form at the 4-th order: there exists a system of symplectic coordinates $((\tilde{\lambda}_j, \Lambda_j)_{1 \leq j \leq n}, (x_j, y_j)_{1 \leq j \leq 2n-1})$ in which the Hamiltonian (1) has the form

$$H_{Kep}(\Lambda) + \epsilon \sum_j \sigma_j(\Lambda)(x_j^2 + y_j^2) + \sum_{j,k} \tau_{j,k}(\Lambda)(x_j^2 + y_j^2)(x_k^2 + y_k^2) + \epsilon O_6(x, y) + O_2(\epsilon), \quad (2)$$

where the symmetric matrix $\tau = (\tau_{j,k}(\Lambda))$ depends analytically of Λ and the masses. The determinant of τ is the ‘‘torsion’’ of the secular system. That the system is strongly non-degenerate (in the sense of Poincaré, or Kolmogorov), means that the torsion does not vanish.

The fourth order terms $\tau_{j,k}$ are formidable to compute. Chierchia-Pinzari use some regularized version of the Deprit coordinates. Those coordinates do not reduce to separate coordinates associated with each planet. The Deprit coordinates were independently rediscovered by G. Pinzari in her PhD thesis [29], before she realized they matched Deprit’s coordinates. Interestingly, Deprit believed that nobody would ever make anything useful with these coordinates. Pinzari managed to compute asymptotics of τ , as before in the well-spaced regime. An induction shows that torsion does not vanish. Chierchia-Pinzari can conclude triumphantly, probably as closely as possible in the spirit of what Arnold had in mind in 1963:

Theorem 4. *The torsion of the planetary problem at the elliptic singularity of the averaged system is non-degenerate, both in restriction to the submanifold obtained by fixing the direction of the angular momentum, and in the system fully reduced by rotations.*

Hence KAM theory can be applied to the planetary problem, with full strength. Chierchia-Pinzari’s theorem gives some additional information on the secular system and allows Chierchia-Pinzari to give better measure estimates than what the weak non-degeneracy would give. (Un)fortunately, the torsion is indefinite (as Herman’s resonance shows), which prevents from using variational methods to (easily?) find invariant sets from Mather theory...

Appendix A. Poincaré coordinates. For the Kepler Hamiltonian

$$K = \frac{\|p\|^2}{2\mu} - \frac{\mu M}{\|q\|}, \quad (q, p) \in \mathbb{R}_*^3 \times \mathbb{R}^3,$$

we will first define some symplectic action-angle coordinates, attributed to Delaunay, which blow up circular or horizontal Keplerian motions; blowing down the singularity will yield the symplectic analytic Poincaré coordinates. For elementary facts about K , we refer to [3]. For other ways to introduce those coordinates, see [36, Chap. VII], or [6, 24, 28, 31, 35].

Notations in the plane Kepler problem with negative energy:
 v = true anomaly = argument of q , measured from the pericenter
 a = semi major axis
 g = argument of the pericenter from Ox
 ϵ = eccentricity.

Restrict to the set $\{(v, a, g, \epsilon), 0 < \epsilon < 1\} \equiv \mathbb{T}^2 \times \mathbb{R}^2$ of non-degenerate, non-circular, elliptical, Keplerian motions. Define coordinate t as the time from the pericenter; it is defined modulo the period T of the orbit. Define ℓ , the *mean anomaly*, as the angle obtained by rescaling time: $\ell := 2\pi t/T \pmod{2\pi}$. Now, if we want an action coordinate $L(K)$ conjugate to ℓ : $dt \wedge dK = d\ell \wedge dL$, we see that

$$L'(K) = \frac{1}{\dot{\ell}} = \frac{T}{2\pi} = \frac{a^{3/2}}{\sqrt{M}} = \frac{\mu^{3/2}M}{(-2K)^{3/2}}.$$

Conventionally choosing $L = 0$ at infinity where $a = +\infty$, we get

$$L = \frac{\mu^{3/2}M}{\sqrt{-2K}} = \mu\sqrt{Ma} \quad \left(\text{so that } K = -\frac{\mu^3 M^2}{2L^2} \right).$$

We now wish to define coordinates on the space of non-circular, non-degenerate, prograde Keplerian ellipses in the plane with fixed L . The *angular momentum*

$$G := \mu\sqrt{Ma(1-\epsilon^2)} = L\sqrt{1-\epsilon^2},$$

is a first integral of K and thus descends to the space of Keplerian orbits. Its Hamiltonian flow acts by 2π -periodic rotations around the origin. Define g as the angle, modulo 2π , measuring time along X_G -orbits, and vanishing when the pericenter meets the Ox -semi-axis.

The coordinates which (ℓ, L, g, G) define are symplectic:

- $\{\ell, L\} = \{g, G\} = 1$ by definition.
- $\{L, g\} = \{L, G\} = 0$ because g and G are first integrals of $K(L)$.
- $\{\ell, G\} = 0$ because the flow of G rotates the Keplerian ellipse without revolving the body along the ellipse.
- $\{\ell, g\} = 0$. Due to the Jacobi identity, $\{L, \{\ell, g\}\} = \{G, \{\ell, g\}\} = 0$. Hence it suffices to show that $\{\ell, g\} = 0$ in restriction to the section $\{\ell = g = 0 \pmod{\pi}\}$ of the L - and G -flows. We may thus assume that the body is on the major axis and that the major axis itself is the x -axis. But then the partial derivatives of ℓ and g with respect of x or p_y are zero, and

$$\{\ell, g\} = \underbrace{\frac{\partial \ell}{\partial x}}_{=0} \frac{\partial g}{\partial p_x} - \frac{\partial \ell}{\partial p_x} \underbrace{\frac{\partial g}{\partial x}}_{=0} + \frac{\partial \ell}{\partial y} \underbrace{\frac{\partial g}{\partial p_y}}_{=0} - \underbrace{\frac{\partial \ell}{\partial p_y}}_{=0} \frac{\partial g}{\partial y} = 0.$$

In the 3-dimensional Kepler problem, choose $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ as a reference plane, called *horizontal*. Temporarily restrict to non-horizontal, non-circular, non-degenerate, prograde elliptic Keplerian motions.

Let $\vec{C} = q \times p$ be the *angular momentum vector* and Θ be its projection on the vertical axis. The flow of X_Θ consists of 2π -periodic rotations in the horizontal plane (diagonally for positions and impulsions), leaving the horizontal 4-plane invariant.

Each Keplerian oriented plane meets the horizontal plane along a half axis, the *ascending line of the node*. Let θ be the angle measuring time along X_Θ -orbits, vanishing when the line of the node is the Ox -semi-axis.

The so-defined coordinates $(\ell, L, g, G, \theta, \Theta)$ are symplectic:

- Poisson brackets with L, G and Θ are what they should : 0, except $\{\ell, L\} = \{g, G\} = \{\theta, \Theta\} = 1$ (we know the flows of L, G and Θ).

– The three Poisson brackets between angles can be checked to vanish as above in the plane. Indeed, on the submanifold $\{\ell = g = \theta = 0 \pmod{\pi}\}$, the partial derivatives of any of these angles with respect to x , p_y or p_z vanish.

Now, define the Poincaré coordinates $(\lambda, \Lambda, \zeta, z)$ by the following formulas (several sign conventions exist):

$$\begin{cases} \lambda = \ell + g + \theta, & \Lambda = L \\ \zeta = \sqrt{2(L - G)} e^{-ig} \\ z = \sqrt{2(G - \Theta)} e^{-i\theta} \end{cases}$$

Knowing that the Delaunay coordinates are symplectic, it is straightforward that the Poincaré coordinates are symplectic too. From the above formulas, one checks that the Poincaré coordinates extend to continuous coordinates at direct circular coplanar motions ($\zeta = z = 0$), since

$$\begin{cases} \zeta = \sqrt{\Lambda/2} \epsilon (1 + O(\epsilon^2)) e^{-ig} \\ z = \sqrt{\Lambda/2} \iota (1 + O(\epsilon^2) + O(\iota^2)) e^{-i\theta}, \quad \cos \iota = \frac{\Theta}{G} \end{cases}$$

(in particular, g and ι need not be defined at the singularity). In fact, their extension is analytic, as one can see by expressing the coordinates as explicit analytic functions of analytic first integrals; see [1] for an elegant choice of first integrals.

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