DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **10**, Number **2&3**, September & October **2008**

Website: http://aimSciences.org

pp. **421–438**

THE FLOW OF THE EQUAL-MASS SPATIAL 3-BODY PROBLEM IN THE NEIGHBORHOOD OF THE EQUILATERAL RELATIVE EQUILIBRIUM

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To Carles Simó for his 32 000 000th minute

Abstract. From a normal form analysis near the Lagrange equilateral relative equilibrium, we deduce that, up to the action of similarities and time shifts, the only relative periodic solutions which bifurcate from this solution are the (planar) homographic family and the (spatial) P_{12} family with its twelfth-order symmetry (see [13, 5]). After reduction by the rotation symmetry of the Lagrange solution and restriction to a center manifold, our proof of the local existence and uniqueness of P_{12} follows that of Hill's orbits in the planar circular restricted three-body problem in [7, 1]. Indeed, near the Lagrange solution, the restrictions of constant energy levels of the reduced flow to a center manifold (actually unique) turn out to be three-spheres. In an annulus of section bounded by relative periodic solutions of each family, the normal resonance along the homographic family entails that the Poincaré return map is the identity on the corresponding connected component of the boundary. Using the reflexion symmetry with respect to the plane of the relative equilibrium, we prove that, close enough to the Lagrange solution, the return map is a monotone twist map.

Consider three point bodies in $\mathbb{R}^3$, with the same mass $\frac{1}{3}$ undergoing Newtonian attraction (it is only in the Appendix that we take general masses).

If $q = (q_0, q_1, q_2) \in (\mathbb{R}^3)^3$ and $p = (p_0, p_1, p_2) \in (\mathbb{R}^3)^3$ respectively denote the configuration, that is the positions of the 3 bodies, and the configuration of the impulsions, that is $p_j = \frac{1}{3}\dot{q}_j,\ j = 0, 1, 2$, the equations of the problem are

$$\ddot{q}_j = \sum_{k \neq j} \frac{1}{3} \frac{q_k - q_j}{||q_k - q_j||^3}, \quad j = 0, 1, 2,$$

or equivalently Hamilton's equations

$$\dot{p}_j = -\frac{\partial H_0}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H_0}{\partial p_j}, \quad j = 0, 1, 2,$$

2000 *Mathematics Subject Classification.* Primary: 70F07, 34C25, 34C14, 34C20.

Key words and phrases. Three-body problem, Lagrange relative equilibrium, relative periodic orbits, Lyapunov family, resonant normal form.

where the Hamiltonian H_0 is defined by

$$H_0(p,q) = \frac{3}{2}\sum_{0\le j\le 2}||p_j||^2 - \frac{1}{9}\sum_{0\le j<k\le 2}\frac{1}{||q_j-q_k||}.$$

We are interested in the so-called *Lagrange relative equilibrium solution* which, after identification of $\mathbb{R}^3$ with $\mathbb{C}\times\mathbb{R}$, is defined by $q^L(t) = \big(q_0^L(t), q_1^L(t), q_2^L(t)\big)$, with

$$q_j^L(t) = \left(\frac{1}{\sqrt{3}}\exp\left[i\left(t-\frac{5\pi}{6}+j\frac{2\pi}{3}\right)\right], 0\right) \tag{L}$$

(this choice of the origin of time is meant to simplify formulae from Jacobi coordinates to polar coordinates below). In this solution, the three bodies rotate uniformly with period 2π in the horizontal plane as a rigid equilateral triangle whose edges have length 1.

1. **Reduction by translations and rotations.** Define the Jacobi coordinates $(Q,P) = \big((Q_0,Q_1,Q_2),(P_0,P_1,P_2)\big)$ by

$$\begin{aligned} Q_0 &= q_0,\ \ Q_1 = q_1 - q_0,\ Q_2 = q_2 - \frac{q_0+q_1}{2}\\ P_0 &= p_0+p_1+p_2,\ P_1 = p_1 + \frac{p_2}{2},\ P_2 = p_2. \end{aligned}$$

The transformation from (q,p) to (Q,P) is symplectic. The total linear momentum P_0 remains constant. We fix it to be 0 and we shall think of (Q_1,Q_2,P_1,P_2) as coordinates in the subspace

$$\sum q_j = 0, \quad \sum p_j = 0$$

of the phase space where the center of mass is fixed at the origin. The Hamiltonian of the (*full*, as opposed to the reduced one, defined below) 3-body problem becomes

$$H(P_1,P_2,Q_1,Q_2) = 3||P_1||^2 + \frac{9}{4}||P_2||^2 - \frac{1}{9||Q_1||} - \frac{1}{9||Q_2+\frac{1}{2}Q_1||} - \frac{1}{9||Q_2-\frac{1}{2}Q_1||},$$

and the Lagrange solution becomes

$$\begin{array}{llll} Q_1^L(t) &= \big(e^{it},0\big) & P_1^L(t) &= \big(\frac{i}{6}e^{it},0\big)\\ Q_2^L(t) &= \left(\frac{\sqrt{3}}{2}ie^{it},0\right) & P_2^L(t) &= \left(-\frac{1}{3\sqrt{3}}e^{it},0\right). \end{array} \tag{L}$$

Notations.

$$\begin{aligned} q_j &= (x_j,y_j,z_j), \quad p_j = (x_j',y_j',z_j'), \quad j=0,1,2\\ Q_1 &= (X_1,Y_1,Z_1) = \big((1+R_1)e^{i\Theta_1}, Z_1\big)\\ Q_2 &= (X_2,Y_2,Z_2) = \left(\left(\sqrt{3}/2+R_2\right)e^{i(\Theta_2+\frac{\pi}{2})}, Z_2\right)\\ P_1 &= (X_1',Y_1',Z_1') = \left(\left(R_1' + i\frac{1/6+\Theta_1'}{1+R_1}\right)e^{i\Theta_1}, Z_1'\right)\\ P_2 &= (X_2',Y_2',Z_2') = \left(\left(R_2' + i\frac{1/6+\Theta_2'}{\sqrt{3}/2+R_2}\right)e^{i(\Theta_2+\frac{\pi}{2})}, Z_2'\right). \end{aligned}$$

Note that the origin of the polar coordinates is chosen at the positions and momenta of the Lagrange solution at time 0.

Consider the symplectic invariant submanifold of vertical angular momentum (there are singularities at the colinear configurations with colinear velocities but we stay far from them)

$$\vec{C} = (C_x, C_y, C_z) = Q_1 \wedge P_1 + Q_2 \wedge P_2 = (C_x^{(1)}, C_y^{(1)}, C_z^{(1)}) + (C_x^{(2)}, C_y^{(2)}, C_z^{(2)}),$$

that is of equations $C_x = C_y = 0$. Provided that $C_z^{(2)}$ does not vanish (and it certainly does not in the neighborhood of the Lagrange relative equilibrium), the latter equations can be solved for (Z_2, Z_2'):

$$Z_2 = -\frac{X_2 C_x^{(2)} + Y_2 C_y^{(2)}}{C_z^{(2)}} = \frac{X_2 C_x^{(1)} + Y_2 C_y^{(1)}}{C_z^{(2)}},$$

$$Z_2' = -\frac{X_2' C_x^{(2)} + Y_2' C_y^{(2)}}{C_z^{(2)}} = \frac{X_2' C_x^{(1)} + Y_2' C_y^{(1)}}{C_z^{(2)}}.$$

Hence we can express the restriction of the Hamiltonian vector field in the (non symplectic) variables

$$R_1, \Theta_1, Z_1, R_2, \Theta_2, R_1', \Theta_1', Z_1', R_2', \Theta_2'.$$

From now on, we focus on the vector field rather than on the Hamiltonian; the symplectic form, which is not standard any longer, does not need to be computed. Invariance under horizontal rotations implies that the vector field depends on Θ_1 and Θ_2 only through $\Theta_1 - \Theta_2$. Up to a change of notations, taking $\Theta_1 - \Theta_2$ as a coordinates amounts to setting $\Theta_2 = 0$. Additionally fix the angular momentum at the value $\frac{1}{3}$ it has for Lagrange, i.e. fix $\Theta_1' + \Theta_2' = 0$. Hence the restriction of the induced vector field can be expressed in the variables $\Theta_1, \Theta_1', R_1, R_1', R_2, R_2', Z_1, Z_1'$. It will be referred to as the *reduced vector field*, in which the Lagrange solution corresponds to a singularity at the origin.

2. **Linear analysis.**

We review some known facts about the linearized equations along a relative equilibrium solution (see [14] in the planar case and [15], unfortunately unpublished, in the spatial case, as well as [12]). They hold for any number of bodies and any masses but we consider only the equilateral relative equilibrium of three equal masses.

2.1. **The splitting of the variational equation.**

It follows from the Pythagoras theorem that, when perturbed in an orthogonal direction, the length of a straightline segment stays constant at the first order of approximation. It follows that the variational equation of the n-body problem along any planar solution splits into two parts which correspond respectively to variations in, or orthogonal to, the plane of motion. Starting with a solution in the horizontal plane, we shall speak of the *horizontal variational equation* (HVE) and the *vertical variational equation* (VVE). In our case, since the triangle edges have length 1, if the variation of $q_i(t)$ is $\delta q_i(t) = \delta q_i^H(t) + \delta q_i^V(t)$, with $\sum_{i=1}^{2} \delta q_i \equiv 0$, (HVE) and (VVE) read respectively

$$\begin{aligned}\delta \ddot{q}_i^H &= \frac{1}{3}\sum_{j\neq i}(\delta q_j^H - \delta q_i^H) - \sum_{j\neq i}\left\langle q_j - q_i, \delta q_j^H - \delta q_i^H\right\rangle (q_j - q_i)\\ &= -\delta q_i^H - \sum_{j\neq i}\left\langle q_j - q_i, \delta q_j^H - \delta q_i^H\right\rangle (q_j - q_i), \quad i = 0,1,2, \qquad \text{(HVE)}\end{aligned}$$

$$\delta \ddot{q}_i^V = \frac{1}{3}\sum_{j\neq i}(\delta q_j^V - \delta q_i^V) = -\delta q_i^V, \quad i = 0,1,2. \qquad \text{(VVE)}$$

2.2. **Vertical variations and the Γ_1-symmetry.** (VVE) is a triple of resonant harmonic oscillators whose solutions have the same period as the relative equilibrium. After fixing the center of mass and switching to the Jacobi coordinates, it becomes $\ddot{\delta}Q_i^V = -\delta Q_i^V$, $i = 1, 2$. Finally, after reducing by the rotations, setting $Q_i^V = (0, 0, Z_i)$ and switching to the variables $(\Theta_1, \Theta_1', R_1, R_1', R_2, R_2', Z_1, Z_1')$, it becomes

$$\ddot{Z}_1 = -Z_1,$$

or,

$$\dot{Z}_1 = 6Z_1', \quad \dot{Z}_1' = -\frac{1}{6}Z_1.$$

We will now describe the solutions of (VVE), but *before the reduction of rotations and in the initial coordinates*. Recall that a vertical variation does not change the shape of a configuration at first order. Hence, the space

$$\{\big((\delta q_1^V, \delta q_2^V, \delta q_3^V), (\delta \dot{q}_0^V, \delta \dot{q}_1^V, \delta \dot{q}_2^V)\big) \in \mathbb{R}^3 \times \mathbb{R}^3, \sum_{i=0}^{2} \delta q_i^V = 0, \sum_{i=0}^{2} \delta \dot{q}_i^V = 0\}$$

of vertical variations at any point $(q^o, p^o) = (q^L(t_0), p^L(t_0))$ of the Lagrange relative equilibrium is the space tangent at (q^o, p^o) to the submanifold

$$\mathcal{V} = \{(q, p),\ q = \rho R q^o,\ p = \sigma S p^o\},$$

where $\rho, \sigma \in \mathbb{R}$ are arbitrary and $R, S \in SO(3)$ are rotations whose axes are horizontal.

If we set

$$e^R(t) = \begin{pmatrix} \operatorname{Re}\zeta e^{i(t+\frac{\pi}{2})} \\ \operatorname{Re}\zeta^2 e^{i(t+\frac{\pi}{2})} \\ \operatorname{Re} e^{i(t+\frac{\pi}{2})} \end{pmatrix}, \qquad e^P(t) = \begin{pmatrix} \operatorname{Re}\zeta^2 e^{i(t+\frac{\pi}{2})} \\ \operatorname{Re}\zeta e^{i(t+\frac{\pi}{2})} \\ \operatorname{Re} e^{i(t+\frac{\pi}{2})} \end{pmatrix}, \qquad \zeta = e^{\frac{2\pi i}{3}},$$

the solutions of (VVE) are of the form $\nu e^R(t+\psi) + \mu e^P(t+\varphi)$, with $\nu, \mu, \psi, \varphi \in \mathbb{R}$. This weird choice of basis is due to the former choice of a Lagrange solution (equation (L) in the introduction).

The solutions of the form $\nu e^R(t+\psi)$ of the (VVE) correspond to solutions obtained from the Lagrange solution by a rotation around the y-axis. The solutions of the form $\mu e^P(t+\varphi)$ are described in the Appendix. We shall now mention the important symmetry properties of the first order solutions of the equations of motion defined by

$$q_\mu^P(t) = q^L(t) + \mu e^P(t),$$

where each element of $e^P(t)$ is identified with a triple of vertical vectors in $\mathbb{R}^3$ (the choice of the phase $\varphi = 0$ is the unique one for which, at time $t = 0$, body 2 lies on the positive y-semi-axis).

Observed in a frame which rotates uniformly at the same angular speed as the Lagrange solution but in the opposite direction, $q_\mu^P(t)$ defines a loop

$$q_\mu^O(t) = \big(q_0^O(t), q_1^O(t), q_2^O(t)\big) = e^{it} \cdot q_\mu^P(t) = e^{it} \cdot q^L(t) + \mu e^P(t)$$

in the configuration space of the 3-body problem (e^{it} acts only on the horizontal component; O is an allusion to *oyster*, see the third part of Figure 1; *the notation q^O should not be confused with q^o, which stands for the configuration $q^L(t_0)$*). This loop is invariant under the action of the twelfth-order group

$$\Gamma_1 = \{s, \sigma \mid s^6 = 1, \sigma^2 = 1, s\sigma = \sigma s^{-1}\}$$

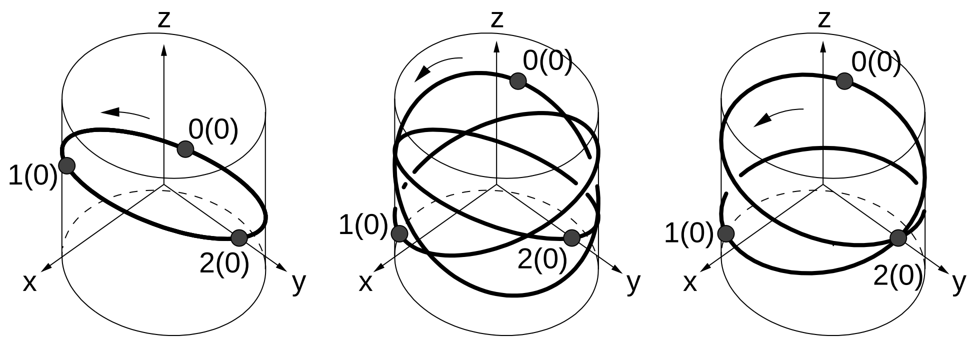

FIGURE 1. From left to right, $q^R(t)$ and $q^P(t)$ in the fixed frame, and $q^P(t)$ in the rotating frame. The positions of the bodies are indicated at time $t = 0$.

with generators acting in the following way (cf. [12, 13], and [5] where the choice of the axes and hence of the action is different: the emphasis there is laid on the Eight which hence lies in the horizontal plane, whereas the Lagrange solution accordingly stands vertically):

$$\begin{aligned}(s \cdot q)(t) &= (\Sigma q_1(t - 2\pi/6), \Sigma q_2(t - 2\pi/6), \Sigma q_0(t - 2\pi/6)) \\ (\sigma \cdot q)(t) &= (\Delta q_1(-t), \Delta q_0(-t), \Delta q_2(-t)),\end{aligned} \tag{A}$$

where Σ denotes the symmetry with respect to the xy-plane, and Δ denotes the symmetry with respect to the y-axis.

We postpone until section 5 a more detailed description of the Γ_1-invariance of q^P and of its projection on the space reduced by rotations.

2.3. **The splitting of (HVE).** Any relative equilibrium solution of the n-body problem in $\mathbb{R}^2$ lies in a four-dimensional symplectic subspace $\mathcal{H}$ of the phase space, invariant under the Newton flow, which is the collection of all homographic solutions with the same configuration up to similarity. If $q^o = q^L(t_0) \in (\mathbb{R}^2)^3$ is the configuration of a relative equilibrium solution at some instant t_0, $\mathcal{H}$ is the set

$$\mathcal{H} = \{(q,p),\ q = \rho R q^o,\ p = \sigma S p^o\},$$

where $\rho, \sigma \in \mathbb{R}$, $R, S \in SO(2)$. Note that, as p^o is the image of q^o under the $\frac{\pi}{2}$-rotation around the vertical, p^o could be replaced by q^o in the above formula.

After the reduction of the rotational symmetry, it becomes the set of homographic (i.e. Keplerian) motions with a given angular momentum, up to rotations. The spectral analysis below entails that, with its symplectic orthogonal, it splits the space of horizontal variations into two invariant subspaces.

2.4. **The spectrum.** The spectrum of the linearization at the Lagrange relative equilibrium of the reduced Newton flow splits into three parts, corresponding to three invariant subspaces, the vertical subspace, the homographic subspace and the symplectic orthogonal of these two, respectively of dimension 2, 2 and 4. Together, the last two subspaces generate the horizontal variations:

- to each of the first two subspaces corresponds a pair of eigenvalues $\pm i$;
- to the last one corresponds a quadruplet of eigenvalues $\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}$ with $\lambda = \frac{1}{\sqrt{2}} + i$ and with horizontal eigenvectors.

The first (inocuous) surprise (which, in fact, can be immediately deduced from the formulas in [15], valid for any set of three masses) is that there are more resonance relations than we expected, since $\lambda = \bar{\lambda} + 2i$.

We shall identify the three invariant subspaces respectively with $\mathbb{C}, \mathbb{C}$ and $\mathbb{C}^2$, and call $\tilde{u}, \tilde{v}, \tilde{h}, \tilde{k}$ complex coordinates:

- $\tilde{u}$ in the invariant subspace of the (horizontal) homographic family.
- $\tilde{v} = m(Z_1 - i\dot{Z}_1) = m(Z_1 - 6iZ_1')$ in the vertical subspace, where $m \in \mathbb{C} \setminus 0$. The coefficient m can be chosen as desired so as to simplify the normal form in the next section. (In our case, $m \approx -0.008 + i\,0.022$.)
- $\tilde{h}$ and $\tilde{k}$ in the hyperbolic part.

The linearized vector field becomes

$$\dot{\tilde{u}} = i\tilde{u},\ \dot{\tilde{v}} = i\tilde{v},\ \dot{\tilde{h}} = \lambda\tilde{h},\ \dot{\tilde{k}} = -\lambda\tilde{k}.$$

In the coordinates $(\Theta_1, \Theta_1', R_1, R_1', R_2, R_2', Z_1, Z_1')$ of the reduced vector field, the orthogonal symmetry with respect to the horizontal plane (which preserves angular momentum when it is vertical) corresponds to changing Z_1, Z_1' into $-Z_1, -Z_1'$. Hence, in the complex coordinates $(\tilde{u}, \tilde{v}, \tilde{h}, \tilde{k})$, the invariance under this symmetry translates into the invariance of the vector field under the transformation

$$\mathcal{T}(\tilde{u}, \tilde{v}, \tilde{h}, \tilde{k}) = (\tilde{u}, -\tilde{v}, \tilde{h}, \tilde{k}),$$

the invariant horizontal subspace being defined by the equation $\tilde{v} = 0$. This remark will play an important role in the sequel.

2.5. **The energy on a center manifold.** We will need to consider a local center manifold of the relative equilibrium (see section 4.1). From our previous description of $\mathcal{H}$ and $\mathcal{V}$, it follows that, at the non reduced level, a center manifold lifts to a submanifold tangent to the submanifold of the phase space generated by rotations and homotheties acting *independently* on the configuration $q^o = q^L(t_0)$ and on the configuration of the impulsions $p^o = p^L(t_0)$. Hence, when pulled back to the non reduced phase space, such a lifted center manifold is tangent to the submanifold

$$\mathcal{C} = \left\{(q,p) \in (\mathbb{R}^3)^3 \times (\mathbb{R}^3)^3,\ q = \rho R q^o,\ p = \sigma S p^o\right\}.$$

where $\rho, \sigma \in \mathbb{R}_+$ and $R, S \in SO(3)$. As in the definition of $\mathcal{H}$ in section 2.3, p^o could be replaced by q^o in this formula (compare to [15]).

The following lemma will be crucial in the last part of the paper:

Lemma 2.1. *As an equilibrium of the restriction of the reduced Hamiltonian to a center manifold, the relative equilibrium solution is a non degenerate minimum. In particular, the restriction to a local center manifold of a constant energy hypersurface close enough to the relative equilibrium is diffeomorphic to the three-sphere.*

Proof. It is enough to show the positivity of the Hessian of the restriction of the energy function H_0 to the submanifold $\mathcal{C}$.

Before reduction, $(\rho, \sigma, R, S) \in (\mathbb{R}_+)^2 \times SO(3) \times SO(3)$ are (generalized) coordinates on $\mathcal{C}$ and the restriction of H_0 is

$$H_0^{\mathcal{C}}(\rho, \sigma, R, S) = \frac{3\sigma^2}{2} \sum_{0 \le j \le 2} ||p_j^o||^2 - \frac{1}{9\rho} \sum_{0 \le j < k \le 2} \frac{1}{||q_j^o - q_k^o||} = \frac{\sigma^2}{6} - \frac{1}{3\rho}.$$

We compute the reduced system by first quotienting by the full group $SO(3)$ and then fixing the length of the angular momentum. This amounts to replacing

(ρ, σ, R, S) by $(\rho, \sigma, R^{-1}S)$ and imposing the relation

$$\rho\sigma || \sum_{i=0}^{2} q_i^o \wedge R^{-1} S p_i^o || = || \sum_{i=0}^{2} q_i^o \wedge p_i^o || := ||\vec{C}^L||.$$

Any element of a neighborhood of the Identity in $SO(3)$ can be uniquely written as $\exp A$, where A is an antisymmetric 3×3 matrix. In particular,

$$\begin{aligned} R^{-1}S &= \exp \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \\ &= Id + \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -(b^2+c^2) & ab & ac \\ ab & -(c^2+a^2) & bc \\ ac & bc & -(a^2+b^2) \end{pmatrix} + \cdots, \end{aligned}$$

where the dots represent terms of order higher than or equal to 3 in a, b, c. Plugging this into the above relation linking ρ, σ and $R^{-1}S$ *and choosing* $t_0 = 0$ *in the definition of* q^o *and* p^o, by a direct computation we get that

$$\sum_{i=0}^{2} q_i^o \wedge R^{-1} S p_i^o = \vec{C}^L + \begin{pmatrix} gb \\ fa \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -gac \\ -fbc \\ -f(c^2+a^2) - g(b^2+c^2) \end{pmatrix} + \cdots$$

where

$$f = \sum_{i=0}^{2} x_i \eta_i = \frac{1}{6},\ g = -\sum_{i=0}^{2} y_i \xi_i = \frac{1}{6},\ q_i^o = (x_i, y_i, 0),\ p_i^o = (\xi_i, \eta_i, 0).$$

(this very simple formula comes from the fact that $\sum_{i=0}^{2} x_i \xi_i = \sum_{i=0}^{2} y_i \eta_i = 0$; notice also that $||\vec{C}^L|| = f + g = \frac{1}{3}$). Hence

$$\frac{1}{\rho} = \left[1 - \frac{1}{2(f+g)^2}\left(fga^2 + fgb^2 + (f+g)^2 c^2 + \cdots\right)\right]\sigma.$$

Finally, we can take $(a, b, c, d = \sigma - 1)$ as local coordinates in $\mathcal{C}$ in the neighborhood of the Lagrange equilibrium. In these coordinates, we get

$$H_0^{\mathcal{C}} = -\frac{1}{6} + \frac{1}{24}(a^2 + b^2) + \frac{1}{6}(c^2 + d^2) + \cdots$$

As all the coefficients are positive, this proves the lemma.

The following consequence was pointed out to us by Rick Moeckel.

Corollary 1. *The local center manifold is unique (and hence $\mathcal{T}$-symmetric).*

Proof. As it stays close to the Lagrange solution, the intersection of any local center manifold with an energy level is a normally hyperbolic compact invariant submanifold of the restriction of the reduced flow to this level. Hence, in a neighborhood, the points whose forward and backward orbits stay close to it belong to the intersection of its stable and unstable manifold, which proves the uniqueness.

3. **A third order normal form.** Using Trip [10] or Maple, we compute a normal form at order 3 of the (reduced) vector field, keeping only the resonant terms. This leads to new complex variables u, v, h, k which are tangent at the first order to $\tilde{u}, \tilde{v}, \tilde{h}, \tilde{k}$. The normal form is not unique at a general order, but the successive changes of variables which eliminate the non resonant terms can be chosen so that, in the resulting (complex) coordinates (u, v, h, k), the vector field is still

invariant under $\mathcal{T}(u,v,h,k)=(u,-v,h,k)$, the invariant horizontal subspace still being defined by the equation $v=0$.

The result is of the following form:

$$\begin{aligned}
\dot{u} &= iu[1+\alpha|u|^2+\beta|v|^2+\gamma hk+\bar{\gamma}\bar{h}\bar{k}]+O_5\\
\dot{v} &= iv[1+a|u|^2+b|v|^2+chk+\bar{c}\bar{h}\bar{k}]+A\bar{v}h\bar{k}+O_5\\
\dot{h} &= \lambda h[1+r|u|^2+s|v|^2+thk+t'\bar{h}\bar{k}]+Rv^2\bar{h}+O_5\\
\dot{k} &= -\lambda k[1+r|u|^2+s|v|^2+thk+t'\bar{h}\bar{k}]-R\bar{v}^2\bar{k}+O_5,
\end{aligned}$$

where the coefficients have the following non-zero values:

$$\begin{gathered}
\alpha=-1,\, \beta=-1, \gamma=\tfrac{9}{2}+6i\sqrt{2},\\
a=-1,\, b=-\tfrac{21}{19},\, c=\tfrac{186}{19}+\tfrac{126\sqrt{2}}{19}i,\, A=-\tfrac{120}{19},\\
r=-\tfrac{11}{12}-\tfrac{\sqrt{2}}{12}i,\, s=-\tfrac{73}{57}+\tfrac{10\sqrt{2}}{57}i,\, t=\tfrac{275}{57}+\tfrac{334\sqrt{2}}{57}i,\\
t'=\tfrac{105}{19}(1-i\sqrt{2}),\, R=\tfrac{5\sqrt{2}}{19}i,
\end{gathered}$$

and where O_5 stands for real analytic functions of order 5 in $u,\bar{u},v,\bar{v},h,\bar{h},k,\bar{k}$.

It calls for a few comments:

- For instance the terms $A\bar{v}h\bar{k}, Rv^2\bar{h}$ and $-R\bar{v}^2\bar{k}$ correspond respectively to the resonances $i=-i+\lambda-\bar{\lambda}, \lambda=2i+\bar{\lambda}$ and $-\lambda=-2i-\bar{\lambda}$.
- The symmetry under $\mathcal{T}$ accounts for the absence of some resonant monomials, e.g. $|u|^2v$ in $\dot{u}$, or $u|v|^2$ and $\bar{u}v^2$ in $\dot{v}$.
- It remains unclear to us why the normal form is also invariant with respect to $u\mapsto -u$ (for instance $\dot{u}$ has no term in $u^2\bar{v}$ or $\bar{u}h\bar{k}$); this symmetry holds at order five.
- At this order, $\mathrm{Re}\,(hk)$ is an approximate first integral.
- Each coefficient is homogeneous for some degree, with respect to the scaling of the eigenvectors of the linearized vector field. For instance, the equality $\alpha=\beta$ is meaningless and disappears if one choses different scalings. On the contrary, the equality $a=\alpha$ has an intrinsic meaning and it will follow from the Appendix that this resonance actually persists in normal forms of every order.

4. **Local existence and uniqueness of a vertical Lyapunov family.** We are going to prove that, in spite of the (1-1)-resonance with the homographic family, a unique (mod similarities) family of relative periodic solutions bifurcates in the vertical direction (see the more precise statement below) from the equilateral Lagrange solution of the equal mass spatial three-body problem. A local expansion of this family is described in Marchal's book [12] under the hypothesis that it possesses the 12th order symmetry of the shape sphere; also, a global continuation is described under the name of P_{12}, in [13], where it is shown how it connects the Lagrange solution to the Eight through periodic solutions which minimize the action in the rotating frame under the constraint of the 12th order symmetry (see also [3]). What is lacking to make the proof rigorous is the uniqueness of such action minimizers. The rigorous proof of the existence of this family near the Eight is given in [5] where it is named the Γ_1-family. This section and the following are devoted to proving the existence and uniqueness of the P_{12} family near the Lagrange relative equilibrium.

We will call a *Lyapunov surface or family* any invariant surface containing the Lagrange equilibrium and foliated by periodic orbits of the reduced three-body problem. We will call it *spatial* if is contained in some conical region $|u| \leq \epsilon|v|$, $\epsilon > 0$, and *vertical* if it is tangent to the vertical subspace $u = h = k = 0$.

Proposition 1. *In the neighborhood of the Lagrange equilibrium, the reduced vector field possesses a unique spatial Lyapunov surface $\mathcal{P}$. This surface is vertical and locally of class C^n for any $n \geq 0$.*

This is certainly not an optimal result with respect to regularity: at the expense of heavy computations, following [7] one should be able to prove analyticity of $\mathcal{P}$ by a direct inspection of the series; the only difference with [7] is that due to the four extra hyperbolic directions, which moreover are resonant with the central part, one must carry out the computations in the full 8-dimensional space. Rather, we shall work in the 4-dimensional center manifold.

Instead of fixing the period as in [13] or [5], in section 1 we have fixed the angular momentum; due to the homogeneity of the Newtonian potential, it is easy to switch from one constraint to the other as long as the angular momentum does not vanish. In our case, the P_{12} family will appear as a Lyapunov family associated to the vertical eigenspace, of equation $u = h = k = 0$.

4.1. **Restriction to the center manifold.** Insofar as we look for the existence and uniqueness of an invariant C^2-surface foliated by periodic solutions, we can restrict our attention to the ($\mathcal{T}$-symmetric because unique) local 4-dimensional center manifold of class C^2, tangent to the subspace of coordinates $\tilde{u}, \tilde{v}$. This center manifold contains all the local recurrence and in particular all the local Lyapunov families.

In the coordinates u, v of such a center manifold, the restriction of the vector field is still invariant under the mapping $\tau : (u, v) \mapsto (u, -v)$ and of the form

$$\begin{aligned} \dot{u} &= iu[1 + \alpha|u|^2 + \beta|v|^2] + O_5 \\ \dot{v} &= iv[1 + a|u|^2 + b|v|^2] + O_5, \end{aligned}$$

with $v = 0$ defining the Lyapunov family of equilateral homographic motions. Moreover, one checks with Maple that the energy becomes (compare to 2.5)

$$H = -\frac{1}{6} + \frac{|u|^2}{9} + \frac{|v|^2}{9} + O_4.$$

The problem is now similar to the planar circular restricted problem in the Lunar case (see [7], [1], [11] or [9] in a more general situation), where the Lyapunov orbits are Hill's direct and retrograde orbits. The proof of existence, local uniqueness and analyticity of the P_{12} family would have gone along the same lines as in [7], had we known that the center manifold were analytic. Instead, we reproduce here the simple proof of [1] which gives the existence and uniqueness but not the analyticity (to get the analyticity we must avoid to restrict to the center manifold, consider formal normal forms at infinite order and use majorant series as in [7]).

4.2. **Blow up.** We take up the above equations which describe the restriction of the vector field to the center manifold. By direct identification one checks that the solutions at time T with initial conditions $u(0) = u$, $v(0) = v$ are of the form

$$
\begin{aligned}
u(T) &= ue^{i\left[1+\alpha|u|^2+\beta|v|^2\right]T} + \tilde{U}(u,v,T)\\
&= ue^{iT}\left(1+i\left[\alpha|u|^2+\beta|v|^2\right]T\right) + U(u,v,T)\\
v(T) &= ve^{i\left[1+a|u|^2+b|v|^2\right]T} + \tilde{V}(u,v,T)\\
&= ve^{iT}\left(1+i\left[a|u|^2+b|v|^2\right]T\right) + V(u,v,T),
\end{aligned}
$$

where $\tilde{U}(u,v,T), U(u,v,T), \tilde{V}(u,v,T), V(u,v,T)$ are of order 5 in $|u|, |v|$ uniformly on a compact interval of time (for what follows, T can be supposed to be close to 2π). Because of the existence of the energy first integral, the equations $u(T) = u$, $v(T) = v$ are a consequence of the two equations

$$\operatorname{Arg} v(T) - \operatorname{Arg} v = 2\pi, \quad u(T) = u.$$

The first equation reads

$$T + \operatorname{Arg}\left[1 + i(a|u|^2 + b|v|^2)T + \frac{V(u,v,T)}{v}\right] = 2\pi,$$

that is

$$T = 2\pi(1 - a|u|^2 - b|v|^2) + O_3.$$

It follows that the second equation becomes

$$2\pi i u\left[(\alpha - a)|u|^2 + (\beta - b)|v|^2\right] = O_4.$$

In order to recover uniqueness of the local solution, localize in a conical region containing the vertical plane $u = 0$, of the form $|u| \le \epsilon|v|$ (some $\epsilon > 0$ being fixed). Such a localization is naturally obtained by means of a complex blow up

$$u = w_1 w_2,\ v = w_2, \quad |w_1| \le \epsilon.$$

The two equations

$$\operatorname{Arg} v(T) - \operatorname{Arg} v = 2\pi, \quad u(T) = u$$

become

$$
\begin{aligned}
-2\pi + T + \operatorname{Arg}\left[1 + i|w_2|^2(a|w_1|^2 + b)T + \frac{V(u,v,T)}{w_2}\right] &= 0\\
\left[e^{iT}\left(1 + i|w_2|^2(\alpha|1|^2 + \beta)T\right) - 1\right] w_1 + \frac{U(u,v,T)}{w_2} &= 0.
\end{aligned}
$$

The functions U and V are C^∞ complex functions which are of order 5 in $|u|$ and $|v|$. It follows that their real and imaginary parts are functions of class C^3 whose three first derivatives vanish along $(w_2 = 0)$. We shall denote such functions by o_3. It follows that the first equation determines T as a C^3-function of $w_1, \bar{w}_1, w_2, \bar{w}_2$ of the form

$$T = 2\pi - 2\pi(b + a|w_1|^2)|w_2|^2 + o_3.$$

The second equation now takes the form

$$\left[2\pi i\left((\beta - b) + (\alpha - a)|w_1|^2\right)|w_2|^2 + o_3\right] w_1 + o_3 = 0.$$

Dividing by $|w_2|^2$ a function o_3, one gets a C^1-function whose first derivative vanishes along $(w_2 = 0)$. As $b \neq \beta$, it follows from the implicit function theorem that this equation is equivalent to

$$w_1 = o_1(w_2, \bar{w}_2),$$

which defines a C^1-submanifold of dimension 2 in the neighborhood of the origin, which is tangent to the w_2-plane. After blowing down, it uniquely defines a surface $\mathcal{P}$ tangent at order 2 to the v-plane and foliated by orbits of period close to 2π.
C^n-regularity. Using normal forms at order three, we have only been able to prove that the surface which supports the family of periodic orbits is of class C^2. But, by considering normal forms at higher orders, the same method allows to prove that, for any integer n, it is of class C^n in some neighborhood of the singularity. Indeed, it is enough to note that the resonant monomials in $\dot{u}$ or $\dot{v}$ are of the form $u^i\bar{u}^j v^k\bar{v}^l$, with $i-j+k-l=1$. After the blow up, such a term becomes $w_1^{(i)}\bar{w}_1^{(j)}w_2^{(i+k)}\bar{w}_2^{(j+l)}$. Since $i+k=j+l+1\geq 1$, any such term stays regular after being divided by w_2. As mentionned, the family is analytic but we will not prove it here.

5. **The Γ_1-symmetric P_{12} family.** In section 2.2 an action of the group Γ_1 was defined on 2π-periodic loops of the configuration space. Yet the invariant surface $\mathcal{P}$ is foliated by periodic orbits whose period varies. So, we will consider the natural extension of the action of Γ_1 to the space of periodic loops of any period $T>0$, obtained by merely replacing every occurence of 2π by T in the formulae (A). (As already mentionned, the scaling symmetry of the n-body problem allows to canonically put in correspondance solutions with a fixed norm of the angular momentum, and solutions with a fixed period; but the reduction by rotations is more tractable by fixing the norm of the angular momentum than the period of periodic orbits.)

Now, a periodic solution of the reduced 3-body problem can be lifted, in the manifold of fixed angular momentum, to a solution of the full 3-body problem which is periodic in a rotating frame. Provided that the rotation is uniform, the angular speed of the frame is unique up to a multiple of 2π per period. If one choses that, for the Lagrange solution, the frame rotates exactly by one turn per period in the direction opposite to the motion, by continuity the rotation of the frame of each orbit of the invariant surface $\mathcal{P}$ becomes well defined. In this section we address the question of existence and uniqueness of the lift of solutions lying in $\mathcal{P}$, to Γ_1-symmetric loops.

Theorem 5.1. *Each leaf of $\mathcal{P}$ is the underlying (non parametrized) orbit of the projection (mod $SO(2)$) of exactly two solutions of the full three-body problem which are Γ_1-symmetric in the rotating frame. These two solutions differ only by a phase shift of half their period.*

The so defined lift of $\mathcal{P}$ corresponds, after normalization of the period, to the P_{12} family described by Marchal in [12] in the fixed frame and [13] in the rotating frame. Marchal found the first terms of the Fourier expansion of the family, which was a strong hint about its existence; but, as mentioned above, he had to a priori assume the 12th order symmetry as an ansatz.

Theorem 5.1 can be proved with an argument involving the action integral, as described in section 7. However, for the sake of simplicity, this section is devoted to proving theorem 5.1 using mainly the description of solutions of (VVE) given in section 2.2.

Proof. Define a curve $\mathcal{C}_2$ on $\mathcal{P}$ through the Lagrange equilibrium by the equation $z_2=0$. That this curve is regular comes from the following observation. Since $\mathcal{P}$ is tangent to the space $\tilde{u}=\tilde{h}=\tilde{k}=0$, one can choose $\tilde{v}=m(Z_1-i\dot{Z}_1)$ as a coordinate on $\mathcal{P}$. Because a permutation of the bodies sends the vertical Lyapunov family to another one, hence to itself by uniqueness, we can as well choose $Z_2-i\dot{Z}_2$. But

$Z_2 = z_2 - (z_0 + z_1)/2 = 3z_2/2$ and $Z_2' = z_2'$. Hence $(z_2, \dot{z}_2) = (z_2, 3z_2')$ are coordinates also, with opposite orientation. (We could of course have directly chosen (Z_2, Z_2') instead of (Z_1, Z_1') in section 1, but nobody is perfect [17].)

Being regular at the origin, the curve $\mathcal{C}_2$ is transverse to the leaves of $\mathcal{P} \setminus 0$ and meets each of them at two points. Let $\mathcal{C}_2^-$ be the component of $\mathcal{C}_2 \setminus 0$ along which $\dot{z}_2 < 0$. (Focusing on $\mathcal{C}_2$ comes from the choice of e^P in section 2.2.) Taking $\mu \in \mathcal{C}_2^-$ as initial condition defines a unique time parametrization $\hat{q}_\mu$ of each leaf of $\mathcal{P}$. Moreover, let q_μ be the unique lift of $\hat{q}_\mu$ to the solution —in the rotating frame which makes it 2π-periodic— of the 3-body problem, satisfying the property that, at time 0, body 2 lies on the positive y-semi-axis. According to section 2.2 we have

$$q_\mu(t) = q_\mu^O(t) + O(\mu^2) = e^{it} \cdot q^L(t) + \mu e^P(t) + O(\mu^2).$$

Since the action integral of the 3-body problem is invariant under the action of Γ_1, the image by an element $\gamma \in \Gamma_1$ of the periodic solution q_μ is itself a periodic solution satisfying

$$\gamma \cdot q_\mu = q_\mu + O(\mu^2),$$

because $\gamma \cdot q_\mu^O = q_\mu^O$ and the action of γ is differentiable. In particular the set of $\gamma \cdot q_\mu$'s projects mod $SO(2)$ on a vertical Lyapunov family which, by uniqueness, lies in $\mathcal{P}$. But the conservation of the energy (or of the period) shows that the projections of q_μ and $\gamma \cdot q_\mu$ have the same underlying (non parametrized) orbit. Hence q_μ and $\gamma \cdot q_\mu$ differ at most by a rotation R and a time shift τ:

$$\gamma \cdot q_\mu(t) = R \cdot q_\mu(t - \tau) = q_\mu(t) + O(\mu^2).$$

Order zero in μ shows that the angle of the rotation R is 2τ. That R acts trivially on e^P and that e^P has a trivial isotropy among phase shifts show, at the order one in μ, that $\tau = 0$ and hence $R = 1$. Hence $\gamma \cdot q_\mu = q_\mu$.

Taking the origin of time rather on $\mathcal{C}_2^+ = \mathcal{C}_2 \cap \{\dot{z}_2 > 0\}$ would have led to another solution of the three-body problem, obtained from q_μ by a phase shift of half a period. This completes the proof.

The remainder of the section is devoted to a few comments. The Γ_1-action defined above is the restriction to the subgroup Γ_1 of the natural action of the group $G = O(3) \times O(2) \times S_3 \ni (\rho, \tau, \sigma)$ on the space of periodic loops in the configuration space of the 3-body problem (cf. [5]):

$$(\rho, \tau, \sigma) \cdot (q_0, q_1, q_2)(t) = \big(\rho q_{\sigma^{-1}(0)}(\tau^{-1}t), \rho q_{\sigma^{-1}(1)}(\tau^{-1}t), \rho q_{\sigma^{-1}(2)}(\tau^{-1}t)\big).$$

Think of $SO(2) \subset O(3)$ as the group of rotations in $\mathbb{R}^3$ around the z-axis. That Γ_1 is contained in the normalizer $N_G(SO(2))$ of $SO(2)$ in G and that the action of Γ_1 on a loop whose angular momentum is vertical preserves this property, show that Γ_1 acts on the space of reduced loops.

On the other hand, the action integral of the 3-body problem is invariant under the action of Γ_1. Hence Γ_1 acts on the set of periodic solutions of the three-body problem. The proof of the theorem above shows that this action is trivial on the set of solutions lying on the vertical Lyapunov surface $\mathcal{P}$ and starting from $\mathcal{C}_2^-$.

One can define $\mathcal{C}_0$ and $\mathcal{C}_1$ as we did for $\mathcal{C}_2$, by exchanging the roles of the bodies. The tangents at the origin to the curves $\mathcal{C}_i$ can be read from the explicit description of $q^P(t)$ in section 2.2. In the coordinates $(z_2, \dot{z}_2)$ in $\mathcal{P}$, they are defined by the following equations:

$$T_0\mathcal{C}_0 : z_2 = -\sqrt{3}\dot{z}_2, \quad T_0\mathcal{C}_1 : z_2 = +\sqrt{3}\dot{z}_2, \quad T_0\mathcal{C}_2 : z_2 = 0.$$

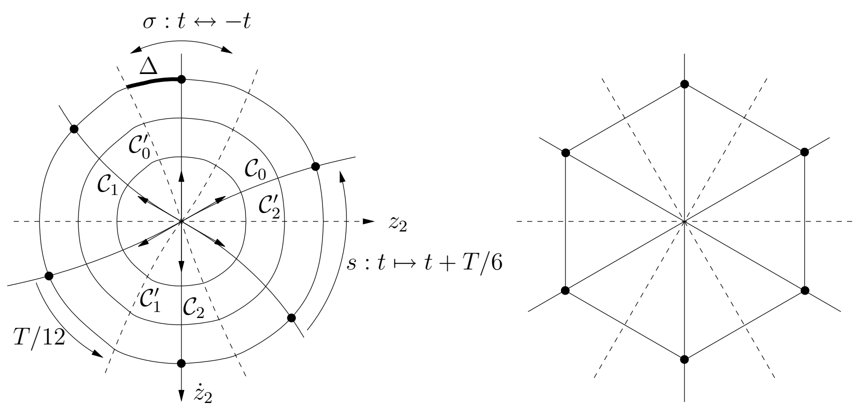

FIGURE 2. The hexagonal structure of the vertical Lyapunov surface $\mathcal{P}$

Similarly, one can introduce the curves $\mathcal{C}'_i$ defined by the equations $\dot{z}_i = 0$. Altogether these six curves constitute a realization on $\mathcal{P}$ of the Γ_1-action on the time circle. The intersections of a periodic solution γ in $\mathcal{P}$ with the three lines $\mathcal{C}_i$ are the vertices of a hexagonal structure on γ such that Γ_1 becomes the group of isometries of the hexagon, the intersections with the $\mathcal{C}'_i$ defining the axes of symmetry orthogonal to the sides of the hexagon (see figure 2, where one shows also a fundamental domain corresponding to the part of the trajectory between an isosceles configuration and a collinear one).

6. **The annulus map and its torsion.**

Theorem 6.1. *Up to the action of similarities and time shifts, exactly two families of relative periodic solutions bifurcate from the equilateral relative equilibrium solution of the equal-mass three-body problem: the periodic homographic family and the quasi-periodic P_{12} family.*

Due to a resonance, the method used to prove the existence and the local uniqueness of P_{12} breaks down for the homographic family; more precisely, the equality $a = \alpha$ prevents from proving as above the existence and uniqueness using the implicit function theorem after an adapted blow-up $u = w_1,\ v = w_1 w_2$. Computing the normal form to a higher order will not help: it follows from lemma 8.1 that for all integers n the coefficients of respectively $u|u|^{2n}$ in $\dot{u}$ and $v|u|^{2n}$ in $\dot{v}$ in higher order normal forms agree.

Of course the homographic family is known to exist (and to be defined by $v = 0$). But the proof of its uniqueness is somewhat more subtle: the same lemma 8.1 is used with the symmetry of the equations under the map τ to prove that the Poincaré return map in an annulus of section has no fixed point in the open annulus. In turn, this implies the absence of any other Lyapunov family.

Proof. First, let $u = \psi(v) = O_3$ be the equation of the C^j-submanifold $\mathcal{P}$. Replacing u by $u - \psi(v)$ and keeping the old name v, we may assume that $u = 0$ and $v = 0$ are invariant submanifolds of the flow, corresponding respectively to the P_{12} family and the homographic family. The equations then take the form:

$$\begin{aligned}\dot u &= iu[1+\alpha|u|^2+\beta|v|^2+O_4]+i\bar u O_4,\\ \dot v &= iv[1+a|u|^2+b|v|^2+O_4]+i\bar v O_4,\end{aligned}$$

where the coefficients α,β,a,b have not changed. Moreover, the equations are invariant under $\tau(u,v)=(u,-v)$ and they preserve the restriction to the center manifold of the energy function

$$H=-\frac{1}{6}+\frac{|u|^2}{9}+\frac{|v|^2}{9}+O_4.$$

Let us take polar coordinates

$$u=r_1e^{i\theta_1},v=r_2e^{i\theta_2}$$

(not to be mixed up with those of section 1). The equations become

$$\begin{aligned}\dot r_1 &= r_1O_4,\\ \dot r_2 &= r_2O_4,\\ \dot\theta_1 &= 1+\alpha r_1^2+\beta r_2^2+O_4,\\ \dot\theta_2 &= 1+ar_1^2+br_2^2+O_4,\end{aligned}$$

where the notation O_4 stands for functions of $r_1,r_2,\theta_1,\theta_2$ which are of order 4 in r_1,r_2.

In each energy surface $H=-\frac{1}{6}+\epsilon^2$ close enough to the origin (i.e. to the Lagrange solution), we define an annulus of section A_ϵ by the equations

$$H=-\frac{1}{6}+\frac{r_1^2}{9}+\frac{r_2^2}{9}+O_4=-\frac{1}{6}+\epsilon^2,\quad \theta_1+\theta_2=0 \bmod (2\pi). \tag{A_ϵ}$$

Starting from initial conditions $r_1,r_2,\theta_1,\theta_2$, the solution after a (bounded) time t is of the form

$$\begin{aligned}r_1(t) &= r_1(1+O_4),\\ r_2(t) &= r_2(1+O_4),\\ \theta_1(t) &= \theta_1+(1+\alpha r_1^2+\beta r_2^2)t+O_4,\\ \theta_2(t) &= \theta_2+(1+ar_1^2+br_2^2)t+O_4.\end{aligned}$$

We are interested in solutions of the equation of period close to 2π. As $(\dot\theta_1,\dot\theta_2)$ is close to $(1,1)$, such a solution will transversally intersect the annulus of section at exactly two points i.e., it corresponds to a fixed point of the *second*-return time T_ϵ in the annulus A_ϵ, defined by

$$4\pi=(\theta_1+\theta_2)(T_\epsilon)=\left[2+(a+\alpha)r_1^2+(b+\beta)r_2^2\right]T_\epsilon+O_4,$$

hence

$$T_\epsilon=2\pi\left[1-\frac{a+\alpha}{2}r_1^2-\frac{b+\beta}{2}r_2^2\right]+O_4.$$

Finally, the Poincaré second-return map in A_ϵ is of the form

$$\begin{aligned}r_1(T_\epsilon) &= r_1(1+O_4),\\ r_2(T_\epsilon) &= r_2(1+O_4),\\ \theta_1(T_\epsilon) &= \theta_1+2\pi\left(1+\frac{\alpha-a}{2}r_1^2+\frac{\beta-b}{2}r_2^2\right)+O_4,\\ \theta_2(T_\epsilon) &= \theta_2+2\pi\left(1-\frac{\alpha-a}{2}r_1^2-\frac{\beta-b}{2}r_2^2\right)+O_4.\end{aligned}$$

Lemma 6.2. *If ϵ is small enough, the equation $\theta_1(T_\epsilon) = \theta_1 + 2\pi$ defines the homographic boundary of the annulus A_ϵ.*

Proof. Since $a = \alpha$ and $b \neq \beta$, the θ_1-component of the map above boils down to

$$\theta_1(T_\epsilon) = \theta_2 + 2\pi\left(1 + \frac{\beta - b}{2} r_2^2\right) + O_4.$$

But we have additional information on the structure of the remainder in this formula:

1. It follows from lemma 8.1 that the structure of the vertical variational equation does not depend on the excentricity of the homographic solution along which it is computed, which implies that the restriction of the Poincaré map to the homographic boundary of A_ϵ, defined by $r_2 = 0$, is the identity, hence the O_4 is a $r_2 O_3$.
2. The flow of the differential equation, hence also the Poincaré map, is equivariant under the tranformation $(r_1, r_2, \theta_1, \theta_2) \mapsto ((r_1, -r_2, \theta_1, \theta_2))$. Hence the O_4 actually is a $r_2^2 O_2$.
3. Finally, since the energy level $H = -\frac{1}{6} + \epsilon^2$ is compact, both r_1 and r_2 are bounded by $c\epsilon$, where c is a constant:

$$\theta_1(T_\epsilon) = \theta_1 + 2\pi\left(1 + r_2^2\left[\frac{\beta - b}{2} + O(\epsilon^2)\right]\right). \tag{*}$$

It follows that the equation of the statement is of the form

$$r_2^2\left(\beta - b + O(\epsilon^2)\right) = 0.$$

For ϵ small enough, it admits only the solution $r_2 = 0$. This completes the proof of the lemma and hence of theorem 6.1.

Theorem 6.3. *If ϵ is small enough, there exist coordinates on the annulus A_ϵ for which the Poincaré return map is a monotone twist map.*

Proof. Since $\theta_1 + \theta_2 = 0 \mod 2\pi$ in A_ϵ we may choose $\psi = \theta_1$ as an angular coordinate. A regular radial coordinate would be $r_1 - r_2$ but choosing $\rho = r_2^2$ is more convenient, albeit singular at $r_1 = 0$. Using (*), we see that the second return map is of the form

$$(\rho, \psi) \mapsto \left(\rho\big(1 + O(\epsilon^4)\big) + O(\rho^2), \psi + 2\pi\rho\left[\frac{\beta - b}{2} + O(\epsilon^2)\right] + O(\rho^2)\right),$$

where the $O(\rho^2)$ are functions of ρ and ψ and the $O(\epsilon^n)$ depend only on ϵ and ψ. As ρ varies from 0 to $\rho_0 = O(\epsilon^2)$, this is indeed a monotone twist map if ϵ is small enough.

7. **Additional comments.**

1. Using the proof of the existence of the P_{12} family as a family of action minimizers among Γ_1-invariants loops (which, as we recalled, leaves open the problem of uniqueness and hence of continuity of the family), one can replace section 5 by the following observation. As the homographic family is not Γ_1-invariant in any rotating frame, it follows from theorem 6.3 that, in the neighborhood of the Lagrange solution, the minimum of the action among Γ_1-symmetric loops of configurations in a rotating frame is unique up to a shift of time by half the period; this defines the two solutions whose existence is asserted by theorem 5.1.

2. Many of the results of this paper hold for an open set of masses in the neighborhood of the diagonal $m_0 = m_1 = m_2$; in general the symmetry group Γ_1 needs to be replaced by the subgroup generated by s^3. However, when two masses are small compared to the third one, it is well known that the Lagrange relative equilibrium becomes linearly stable and additional resonances may appear.

8. **Appendix: The vertical variational equation along the equilateral homographic family.** This section is devoted to solving the vertical variational equation of homographic motions in the three-body problem. It is used in the proof of theorem 6.1. For the sake of generality, we assume that the masses m_i are arbitrary. Also, we give more details than is necessary. Here we do not seek originality, but only a geometric description of the solutions.

A homographic solution of the three-body problem is a solution of the form

$$q(t) = (q_0(t), q_1(t), q_2(t)) = \rho(t)\hat{q} = \rho(t)\,(\hat{q}_0, \hat{q}_1, \hat{q}_2)\,,$$

where $\rho(t) \in \mathbb{C}$ satisfies the Kepler equation $\ddot{\rho} = -\frac{c\rho}{|\rho|^3}$ and $\hat{q}$ is a plane central configuration.

More precisely we are interested in the case where the configuration is equilateral. If $r(t)$ is the length of the sides of the triangle at time t, for such an equilateral homographic solution $q(t) \in (\mathbb{R}^2)^3$, we have for $i = 0, 1, 2$,

$$\ddot{q}_i = \sum_{j \neq i} m_j \frac{q_j - q_i}{r(t)^3} = -\frac{M}{r(t)^3} q_i\,, \quad \text{where} \quad M = \sum_{i=0}^{2} m_i$$

because $\sum_{i=0}^{2} m_i q_i(t) = 0$. The vertical variational equation is

$$\ddot{z} = -\frac{M}{r(t)^3} z \tag{VVE}$$

and we want to solve it in the subspace

$$\mathcal{D} = \left\{ z = (z_0, z_1, z_2) \in \mathbb{R}^3,\ \sum_{i=0}^{2} m_i z_i = 0 \right\}.$$

Let $\langle , \rangle$ denote the standard Euclidean product in $\mathbb{R}^2$.

Lemma 8.1 ([8])**.** *The solutions of (VVE) are paths of the form*

$$z_i(t) = \langle q_i(t), d_i \rangle\,, \quad i = 0, 1, 2, \tag{S}$$

where the triple $(d_0, d_1, d_2) \in (\mathbb{R}^2)^3$ is such that

$$\sum_{i=0}^{2} m_i \langle q_i(t_0), d_i \rangle = 0 \quad \text{and} \quad \sum_{i=0}^{2} m_i \langle \dot{q}_i(t_0), d_i \rangle = 0 \tag{C}$$

for some time t_0 (and hence for all).

Proof. For any choice of triple $(d_0, d_1, d_2) \in (\mathbb{R}^2)^3$, formula (S) gives a solution of (VVE) in the space of unrestricted triples $(z_0, z_1, z_2) \in \mathbb{R}^3$. Now, the 4-dimensional subspace of the space $\{(z_i, \dot{z}_i),\ i = 0, 1, 2\} = \mathbb{R}^6$ defined by the equations

$$\sum_{i=0}^{2} m_i z_i = 0,\ \sum_{i=0}^{2} m_i \dot{z}_i = 0$$

is left invariant by such solutions of (VVE). Hence, if one chooses the d_i such that condition (C) be satisfied at some instant t_0, we get a solution $z(t)$ of (VVE) in $\mathcal{D}$. Moreover the set of so-obtained solutions is 4-dimensional hence we have got all of them.

In this 4-dimensional space of solutions, one can distinguish four 2-dimensional vector subspaces, which generally are pairwise transverse (not when the masses are all equal to each other, though; see below):

1. solutions corresponding to $d_0 = d_1 = d_2 = d \in \mathbb{R}^2$;
2. solutions which preserve the verticality of the angular momentum $\mathcal{C}$, that is
$$\sum_{i=0}^{2} m_i \left(\langle d_i, \dot{q}_i \rangle q_i - \langle d_i, q_i \rangle \dot{q}_i \right) = 0;$$
3. solutions such that the d_i are aligned with the major axis of the ellipse described by the corresponding body, that is, if t_0 denotes the time at which the bodies are at the perihelia of their respective ellipses,
$$d_i = \delta_i q_i(t_0) \quad \text{with} \quad \sum_{i=0}^{2} m_i \delta_i ||q_i(t_0)||^2 = 0;$$
4. solutions corresponding to the d_i orthogonal to the major axis of the corresponding ellipse, that is
$$d_i = \delta_i \dot{q}_i(t_0) \quad \text{with} \quad \sum_{i=0}^{2} m_i \delta_i ||\dot{q}_i(t_0)||^2 = 0.$$

To get a feeling of these solutions, notice that defining $z_i(t)$ by projecting the elliptic motion of body i on some axis in the plane (this is essentially what amounts to the scalar product with d_i) amounts to infinitesimally rotating the plane of the corresponding ellipse around an axis through the origin, orthogonal to d_i. Hence, the first 2-dimensional space of solutions of (VVE) corresponds to infinitesimal rotations of the plane of the whole homographic solution while the last two correspond to different infinitesimal rotations of the plane for each body, around axes of rotation othogonal or parallel to the major axes of the ellipses. Finally, *all the solutions are obtained by inclining the plane in an appropriate way for each body. They are all periodic with the same period as the homographic solution (1:1 resonance).* This explains the equality $a = \alpha$ and its higher order analogues.

When the eccentricity is zero i.e. for the relative equilibrium solutions, the equilateral solution is of the form (after identification of the horizontal plane with $\mathbb{C}$:
$$q_j(t) = \rho_j e^{i(\omega t + \varphi_j)}, \quad j = 0, 1, 2, \quad \sum_{j=0}^{2} m_j \rho_j e^{i\varphi_j} = 0.$$
The two equations which express that the center of mass of the z_j is at the origin are equivalent to
$$\sum_{j=0}^{2} m_j \rho_j d_j e^{-i\varphi_j} = 0.$$
Apart from the trivial case of a global rotation (all d_j equal), it is obviously satisfied by $d_j = e^{2i\varphi_j}$ which, in the case of equal masses gives d_0, d_1, d_2 respectively aligned with the principal axis of the ellipse described by body $0, 2, 1$ (note the permutation of 1 and 2).

In case the eccentricity is arbitrary but the masses are all equal, one checks immediately that choosing as above d_0, d_1, d_2 respectively aligned with the principal axis of the ellipse described by bodies $0, 2, 1$ gives a solution for which the angular momentum stays vertical.

Thanks to Alain Albouy for illuminating discussions, to Rick Moeckel for the beautiful handwritten notes [15] *of his course in Trieste and for pointing out Corollary 1, to Christian Marchal for having been the first to understand the interest of the vertical variational equation of the Lagrange equilateral solution for equal masses and to Richard Montgomery for asking about some Hessian. Thanks to Mickaël Gastineau for so kindly adapting Trip to our needs.*

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Received October 2006; revised June 2007.

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