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Averaging the planar three-body problem in the neighborhood of double inner collisions

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Levi-Civita's regularization procedure for the two-body problem easily extends to a regularization of double inner collisions in the system consisting of two uncoupled Newtonian two-body problems. Some action-angle variables are found for this regularization, and the inner body is shown to describe ellipses on all energy levels. This allows us to define a second projection of the phase space onto the space of pairs of ellipses with fixed foci. It turns out that the initial and regularized averaged Hamiltonians of the three-body problem agree, when seen as functions on the space of pairs of ellipses. After the reduction of the problem by the symmetry of rotations, the initial and regularized averaged planar three-body problems are shown to be orbitally conjugate, up to a diffeomorphism in the parameter space consisting of the masses, the semi major axes and the angular momentum.

Key Words: three-body problem, regularization, averaging, secular system

In the classical three-body problem, if one of the masses is far from or large compared to the other two, each of the masses approximately describes a Keplerian ellipse whose elements slowly vary with time. At the limit where one mass is infinite or infinitely far away, the frequencies of these slow *secular* deformations vanish; the system then is the product of two uncoupled Keplerian problems and is thus completely integrable and dynamically degenerate. In the neighborhood of this limit, if furthermore the Keplerian frequencies satisfy a finite number of non-resonance conditions, the *averaged system*, which is obtained by averaging the initial vector field over the Keplerian ellipses, is the first of the normal forms of the full system [1]. As such, it is also called the *first order secular system* and, for the planar three-body problem, it is completely integrable.

Since Lagrange and Laplace first tried to solve the problem of the stability of the solar system, this averaged system has been extensively studied. However, due to astronomical reasons, existing studies mainly examine the neighborhood of circular and coplanar orbits. Jefferys-Moser's [6] and Lieberman's [8] results are a first step towards a more global study. Yet they did not describe the dynamics up to collisions.

It turns out in certain conditions, especially when the angular momentum is small enough and when the energy is sufficiently negative, that the conservation of these two first integrals does not prevent the two inner bodies from colliding [7].

After having taken advantage of the Galilean invariance and having fixed the center of mass, the averaged system is a priori defined on the space of pairs of oriented ellipses with fixed foci which do not intersect one another. This space can be compactified by adding degenerate eccentricity-one ellipses at infinity. Such an ellipse corresponds to a collision orbit where the body goes back and forth along a line segment between its pericenter and its apocenter [2, 10]). A striking feature of the averaged system is that it extends to an analytic function where the inner ellipse is degenerate [4]. Thus, understanding the global structure of the fixed points of the averaged system after its reduction by the symmetry of rotation demands taking into account these double inner collisions.

The non-averaged perturbing function of the three-body problem extends to a continuous function at collisions. Unfortunately, this extension is not even differentiable. So the extension of the averaged system itself appears to be dynamically irrelevant. This paper proves that, in the case of the planar problem, the extension of the averaged system actually is the averaged system associated to the *regularized* problem, up to some diffeomorphism in the parameter space. The Levi-Civita regularization is used [2, 10]. It substitutes the eccentric anomaly for the mean anomaly. The perturbing function is indeed an analytic function of the eccentric anomaly, which is thus a better adapted angle in the neighborhood of collisions.

A noteworthy consequence of this formal study of the averaged system in the neighborhood of double inner collisions is the existence of quasiperiodic invariant "punctured tori" on which the two inner bodies get arbitrarily close to one another an infinite number of times. These motions generalize those which Chenciner-Llibre had found in the case of the circular restricted planar problem [3]. We give an outline of this result in section 4. The complete proof, which requires to build higher order secular systems and to apply some sophisticated version of KAM theorem à la Herman, will be given in a forthcoming paper [5].

1. SETTING AND NOTATIONS

First, symplectically reduce the three-body problem by the Galilean symmetry. The new system describes two fictitious bodies turning around a fixed center of attraction (cf. § 24, chap. II, first volume of the *Leçons* [9]). Note that the fixed center does not attract both bodies with the same mass. Let $q_j \in \mathbf{C}\setminus 0$ ($j = 1,2$) be the position vectors of the two fictitious masses μ_1 and μ_2, and $p_j \in \mathbf{C}^*$ be their linear momentum covectors. Neglecting the mutual interaction of the two fictitious moving masses (*ibid.*) leads to the system defined on the space of quadruplets (q_1, q_2, p_1, p_2) i.e., the cotangent bundle $T^*(\mathbf{C}\setminus 0)^2 \simeq (\mathbf{C}\setminus 0)^2 \times \mathbf{C}^2$ by the Hamiltonian

$$F_k = \frac{|p_1|^2}{2\mu_1} - \frac{\mu_1 M_1}{|q_1|} + \frac{|p_2|^2}{2\mu_2} - \frac{\mu_2 M_2}{|q_2|}$$

and the symplectic form

$$\omega = \Re(dp_1 \wedge d\bar{q}_1 + dp_2 \wedge d\bar{q}_2),$$

where $\Re$ stands for the real part of a complex number. It is the direct product of two uncoupled Keplerian problems and thus defines a *Keplerian action* of the torus $\mathbf{T}^2$ on an open set of the phase space.

Let's restrict ourselves to pairs of elliptic motions such that the two ellipses do not meet one another and let's call the inner ellipse the first ellipse. (As long as we do not take the interaction between the two bodies into account, we actually do not really care if the two ellipses meet one another.) The relevant part of the phase space is diffeomorphic to

$$T^*(\mathbf{C}\setminus 0) \times \Big(T^*(\mathbf{C}\setminus 0) \setminus ((\mathbf{C}\setminus 0)\times \mathbf{R})\Big) \simeq \left(\mathbf{S}^1 \times \mathbf{R}^3\right) \times \mathbf{A}^4,$$

where $\mathbf{A}^4 \simeq \mathbf{R}\times\mathbf{S}^1\times\mathbf{R}^2\times\mathbf{S}^0$ is the phase space of the outer body; the factor $\mathbf{S}^0$ corresponds to the two possible ways the outer body can move around the inner ellipse. Let L.C. be the two-sheeted covering of Levi-Civita, defined as the product of the cotangent map of $z \mapsto z^2$ by $id_{\mathbf{A}^4}$:

$$\begin{array}{rccl} \text{L.C.}: & T^*(\mathbf{C}\setminus 0)\times\mathbf{A}^4 & \longrightarrow & T^*(\mathbf{C}\setminus 0)\times\mathbf{A}^4 \\ & ((z,w),a) & \longmapsto & ((q_1,p_1),a) = \left(\left(z^2, \frac{w}{2\bar{z}}\right), a\right). \end{array}$$

L.C. is symplectic:

$$\text{L.C.}^*\omega = \Re(dw \wedge d\bar{z} + dp_2 \wedge d\bar{q}_2).$$

Lemma 1.1. *For any real number $f > 0$, the Hamiltonian*

$$\text{L.C.}^*\left(|q_1|(F_k + f)\right)$$

extends to an $\mathbf{R}$*-analytic Hamiltonian on* $T^*\mathbf{C} \times \mathbf{A}^4$.

The proof is obvious : this Hamiltonian can be written

$$|z|^2 \mathrm{L.C.}^*(F_k + f) = \frac{|w|^2}{8\mu_1} + \left(f + \frac{|p_2|^2}{2\mu_2} - \frac{\mu_2 M_2}{|q_2|} \right) |z|^2 - \mu_1 M_1.$$

A direct consequence of Leibniz rule is that on the energy surface

$$\mathrm{L.C.}^*(F_k + f) = 0,$$

outside collisions, the Hamiltonian vector fields associated to

$$\mathrm{L.C.}^* \left(|q_1|(F_k + f) \right) \quad \text{and} \quad \mathrm{L.C.}^* F_k$$

define the same oriented straight line field.

2. REGULARIZED KEPLERIAN DYNAMICS

The (pull-back by L.C. of the) angular momentum is a first integral for the regularized Hamiltonian $\mathrm{L.C.}^* \left(|q_1|(F_k + f) \right)$ because it was one for F_k and the slow-down function $|q_1|$ is invariant by rotations. Moreover, since $|q_1|$ only depends on the inner body, orbits of the outer body remain unchanged. If $(\Lambda_2, \lambda_2, \xi_2, \eta_2)$ are Poincaré coordinates (cf. § 58, chap. III, first vol., *Leçons* [9]) of this body on $\mathbf{A}^4$, the functions Λ_2, ξ_2, and η_2 are first integrals, as well as

$$f_1(\Lambda_2) := f - \frac{\mu_2^3 M_2^2}{2\Lambda_2^2},$$

which, on the energy level $\mathrm{L.C.}^*(|q_1|(F_k + f)) = 0$, is the opposite of the energy of the inner body. Thus the Hamiltonian vector field of

$$\mathrm{L.C.}^* \left(|q_1|(F_k + f) \right) = \frac{|w|^2}{8\mu_1} + f_1(\Lambda_2)|z|^2 - \mu_1 M_1$$

is the skew-product of a rotator (outer body) slowed down by a pair of $(1,1)$-resonant harmonic oscillators (inner body).

The point $z = w = 0$ from the phase space can be ignored, because it corresponds to an infinite energy for the initial problem. The phase space is then diffeomorphic to

$$(T^*\mathbf{C}) \setminus 0 \times \mathbf{A}^4 \simeq \mathbf{S}^3 \times \mathbf{R} \times \mathbf{A}^4.$$

Since the L.C. mapping is a two-sheeted covering, the pull-backs by L.C. of all the initial observables (e.g. q_1, the angular momentum G_1 of the first ellipse, etc.) descend through the antipodal mapping of the sphere

$$\mathbf{S}^3 \times \mathbf{R} \times \mathbf{A}^4 \overset{(z,w,a)\sim(-z,-w,a)}{\longrightarrow} \mathrm{SO}_3 \times \mathbf{R} \times \mathbf{A}^4$$

and in general they extend to $\mathrm{SO}_3 \times \mathbf{R} \times \mathbf{A}^4$ by continuity. We will generally denote the extensions by the name of the initial observable. Let $\mathcal{F}_k$ be the direct image of L.C.* $(|q_1|(F_k + f))$ by the antipodal mapping.

Lemma 2.1. *There exist a blow-up*

$$b : \mathbf{T}^2 \times \mathbf{R}^2 \times \mathbf{A}^4 \to \mathrm{SO}_3 \times \mathbf{R} \times \mathbf{A}^4$$

of the phase space and coordinates $((\mathcal{L}_1, \delta_1, \mathcal{G}_1, \gamma_1), (\Lambda_2, \delta_2, \xi_2, \eta_2))$ *on each of the two connected components of* $(\mathbf{R}\times\mathbf{S}^1\times\mathbf{R}\times\mathbf{S}^1)\times\mathbf{A}^4 \simeq \mathbf{T}^2\times\mathbf{R}^2\times\mathbf{A}^4$ *such that*

$$b^*\mathcal{F}_k = \mathcal{L}_1\sqrt{\frac{2f_1(\Lambda_2)}{\mu_1}} - \mu_1 M_1$$

and

$$b^*\omega = d\mathcal{L}_1 \wedge d\delta_1 + d\mathcal{G}_1 \wedge d\gamma_1 + d\Lambda_2 \wedge d\delta_2 + d\xi_2 \wedge d\eta_2.$$

Moreover, we have

$$b^*q_1 = \frac{e^{i\gamma_1}}{\sqrt{2\mu_1 f_1(\Lambda_2)}}\left(-\sqrt{\mathcal{L}_1^2 - \mathcal{G}_1^2} + \mathcal{L}_1\cos\delta_1 + i\mathcal{G}_1\sin\delta_1\right)$$

and

$$\delta_2 = \lambda_2 + \frac{\alpha_2^{kep}(\Lambda_2)}{2f_1(\Lambda_2)}\sqrt{\mathcal{L}_1^2 - \mathcal{G}_1^2}\sin(\delta_1),$$

where $\alpha_2^{kep}(\Lambda_2) = f_1'(\Lambda_2)$*, and the angular momentum* G_1 *of the inner body is unchanged:* $b^*G_1 = \mathcal{G}_1$*.*

Note: the Poincaré coordinates of the outer body and their pull-back by b are designated by the same letters.

Proof. The approach is to straighten the ellipsoids of constant energy L.C.$^*(|q_1|(F_k + f))$, diagonalize the vector field and eventually use symplectic polar coordinates. This is completed by the blow-up

$$\begin{array}{rccc} b_2 : & (\mathbf{R}^+ \times \mathbf{T}^1)^2 \times \mathbf{A}^4 & \longrightarrow & (T^*\mathbf{C} \setminus 0) \times \mathbf{A}^4 \simeq \mathbf{S}^3 \times \mathbf{R} \times \mathbf{A}^4 \\ & ((r_j, \theta_j)_{j=1,2}, (\Lambda_2, \delta_2, \xi_2, \eta_2)) & \longmapsto & ((w, z), (\Lambda_2, \lambda_2, \xi_2, \eta_2)) \end{array}$$

defined by

$$\begin{cases} \dfrac{w}{\sqrt{2}\sqrt[4]{8\mu_1 f_1(\Lambda_2)}} + \dfrac{i}{\sqrt{2}}\sqrt[4]{8\mu_1 f_1(\Lambda_2)}z = \sqrt{2r_1}e^{i\theta_1} \\ \dfrac{\bar{w}}{\sqrt{2}\sqrt[4]{8\mu_1 f_1(\Lambda_2)}} + \dfrac{i}{\sqrt{2}}\sqrt[4]{8\mu_1 f_1(\Lambda_2)}\bar{z} = \sqrt{2r_2}e^{i\theta_2} \end{cases}$$

and

$$\delta_2 = \lambda_2 + \frac{\alpha_2(\Lambda_2)}{2f_1(\Lambda_2)}\Re(p_1\bar{q}_1)$$

where $\alpha_2(\Lambda_2) := f_1'(\Lambda_2)$. We indeed have to subsitute δ_2 for λ_2 if we want to straighten the ellipsoids and still preserve the symplectic form. The pull-back of the Hamiltonian and of the symplectic form are

$$b_2{}^*\mathrm{L.C.}^*(|q_1|(F_k + f)) = \sqrt{\frac{f_1(\Lambda_2)}{2\mu_1}}\,(r_1 + r_2) - \mu_1 M_1$$

and

$$b_2{}^*\mathrm{L.C.}^*\omega = dr_1 \wedge d\theta_1 + dr_2 \wedge d\theta_2 + d\Lambda_2 \wedge d\delta_2 + d\xi_2 \wedge d\eta_2.$$

The Levi-Civita mapping is a two-sheeted covering. Let's now go back downstairs, outside circular ellipses, through the two-sheeted covering

$$\begin{array}{ccc} (\mathbf{R}_*^+ \times \mathbf{T}^1)^2 \times \mathbf{A}^4 & \longrightarrow & (\mathbf{R} \times \mathbf{T}^1)^2 \times \mathbf{A}^4 \\ ((r_j, \theta_j)_{j=1,2}, a) & \longmapsto & ((\mathcal{L}_1, \delta_1, \mathcal{G}_1, \gamma_1), a) \end{array}$$

defined by

$$\begin{cases} \mathcal{L}_1 = \dfrac{r_1 + r_2}{2} \\ \delta_1 = \theta_1 + \theta_2 \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{G}_1 = \dfrac{r_1 - r_2}{2} \\ \gamma_1 = \theta_1 - \theta_2 + \pi. \end{cases}$$

The expression of q_1 given in the proposition is proved by a straightforward computation (cf. [2]), as well as those of δ_2 and $\mathcal{G}_1$. The translation by π in the definition of γ_1 is due to historical reasons: as a coordinate for the orientation of an ellipse, people usually use the argument of the pericenter rather than that of the apocenter. Let eventually b denote the direct image of the blow-up b_2 by the 2-sheeted covering previously defined. Then for a fixed $a \in \mathbf{A}^4$, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{S}^3 \times \mathbf{R} & \overset{b_2}{\longleftarrow} & \mathbf{T}^2 \times \mathbf{R}^2 \\ (w,z) \sim (-w,-z) \Big\downarrow & & \Big\downarrow (\theta_1, \theta_2) \sim (\theta_1 + \pi, \theta_2 + \pi) \\ \mathrm{SO}_3 \times \mathbf{R} & \overset{b}{\longleftarrow} & \mathbf{T}^2 \times \mathbf{R}^2. \end{array}$$

The left vertical arrow is the direct product of the identity of $\mathbf{R}$ by the universal covering $\mathbf{S}^3 \to \mathbf{RP}^3 = \mathrm{SO}_3$. When restricted to each energy level $\mathbf{T}^2 \times \mathbf{I}$, the horizontal arrows identify to one point the circles of the boundary which are obtained by fixing one of the two angles, depending on the connected component of the boundary. ▮

The expression of q_1 in the previous lemma yields:

Corollary 2.1. *The orbits of the inner body in the physical plane (coordinate q_1) under the regularized Keplerian flow are ellipses for any value of the energy.*

Definition 2.1. Let (q_1, p_1, q_2, p_2) be a point of the phase space. There exists a unique point (q_1, p_1', q_2, p_2) which describes the same pair of ellipses in the configuration space under the flow of F_k as (q_1, p_1, q_2, p_2) under the flow of $\mathcal{F}_k$. Let k_f be the diffeomorphism of the phase space to itself defined by

$$k_f : (q_1, p_1, q_2, p_2) \longmapsto k_f(q_1, p_1, q_2, p_2) = (q_1, p_1', q_2, p_2).$$

Also, let π_{F_k} (resp. $\pi_{\mathcal{F}_k}$) the quotient maps by the Keplerian action of $\mathbf{T}^2$ (resp. by its regularized action).

The diffeomorphism k_f induces the identity on the energy level $\mathcal{F}_k = 0$ and the action-angle coordinates of the inner body for the regularized problem, $(\mathcal{L}_1, \delta_1, \mathcal{G}_1, \gamma_1)$, agree on this level with the Delaunay coordinates (L_1, l_1, G_1, g_1) (cf. § 58, chap. III, first vol. of the *Leçons* [9]) once the mean anomaly l_1 has been replaced by the eccentric anomaly u_1.

3. AVERAGING

Let $F = F_k + F_p$ be a perturbation of F_k, chosen among Hamiltonians which are defined and of class $\mathcal{C}^\infty$ over an open set $\mathcal{O}$ of the phase space. Suppose that the following three properties hold: (1) F_p is a function on the configuration space (i.e. it does not depend on the velocities); (2) $\mathcal{O}$ is invariant by the Keplerian action of the torus $\mathbf{T}^2$; (3) $\mathcal{O}$ contains an open subset of the set of double inner collisions $q_1 = 0$. Let then $\mathcal{F}_k + \mathcal{F}_p$ be the regularized analogue of $F_k + F_p$, with $\mathcal{F}_k = |q_1|(F_k + f)$ and $\mathcal{F}_p = |q_1| F_p$. Typically, F_p is the perturbing function of the planar three-body problem for the Jacobi decomposition (cf. § 42, chap. II, first vol. of the *Leçons* [9]).

Recall that up to the terms F_k and $\mathcal{F}_k$ –which, as functions of L_1 and Λ_2, are constants after the symplectic reduction by fast angles (l_1, λ_2) or (δ_1, δ_2), the first order secular systems of the initial and regularized problems are the averages of F_p and $\mathcal{F}_p$ over the orbits of the Keplerian actions (cf. the encyclopædia [1]):

$$\langle F_p\rangle = \frac{1}{4\pi^2}\int_{\mathbf{T}^2} F_p\, dl_1\, dl_2 \quad \text{and} \quad \langle \mathcal{F}_p\rangle = \frac{1}{4\pi^2}\int_{\mathbf{T}^2} \mathcal{F}_p\, d\delta_1\, d\delta_2.$$

$\langle F_p\rangle$ and $\langle \mathcal{F}_p\rangle$ can be factorized through the space of pairs of oriented ellipses with fixed foci. This space can be thought of as the quotient space of the phase space either by π_{F_k} or by $\pi_{\mathcal{F}_k}$. So let $\{F_p\}$ and $\{\mathcal{F}_p\}$ be functions on an open subset of the space of pairs of oriented ellipses with fixed foci defined by

$$\langle F_p\rangle = \{F_p\}\circ \pi_{F_k} \quad \text{and} \quad \langle \mathcal{F}_p\rangle = \{\mathcal{F}_p\}\circ\pi_{\mathcal{F}_k}.$$

Elliptic elements such as the semi major axes a_1, a_2, or eccentricities e_1, e_2, can naturally be thought of as functions over the space of pairs of ellipses, whereas the functions $L_1, \mathcal{L}_1, f(\Lambda_2)$, ..., will be thought of as being defined over the phase space, once the masses and the parameter f are fixed. For instance we can write

$$L_1 = \mu_1\sqrt{M_1}(\sqrt{a_1}\circ\pi_{F_k}) \quad \text{and} \quad \mathcal{L}_1 = \sqrt{2\mu_1 f_1}(a_1\circ\pi_{\mathcal{F}_k}).$$

Proposition 3.1. *The initial and regularized secular Hamiltonians satisfy*

$$\{\mathcal{F}_p\} = a_1\{F_p\}$$

i.e.,

$$\langle\mathcal{F}_p\rangle = \left(\frac{L_1^2}{\mu_1^2 M_1}\langle F_p\rangle\right)\circ\, k_f.$$

Proof. In the integral defining $\langle\mathcal{F}_p\rangle$, after having taken into account the fact that $d\delta_1\wedge d\delta_2 = d\delta_1\wedge d\lambda_2 = d\delta_1\wedge dl_2$, let's carry out the change of variables corresponding to k_f :

$$\langle\mathcal{F}_p\rangle = \frac{1}{4\pi^2}\int_{k_f(\mathbf{T}^2)} \mathcal{F}_p\circ k_f^{-1}\, d(\delta_1\circ k_f^{-1})\, d\lambda_2.$$

Since we have supposed that F_p is a function over the configuration space and since k_f is precisely fibered over this space, $\mathcal{F}_p = |q_1|F_p$ is invariant by k_f. Moreover, since δ_1 is the eccentric anomaly of the ellipse of the regularization, $\delta_1\circ k_f^{-1} = u_1$. Thus

$$\langle\mathcal{F}_p\rangle = \frac{1}{4\pi^2}\int_{k_f(\mathbf{T}^2)} F_p\,|Q_1|\, du_1\, d\lambda_2.$$

After Kepler's equation we have $|Q_1|\, du_1 = (a_1 \circ \pi_{F_k})\, d\lambda_1$ and

$$\langle \mathcal{F}_p \rangle = \frac{a_1 \circ \pi_{F_k}}{4\pi^2} \int_{k_f(\mathbf{T}^2)} F_p \, d\lambda_1 \, d\lambda_2 = a_1 \circ \pi_{F_k} \, \langle F_p \rangle \circ k_f.$$

∎

This proposition calls for a few remarks:

• The averaged system does not depend on the mean anomalies λ_j, so under its flow the conjugate variables Λ_j, or, equivalently, the semi major axes a_j, are first integrals. Hence the constant factor a_1 is not important. It could actually have been removed with a different normalization of the slow-down function; e.g. consider $f_1(\Lambda_2)|Q_1|(F+f)$ instead of $\mathcal{F}$.

• This result holds for the restricted problems (see my thesis [4] for more details on the link between the restricted and the full problems at the secular level).

• For the spatial problem, Moser's regularization may be used, instead of that of Levi-Civita. Explicit computations are more complicated. But if we are only to prove that the previous result holds for the spatial problem, it suffices to notice that before being perturbed the motion of each of the two bodies, separately, is planar, and to know that Levi-Civita's regularization is just a two-sheeted covering of Moser's.

We are now going to confront the unfortunate fact that the diffeomorphism k_f is not symplectic. After the reduction by the fast angles and by the symmetry of rotation, the space of pairs of ellipses is a sphere $\mathbf{S}^2$, and the parameters are the masses, the energy level f, the semi major axes and the angular momentum [4]. Therefore, the phase space is two-dimensionnal and the orbits are just the energy levels.

In the next theorem, the perturbation F_p is the Jacobi perturbing function of the planar three-body problem [4]. All we actually use is that its average satisfies

$$a_1\{F_p\} = \{\mathcal{F}_p\} = \mu_1 m_2 h(e_1, e_2, a_1, a_2, g, m_0, m_1)$$

for some $\mathbf{R}$-analytic function h, where m_0, m_1 and m_2 are the masses of the three initial bodies. In particular, $\{F_p\}/m_2$ does not depend on m_2. Also, notice that the four masses which intervene in the expression of F_k satisfy the following equalities:

$$\frac{1}{\mu_1} = \frac{1}{m_0} + \frac{1}{m_1}, \; \frac{1}{\mu_2} = \frac{1}{M_1} + \frac{1}{m_2}, \; M_1 = m_0 + m_1 \text{ and } M_2 = M_1 + m_2.$$

Is all the dynamics of $\langle \mathcal{F}_p \rangle$ contained, in some sense, in that of $\langle F_p \rangle$?

Theorem 3.1. *After reduction by the symmetry of rotation and by the Keplerian actions of* $\mathbf{T}^2$*, once the masses* m_0*,* m_1 *and* m_2*, the semi major axes* a_1 *and* a_2*, the energy* f *and the angular momentum* C *have been fixed, there exists a fictitious value* m_2' *of the outer mass such that the averaged regularized system* $\langle \mathcal{F}_p \rangle$ *is* $\mathbf{R}$*-analytically orbitally conjugate to the averaged initial system* $\langle F_p \rangle$ *in which* m_2' *substitutes for* m_2*.*

Proof. Consider the phase space after it has been symplectically reduced by the fast angles and once it has been quotiented by rotations. The mappings π_{F_k} and $\pi_{\mathcal{F}_k}$ induce two diffeomorphisms from this space into the space of pairs of ellipses which do not meet and with fixed energy and foci, mod rotations. We will call $\mathcal{E}$ the latter space. It is diffeomorphic to $\mathbf{S}^2 \times \mathbf{I} \times \mathbf{S}^0$. Local coordiates almost everywhere on this space –precisely, where the ellipses are neither circular nor degenerate [4], are given by $(e_1, e_2, g = g_1 - g_2)$. The semi major axes a_j, the masses m_0, m_1 and m_2 and, in the case of the regularized problem, f, are the parameters.

Proposition 3.1 asserts that the initial and regularized secular Hamiltonians define the same function on $\mathcal{E}$, up to the multiplicative constant a_1. In order to complete the symplectic reduction by the symmetry of rotation, we still need to restrict ourselves to a (regular) level of the angular momentum.

In each of the two problems, the conservation of the angular momentum $C = \mathcal{C}$ is equivalent, on $\mathcal{E}$, to that of

$$\frac{C}{\Lambda_2} \circ \pi_{F_k}^{-1} = \frac{\mu_1 \sqrt{M_1 a_1}}{\mu_2 \sqrt{M_2 a_2}} \epsilon_1 + \epsilon_2$$

and

$$\frac{\mathcal{C}}{\Lambda_2} \circ \pi_{\mathcal{F}_k}^{-1} = \sqrt{2\mu_1 \left(f - \frac{\mu_2 M_2}{2a_2} \right)} \frac{\epsilon_1}{\mu_2 \sqrt{M_2 a_2}} + \epsilon_2$$

repectively (where $\epsilon_j := \sqrt{1 - e_j^2})$). These two functions have the same level surfaces when the coefficient in front of ϵ_1 is the same. So it suffices to notice that the function

$$m_2 \mapsto \frac{\mu_1 \sqrt{M_1 a_1}}{\mu_2 \sqrt{M_2 a_2}}$$

is a diffeomorphism of $]0, +\infty[$ into itself and that the level curves of $a_1 \{F_p\} = \{\mathcal{F}_p\}$ do not depend on m_2. ∎

Therefore it is dynamically relevant to globally study the singularities of the secular Hamiltonian, including up to inner collisions.

4. INVARIANT PUNCTURED TORI

Theorem 3.1 is a key step towards proving the existence of quasiperiodic invariant "punctured" 4-tori in the planar three-body problem. Here, by "punctured" 4-torus we mean a torus minus a finite number of 2-tori [3]. Along trajectories of such punctured tori, the two inner bodies get arbitrarily close to one another an infinite number of times. Each time, these bodies miss a double collision because when the eccentricities of their ellipses reach the value one, they are not quite at the pericenters.

Theorem 4.1. *In the planar three-body problem, for any given set of masses, there exists a transversally Cantor set of positive Liouville measure which consists of diophantine quasiperiodic punctured tori, such that along its trajectories the two inner bodies get arbitrarily close to one another an infinite number of times.*

The proof consists of four steps :

1. regularize double inner collisions;
2. build the secular systems of the regularized problem;
3. apply some adapted KAM theorem to find a positive measure of invariant tori for the regularized problem;
4. check that most of the found quasiperiodic motions do not meet the double- inner-collision set.

The first step was described in § 2. Thanks the coordinates of Lemma 2.1 and to Theorem 3.1, the second and third steps are very similar to building the secular systems of the non-regularized planar three-body problem. Surprisingly enough, the fourth step does not concern the secular systems proper, but rather the conjugacy diffeomorphism between the regularized problem and its secular systems. In some specific region of the phase space, which in [5] I call the *asynchronous region*, this diffeomorphism can be computed by quadrature of trigonometric polynomials, which makes the transversality condition easy to check.

These motions generalize the solutions that Chenciner-Llibre [3] had found in the planar circular restricted problem. Indeed, the averaged systems of the restricted problems are the limits of the averaged system of the full problem when the adequate mass goes to zero [5]. So the restricted problems are particular limit cases of our study, while on the other hand the proof of Chenciner-Llibre cannot be easily extended to the full 3-body problem.

A forthcoming paper [5] gives the complete proof of theorem 4.1, together with the proof of the existence of some other kinds of periodic or quasiperiodic motions in the planar three-body problem.

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