

# Quasiperiodic motions in the planar three-body problem

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In the direct product of the phase and parameter spaces, we define the *perturbing region*, where the Hamiltonian of the planar three-body problem is  $C^k$ -close to the dynamically degenerate Hamiltonian of two uncoupled two-body problems. In this region, the *secular systems* are the normal forms that one gets by trying to eliminate the mean anomalies from the perturbing function. They are Pöschel-integrable on a transversally Cantor set. This construction is the starting point for proving the existence of and describing several new families of periodic or quasiperiodic orbits: short periodic orbits associated to some secular singularities, which generalize Poincaré's periodic orbits of the second kind ("Les Méthodes nouvelles de la mécanique céleste", first Vol., Gauthiers-Villars, Paris, 1892–1899); quasiperiodic motions with three (resp. two) frequencies in a rotating frame of reference, which generalize Arnold's solutions (*Russian Math. Survey* **18** (1963), 85–191) (resp. Lieberman's solutions; *Celestial Mech.* **3** (1971), 408–426); and three-frequency quasiperiodic motions along which the two inner bodies get arbitrarily close to one another an infinite number of times, generalizing the Chenciner-Llibre's invariant "punctured tori" (*Ergodic Theory Dynamical Systems* **8** (1988), 63–72). The proof relies on a sophisticated version of KAM theorem, which itself is proved using a normal form theorem of M. Herman ("Démonstration d'un Théorème de V.I. Arnold," Séminaire de Systèmes Dynamiques and Manuscripts, 1998).

*Key Words:* three-body problem, secular system, averaging, regularization, KAM theorem, periodic orbits

Perturbative studies of the three-body problem split the dynamics into two parts: a fast, *Keplerian* dynamics, which describes the motion of the bodies along three ellipses as if each body underwent the attraction of only one fictitious center of attraction; and a slow, *secular* dynamics, which describes the deformations of these Keplerian ellipses, due to the fact that each body actually undergoes the attraction of the other two. This splitting is not unique. If we want to keep the symmetry of translations though, the choice of the splitting boils down to that of only two 2-body problems. The

splitting

$$F = F_{Kep} + F_{per}$$

of the Hamiltonian, such as we define it in § 1.1, is inherited from the Jacobi coordinates. It will turn out to be dynamically relevant in § 2, when it proves adapted to both the planetary and the lunar problems. This is not the case for the heliocentric splitting used by Lieberman [18] for instance, because heliocentric coordinates do not diagonalize the metric of the kinetic energy. In § 1.2, some notations relative to the Keplerian dynamics are defined. In order to study the dynamics globally in the eccentricities of the inner bodies, in § 1.3 the perturbing function is expanded in the powers of the ratio of the distances of the bodies from the center of mass of the two inner bodies.

In the 18th century, when Lagrange and Laplace tried to prove the stability of the system consisting of the Sun, Jupiter and Saturn, they introduced the *averaged system*. Its Hamiltonian

$$\langle F \rangle = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} F d\lambda_1 d\lambda_2$$

is obtained by averaging the initial Hamiltonian along the Keplerian ellipses which are parametrized by the mean anomalies  $\lambda_1$  and  $\lambda_2$  of the two fictitious Kepler problems. This averaged system agrees on some transversally Cantor set with the first of the normal forms which I denote by  $F_\pi^n$ ,  $n \geq 1$ , which are obtained by trying to eliminate the fast angles from the Hamiltonian, up to increasing orders of smallness. These normal forms are called the *secular systems* of the planar three-body problem. They are completely integrable in the sense of Pöschel [23] on the transversally Cantor set where the mean anomalies have actually been eliminated. The purpose of this paper is to show that studying the global secular dynamics—globally both in the parameter and in the phase spaces—is one of the few ways we have to understand the global dynamics of the planar three-body problem.

For the sake of simplicity, the secular systems are usually studied in the separate cases of the planetary and lunar regions (cf. § 2.1)<sup>1</sup>. If the outer body is given the number 2, the small parameters  $\varepsilon$  for these regions respectively are some mass ratio, for instance  $\varepsilon = (m_1 + m_2)/m_0$ , and the ratio  $\varepsilon = a_1/a_2$  of the semi major axes. In § 2, the secular systems are built more globally at any order. In the direct product of the phase and parameter spaces, I abstractly define a *perturbing region*  $\Pi_\varepsilon^k$  (§ 2.1), where

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<sup>1</sup>An exception is the paper by Lidov and Ziglin [17]. However in this paper, the dynamics of only the first term of the averaged system is studied. Also, although Lidov and Ziglin do not assume that the angular momentum is large, their study is not relevant when excentricities get close to one.

the perturbing function is  $\varepsilon$ -small in some  $C^k$ -norm which is well suited to the coming application of KAM theory. Equivalently, the perturbing region is a region where the vector field is  $\varepsilon$ -close to that of the two uncoupled Kepler problems and thus where a perturbative study of the planar three-body problem is possible. In § 2.2, I prove that the secular systems  $F_\pi^n$  are  $\varepsilon^{1+n}$ -close to the conjugacy class of the initial system.

In § 2.1, I also define the *asynchronous region*  $A_\varepsilon^k$ , as the subset of the perturbing region where the ratio  $\nu_2/\nu_1$  of the Keplerian frequencies is  $\varepsilon$ -small. It extends the lunar region. As Jefferys-Moser [15] had already noticed it, the fact that in  $A_\varepsilon^k$  the two Keplerian frequencies do not interfere makes it possible to build some Liouville-integrable modified secular systems  $F_a^n$  over  $A_\varepsilon^k$  (§ 2.3). Using the trick explained in appendix C, these asynchronous secular systems can even be computed to any order by mere quadratures of trigonometric polynomials.

The global secular dynamics proper will be studied in another paper [8], where we will describe the bifurcation diagram of the secular systems in the perturbing region provided that the semi major axes ratio  $a_1/a_2$  is small enough. For instance, it will be proved that it is not all the secular singularities, which lie on the submanifold of aligned ellipses, contrary to what one might think by just investigating the lunar or planetary problems. Also, paper [7] explains why it is relevant to study this secular dynamics up to double inner collisions, whereas, for astronomical reasons, most existing studies focus on the neighborhood of circular orbits.

We eventually describe and apply KAM techniques due to M. Herman (§ 3.1), which allow us to meet the following two specific difficulties: weak diophantine conditions, arising from the proper degeneracy of the Newtonian potential, and isotropic invariant tori which may not have the maximum dimension, arising from secular limit degeneracies. As a result, we prove that, in the perturbing region: a positive measure of regular secular orbits persists in the planar three-body problem as quasiperiodic invariant 3-tori; and a positive measure of secular non-degenerate singularities persists in the planar three-body problem as quasiperiodic invariant 2-tori, the secular limit degeneracy surviving the break down of the proper degeneracy of the Newtonian potential (§ 3.2).

Also, using the result proved in [7] concerning the averaging in the neighborhood of double inner collisions, we prove that regular secular orbits which are transverse to the collision set give rise to 3-frequency quasiperiodic motions for which the two inner bodies get arbitrarily close to one another an infinite number of times (§ 3.3). These solutions generalize to the elliptic restricted problem and to the full problem the invariant “punctured tori” that Chenciner-Llibre had found in the circular restricted problem [5].

## 1. SETTING AND NOTATIONS

### 1.1. Jacobi's splitting

Consider three points of masses  $m_0$ ,  $m_1$  and  $m_2$  undergoing gravitational attraction in the plane. Identify the plane to  $\mathbf{R}^2$  by choosing a frame of reference. The phase space is the space

$$\left\{ (p_j, q_j)_{0 \leq j \leq 2} \in (\mathbf{R}^{2*} \times \mathbf{R}^2)^3 \mid \forall 0 \leq j < k \leq 2, q_j \neq q_k \right\}$$

of linear momentum covectors  $(p_0, p_1, p_2)$  and position vectors  $(q_0, q_1, q_2)$  of each body. It is the open set of the cotangent bundle  $T^*\mathbf{R}^6$  which is obtained by ruling out collisions. Hence it is naturally endowed with the symplectic form  $\omega = \sum_{j,l} dp_j^l \wedge dq_j^l$  and the Euclidean metric, whose norm will be denoted by  $|\cdot|$ . If the frame of reference is Galilean, the Hamiltonian is

$$\frac{1}{2} \sum_{0 \leq j \leq 2} \frac{|p_j|^2}{m_j} - \gamma \sum_{0 \leq j < k \leq 2} \frac{m_j m_k}{|q_j - q_k|},$$

where  $\gamma$  is the universal constant of gravitation. Thanks to the invariance of Newton's equations with respect to change of the time unit, we may suppose that  $\gamma = 1$ .

In order to carry out the reduction by the symmetry of translations, consider the Jacobi coordinates  $(P_j, Q_j)_{j=0,1,2}$  (cf. chap. II, first Vol. of the *Leçons* [22]), defined by

$$\begin{cases} P_0 = p_0 + p_1 + p_2 \\ P_1 = p_1 + \sigma_1 p_2 \\ P_2 = p_2, \end{cases} \quad \begin{cases} Q_0 = q_0 \\ Q_1 = q_1 - q_0 \\ Q_2 = q_2 - \sigma_0 q_0 - \sigma_1 q_1, \end{cases}$$

where  $1/\sigma_0 = 1 + m_1/m_0$  and  $1/\sigma_1 = 1 + m_0/m_1$ . The phase space reduced by translations can be identified to the open set of  $T^*\mathbf{R}^4$  which is described by the Jacobi coordinates  $(P_j, Q_j)_{j=1,2}$  outside collisions. If the frame of reference is attached to the center of mass, i.e. if  $P_0 = 0$ , and if  $Q_2 \neq 0$ , the reduced Hamiltonian can be written as

$$F = F_{Kep} + F_{per},$$

where, up to the choice of the masses  $M_1$  and  $M_2$ ,  $F_{Kep}$  and  $F_{per}$  are defined by

$$F_{Kep} = \frac{|P_1|^2}{2\mu_1} + \frac{|P_2|^2}{2\mu_2} - \frac{\mu_1 M_1}{|Q_1|} - \frac{\mu_2 M_2}{|Q_2|},$$

$$F_{per} = -\frac{m_0 m_1 - \mu_1 M_1}{|Q_1|} - \frac{m_1 m_2}{|Q_2 - \sigma_0 Q_1|} - \frac{m_0 m_2}{|Q_2 + \sigma_1 Q_1|} + \frac{\mu_2 M_2}{|Q_2|}$$

with the reduced masses  $\mu_1$  and  $\mu_2$  themselves defined by

$$\frac{1}{\mu_1} = \frac{1}{m_0} + \frac{1}{m_1} \quad \text{and} \quad \frac{1}{\mu_2} = \frac{1}{m_0 + m_1} + \frac{1}{m_2}.$$

### 1.2. Keplerian dynamics

The Hamiltonian  $F_{Kep}$  is the *Keplerian Hamiltonian*. We will exclusively pay attention to bounded motions and their perturbations. Then  $F_{Kep}$  is the completely integrable Hamiltonian of two fictitious bodies of masses  $\mu_1$  and  $\mu_2$  which revolve along ellipses around a fixed center of attraction, without mutual interaction. The Keplerian dynamics is a direct product and induces a *Keplerian action* of the 2-torus on the phase space, up to collision orbits.

For the  $j$ -th fictitious body, with  $j = 1$  or  $2$ , the mean longitude will be designated by  $\lambda_j$ , the semi major axis by  $a_j$ , the eccentricity by  $e_j$ , the ‘‘centricity’’  $\sqrt{1 - e_j^2}$  by  $\epsilon_j$ , the argument of the pericenter by  $g_j$  and the mean motion by  $\nu_j$  (cf. chap. III, first Vol. of the *Leçons* [22]). Let also  $g = g_1 - g_2$  be the difference of the arguments of the pericenters and  $(\Lambda_j, \lambda_j, \xi_j, \eta_j)$  be the Poincaré coordinates, where

$$\begin{cases} \Lambda_j = \mu_j \sqrt{M_j a_j} \\ \xi_j + i\eta_j = \sqrt{2L_j(1 - e_j^2)} e^{-ig_j}. \end{cases}$$

The longitudes  $\lambda_j$  are the *fast* angles and their conjugate variables  $\Lambda_j$  are the *fast* actions, whereas the other coordinates are called the *slow* variables. The Keplerian part and the mean motions can be written

$$F_{Kep} = -\frac{\mu_1^3 M_1^2}{2\Lambda_1^2} - \frac{\mu_2^3 M_2^2}{2\Lambda_2^2} \quad \text{and} \quad \nu_j = \frac{\partial F_{Kep}}{\partial \Lambda_j} = \frac{\sqrt{M_j}}{a_j^{3/2}}.$$

Under the Keplerian flow, the (real) bodies describe ellipses whose foci all are the moving center of mass of  $m_0$  and  $m_1$ . In particular, the two ellipses of  $m_0$  and  $m_1$  are described by  $\sigma_1 Q_1$  and  $-\sigma_0 Q_1$ . Hence they have the same excentricity and are in opposition.

### 1.3. The perturbing function and Legendre’s polynomials

The Hamiltonian  $F_{per}$  is the *perturbing function*. It is real analytic outside collisions of the bodies *and* outside collisions of the fictitious body  $Q_2$  with the center, which is not bothering insofar as we will suppose that the ellipse which is described by  $Q_2$  is the outer ellipse. In order for  $F_{per}$  to be as small as possible, the proof of the coming lemma shows that the optimal

choice for  $M_1$  and  $M_2$  is to set  $M_1 = m_0 + m_1$  and  $M_2 = m_0 + m_1 + m_2$ . The perturbing function can then be written

$$F_{per} = -\mu_1 m_2 \left[ \begin{array}{l} \frac{1}{\sigma_0} \left( \frac{1}{|Q_2 - \sigma_0 Q_1|} - \frac{1}{|Q_2|} \right) \\ + \frac{1}{\sigma_1} \left( \frac{1}{|Q_2 + \sigma_1 Q_1|} - \frac{1}{|Q_2|} \right) \end{array} \right].$$

Let  $P_n$  be the  $n$ -th Legendre polynomial (the risk is small of mixing it up with a linear momentum) and let  $\zeta$  be the oriented angle  $(\widehat{Q_1, Q_2})$ . Let also  $\hat{\sigma} = \max(\sigma_0, \sigma_1)$ .

LEMMA 1.1. *The expansion*

$$F_{per} = -\frac{\mu_1 m_2}{|Q_2|} \sum_{n \geq 2} \sigma_n P_n(\cos \zeta) \left( \frac{|Q_1|}{|Q_2|} \right)^n, \quad \sigma_n = \sigma_0^{n-1} + (-1)^n \sigma_1^{n-1}$$

of the perturbing function in the powers of  $|Q_1|/|Q_2|$  is convergent in the complex disc

$$\frac{|Q_1|}{|Q_2|} < \frac{1}{\hat{\sigma}} \in [1, 2].$$

*Proof.* The Legendre polynomials can be defined, if  $|Q_1|/|Q_2| < 1$ , by

$$\frac{1}{|Q_2 - Q_1|} = \frac{1}{|Q_2|} \sum_{n \geq 0} P_n(\cos \zeta) \left( \frac{|Q_1|}{|Q_2|} \right)^n$$

(cf. § 38, chap. II, first Vol. of the *Leçons* [22]). Expanding the perturbing function similarly and noticing that the Legendre polynomials are odd or even according to their own degree yields the result.  $\blacksquare$

## 2. GLOBAL SECULAR SYSTEMS

### 2.1. Perturbing and asynchronous regions

Recall that  $\hat{\sigma} = \max(\sigma_1, \sigma_2)$  and let

$$\Delta = \max_{(\lambda_1, \lambda_2, g) \in \mathbf{T}^3} \frac{\hat{\sigma} |Q_1|}{|Q_2|} = \hat{\sigma} \frac{a_1(1 + e_1)}{a_2(1 - e_2)}$$

be a measure of how close the outer ellipse is from the inner ellipses when they are in opposition ( $g = \pi \pmod{2\pi}$ ). We will suppose that  $\Delta < 1$ , i.e.

that the outer ellipse does not meet the other two, whatever the difference  $g$  of the arguments of the pericenters. In particular, for given semi major axes, the eccentricity  $e_2$  of the outer ellipse cannot be arbitrarily close to 1.

We will also assume that the eccentricity of the inner ellipses is upper bounded from 1. Indeed, the neighborhood of inner collision orbits ( $e_1 = 1$ ) requires some special care, which is dealt with in another paper [7]. We will use the results of the latter paper in the last section.

With these two assumptions, let  $\mathcal{P}$  be the restricted phase space. It is diffeomorphic to  $(\mathbf{S}^1 \times \mathbf{R}^3 \times \mathbf{S}^0)^2$  and it can be thought of as a fiber bundle over  $\mathbf{R}^6 \times (\mathbf{S}^0)^2$  whose fibers are the orbits of the Keplerian  $\mathbf{T}^2$ -action. The four connected components correspond to the two possible orientations of the two fictitious ellipses. On each component  $(\mathbf{S}^1 \times \mathbf{R}^3)^2$ , some coordinates are given by the Poincaré variables  $(\Lambda_j, \lambda_j, \xi_j, \eta_j)_{j=1,2}$ .

Let also  $\mathcal{M} \simeq \mathbf{R}^3$  be the space described by the three mass parameters  $m_0, m_1$  and  $m_2$ .

**DEFINITION 2.1.** Let  $\varepsilon$  be a positive real number and  $k$  be a non-negative integer. The *perturbing region* of parameters  $\varepsilon$  and  $k$ , designated by  $\Pi_\varepsilon^k$ , will be the open subset of  $\mathcal{P} \times \mathcal{M}$  defined by the following inequality:

$$(\Pi_\varepsilon^k) \quad \max \left( \frac{m_2}{M_1} \left( \frac{a_1}{a_2} \right)^{3/2}, \frac{\mu_1 \sqrt{M_2}}{M_1^{3/2}} \left( \frac{a_1}{a_2} \right)^2 \right) \frac{1}{\varepsilon_2^{3(2+k)} (1 - \Delta)^{2k+1}} < \varepsilon.$$

(The notations for elliptical elements have been defined in § 1.2.)

Appendix A justifies this definition by proving that inside the perturbing region the perturbing function is  $\varepsilon$ -small in the  $C^k$ -norm of proposition 2.1. The inequality is not optimal and the given powers are not meaningful. The factor  $1/\varepsilon_2^{3(2+k)}$  prevents the outer body from getting too close from collisions with the fictitious center of attraction ( $Q_2 = 0$ ), and the factor  $1/(1 - \Delta)^{2k+1}$  prevents the two outer bodies from getting too close from each other ( $q_2 = q_0$  or  $q_1$ ).

In order to get a rough idea of the meaning of the definition of the perturbing region, temporarily assume that the outer eccentricity is upper bounded ( $e_2 \leq \text{Cst} < 1$ ) and that the semi major axes ratio is small enough (say,  $a_1/a_2 < 1/2$ ). Then a sufficient condition for being in the perturbative region is

$$\frac{(\mu_1 + m_2)M_2}{M_1^2} \frac{a_1}{a_2} < \varepsilon;$$

this fact is elementary to check. Traditionally, two sub-regions are given specific names:

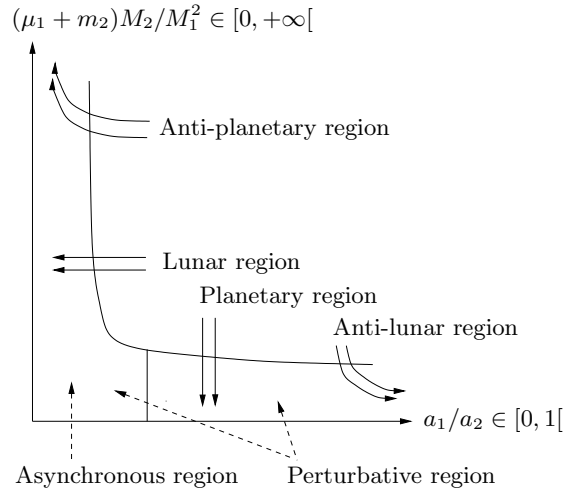
- the *planetary* region, where the eccentricity of the outer ellipse and both semi major axes are in a small compact set, and where two masses out of three, including the outer mass, are  $\varepsilon$ -small compared to the third mass ( $(\mu_1 + m_2)/M_1 \leq \text{Cst } \varepsilon$ );

- the *lunar* region, where the masses are in a compact set, and where the outer body is  $1/\varepsilon$ -far away from the other two ( $a_1/a_2 < \text{Cst } \varepsilon$ ).

As it is shown on Figure 1 when  $e_2 \leq \text{Cst} < 1$ , our generalization quantifies:

- in the *anti-planetary* region, how the perturbing region sharpens when the outer body has a large mass ( $m_2 \simeq M_2$ ); in other words, to which extent the outer mass may be large provided that the outer ellipse is far from the other two;

- in the *anti-lunar* region, how the perturbing region sharpens when the outer ellipse is close to one of the inner ellipses; in other words, to which extent the outer ellipse may be close to the other two provided that one of the two inner bodies has a large mass.



**FIG. 1.** The perturbing region

Recall that  $\nu_j = \sqrt{M_j}/a_j^{3/2}$  is the Keplerian frequency of the  $j$ -th body:

**DEFINITION 2.2.** Let  $\varepsilon$  be a positive real number and  $k$  be a positive integer. The *asynchronous* region with parameters  $\varepsilon$  and  $k$ , designated by  $A_\varepsilon^k$ , will be the open set of  $\Pi_\varepsilon^k$  which is bounded by the following additional



inequality:

$$(A_\varepsilon^k) \quad \frac{\nu_2}{\nu_1} = \sqrt{\frac{M_2}{M_1}} \left( \frac{a_1}{a_2} \right)^{3/2} < \varepsilon.$$

If this inequality is satisfied, the inner bodies revolve quickly when compared to the outer body. Hence the two Keplerian frequencies do not come into play at the same level of smallness and it is possible to solve the cohomological equations of elimination of the fast angles without non-resonance conditions, as we will make it more precise in proposition 2.2. The asynchronous region extends the lunar region.

(The reader can check that the *second* asynchronous region, where the inner bodies would revolve much slower than the outer body, is empty.)

We need some more notations. If  $p \geq 1$  is an integer and  $\gamma > 0$  and  $\tau \geq p - 1$  are real numbers, let

$$\begin{cases} HD_{\gamma,\tau}(p) = \left\{ \alpha \in \mathbf{R}^p : \forall k \in \mathbf{Z}^p \setminus 0, |k \cdot \alpha| \geq \frac{\gamma}{|k|^\tau} \right\} \\ hd_{\gamma,\tau} = \{(x, m) \in \mathcal{P} \times \mathcal{M} : (\nu_1(x, m), \nu_2(x, m)) \in HD_{\gamma,\tau}(2)\}, \end{cases}$$

where, for  $p$ -uplets  $k$  of  $\mathbf{Z}^p$ ,  $|\cdot|$  stands for the  $l_2$ -norm:

$$|k| = \sqrt{k_1^2 + \dots + k_p^2};$$

$HD_{\gamma,\tau}(p)$  is the transversally Cantor set of frequency vectors in  $\mathbf{R}^p$  which satisfy homogeneous diophantine conditions of constants  $\gamma, \tau$  and  $hd_{\gamma,\tau}$  is the inverse image of  $HD_{\gamma,\tau}(2)$  by the Keplerian frequency map  $(\nu_1, \nu_2)$  in the space  $\mathcal{P} \times \mathcal{M}$ . In the definition of  $hd_{\gamma,\tau}$ , nothing prevents  $\gamma$  or  $\tau$  to be functions on  $\mathcal{P} \times \mathcal{M}$ . Besides, let

$$hd = \bigcup_{\gamma>0, \tau \geq 1} hd_{\gamma,\tau}.$$

If  $x_1$  and  $x_2$  are two quantities, let  $\overset{\vee}{x} = \min(x_1, x_2)$ . When  $\varepsilon \rightarrow 0$ , let

$$\begin{cases} \tilde{\Pi}_\varepsilon^k = (\Pi_\varepsilon^k \times \mathbf{R}^2) \cap \left\{ \overset{\vee}{\Lambda} = O(\overset{\vee}{\Lambda}_0) \right\} \cap \left\{ \overset{\vee}{\nu} = O(\overset{\vee}{\nu}_0) \right\} \\ \tilde{A}_\varepsilon^k = (A_\varepsilon^k \times \mathbf{R}^2) \cap \left\{ \overset{\vee}{\Lambda} = O(\overset{\vee}{\Lambda}_0) \right\} \cap \left\{ \overset{\vee}{\nu} = O(\overset{\vee}{\nu}_0) \right\} \end{cases}$$

be some open sets of  $\Pi_\varepsilon^k \times \mathbf{R}^2$  and of  $A_\varepsilon^k \times \mathbf{R}^2$ , where  $(\overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0)$  stand for coordinates of  $\mathbf{R}^2$ . These open sets can be thought of as fiber bundles

over the parameter space  $\mathcal{M} \times \mathbf{R}^2$ . The additional parameters  $(\check{\Lambda}_0, \check{\nu}_0)$  are meant to localize the particular region on which we focus in the phase space.

## 2.2. Resonant elimination of the fast angles

PROPOSITION 2.1. *Let  $n \geq 0$  and  $k \geq 0$  be integers and  $\gamma > 0$  and  $\tau \geq 1$  be real numbers. For every  $\varepsilon > 0$  there exist*

- an open set  $\dot{\Pi}_\varepsilon^k$  of  $\tilde{\Pi}_\varepsilon^k$ , with fiber  $\dot{\Pi}_\varepsilon^k(m, \check{\Lambda}_0, \check{\nu}_0)$  over the base point  $(m, (\check{\Lambda}_0, \check{\nu}_0))$  of  $\mathcal{M} \times \mathbf{R}^2$ ;
- for every  $(m, \check{\Lambda}_0, \check{\nu}_0) \in \mathcal{M} \times \mathbf{R}^2$ , a  $C^\infty$ -Hamiltonian

$$F_\pi^n : \dot{\Pi}_\varepsilon^k(m, \check{\Lambda}_0, \check{\nu}_0) \rightarrow \mathbf{R}$$

and a  $C^\infty$ -symplectomorphism

$$\phi^n : \dot{\Pi}_\varepsilon^k(m, \check{\Lambda}_0, \check{\nu}_0) \rightarrow \phi^n \left( \dot{\Pi}_\varepsilon^k(m, \check{\Lambda}_0, \check{\nu}_0) \right)$$

which is  $\varepsilon$ -close to the identity in some  $C^k$ -norm  $\|\cdot\|_k$ ;

such that

- for the Liouville measure associated to the symplectic form  $\omega/\check{\Lambda}_0$ , the relative measure of  $\dot{\Pi}_\varepsilon^k$  in  $\tilde{\Pi}_\varepsilon^k$  tends to 1 when  $\varepsilon$  tends to 0 ;
- there exists a constant  $C > 0$  such that, for every  $\varepsilon > 0$ ,

$$\frac{1}{\check{\nu}_0 \check{\Lambda}_0} \|F \circ \phi^n - F_\pi^n\|_k \leq C\varepsilon^{1+n} \quad \text{over } \dot{\Pi}_\varepsilon^{k+n(\tau+4)};$$

- the restriction of the infinite jet of  $F_\pi^n$  to the transversally Cantor set  $hd_{\gamma\check{\nu}_0, \tau}$  is invariant by the Keplerian action of the 2-torus and by the diagonal action of the circle making the two bodies rotate simultaneously, hence  $F_\pi^n$  is completely integrable on  $hd_{\gamma\check{\nu}_0, \tau}$ .

*Proof.* Recall that  $\check{\Lambda} = \min(\Lambda_1, \Lambda_2)$ . Let  $\check{\Lambda}_0 > 0$  be a real number and assume that  $\check{\Lambda}/\check{\Lambda}_0$  is bounded in  $]0, +\infty[$  when  $\varepsilon$  goes to zero. Let  $(\check{\Lambda}_j, \lambda_j, \check{\xi}_j, \check{\eta}_j)_{j=1,2}$  be the rescaled Poincaré coordinates defined by

$$(\Lambda_j, \lambda_j, \xi_j, \eta_j)_{j=1,2} = (\check{\Lambda}_0 \check{\Lambda}_j, \lambda_j \sqrt{\check{\Lambda}_0} \check{\xi}_j, \sqrt{\check{\Lambda}_0} \check{\eta}_j)_{j=1,2}.$$

For  $m \in \mathcal{M}$  and  $\tilde{\Lambda}_j, \tilde{\xi}_j, \tilde{\eta}_j > 0$ , define  $\|\cdot\|_0$  by

$$\|F\|_0 = \sup \left\{ F \left( \left( \tilde{\Lambda}_j, \lambda_j, e^{i\gamma_j} (\tilde{\xi}_j + i\tilde{\eta}_j) \right)_{j=1,2}; m \right) : (\lambda_j, \gamma_j)_{j=1,2} \in \mathbf{T}^4 \right\},$$

where  $F$  is thought of as a function of  $\left( \left( \tilde{\Lambda}_j, \lambda_j, \tilde{\xi}_j + i\tilde{\eta}_j \right); m \right)$ . So  $\|F\|_0$  is a  $C^0$ -estimate of  $F$  which depends on  $m$  and  $(\Lambda_j, \xi_j^2 + \eta_j^2)_{j=1,2}$ , or, equivalently, on  $m$ , the semi major axes  $a_j$  and the excentricities  $e_j$ . Now, let  $\|F\|_k$  be the sup of the  $\|\cdot\|_0$ -norms of all the derivatives of  $F$  of order less than or equal to  $k$ , in the rescaled Poincaré coordinates.

From now on in this proof, the symplectic form will be the standard symplectic form for the rescaled coordinates  $(d\tilde{\Lambda}_1 \wedge d\lambda_1 + \dots)$ . In order to keep the same Hamiltonian vector field, we need to replace  $F$  by  $F/\tilde{\Lambda}_0$ . The nice thing about the coordinate chart  $(\tilde{\Lambda}_j, \lambda_j, \tilde{\xi}_j, \tilde{\eta}_j)_{j=1,2}$  and the associated norms is that when  $\varepsilon$  goes to 0 the  $C^k$ -norm of the perturbing function  $F_{per}$  now has the same behavior as the  $C^0$ -norm of  $F_{per}$ , in the sense that the ratios  $\|F_{per}\|_k / \|F_{per}\|_0$  are upper bounded over  $\Pi_\varepsilon^k$  (cf. Appendix A).

The symplectomorphism  $\phi^n$  will be obtained as the composition of  $n$  time-one maps  $\psi_1$  of small autonomous Hamiltonian vector fields.

We want to eliminate the fast angles from the perturbing function  $F_{per}$ . Let  $H$  be a Hamiltonian to be determined,  $X_H$  its vector field for the symplectic form  $d\tilde{\Lambda}_1 \wedge d\lambda_1 + \dots$  and  $\psi_t$  its flow. Define the (first order) complementary part  $F_{comp}^1$  of  $F$  by the equality

$$\psi_1^* F = F_{Kep} + (F_{per} + X_H \cdot F_{Kep}) + F_{comp}^1,$$

where  $X_H$  is seen as a derivation operator. Let

$$\langle F_{per} \rangle = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} F_{per} d\lambda_1 d\lambda_2$$

be the average of  $F_{per}$  and  $\tilde{F}_{per} = F_{per} - \langle F_{per} \rangle$  be its part of zero-average. Eliminating the fast angles from  $F_{per}$  modulo the complementary part  $F_{comp}^1$  means choosing  $H$  so that the cohomological equation

$$-X_H \cdot F_{Kep} = X_{F_{Kep}} \cdot H = \tilde{F}_{per}$$

is satisfied. The Hamiltonian  $F$  then is conjugate to

$$\psi_1^* F = F_{Kep} + \langle F_{per} \rangle + F_{comp}^1,$$

with

$$F_{comp}^1 = (\psi_1^* - id - X_H) F_{Kep} + (\psi_1^* - id) F_{per}.$$

The cohomological equation can be thought of as a family of partial differential equations on the Keplerian tori. The coming lemma, an easy refinement of lemma 1.6 in Bost's exposé [2], asserts that these partial differential equations have a solution on the transversally Cantor set  $hd = \bigcup_{\gamma > 0, \tau \leq 0} hd_{\gamma, \tau}$  of diophantine tori. Let

$$C_0^\infty(\mathbf{T}^p, \mathbf{R}^q) = \{f \in C^\infty(\mathbf{T}^p, \mathbf{R}^q), f(0) = 0\}$$

and

$$C_*^\infty(\mathbf{T}^p, \mathbf{R}^q) = \left\{ g \in C^\infty(\mathbf{T}^p, \mathbf{R}^q), \int_{\mathbf{T}^p} g(\theta) d\theta = 0 \right\}.$$

These two sets are tame Fréchet spaces, in the sense of Hamilton [11].

LEMMA 2.1 (Bost [2]). *Let  $\alpha \in HD_{\gamma, \tau}(p)$ . The Lie derivation*

$$\begin{aligned} \mathcal{L}_\alpha : C_0^\infty(\mathbf{T}^p, \mathbf{R}^q) &\rightarrow C_*^\infty(\mathbf{T}^p, \mathbf{R}^q) \\ f &\mapsto df \cdot \alpha \end{aligned}$$

*is a tame isomorphism. There exist constants  $A_k$  which are independent of  $\gamma$  and  $\tau$  such that for every function  $g \in C_*^\infty(\mathbf{T}^p, \mathbf{R}^q)$  and for every positive integer  $k$  the following estimate holds:*

$$\|\mathcal{L}_\alpha^{-1}g\|_k \leq \frac{A_k}{\gamma} \|g\|_{k+p+\tau+1}.$$

*Moreover, if  $g$  depends smoothly (resp. analytically) on some parameters,  $\mathcal{L}_\alpha^{-1}g$  depends smoothly (resp. analytically) on the same parameters.*

Consider the successive derivatives of the cohomological equation in the directions which are normal to the Keplerian tori in the phase space. These derivatives yield not only a family indexed by  $HD$  of functions on the 2-torus, but a whole infinite jet  $H$  along  $hd$  (cf. [4] for instance).

In order to get some finite estimates of this jet, we need to consider a subset of  $hd$  where we have some control over  $\gamma$  and  $\tau$ . But in order to get a positive measure of invariant tori, we cannot be too restrictive on the constants  $\gamma$  and  $\tau$ . A compromise is to focus on Keplerian tori whose frequency vector satisfy homogeneous diophantine conditions with constants  $\gamma = O(\check{\nu})$  and  $\tau = O(1)$  when the small parameter  $\varepsilon$  goes to 0.<sup>2</sup> So, let  $\check{\nu}_0$  be another additional parameter and assume that  $\check{\nu} = O(\check{\nu}_0)$ . In

<sup>2</sup>It is indeed the smallest of the two Keplerian frequencies which comes into play; thanks to M. Herman for having reminded me of it.

other words, focus on

$$\left[ \left( \Pi_\varepsilon^{k+\tau+4} \cap hd_{\gamma \check{\nu}_0, \tau} \right) \times \left\{ (\check{\Lambda}_0, \check{\nu}_0) \right\} \right] \cap \left\{ \check{\Lambda} = O(\check{\Lambda}_0), \check{\nu} = O(\check{\nu}_0) \right\}.$$

According to the previous lemma and the lemma A.2 given in appendix A, if  $\gamma$  is fixed, the jet  $H$  is such that, on this set,

$$\|H\|_{k+1} \leq \frac{\text{Cst}}{\check{\nu}_0} \left\| \frac{\tilde{F}_{per}}{\check{\Lambda}_0} \right\|_{k+\tau+4} \leq \text{Cst } \varepsilon.$$

The jet  $H$  is of class  $C^\infty$ -Whitney. Hence it extends to a  $C^\infty$ -Hamiltonian which satisfies the same estimate over

$$\tilde{\Pi}_\varepsilon^{k+\tau+4} = \left( \Pi_\varepsilon^{k+\tau+4} \times \mathbf{R}^2 \right) \cap \left\{ \check{\Lambda} = O(\check{\Lambda}_0) \right\} \cap \left\{ \check{\nu} = O(\check{\nu}_0) \right\}$$

(cf. Pöschel [23], extension theorem p. 664). Of course, we should and can build this extension consistently with the symmetry of rotations. Another way to proceed would be to extend the jet only once it has been reduced by the symmetry of rotations.

The Hamiltonian  $H$  is  $\varepsilon$ -small on  $\tilde{\Pi}_\varepsilon^{k+\tau+4}$ . Hence its vector field  $X_H$  defines a flow  $(\psi_t)_{0 \leq t \leq 1}$  up to time 1 on some open subset  $\dot{\Pi}_\varepsilon^{k+\tau+4}$  of  $\tilde{\Pi}_\varepsilon^{k+\tau+4}$  such that the complement of  $\dot{\Pi}_\varepsilon^{k+\tau+4}$  is of Liouville measure  $O(\varepsilon)$ . Note that  $\dot{\Pi}_\varepsilon^{k+\tau+4}$  can be chosen so that it is some union of Keplerian  $\mathbf{T}^2$ -orbits.

Let's now evaluate the size of the complementary part  $F_{comp}^1$ . This part is equal to

$$F_{comp}^1 = \int_0^1 (1-t) \psi_t^* (X_H^2 \cdot F_{Kep}) dt + \int_0^1 \psi_t^* (X_H \cdot F_{per}) dt.$$

Since  $X_H \cdot F_{Kep} = -\tilde{F}_{per}$  satisfies the same estimate as  $F_{per}$ , the inequality

$$\frac{\|F_{comp}^1\|_k}{\check{\Lambda}_0 \check{\nu}_0} \leq \text{Cst } \|H\|_{k+1} \left( \frac{\|\tilde{F}_{per}\|_{k+1}}{\check{\Lambda}_0 \check{\nu}_0} + \frac{\|F_{per}\|_{k+1}}{\check{\Lambda}_0 \check{\nu}_0} \right) \leq \text{Cst } \varepsilon^2$$

holds on  $\dot{\Pi}_\varepsilon^{k+\tau+4}$ . Define the *first order secular system*  $F_\pi^1$  and *resonant part*  $F_{res}^1$  by

$$\begin{cases} \psi_1^* F = F_\pi^1 + F_{comp}^1 \\ F_\pi^1 = F_{Kep} + \langle F_{per} \rangle + F_{res}^1. \end{cases}$$

The resonant part is such that

$$\left\| \frac{F_{res}^1}{\nu_0 \Lambda_0} \right\|_k \leq \text{Cst } \varepsilon \quad \text{over } \dot{\Pi}_\varepsilon^{k+\tau+4}.$$

Moreover, its infinite jet vanishes over  $hd_{\gamma\nu_0, \tau}$  :

$$j^\infty F_{res}^1|_{hd_{\gamma\nu_0, \tau}} = 0.$$

Let's sketch the second order averaging ( $n = 2$ ). Let  $H^2$  be a new Hamiltonian to be determined and  $\psi_t^2$  be its flow. The second order complementary part  $F_{comp}^2$  can be defined by

$$(\psi_1 \circ \psi_1^2)^* F = F_{Kep} + \langle F_{per} \rangle + F_{res}^1 + (F_{comp}^1 + X_{H^2} \cdot F_{Kep}) + F_{comp}^2,$$

or

$$F_{comp}^2 = (\psi_1^{2*} - id - X_{H^2})F_{Kep} + (\psi_1^{2*} - id)(\langle F_{per} \rangle + F_{res}^1 + F_{comp}^1).$$

There exists a unique Hamiltonian  $H^2$  such that

$$X_{H^2} \cdot F_{Kep} = -\tilde{F}_{comp}^1$$

over the *same* transversally Cantor set  $hd_{\gamma\nu_0, \tau}$  as the one on which the jet of  $F_{res}^1$  vanishes.  $H^2$  is of class Whitney- $C^\infty$  and thus can be extended into a  $C^\infty$ -function which is rotation-invariant. Then the second order complementary part  $F_{comp}^2$  of  $(\psi_1 \circ \psi_1^2)^* F$ , i.e. the part which actually depends on the fast angles, is of size  $O(\varepsilon^3)$ .

The induction which proves the proposition is a repeat of the same arguments.  $\blacksquare$

So, in the perturbing region, the pull-back of the Hamiltonian of the three-body problem by  $\phi^n$  may be written as the sum of the secular Hamiltonian and of the complementary part:

$$\phi^{n*} F = F_\pi^n + F_{comp}^n,$$

where  $F_{comp}^n$  is of size  $O(\varepsilon^{1+n})$ . In turn, the secular Hamiltonian, which is Pöschel-integrable, can be split into a Liouville-integrable part and a resonant part of size  $O(\varepsilon)$  and whose infinite jet vanishes along  $hd_{\gamma\nu_0, \tau}$ :

$$F_\pi^n = F_{int}^n + F_{res}^n, \quad \text{with} \quad \begin{cases} F_{int}^n = F_{Kep} + \langle F_{per} \rangle + \dots + \langle F_{comp}^{n-1} \rangle \\ j^\infty F_{res}^n|_{hd_{\gamma\nu_0, \tau}} = 0. \end{cases}$$

The Keplerian Hamiltonian  $F_{Kep}$  can thus be thought of as the zero-th order secular system, the perturbing function  $F_{per}$  as the zero-th order complementary part, and the averaged system  $F_{Kep} + \langle F_{per} \rangle$  as the integrable part of the first order secular system.

### 2.3. Non resonant elimination

In the asynchronous region, the infinite jet of the resonant part  $F_{res}^n$  can be chosen to vanish not only on a transversally Cantor set, but everywhere:

PROPOSITION 2.2. *Let  $n \geq 0$  and  $k \geq 0$  be integers. There exist*

- an open set  $\dot{A}_\varepsilon^k$  of  $\tilde{A}_\varepsilon^k$ , with fiber  $\dot{A}_\varepsilon^k(m, \overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0)$  over the base point  $(m, (\overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0))$  of  $\mathcal{M} \times \mathbf{R}^2$ ;
- for every  $(m, \overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0) \in \mathcal{M} \times \mathbf{R}^2$ , a  $C^\infty$ -Hamiltonian

$$F_a^n : \dot{A}_\varepsilon^k(m, \overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0) \rightarrow \mathbf{R}$$

and a  $C^\infty$ -symplectomorphism

$$\phi_a^n : \dot{A}_\varepsilon^k(m, \overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0) \rightarrow \phi_a^n \left( \dot{A}_\varepsilon^k(m, \overset{\vee}{\Lambda}_0, \overset{\vee}{\nu}_0) \right)$$

which is  $\varepsilon$ -close of the identity in the  $C^k$ -norm  $\|\cdot\|_k$  of proposition 2.1;

such that

- for the Liouville measure associated to the symplectic form  $\omega/\overset{\vee}{\Lambda}_0$ , the relative measure of  $\dot{A}_\varepsilon^k$  in  $\tilde{A}_\varepsilon^k$  goes to 1 when  $\varepsilon$  goes to 0;
- there exists a constant  $D > 0$  which is independent of  $\varepsilon$  such that for every  $\varepsilon > 0$

$$\frac{1}{\overset{\vee}{\nu}_0 \overset{\vee}{\Lambda}_0} \|F \circ \phi_a^n - F_a^n\|_k \leq D\varepsilon^{1+n} \quad \text{over } \dot{A}_\varepsilon^{k+n};$$

- $F_a^n$  is invariant by the Keplerian action of  $\mathbf{T}^2$  and by the diagonal action of the circle which makes the two bodies rotate simultaneously; hence it is completely integrable over  $\dot{A}_\varepsilon^{k+n}$ .

*Proof.* Recall that in the asynchronous region we have  $\overset{\vee}{\nu} = \nu_2$ . Consider the cohomological equation of the latter proof (proposition 2.1). Let  $H =$

$H_1(\lambda_1, \lambda_2) + H_2(\lambda_2)$ . Rather than solving the exact cohomological equation

$$\nu_1 \partial_{\lambda_1} H + \nu_2 \partial_{\lambda_2} H = \frac{\tilde{F}_{per}}{\check{\Lambda}_0},$$

we are going to solve the perturbed equation

$$\nu_1 \partial_{\lambda_1} H + \nu_2 \partial_{\lambda_2} H = \frac{\tilde{F}_{per}}{\check{\Lambda}_0} + \nu_2 \partial_{\lambda_2} H_1,$$

where the term  $\nu_2 \partial_{\lambda_2} H_1$  will prove small (recall that  $\nu_2 \ll \nu_1$ ). In this purpose, let

$$\begin{cases} H_1(\lambda_1, \lambda_2) = \frac{1}{\check{\Lambda}_0 \nu_1} \int_0^{\lambda_1} \left( \tilde{F}_{per} - \int_{\mathbb{S}^1} \tilde{F}_{per} d\lambda_1 \right) d\lambda_1 \\ H_2(\lambda_2) = \frac{1}{\check{\Lambda}_0 \nu_2} \int_0^{\lambda_2} \left( \int_{\mathbb{S}^1} \tilde{F}_{per} d\lambda_1 \right) d\lambda_2. \end{cases}$$

$H_2$  eliminates the harmonic components of  $F_{per}$  which do not depend on  $\lambda_1$  and  $H_1$  eliminates the harmonic components which depend on  $\lambda_1$  modulo the small error term  $\nu_2 \partial_{\lambda_2} H_1$ . Thus we have

$$-X_H \cdot F_{Kep} = X_{F_{Kep}} \cdot H = \tilde{F}_{per} + \nu_2 \check{\Lambda}_0 \partial_{\lambda_2} H_1.$$

Define the (first order) complementary part  $F_{comp}^1$  of  $\psi_1^* F$  by

$$\psi_1^* F = F_{Kep} + \langle F_{per} \rangle + F_{comp}^1,$$

or,

$$F_{comp}^1 = \int_0^1 (1-t) \psi_t^* (X_H^2 \cdot F_{Kep}) dt + \int_0^1 \psi_t^* (X_H \cdot F_{per}) dt - \nu_2 \check{\Lambda}_0 \partial_{\lambda_2} H_1.$$

The complementary part now has an additional term which satisfies

$$\begin{aligned} \frac{\left\| \nu_2 \check{\Lambda}_0 \partial_{\lambda_2} H_1 \right\|_k}{\check{\nu}_0 \check{\Lambda}_0} &\leq \frac{\left\| \tilde{F}_{per} \right\|_{k+1}}{\nu_1 \check{\Lambda}_0} = \frac{\nu_2}{\nu_1} \frac{\left\| \tilde{F}_{per} \right\|_{k+1}}{\nu_2 \check{\Lambda}_0} \\ &\leq \text{Cst } \varepsilon^2 \end{aligned}$$

in  $A_\varepsilon^{k+1}$ . Let  $\dot{A}_\varepsilon^{k+1}$  be a sufficiently large open subset of  $A_\varepsilon^{k+1}$  over which the flow of  $X_H$  is defined up to time one.



$F_{comp}^1$  satisfies the estimate

$$\left\| \frac{F_{comp}^1}{\nu_0 \Lambda_0} \right\|_k \leq \text{Cst } \varepsilon^2$$

over  $\dot{A}_\varepsilon^{k+1}$  and  $F_a^1$  can be defined by  $F_a^1 = F_{Kep} + \langle F_{per} \rangle$ .

The induction is similar to that of proposition 2.1.  $\blacksquare$

The asynchronous secular systems are Liouville-integrable normal forms of the planar three-body problem. Periodic orbits can be shown to exist using the elementary implicit function theorem in the neighborhood of every non degenerate secular singularity. By secular singularity, we mean a fixed point of the Hamiltonian  $F_a^n$  after the symplectic reduction by the symmetry of rotation and by the fast angles.

**PROPOSITION 2.3** (Short periodic orbits). *There exist integers  $k$  and  $n$  and a real number  $\varepsilon > 0$  such that every non degenerate secular singularity of  $F_a^n$  in  $\dot{A}_\varepsilon^k$  gives rise to a family of short periodic orbits in the planar three-body problem, indexed by rationally dependent mean motions.*

The proof, which is standard (cf. Moser [20]), is left to the reader. Along such periodic orbits, the elliptical elements of the ellipses in a rotating frame of reference undergo some small oscillations which vanish precisely when each body has made some given integral number of revolutions. The particular case corresponding to a large angular momentum and one of the ellipses being almost circular yields Poincaré's periodic orbits of the second kind in the asynchronous region [21, 8].

On the other hand, regular secular orbits in the asynchronous region give rise to long periodic orbits, where the ellipses oscillate or fully rotate with respect to one another. The relative motion of the ellipses is a rotation or a libration according to the homotopy class of the secular orbit on the secular sphere of constant angular momentum minus the two points corresponding to circular ellipses [8].

### 3. DIOPHANTINE INVARIANT TORI

#### 3.1. KAM theorem

For  $p \geq 1$  and  $q \geq 0$ , consider the phase space  $T^*\mathbf{T}^p \times T^*\mathbf{R}^q = \mathbf{R}_r^{p*} \times \mathbf{T}_\theta^p \times \mathbf{R}_x^{q*} \times \mathbf{R}_y^q$ , endowed with the natural coordinates  $(r, \theta, x, y)$  and symplectic form  $\omega = dr \wedge d\theta + dx \wedge dy$ . All the mappings here are of class  $C^\infty$ .

Let  $\delta > 0$ ,  $\alpha \in \mathbf{R}^p$ ,  $\beta \in \mathbf{R}^q$  and  $s \in \{\pm 1\}^q$ . Let  $\mathbf{N}_{\alpha,\beta,s}$  be the space defined by

$$\mathbf{N}_{\alpha,\beta,s} = \left\{ \begin{array}{l} N \in C^\infty(\mathbf{T}^p \times \mathbf{B}_\delta^{p+2q}, \mathbf{R}) : \\ N = \langle \alpha, r \rangle + \sum_{j=1}^q \beta_j (x_j^2 + s_j y_j^2) + \langle A_1(\theta), r \otimes r \rangle \\ \quad + \langle A_2(\theta), r \otimes z \rangle + O_3(r, z), \\ A_1 \in C^\infty(\mathbf{T}^p, \otimes^2 \mathbf{R}^{n*}), \\ A_2 \in C^\infty(\mathbf{T}^p, \mathbf{R}^p \otimes T^*(\mathbf{R}^q)^*) \end{array} \right\},$$

where  $\mathbf{B}_\delta^{n+2q}$  is the Euclidean ball of  $\mathbf{R}^{n+2q}$  centered at the origin and of radius  $\delta$ , and where  $z = (x, y)$ ;  $\mathbf{N}_{\alpha,\beta,s}$  is a space of first order normal forms with fixed frequency: for the flow of a Hamiltonian of  $\mathbf{N}_{\alpha,\beta,s}$ , the isotropic torus  $\mathbf{T}^p \times \{0\} \times \{0\}$  is invariant  $\alpha$ -quasiperiodic and its normal dynamics is elliptic, hyperbolic, or a mixture of both cases according to the signs of  $s$ , with normal frequency vector  $\beta$ .

Let  $\gamma > 0$  and  $\tau > p - 1$  be real numbers and  $|\cdot|$  be the  $l_2$ -norm. Let  $HD_{\gamma,\tau}(p, q, s)$  be the set

$$\left\{ \begin{array}{l} (\alpha, \beta) \in \mathbf{R}^p \times \mathbf{R}^q : \forall k \in \mathbf{Z}^p \forall i, j \in \mathbf{Z}^q \\ \text{if } k \neq 0, \quad |k \cdot \alpha| \geq \frac{\gamma}{|k|^\tau}, \\ \text{if } s_j = 1, \quad |k \cdot \alpha + 2\beta_j| \geq \frac{\gamma}{(1 + |k|)^\tau}, \\ \text{if } i \neq j, \quad s_i, s_j = 1, \quad |k \cdot \alpha \pm \beta_i \pm \beta_j| \geq \frac{\gamma}{(1 + |k|)^\tau}, \\ \text{if } s_i, s_j = 1, \quad |\beta_i \pm \beta_j| \geq \gamma \end{array} \right\},$$

of frequency vectors satisfying some homogeneous diophantine conditions.

**THEOREM 3.1** (Herman [13]). *There exists a Lie group  $\mathbf{G}$  of symplectomorphisms such that whenever  $(\alpha, \beta) \in HD_{\gamma,\tau}(p, q, s)$  for some constants  $\gamma$  and  $\tau$ , the map*

$$\begin{aligned} \Phi_{\alpha,\beta} : \mathbf{N}_{\alpha,\beta,s} \times \mathbf{G} \times \mathbf{R}^p \times \mathbf{R}^q &\longrightarrow C^\infty(\mathbf{T}^p \times \mathbf{B}_\delta^{p+2q}, \mathbf{R}) \\ (N, G, \hat{\alpha}, \hat{\beta}) &\longmapsto G^*N + \langle \hat{\alpha}, r \rangle + \sum_1^q \hat{\beta}_j (x_j^2 \pm y_j^2) \end{aligned}$$

is a tame local diffeomorphism in the neighborhood of  $(N, id, 0, 0)$ .

Furthermore, there exist integers  $k \geq 1$  and  $n \geq 2$  such that if  $(\alpha, \beta) \in HD_{\varepsilon\gamma,\tau}(p, q, s)$  and if  $\varepsilon$  is small enough, the local image of  $\Phi_{\alpha,\beta}$  contains some  $C^k$ -semi-ball of radius  $Cst \varepsilon^{1+n}$ . The constant  $Cst$  depends continuously on

$$N \in \bigcup_{(\alpha,\beta) \in CDH_{\varepsilon\gamma,\tau}(p,q,s)} \mathbf{N}_{\alpha,\beta,s},$$

but is independent from  $(\alpha, \beta) \in CDH_{\varepsilon\gamma,\tau}(p, q, s)$ .

The first part of the theorem is proved by Herman [13] using Hamilton's inverse function theorem in tame Fréchet spaces [11, 2]. The second part is a consequence of Hamilton's proof of this inverse function theorem. M. Herman has actually proved that the sharp exponent is  $1 + n = 2 + \zeta$ ,  $\zeta > 0$ .

Now we show how to apply this theorem to the existence of invariant tori for some class of Hamiltonians which are completely integrable on a transversally Cantor set. In the next subsection (§ 3.2) we will show that the Hamiltonian  $F$  of the planar three-body problem in the neighborhood of regular or non degenerate singular secular invariant tori falls into this category. Neighborhoods of secular singularities and of regular secular orbits respectively correspond to  $(p, q) = (2, 1)$  and  $(3, 0)$ . It will be fundamental that we may choose the order  $n$  of the secular system as large as we want, so that the perturbation has an arbitrarily large order of smallness compared to the size of the terms which break down the proper degeneracy of the Keplerian part.

**THEOREM 3.2.** *Let  $N_h, R_h$  and  $P_h$  be  $C^\infty$ -Hamiltonians on  $\mathbf{T}^p \times \mathbf{B}_\delta^{p+2q}$  depending smoothly on some parameter  $h \in \mathbf{B}_\varepsilon^t$ , with  $\varepsilon > 0$  and  $t = p + q$ . Assume the following properties hold for large enough integers  $n$  and  $k$ :*

- *For every  $h \in \mathbf{B}_1^t$ , the torus  $\mathbf{T}_0^p = \mathbf{T}^p \times 0$  is an invariant quasiperiodic torus of  $N_h$  and there exist a frequency vector  $(\alpha_0(h), \beta_0(h))$  and signs  $s \in \{\pm 1\}^q$  such that*

$$N_h \in \mathbf{N}_{\alpha_0(h), \beta_0(h), s}.$$

- *The frequency map  $h \mapsto (\alpha_0(h), \beta_0(h))$  is a local diffeomorphism such that for every  $\varepsilon > 0$  and  $h \in \mathbf{B}_\varepsilon^t$  its image contains a  $t$ -ball of radius  $\text{Cst} \varepsilon$  for some constant  $\text{Cst}$ .*

- *Whenever  $(\alpha_0(h), \beta_0(h)) \in HD_{\varepsilon\gamma, \tau}(p, q, s)$  the infinite jet of the resonant part  $R_h$  vanishes on  $\mathbf{T}_0^p$ :*

$$j^\infty R_h|_{\mathbf{T}_0^p} = 0.$$

- *The perturbation  $P_h$  satisfies*

$$\|P_h\|_k \leq \text{Cst} \varepsilon^{1+n}$$

for some  $C^k$ -norm  $\|\cdot\|_k$ .

*There exist a real number  $\varepsilon > 0$  and a  $C^\infty$ -map  $h \in \mathbf{B}_\varepsilon^t \mapsto (\alpha_1(h), \beta_1(h))$  which is  $C^k$ -close to  $(\alpha_0, \beta_0)$ , such that whenever*

$$(\alpha_1(h), \beta_1(h)) \in HD_{\varepsilon\gamma, \tau}(p, q, s)$$

the Hamiltonian

$$F_h = N_h + R_h + P_h$$

has an invariant isotropic  $p$ -torus with frequency vector  $(\alpha_1(h), \beta_1(h))$ .

We are going to prove of this theorem, assuming theorem 3.1. Many other theorems are described and proved by Herman [13] in a unified way, using Hamilton's inverse function theorem between some well chosen functional tame Fréchet space. In particular, the non degeneracy hypothesis on the frequency map  $(\alpha_0, \beta_0)$  may be weakened by only requiring that the map is non-planar in the sense of Pyartli [24]. However, in the planar three-body problem there are enough parameters, so we do not need this refinement. Moreover, under this weak hypothesis, the end of the coming proof would demand to be modified.

*Proof.* First, assume that  $q = 0$  and that the resonant part  $R$  is equal to zero. In this particular case, the proof of the result will not require the non degeneracy hypothesis on the frequency map. Let  $h \in \mathbf{B}_\varepsilon^t$  and  $\alpha \in HD_{\varepsilon\gamma, \tau}$ . The Hamiltonian

$$\hat{N}_h = N_h + \langle \alpha - \alpha_0(h), r \rangle$$

is in  $\mathbf{N}_\alpha$ . Theorem 3.1 asserts that if  $\alpha$  is close enough to  $\alpha_0(h)$  and if  $\varepsilon$  is small enough,  $F_h = N_h + P_h$  is in the local image of  $\Phi_\alpha$  in the neighborhood of  $(\hat{N}_h, id, 0)$  and hence can be written

$$F_h = G^* (N'_h + \langle \alpha - \alpha_0(h), r \rangle) + \langle \hat{\alpha}, r \rangle$$

for some Hamiltonian  $N'_h$  close to  $N_h$  in  $\mathbf{N}_{\alpha_0(h)}$ , some symplectomorphism  $G \in \mathbf{G}$  close to the identity and some small correction  $\hat{\alpha} \in \mathbf{R}^p$  in the frequencies. This defines a map  $\alpha \mapsto \hat{\alpha}$  which is  $C^\infty$ -Whitney and which can thus be smoothly extended to possibly non diophantine vectors  $\alpha$ . Since

$$\left. \frac{\partial \hat{\alpha}}{\partial \alpha} \right|_{G=id} = -id_{\mathbf{R}^p},$$

if  $n$  is large enough and  $\varepsilon$  small enough, this extension is a local diffeomorphism in the neighborhood of  $\alpha_0(h)$ , whose local image contains some given ball of size  $\text{Cst } \varepsilon^{1+n}$ . Hence for every  $h \in \mathbf{B}_\varepsilon^t$  there exists a unique  $\alpha(h)$  such that  $\hat{\alpha}(\alpha(h)) = 0$ . Whenever  $\alpha(h) \in HD_{\varepsilon\gamma, \tau}$ , this means that  $F_h$  is conjugate to  $N'_h + \langle \alpha(h) - \alpha_0(h), r \rangle$ :

$$F_h = G^* (N'_h + \langle \alpha(h) - \alpha_0(h), r \rangle).$$

Hence  $F_h$  has an invariant isotropic torus, namely the pull-back by  $G$  of the zero section  $\mathbf{T}_0^p$ . The frequency vector, a conjugacy invariant, is  $\alpha_1(h) := \alpha(h)$ , which can be smoothly extended by Whitney's extension theorem.

Now we still assume that  $q = 0$ , but we generalize the preceding proof to the case where the resonant part  $R$  is non trivial. It is not enough to apply  $\Phi_\alpha$  to  $N_h + R_h$  because rather than obtaining a transversally Cantor set of invariant tori, in general we would only get the intersection of two transversally Cantor sets of invariant tori, which might well be of zero measure.

Let  $H \in \mathbf{B}_{\text{CSt}\varepsilon^{1+n}}^t$  be a parameter shift to be determined. Let  $h \in \mathbf{B}_\varepsilon^t$  such that  $\alpha_0(h + H) \in HD_{\varepsilon\gamma, \tau}$ . Thus we have

$$j^\infty R_{h+H}|_{\mathbf{T}_0^p} = 0.$$

Hence for every  $\alpha \in \mathbf{R}^p$ ,

$$\hat{N}_h = N_h + \langle \alpha - \alpha_0(h), r \rangle + R_{h+H} \in \mathbf{N}_\alpha.$$

Now, the Hamiltonian  $F_h$  can be artificially split as

$$F_h = [N_h + R_{h+H}] + [P_h + R_h - R_{h+H}].$$

The second bracket is of  $C^k$ -size  $O(\varepsilon^{1+n})$ . Hence, by theorem 3.1, if  $\alpha$  is in  $HD_{\varepsilon\gamma, \tau}$  and if  $n$  is large enough and  $\varepsilon$  small enough, we have

$$F_h = G^* (N'_h + \langle \alpha - \alpha_0(h), r \rangle + R_{h+H}) + \langle \hat{\alpha}, r \rangle$$

for some Hamiltonian  $N'_h \in \mathbf{N}_{\alpha_0(h)}$ , some symplectomorphism  $G$  and some correction  $\hat{\alpha}$  in the frequencies. The map  $\alpha \mapsto \hat{\alpha}$  is  $C^\infty$ -Whitney and can thus be extended into a smooth map, which actually is local diffeomorphism for small  $\varepsilon$ 's. So there is a unique  $\alpha(h, H)$  such that  $\hat{\alpha}(\alpha(h, H)) = 0$ . We have assumed that  $\alpha_0(h + H) \in HD_{\varepsilon\gamma, \tau}$ . If moreover

$$\alpha(h, H) \in HD_{\varepsilon\gamma, \tau},$$

$F_h$  is the pull-back by  $G$  of

$$N'_h + \langle \alpha - \alpha_0(h), r \rangle + R_{h+H}$$

and thus has an invariant torus with frequency vector  $\alpha(h, H)$ . Assume that there is an  $H$  such that the two frequency vectors agree:

$$\alpha_0(h + H) = \alpha(h, H).$$

Then we smoothly extend the function  $h \mapsto H$  and set  $\alpha_1(h) = \alpha_0(h + H(h))$ . So we want to prove that such a function  $H(h)$  exists. For  $h$ 's

such that  $\alpha_0(h+H)$  is in  $HD_{\varepsilon\gamma,\tau}$ , the infinite jet of  $R_{h+H}$  vanishes along  $\mathbf{T}_0^p$  and so  $\alpha(h, H)$  is a flat function of  $H$ . Hence, by the non degeneracy assumption on  $\alpha_0$  and by the implicit function theorem, the equation  $\alpha_0(h+H) = \alpha(h, H)$  indeed has a unique solution  $H$ , provided  $\varepsilon$  is small enough. Furthermore  $H$  is in a ball of radius  $O(\varepsilon^{1+n})$  as assumed. This completes the proof in the case  $q = 0$ .

The generalization to  $q \geq 1$  is straightforward.  $\blacksquare$

### 3.2. Perturbation of non collision secular orbits

As a consequence of theorem 3.2, of the construction of the secular systems, of the estimate of proposition 2.1 and of the computation of the averaged Hamiltonian which was carried out in appendix C, we are going to prove the existence of some invariant two- and three-dimensional diophantine tori in the planar three-body problem in a rotating frame of reference. In a Galilean frame of reference, i.e. before the symplectic reduction by the symmetry of rotations, these quasiperiodic motions have one additional frequency, namely the angular speed of the simultaneous rotation of the three ellipses.

Lagrangian tori correspond to regular secular orbits, whereas lower-dimensional isotropic tori correspond to secular singularities after the symplectic reduction by the symmetry of rotations and by the fast angles  $\lambda_1$  and  $\lambda_2$ . In this paper, as an example, we will focus on the well-known secular singularities where either the inner or the outer ellipses are almost circular. When the angular momentum is large enough, these singularities were already mentioned by Tisserand in his *Traité de Mécanique Céleste* and used by Poincaré to find his period orbits of the second kind [21].

Lieberman's paper [18] focuses on the singularity  $e_1 \simeq 0$  (first part of lemma 3.1) in the particular case of the lunar problem, where the saddle-node bifurcation mentioned in the proof of lemma 3.1 is not visible since  $b$  goes to 0.

The angular momentum  $C$  equals

$$C = \pm\Lambda_1\sqrt{1-e_1^2} \pm \Lambda_2\sqrt{1-e_2^2},$$

where sign of  $b$  is positive if the corresponding body revolves counterclockwise and negative otherwise. Let

$$a = \frac{a_1}{a_2}, \quad b = \pm\frac{\Lambda_1}{\Lambda_2} \quad \text{and} \quad c = b\sqrt{1-e_1^2} + \sqrt{1-e_2^2},$$

where the sign is positive or negative according to whether the two fictitious bodies turn in the same direction or not. The three functions  $a$ ,  $b$  and  $c$  are constant after the symplectic reduction by the symmetry of rotation

and by the fast angles  $\lambda_1$  and  $\lambda_2$ . In the following lemma, we consider the system reduced by the rotations only:

LEMMA 3.1. *If  $c \neq 2b$  and if  $a$  and  $\varepsilon$  are small enough, there exist some variables  $(x, y)$  which are close to  $(\xi_1, \eta_1)$ , such that  $(\Lambda_1, \lambda_1, \Lambda_2, \lambda_2, x, y)$  are local symplectic coordinates in the neighborhood of  $(\xi_1, \eta_1) = (0, 0)$ , and the Liouville-integrable part of the  $n$ -th order secular system  $F_\pi^n$  (cf. the end of § 2.3) is*

$$F_{int}^n = F_{Kep}(\Lambda_1, \Lambda_2) - \frac{\mu_1 m_2 a^2}{4} \frac{1}{a_2 (c-b)^3} \times$$

$$\left[ 1 + \frac{3}{2} \left( \frac{c-2b}{c-b} \right) (1 + O^\circ(a, \varepsilon)) \frac{x^2 + y^2}{\Lambda_1} \right] + O_3(x, y),$$

where the term  $O^\circ$  only depends on the semi major axes, the masses and the angular momentum.

Similarly, if

$$c \neq \frac{1}{3} \left( 2 + \sqrt{1 + 15b^2} \right)$$

and if  $a$  and  $\varepsilon$  are small enough, there exist some variables  $(X, Y)$  which are close to  $(\xi_2, \eta_2)$ , such that  $(\Lambda_1, \lambda_1, \Lambda_2, \lambda_2, X, Y)$  are local symplectic coordinates in the neighborhood of  $(\xi_2, \eta_2) = (0, 0)$ , and the Liouville-integrable part of the  $n$ -th order secular system  $F_\pi^n$  is

$$F_{int}^n = F_{Kep}(\Lambda_1, \Lambda_2) - \frac{\mu_1 m_2 a^2}{8a_2 b^2} (5b^2 - 3c^2 + 6c - 3) \times$$

$$\left[ 1 + \frac{3}{2} \left( \frac{5b^2 - 3c^2 + 4c - 1}{5b^2 - 3c^2 + 6c - 3} \right) (1 + O^\circ(a, \varepsilon)) \frac{X^2 + Y^2}{\Lambda_1} \right] + O_3(X, Y),$$

where the term  $O^\circ$  only depends on the semi major axes, the masses and the angular momentum.

*Proof.* From the last formula of appendix C, the first non constant term of the averaged system  $\langle F_{per} \rangle$  is

$$\mu_1 m_2 f_0 a^2 / (8a_2), \quad \text{where } f_0 = -\frac{2 + 3e_1^2}{(c - b\sqrt{1 - e_1^2})^3}.$$

Its Taylor expansion in the neighborhood of  $e_1 = 0$ ,

$$f_0 = -\frac{2}{(c-b)^3} \left( 1 + \frac{3}{2} \frac{c-2b}{c-b} e_1^2 \right) + O(e_1^4),$$

reminds us of the facts that  $c = b$  corresponds to a degenerate eccentricity-one outer ellipse (which is impossible in the perturbing region  $\tilde{\Pi}_\varepsilon^k$ ) and that, when  $a$  goes to 0,  $c = 2b$  is the limit value of the saddle-node bifurcation which the singularity  $e_1 = 0$  takes part in [6, 8]. Let's keep away from this difficulty by choosing  $c \neq 2b$ . Since

$$\xi_1 + i\eta_1 = \sqrt{2\Lambda_1} \sqrt{1 - \sqrt{1 - e_1^2} e^{-ig_1}},$$

in symplectic coordinates  $f_0$  equals

$$f_0 = -\frac{2}{(c-b)^3} \left( 1 + \frac{3}{2} \left( \frac{c-2b}{c-b} \right) \frac{\xi_1^2 + \eta_1^2}{\Lambda_1} \right) + O((\xi_1^2 + \eta_1^2)^2).$$

So the Liouville-integrable part of  $F_\pi^n$  is

$$\begin{aligned} F_{int}^n = F_{Kep}(\Lambda_1, \Lambda_2) - \frac{\mu_1 m_2 a^2}{4} \frac{1}{a_2 (c-b)^3} \left( 1 + \frac{3}{2} \left( \frac{c-2b}{c-b} \right) \frac{\xi_1^2 + \eta_1^2}{\Lambda_1} \right) \\ + O((\xi_1^2 + \eta_1^2)^2) + O\left(\frac{a^3}{a_2}\right) + O(\varepsilon). \end{aligned}$$

The terms  $O(a^3/a_2)$  and  $O(\varepsilon)$  a priori contain some terms which are linear in  $(\xi_1, \eta_1)$ . According to the implicit function theorem, the singularity  $e_1 = 0$  of  $f_0$  persists for  $F_{int}^n$  if  $a$  and  $\varepsilon$  are small enough. Hence a translation in the  $(\xi_1, \eta_1)$ -variables suffices to get rid of the linear part; then by a rotation in the plane of these variables we can diagonalize the part which is quadratic in the  $(\xi_1, \eta_1)$ -variables; eventually, rescaling these two variables with inverse factors for one variable and the other lets us straighten the energy levels from ellipses into circles.

In the symplectic planes which are parametrized by  $(\xi_1, \eta_1)$ , these transformations are symplectic and close to the identity. Let  $(x, y)$  be the new variables. The symplectic diffeomorphism  $(\xi_1, \eta_1) \mapsto (x, y)$  can be lifted to a symplectic diffeomorphism

$$(\Lambda, \lambda, \xi_1, \eta_1) \mapsto (r = \Lambda, \theta, x, y)$$

of the total space without modifying the slow actions, with  $\Lambda = (\Lambda_1, \Lambda_2)$ ,  $\lambda = (\lambda_1, \lambda_2)$  and  $\theta = (\theta_1, \theta_2)$ . Indeed, let  $S_\Lambda(\xi_1, y)$  be a generating function of the diffeomorphism  $(\xi_1, \eta_1) \mapsto (x, y)$ , i.e. a primitive of the closed 1-form

$$\eta_1 d\xi_1 + x dy$$



when  $\Lambda$  is fixed. Then the function  $S(\Lambda, \theta, \xi_1, y) = S_\Lambda(\xi_1, y)$  generates the diffeomorphism which we were looking for, and which is defined by

$$dS = (\lambda - \theta) \cdot d\Lambda + (r - \Lambda) \cdot d\theta + \eta_1 d\xi_1 + xdy,$$

or,

$$\begin{cases} \lambda = \theta + \partial_\Lambda S \\ r = \Lambda \\ \eta_1 = \partial_{\xi_1} S \\ x = \partial_y S. \end{cases}$$

In the new variables,  $F_{int}^n$  has the wanted expression.

The proof of the second part of the lemma is a repeat of the same kind of computations.  $\blacksquare$

**THEOREM 3.3.** *There are integers  $k \geq 1$  and  $n \geq 1$  and a real number  $\varepsilon > 0$  such that inside the perturbing region  $\dot{\Pi}_\varepsilon^{k+n(\tau+4)}$ :*

- *a positive measure of normally elliptic quasiperiodic isotropic tori of  $F_\pi^n$  for which the inner or outer ellipses are almost circular (cf. lemma 3.1)*
- *and a positive measure of quasiperiodic Lagrangian tori of  $F_\pi^n$*

*survive in the dynamics of the planar three-body problem.*

*Proof.* The first part of the theorem is a straightforward consequence of proposition 2.1, lemma 3.1 and theorem 3.2 with  $p = 2$  and  $q = 1$  and with

$$F_h = F \circ \phi^n, N_h = F_{int}^n, R_h = F_{res}^n, P_h = F \circ \phi^n - F_\pi^n$$

at any non degenerate secular-invariant 2-torus of parameters

$$h = (a_1, a_2, c).$$

(The definition of  $\varepsilon$  is not quite the same in both settings, though. In theorem 3.2,  $\varepsilon$  is the size of the smallest frequency whereas here the secular frequency is some power of  $\varepsilon$ .) In particular, it is easy to check that, except for a set of parameters of finite measure, the frequency map is a local diffeomorphism when the frequency vector

$$\left( \frac{\sqrt{M_1}}{a_1^{3/2}}, \frac{\sqrt{M_2}}{a_2^{3/2}}, \frac{3}{8} \mu_1 m_2 \frac{a_1^2}{a_2^3} \frac{c - 2b}{(c - b)^4} (1 + O^\sigma(a, \varepsilon)) \right)$$

is seen, for instance, as a function of the semi major axes  $a_j$  and of the angular momentum  $c$ . Rather than adjusting the semi major axes, it would

be possible to adjust the masses of the bodies. Note that once we know that the secular system  $F_\pi^n$  has an elliptical or hyperbolic torus for  $a$  and  $\varepsilon$  small enough, we do not need to let  $a$  go to zero anymore in order to apply theorem 3.2.

The second part of the theorem is similar to the first one, with  $p = 3$  and  $q = 0$  in theorem 3.2. For instance in the neighborhood of each of the two singularities of lemma 3.1—but not at the singularity proper—, there exist some coordinates which are close to the Delaunay variables and which are action-angle variables for the  $n$ -th order secular system  $F_\pi^n$ . This is because the first term of the expansion of  $\langle F_{per} \rangle$  (cf. appendix C) does not depend on the difference of the arguments of the pericenters. Along the regular invariant 3-tori, the two Keplerian ellipses slowly rotate with respect to one another, with almost constant eccentricities. The details are left to the reader. ■

This theorem could be generalized in several ways:

1. When  $c = b$  or  $c = \sqrt{2 + 15b^2}/3$  (cf. lemma 3.1), the singularities looked at here are degenerate and take part in what in general is a saddle-node bifurcation [6, 8]. The normal frequency of the unperturbed invariant torus of  $F_\pi^n$  vanishes, and in general it is hopeless to try to perturb such tori [13]. However, under appropriate transversality conditions, parabolic tori persist and furthermore the whole saddle-node bifurcation persists, with all lower dimensional invariant tori parametrized by pertinent transversally Cantor sets [12].

2. Giving the complete picture of all 2- or 3-dimensional tori arising from secular orbits would actually require to first describe the bifurcation diagram of the global secular system. This diagram will be described in a forthcoming paper [8] when the semi major axes ratio is small enough. It will be proved that in general the secular systems have either 2, 4 or 6 non degenerate singularities, according to the values of parameters. These singularities do not necessarily belong to the submanifold of aligned ellipses, contrary to the singularities which were known before. In order to prove the existence of corresponding invariant 2-tori in the planar three-body problem, we need to check that these singularities are non degenerate. Unfortunately, since the first terms of the averaged system sometimes are degenerate, the non degenerate leading terms of  $F_\pi^n$  may come from higher order averaging [6, 8]. Furthermore, the hyperbolic tori may be the source of some Hamiltonian instability [9].

3. Another possible generalization of theorem 3.3 would be to consider the spatial three-body problem. Although the spatial secular systems are not completely integrable, when we consider the expansion of the averaged system in the powers of the semi major axis ratio it turns out that the first term is integrable because it does not depend on the argument of the peri-

center of the outer ellipse. We would need first to apply a theorem similar to theorem 3.2 in order to get some Lagrangian or lower dimensional tori for the  $n$ -th order secular system  $F_\pi^n$  reduced by the fast angles, and then to apply the same theorem to get similar tori in the full spatial three-body problem. We cannot apply the theorem similar to theorem 3.2 only once: because of the proper degeneracy of the Keplerian part, the frequency vector only satisfies homogeneous diophantine conditions of constants  $(\varepsilon\gamma, \tau)$ ; so, according to theorem 3.1, the perturbed Hamiltonian, which is only  $\varepsilon^2$ -close to the integrable part of the secular system, may be out of the local image of the tame diffeomorphism  $\Phi_{\alpha,\beta}$ . Note that we would need the full power of Herman's theorem 3.1 to know that the lower dimensional invariant tori of  $F_\pi^n$  are normally non degenerate. Most KAM theorems do not provide this property of the perturbed tori and thus cannot be applied twice in a row.

### 3.3. Perturbation of almost-collision orbits

In certain conditions, especially when the angular momentum is small enough and when the energy is sufficiently negative, the conservation of these two first integrals does not prevent the two inner bodies from colliding [16].

After the symplectic reduction by the fast angles, the averaged system  $\langle F_{per} \rangle$  is a priori defined on the space of pairs of oriented ellipses with fixed foci and semi major axes, which do not intersect one another. This space can be compactified by adding degenerate eccentricity-one ellipses at infinity. Such an ellipse corresponds to a collision orbit where the body goes back and forth along a line segment between its pericenter and its apocenter [3, 25].

A striking feature of the averaged system is that it extends to an analytic function where the inner ellipse is degenerate [7]. On the other hand, the non-averaged perturbing function  $F_{per}$  extends to a continuous function which, unfortunately, is not even differentiable. So at first sight the extension of the averaged system itself appears to be dynamically irrelevant. But in [7] it is proved that the extension of the averaged system actually is the averaged system associated to the *regularized* problem, up to some diffeomorphism in the parameter space. More precisely, if  $\mathcal{F}_{per}$  denotes the regularized perturbing function of the planar three-body problem and  $\langle \mathcal{F}_{per} \rangle$  its average with respect to the fast angles of the regularized Keplerian dynamics (cf. [7]), the following theorem holds:

**THEOREM 3.4** ([7]). *After reduction by the symmetry of rotation and by the initial and regularized Keplerian actions of  $\mathbf{T}^2$ , once the masses  $m_0$ ,  $m_1$  and  $m_2$ , the semi major axes  $a_1$  and  $a_2$ , the energy  $f$  of the regularized energy level and the angular momentum  $C$  have been fixed, there exists*

a fictitious value  $m'_2$  of the outer mass such that the averaged regularized system  $\langle \mathcal{F}_{per} \rangle$  is  $\mathbf{R}$ -analytically orbitally conjugate to the averaged initial system  $\langle F_{per} \rangle$  in which  $m'_2$  substitutes for  $m_2$ .

Define the extended perturbing and asynchronous regions  $E\Pi_\varepsilon^k$  and  $EA_\varepsilon^k$  by dropping the condition assuming that the inner eccentricity has an upper bound in the definition of the perturbing and asynchronous regions  $\Pi_\varepsilon^k$  and  $A_\varepsilon^k$  (§ 2.1).

Theorem 3.4 is a key step towards proving the following result, which asserts the existence of quasiperiodic invariant “punctured tori” in the planar three-body problem:

**THEOREM 3.5.** *If  $k$  is large enough and  $\varepsilon > 0$  small enough, there is a transversally Cantor set in the extended asynchronous region  $EA_\varepsilon^k$ , which has positive Liouville measure and which consists of diophantine quasiperiodic punctured tori of  $F$ , such that along its trajectories the two inner bodies get arbitrarily close to one another an infinite number of times without ever colliding.*

*Proof.* The proof consists of four steps :

1. regularize double inner collisions, i.e. build a Hamiltonian  $\mathcal{F}$  which extends to an analytic Hamiltonian at collisions  $Q_1 = 0$ , and which is orbitally conjugate to  $F$  on some given manifold of constant energy;
2. build the secular systems  $\mathcal{F}_\pi^n$  of the regularized problem and their asynchronous analogue  $\mathcal{F}_a^n$ ;
3. apply theorem 3.2 to find a positive measure of invariant tori on the regularized energy surface for the regularized dynamics;
4. check some transversality condition which ensures that almost all orbits on these tori never go through collisions.

The first step was described in [7]: for any given energy manifold of equation

$$F_{Kep} = -f, \quad f > 0,$$

the regularized Hamiltonian  $\mathcal{F}$  is defined as

$$\mathcal{F} = \text{L.C.}^*(|Q_1|(F + f)),$$

where L.C. is the two-sheeted Levi-Civita covering.<sup>3</sup> The regularized Keplerian part and the perturbing function of  $\mathcal{F}$  are

$$\mathcal{F}_{Kep} = \text{L.C.}^*(|Q_1|(F_{Kep} + f)) \quad \text{and} \quad \mathcal{F}_{per} = \text{L.C.}^*(|Q_1|(F_{per})).$$

<sup>3</sup>As Alain Albouy has noticed, Goursat [10] introduced the transformation previously to Levi-Civita.

The regularized Keplerian part has some action-angle coordinates

$$(\mathcal{L}_1, \delta_1, \mathcal{G}_1, \gamma_1, \Lambda_2, \delta_2, \xi_2, \eta_2)$$

such that on the energy manifold  $\mathcal{F}_{Kep} = 0$  the variables  $(\mathcal{L}_1, \delta_1, \mathcal{G}_1, \gamma_1)$  agree with the Delaunay variables  $(L_1, u_1, G_1, g_1)$  where the eccentric anomaly  $u_1$  substitutes for the mean anomaly  $l_1$ .

The second step is very similar to building the secular systems of the non-regularized problem. Indeed, if we rescale the action-angle variables  $(\mathcal{L}_1, \delta_1, \mathcal{G}_1, \gamma_1, \Lambda_2, \delta_2, \xi_2, \eta_2)$  in the same way as in the proof of theorem 2.1—take  $\check{\Lambda}_0 = \min(\mathcal{L}_1, \Lambda_2)$ ,  $\check{\mathcal{L}}_1 = \check{\Lambda}_0 \tilde{\mathcal{L}}_1$ , etc.—, since  $\text{L.C.}^*|Q_1|$  is a  $C^k$ -bounded function, the regularized perturbing function satisfies the same estimates as those proved in appendix A for the initial perturbing function, except that we do not need to suppose that the eccentricity of the inner ellipses is upper bounded. As a consequence, the secular systems  $\mathcal{F}_\pi^n$  and  $\mathcal{F}_a^n$  of the regularized problem satisfy the same estimates as those in theorems 2.1 and 2.2, over the extended perturbing and asynchronous regions.

Thanks to theorem 3.4, we know at once what the averaged regularized dynamics is. In particular we know what the frequency vector is for free. Hence step three too is very similar to the proof of theorem 3.3. Theorem 3.2 yields the existence of a positive measure of quasiperiodic Lagrangian 3-tori which are perturbations of invariant regular 3-tori of the secular systems  $\mathcal{F}_\pi^n$  or  $\mathcal{F}_a^n$ . Using the semi major axes, the angular momentum and the masses of the bodies to adjust the frequency vector and the energy, it is possible to get a positive measure of such perturbed tori on each regularized energy manifold. Such tori correspond to the compactification of invariant 3-manifolds of the non-regularized problem.

In the extended asynchronous region  $EA_\varepsilon^k$ , both the secular systems  $\mathcal{F}_a^n$  and the conjugacy diffeomorphisms  $\phi_a^n$  such that  $\|\phi_a^{n*}\mathcal{F} - \mathcal{F}_a^n\| = O(\varepsilon^{1+n})$  can be computed by quadrature of trigonometric polynomials, which makes the transversality condition of step 4 easy to check. Lemma 3.2 below yields the result, because by choosing  $n$  large enough the perturbation  $\mathcal{F}_{comp}^n$  can be made so small that the result of the lemma holds for  $\phi_a^{n*}\mathcal{F}$ . ■

LEMMA 3.2. *Consider the asynchronous secular system  $\mathcal{F}_a^n$  reduced by the symmetry of rotations. For any  $n \geq 1$ , for every  $k$  large enough and  $\varepsilon > 0$  small enough, there is an invariant open subset of the extended asynchronous region  $EA_\varepsilon^k$ , which has positive measure, whose closure consists of invariant tori, and whose orbits do not meet the collision set.*

Note that the collision set of  $\mathcal{F}_a^n$  is the set where the *physical* mean longitude  $\phi_a^{n*}\lambda_1$  is equal to the *physical* argument of pericenter  $\phi_a^{n*}g_1$ ,

which means that the inner bodies are at their pericenters, and where the *physical* eccentricity  $\phi_a^{n*}e_1$  is equal to one.

*Proof.* We have

$$\mathcal{F}_a^n = \phi_a^{n*} \text{L.C.}^*(|Q_1|(F + f)) + O(\varepsilon^{1+n}).$$

We want to check that almost all orbits of  $\mathcal{F}_a^n$  do not meet the collision set.

The first term of the expansion of  $\langle F_{per} \rangle$  in the powers of the semi major axes ratio does not depend on the difference  $g$  of the arguments of the pericenters. Thus, up to higher order terms when  $a$  is small, in the averaged dynamics ellipses rotate relatively to one another with constant eccentricities  $e_1$  and  $e_2$ . From theorem 3.4, this is also the case for the averaged regularized system  $\langle \mathcal{F}_{per} \rangle$ , and hence for the  $n$ -th order secular system  $\mathcal{F}_a^n$ , up to order  $a^3$ .

Variables  $(\mathcal{L}_1, \delta_1, \Lambda_2, \delta_2, \xi_2, \eta_2)$  form a coordinate system almost everywhere for the system reduced by the symmetry of rotations [8]. Invariant Lagrangian tori of  $\mathcal{F}_a^n$  are parametrized by  $(\delta_1, \delta_2, g)$ . For such a torus, consider the Poincaré section defined by  $\phi_a^{n*}(\lambda_1 - g_1) = 0$ . The section is a 2-torus parametrized by  $(\delta_2, g)$ . The first-return map is a rotation which is  $\varepsilon$ -close to leaving  $g$  invariant.

We now need to compute an approximation of the conjugacy diffeomorphism  $\phi_a^n$ . Adapting the proof of proposition 2.2 to the case of the regularized problem,  $\phi_a^n$  is obtained as the composition of some time-one map  $\psi_1$  of an autonomous Hamiltonian  $\mathcal{H} = \mathcal{H}_1(\delta_1, \delta_2) + \mathcal{H}_2(\delta_2)$  with some other time-one maps which are closer to the identity than  $\psi_1$  when  $\varepsilon$  is small. Besides, if  $\nu_2/\nu_1$  is small, the Hamiltonian  $\mathcal{H}_1$  is small compared to  $\mathcal{H}_2$  (cf. the proof of proposition 2.2).

A straightforward computation shows that the regularized Keplerian frequency vector is  $a_1(\nu_1, \nu_2)$ . So  $\mathcal{H}_2$  is defined by

$$\mathcal{H}_2 = \frac{1}{4\pi^2 a_1 \nu_2 \Lambda_0} \int_0^{\delta_2} \left( \int_{\mathbf{S}^1} |Q_1|_{F_{per}} d\delta_1 - \frac{1}{4\pi^2} \int_{\mathbf{T}^2} |Q_1|_{F_{per}} d\delta_1 d\delta_2 \right) d\delta_2.$$

Its analogue  $H_2$  for the non-regularized problem is

$$H_2 = \frac{1}{\nu_2 \Lambda_0} \int_0^{\lambda_2} \left( \int_{\mathbf{S}^1} F_{per} d\lambda_1 - \frac{1}{4\pi^2} \int_{\mathbf{T}^2} F_{per} d\lambda_1 d\lambda_2 \right) d\lambda_2.$$

As already mentionned it is proved in [7] that

$$\delta_1 = u_1 + O(\mathcal{F}_{Kep}),$$

where  $u_1$  is the eccentric anomaly of the fictitious inner body. By the Kepler equation,  $|Q_1| du_1 = d\lambda_1$ . So,

$$|Q_1| d\delta_1 = du_1 + O(\mathcal{F}_{Kep}).$$

Besides, it is proved in the same paper that

$$\delta_2 = \lambda_2 + O\left(\frac{\nu_2}{\nu_1}\right).$$

Hence,

$$\mathcal{H}_2 = \frac{1}{a_1} H_2 + O(\mathcal{F}_{Kep}) + O\left(\frac{\nu_2}{\nu_1}\right).$$

(In particular it is the case in the lunar region that the three quantities  $a$ ,  $\nu_2/\nu_1$  and  $\varepsilon$  are small.) The factor  $1/a_1$  can actually be eliminated, by choosing a better regularization.

Let  $\psi_1^0$  be the time-one map of  $H_2$ . We have

$$\psi_1^{0*} e_1 = e_1 + X_{H_2} \cdot e_1 + O(\varepsilon^2) + O(\nu_2/\nu_1) + O(a).$$

So the points which belong to both a trajectory such that  $e_1(t) \equiv 1 + O(a^3)$  and to the collision set  $\psi_1^{0*} e_1 = 1$  are the solutions of some equation

$$\frac{\partial H_2}{\partial g} + O(\varepsilon^2) + O(\nu_2/\nu_1) + O(a) = 0.$$

The transversality condition thus boils down to the non trivial dependence of  $H_2$  on  $g$ , in the sense that all the zeros of the function

$$v_2 \mapsto \frac{\partial H_2}{\partial g}$$

should be isolated. Now, using the same trick as in appendix C, a mere quadrature of trigonometric polynomials yields

$$H_2 = -\frac{\mu_1 m_2}{32\pi\nu_2} \frac{a^2}{a_2 \varepsilon_2^3} \left( \begin{array}{l} 15e_1^2 \sin(2v_2 - 2g) + 15e_1^2 e_2 \sin(v_2 - 2g) \\ + 5e_1^2 e_2 \sin(3v_2 - 2g) + 2(2 + 3e_1^2) e_2 \sin v_2 \\ + 5e_1^2 (3 + 4e_2) \sin(2g) \end{array} \right) + O\left(\frac{a^3}{\nu_2 a_2 \varepsilon_2^5}\right),$$

which shows that the analytic function  $v_2 \mapsto \frac{\partial H_2}{\partial g}$  is certainly nowhere locally constant.  $\blacksquare$

Appendix B shows that at the secular level the restricted problems are mere limiting cases of the full problem. So the method of proof of theorem 3.5 works for the restricted problems too.

### APPENDIX A Estimate of the perturbing function

LEMMA A.1. *The perturbing function and its average satisfy the estimate*

$$\|F_{per}\|_k, \|\langle F_{per} \rangle\|_k \leq \text{Cst } \mu_1 m_2 \frac{a_1^2}{a_2^3 \epsilon_2^6} \frac{1}{\epsilon_2^{3k}} \frac{1}{(1-\Delta)^{2k+1}}$$

over  $\mathcal{P} \times \mathcal{M}$ .

Recall that the parameter  $\Delta = \hat{\sigma} \max_{\mathbf{T}^3} \frac{|Q_1|}{|Q_2|}$  measures how close the outer ellipse is from the other two, and that the norm  $\|\cdot\|_k$  was defined in the proof of proposition 2.1.

*Proof.* The perturbing function equals

$$F_{per} = -\frac{\mu_1 m_2}{|Q_2|} \left( \frac{1/\sigma_0}{|1-\sigma_0 z|} + \frac{1/\sigma_1}{|1+\sigma_1 z|} - \frac{M_1}{\mu_1} \right),$$

where  $z = Q_1/Q_2 \in \mathbf{C}$ . Reduce the factor between brackets to the same denominator. Up to some multiplicative constant, the numerator is upper bounded by  $|z|^2$  and the denominator is lower bounded by  $1-\Delta$ . Using the inequalities

$$|Q_1| \leq a_1(1+e_1) \leq 2a_1 \quad \text{and} \quad |Q_2| \geq a_2(1-e_2) \geq \frac{1}{2}a_2\epsilon_2^2,$$

we get the  $C^0$ -estimate

$$\|F_{per}\|_0 \leq \text{Cst } \mu_1 m_2 \frac{a_1^2}{a_2^3 \epsilon_2^6} \frac{1}{1-\Delta}.$$

The coordinates

$$(\tilde{\Lambda}_j, \lambda_j, \tilde{\xi}_j, \tilde{\eta}_j)_{j=1,2} = \left( \frac{\Lambda_j}{\Lambda_0}, \lambda_j, \frac{\xi_j}{\sqrt{\Lambda_0}}, \frac{\eta_j}{\sqrt{\Lambda_0}} \right)_{j=1,2}$$

have precisely been chosen so that the successive derivatives will not ruin the estimates, at least outside a neighborhood of the boundary of the real



analyticity domain of  $F_{per}$ . For instance, we have

$$\frac{\partial}{\partial \tilde{\Lambda}_1} = 2 \frac{\check{\Lambda}_0}{\Lambda_1} a_1 \frac{\partial}{\partial a_1};$$

since  $\check{\Lambda} = O(\check{\Lambda}_0)$ , where  $\check{\Lambda} = \min(\Lambda_1, \Lambda_2)$ , we get

$$\frac{\check{\Lambda}_0}{\Lambda_1} \leq \text{Cst.}$$

So, by derivating and then proceeding as for the upper bound of  $\|F_{per}\|_0$ , we get

$$\left\| \frac{\partial F_{per}}{\partial \tilde{\Lambda}_1} \right\|_0 \leq \text{Cst} \mu_1 m_2 \frac{a_1^2}{a_2^3 \epsilon_2^6} \frac{1}{(1 - \Delta)^3}.$$

The derivation in the direction of  $\tilde{\Lambda}_1$  amounts to multiplying the bound by  $1/(1 - \Delta)^2$ .

The derivatives in the other directions can be estimated similarly. In  $\mathcal{P} \times \mathcal{M}$ , the inner ellipses cannot have a large eccentricity. On the other hand, the outer ellipse may have an eccentricity close to 1. So, for instance, if  $v_2$  is the true anomaly of the outer ellipse, the second Kepler law shows that

$$\frac{\partial v_2}{\partial \lambda_2} \leq \frac{2}{\epsilon_2^3}$$

(cf. appendix C), so each derivation with respect to  $\lambda_2$  yields a factor  $1/\epsilon_2^3$  in the estimates. The final estimate of  $F_{per}$  in the statement of the lemma is a straightforward consequence of these remarks.

The average

$$\langle F_{per} \rangle = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} F_{per} d\lambda_1 d\lambda_2$$

obviously satisfies the same  $C^0$ -estimate—which does not depend on the mean anomalies—, and hence the same  $C^k$ -estimates. ■

Consider the condition

$$\max \left( \frac{m_2}{M_1} \left( \frac{a_1}{a_2} \right)^{3/2}, \frac{\mu_1 \sqrt{M_2}}{M_1^{3/2}} \left( \frac{a_1}{a_2} \right)^2 \right) \frac{1}{\epsilon_2^{3(2+k)} (1 - \Delta)^{2k+1}} < \varepsilon$$

which was used to define the perturbing and asynchronous regions  $\Pi_\varepsilon^k$  and  $A_\varepsilon^k$  in § 3.1. The following lemma shows where this inequality holds, the perturbing function and its average satisfy the  $C^k$ -estimates which come into play in the proof of proposition 2.1.

LEMMA A.2. *Let  $k$  be a fixed positive integer. The perturbing function and its average satisfy*

$$\left\| \frac{F_{per}}{\nu \Lambda} \right\|_k, \left\| \frac{\langle F_{per} \rangle}{\nu \Lambda} \right\|_k \leq \text{Cst } \varepsilon$$

over  $\Pi_\varepsilon^k$  and  $A_\varepsilon^k$ .

*Proof.* By choosing adequate length and mass units, we may assume that  $a_1 = M_2 = 1$ . Suppose that

$$\left\| \frac{F_{per}}{\nu \Lambda} \right\| \leq \text{Cst } \varepsilon$$

and look for sufficient conditions. If we let

$$\delta_k = \frac{1}{\varepsilon_2^{3(2+k)} (1 - \Delta)^{2k+1}} (> 1),$$

we have

$$\frac{\mu_1 m_2}{a_2^3} \max \left( \frac{1}{\mu_1 \sqrt{M_1}}, \frac{1}{\mu_2 \sqrt{a_2}} \right) \max \left( \frac{1}{\sqrt{M_1}}, a_2^{3/2} \right) \delta_k < \text{Cst } \varepsilon,$$

or, by splitting the max into two parts,

$$\begin{cases} \max \left( \frac{\delta_k m_2}{M_1 a_2^3}, \frac{\delta_k m_2}{\sqrt{M_1} a_2^{3/2}} \right) < \varepsilon \\ \max \left( \frac{\delta_k \mu_1}{M_1^{3/2} a_2^{7/2}}, \frac{\delta_k \mu_1}{M_1 a_2^2} \right) < \varepsilon. \end{cases}$$

These inequalities are consequences of the stronger inequalities

$$\begin{cases} \frac{\delta_k m_2}{M_1 a_2^{3/2}} < \varepsilon \\ \frac{\delta_k \mu_1}{M_1^{3/2} a_2^2} < \varepsilon. \end{cases}$$

In order to get the inequality defining the perturbing region, it suffices to come back to general length and mass units. ■

APPENDIX B

Averaging the restricted three-body problems

The *restricted* problems are the particular cases where one of the masses equals zero. The restricted problems are *not* the Hamiltonian limit of the full problem when the corresponding mass goes to zero, although their first order secular systems are:

LEMMA B.1. *The averaged Hamiltonians of the restricted three-body problems are the limit of the averaged Hamiltonians of the full problem, when the corresponding mass goes to zero.*

*Proof.* For instance, consider the case when one of the inner bodies has zero mass :  $\mu_1 = 0$ . (The other case, when  $m_2 = 0$ , is similar.)

To begin with, assume that the inner fictitious body has a small but positive mass  $\mu_1$ . In the spirit of the beginning of the proof of proposition 2.1, let

$$\Lambda_j = \mu_1 \tilde{\Lambda}_j \quad \text{and} \quad \xi_j + i\eta_j = \sqrt{\mu_1} (\tilde{\xi}_j + i\tilde{\eta}_j).$$

This rescaling is necessary to have some  $C^k$ -estimates of the perturbing function with  $k \geq 1$ , which do not explode when  $\mu_1$  goes to zero. For the new standard symplectic form  $d\tilde{\Lambda}_1 \wedge d\lambda_1 + \dots$ , the new Hamiltonian is  $F/\mu_1$ , or

$$\frac{M_1^2}{2\tilde{\Lambda}_1^2} - \frac{\mu_2^3}{\mu_1^3} \frac{M_2^2}{2\tilde{\Lambda}_2^2} - m_2 \left[ \frac{1}{\sigma_0} \left( \frac{1}{|Q_2 - \sigma_0 Q_1|} - \frac{1}{|Q_2|} \right) + \frac{1}{\sigma_1} \left( \frac{1}{|Q_2 + \sigma_1 Q_1|} - \frac{1}{|Q_2|} \right) \right].$$

Every term has a finite limit when  $\mu_1$  goes to 0, but the second term. At the limit, when  $\mu_1$  goes to 0, the dynamics of the inner body is determined by the Hamiltonian

$$F_1 = -\frac{M_1^2}{2\tilde{\Lambda}_1^2} - m_2 \left[ \frac{1}{\sigma_0} \left( \frac{1}{|Q_2(t) - \sigma_0 Q_1|} - \frac{1}{|Q_2(t)|} \right) + \frac{1}{\sigma_1} \left( \frac{1}{|Q_2(t) + \sigma_1 Q_1|} - \frac{1}{|Q_2(t)|} \right) \right],$$

where the coordinates  $(P_2, Q_2)$  of the outer body are periodic functions of time  $t$ , and where  $\sigma_0$  and  $\sigma_1$  need to be replaced by 0 or 1 according to whether it is the body 1 or 2 which has zero mass. The phase space is the direct product of that of the inner body by the cylinder  $\mathbf{S}^1 \times \mathbf{R}$  which is parametrized by time  $t$  and its symplectically conjugate variable  $\tau$ .

The elimination of the mean longitude of the inner body and of the time in the perturbing function of the Hamiltonian  $(F_1 + \tau)|_{\mu_1=0}$  leads to the

restriction of the extension of the averaged system  $\langle F \rangle / \mu_1$  to the boundary  $\mu_1 = 0$ . ■

### APPENDIX C Averaged Hamiltonian

In this appendix, we expand the average

$$\langle F_{per} \rangle = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} F_{per} dl_1 dl_2.$$

of the perturbing function in the powers of the semi major axes ratio.

We will need the following notations. Let  $u_j$  and  $v_j$  be the eccentric and the true anomalies of the fictitious body  $j$  (cf. chap. III of the *Leçons* [22]). The distance of this body from the origin is

$$\begin{cases} |Q_j| = a_j \rho_j, & \rho_j = 1 - e_j \cos u_j \\ \text{or} \\ |Q_j| = a_j \epsilon_j^2 \varrho_j, & 1/\varrho_j = 1 + e_j \cos v_j. \end{cases}$$

LEMMA C.1. *There exist some polynomials  $Q_n$ , for  $n \geq 2$ , such that the average  $\langle F_{per} \rangle$  of the perturbing function is*

$$\langle F_{per} \rangle = -\mu_1 m_2 \frac{\epsilon_2}{a_2} \sum_{n \geq 2} \sigma_n Q_n(e_1, e_2, \cos g) \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^n.$$

For every integer  $n \geq 2$ ,  $Q_n$  is a polynomial in three variables with rational coefficients, such that:

$$\begin{cases} Q_n(\cos(g + \pi)) = (-1)^n Q_n(\cos g) \\ Q_n(-e_1, e_2, \cos g) = Q_n(e_1, -e_2, \cos g) = Q_n(e_1, e_2, \cos(g + \pi)) \\ Q_n(0, e_2, \cos g) = Q_n(0, e_2, 1) \quad \text{et} \quad Q_n(e_1, 0, \cos g) = Q_n(e_1, 0, 1). \end{cases}$$

Moreover, there exist some polynomials  $\tilde{Q}_{2n}$ , for  $n \geq 1$ , such that if  $m_1 = m_2$ ,

$$\langle F_{per} \rangle = -\mu_1 m_2 \frac{\epsilon_2}{a_2} \sum_{n \geq 1} \frac{1}{2^{n-2}} \tilde{Q}_{2n}(e_1, e_2, \cos(2g)) \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^{2n}.$$

The  $\tilde{Q}_{2n}$ 's are three-variable polynomials with rational coefficients too.

*Proof.* Start with the expansion of the perturbing function using the Legendre polynomials (cf. lemma 1.1):

$$F_{per} = -\mu_1 m_2 \sum_{n \geq 2} \sigma_n P_n(\cos \zeta) \frac{\rho_1^n}{\varrho_2^{n+1}} \frac{a_1^n}{(a_2 \epsilon_2^2)^{n+1}},$$

where  $\rho_1 = 1 - e_1 \cos u_1$  and  $1/\varrho_2 = 1 + e_2 \cos v_2$ . The Kepler equation  $l_1 = u_1 - e_1 \sin u_1$  and the second Kepler law yield

$$dl_1 = \rho_1 du_1 \quad \text{and} \quad dl_2 = \varrho_2^2 \epsilon_2^3 dv_2.$$

In order to use  $(u_1, v_2)$  as integrating variables, note that

$$P_{n-1}(\cos \zeta) \rho_1^n$$

is the sum of several terms of the type

$$\rho_1^n \cos^k \zeta = \rho_1^{n-k} (\rho_1 \cos \zeta)^k, \quad n > k;$$

to expand  $\cos \zeta$  by splitting  $\zeta$  into  $\zeta = v_1 + (g - v_2)$ , it suffices to notice that

$$\rho_1 \cos v_1 = \cos u_1 - e_1 \quad \text{and} \quad \rho_1 \sin v_1 = \epsilon_1 \sin u_1,$$

which lets us eliminate the true anomaly  $v_1$ . The computation of the first terms of

$$\langle F_{per} \rangle = -\mu_1 m_2 \frac{\epsilon_2}{a_2} \sum_{n \geq 2} \sigma_n Q_n \cdot \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^n,$$

with

$$Q_n = \frac{1}{4\pi^2} \int_{\mathbf{T}^2} P_n(\cos \zeta) \frac{\rho_1^{n+1}}{\varrho_2^{n-1}} du_1 dv_2,$$

boils down to the quadrature of some trigonometric polynomials.

At first sight,  $Q_n$  is a polynomial function of  $e_1, \epsilon_1, e_2, \cos g$  and  $\sin g$ . But the terms with an odd power of  $\epsilon_1$  have zero average in the  $u_1$  variable. Hence  $Q_n$  depends only on  $\epsilon_1^2 = 1 - e_1^2$ , which let us think of it as a polynomial in  $e_1$ . Moreover, thanks to the invariance of Newton's equations by the change of orientation of the physical plane,  $Q_n$  is even in the angle  $g$ , so it depends only on  $\cos g$ .

Since  $|Q_1| = a_1 \rho_1$ , the perturbing function  $F_{per}$  is invariant by  $(a_1, g) \mapsto (-a_1, g + \pi)$ . In other words,  $P_n$  is odd or even according to its own degree. The polynomials  $Q_n$  inherit this invariance in that

$$Q_n(\cos(g + \pi)) = (-1)^n Q_n(\cos g).$$

Similarly, in the formula defining  $Q_n$  as an integral, the term under the integral is invariant by

$$(v_1, e_1, g) \mapsto (v_1 + \pi, -e_1, g + \pi).$$

Hence  $Q_n$  itself is invariant by  $(e_1, g) \mapsto (-e_1, g + \pi)$ .  $Q_n$  satisfies the analogous invariance with  $e_2$ .

The third invariance property arises from the fact that the variables  $(e_1, e_2, g)$  are only defined on a blown-up space: when an ellipse is circular, its argument of pericenter is not physically defined and thus the averaged system cannot depend on it.

Lastly, the perturbing function is invariant by

$$(Q_1, \sigma_0, \sigma_1) \mapsto (-Q_1, \sigma_1, \sigma_0).$$

If the two inner masses are equal to one another, i.e. if  $\sigma_0 = \sigma_1 = 1/2$ , it is invariant by  $Q_1 \mapsto -Q_1$ . Its average is then invariant by  $g \mapsto g + \pi$  and it depends on  $g$  only through  $\cos(2g)$ . ■

Lieberman [18] for instance uses the true anomaly for both bodies, which leads to more complicated computations. There actually are two small mistakes in the term in  $a_1^4$  that he gives. We find

$$\langle F_{per} \rangle = -\frac{\mu_1 m_2 \epsilon_2}{a_2} \left[ \begin{array}{l} \frac{2 + 3e_1^2}{8} \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^2 \\ - \frac{15}{64} (\sigma_0 - \sigma_1) (4 + 3e_1^2) e_1 e_2 \cos g \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^3 \\ + \frac{9}{1024} \sigma_4 \left( \begin{array}{l} 70e_1^2 e_2^2 (2 + e_1^2) \cos(2g) \\ + 45e_1^4 e_2^2 + 30e_1^4 \\ + 120e_1^2 e_2^2 + 80e_1^2 \\ + 24e_2^2 + 16 \end{array} \right) \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^4 \\ + O \left( \left( \frac{a_1}{a_2 \epsilon_2^2} \right)^5 \right) \end{array} \right].$$

In this paper we only use the term in  $a_1^2$ , but all the terms given here are needed to determine the global bifurcation diagram of the secular systems [8].

### ACKNOWLEDGMENT

These results are part of my PhD thesis, which I prepared at the Bureau des Longitudes and Université Paris 13. The paper was written while at Northwestern University; I express my gratitude to D. Saari and E. Friedlander for their hospitality. Special thanks to A. Chenciner for his invaluable advice and to M. Herman for his help with KAM theory, and to both for their critical reviewing of my thesis.

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