A SUMMARY OF M. HERMAN'S PROOF OF 'ARNOLD'S THEOREM' IN CELESTIAL MECHANICS

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Consider $1+n$ point bodies with masses $m_0, \epsilon m_1, ..., \epsilon m_n > 0$ ($\epsilon > 0$) and position vectors $x_0, x_1, ..., x_n \in \mathbb{R}^3$. According to Newton's equations we have

$$\ddot{x}_j = m_0 \frac{x_0 - x_j}{||x_0 - x_j||^3} + \epsilon \sum_{k \neq j} m_k \frac{x_k - x_j}{||x_k - x_j||^3} \quad (j = 1, ..., n).$$

These equations have a limit when $\epsilon \to 0$, for which each planet (masses ϵm_j) undergoes the only attraction of the sun (mass m_0). If their energies are negative, planets describe Keplerian ellipses with some given semi major axes and excentricities. As a whole, the system is quasiperiodic with n frequencies. In 1963, V. Arnold [A] published the following remarkable result.

Theorem 1. *For every $m_0, m_1, ..., m_n > 0$ and for every $a_1 > ... > a_n > 0$ there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, in the phase space in the neighborhood of circular and coplanar Keplerian motions with semi major axes $a_1, ..., a_n$, there is a subset of positive Lebesgue measure of initial conditions leading to quasiperiodic motions with $3n - 1$ frequencies.*

The proof of this theorem is rendered difficult by the multitudinous degeneracies of the planetary problem. Arnold's initial proof does not fully describe these degeneracies and actually misses one of them. Hence it is wrong in the case of $n \geq 3$ planets in space. In 1998, in a series of lectures M. Herman sketched a complete and more conceptual proof of this theorem [F]. I will now review a couple of ideas which make this proof so powerful and, I believe, elegant. These ideas mainly pertain to some normal forms of Hamiltonians, which might not surprise the specialists but which epitomize the structure of KAM theory as understood by M. Herman.

Let $X = \mathbb{T}^p \times \bar{\mathbb{B}}^p$, $\mathbb{T}^p = \mathbb{R}^p/\mathbb{Z}^p$ and $\bar{\mathbb{B}}^p$ be the closed p-dimensional unit Euclidean ball. Endow X with the natural coordinates (θ, r) and the standard symplectic form $\omega = \sum_{j=1}^p d\theta_j \wedge dr_j$. If $H \in C^\infty(X)$ is a smooth Hamiltonian, its Hamiltonian vector field is $\dot{\theta} = \partial_r H, \quad \dot{r} = -\partial_\theta H$. Denote by R_α, $\alpha \in \mathbb{R}$, the Hamiltonian defined by $R_\alpha = \alpha \cdot r$. Let $\mathscr{N}_\alpha = \{R_\alpha + O(r^2)\}$ be the space of Hamiltonians for the flow of whom the torus $\mathbb{T}_0^p = \mathbb{T}^p \times \{0\}$ is invariant and quasiperiodic with frequency vector α. Let also $\mathscr{G}$ be some space of Hamiltonian diffeomorphisms, which we will not fully describe here, but which is diffeomorphic to a neighborhood of $(0, id)$ in the product $\mathbb{B}_1^\infty(\mathbb{T}^p) \times \mathrm{Diff}_o^\infty(\mathbb{T}^p)$, where $\mathbb{B}_1^\infty(\mathbb{T}^p)$ is the space (acting by translation in the r direction) of closed one-forms on $\mathbb{T}^p$ and $\mathrm{Diff}_o^\infty(\mathbb{T}^p)$ is the space (acting contragrediently) of diffeomorphisms of the torus which fix the origin. Let ϕ_α be the map

$$\begin{array}{rlcl} \phi_\alpha: & \mathscr{N}_\alpha \times \mathscr{G} \times \mathbb{R}^p & \to & C_+^\infty(X) \\ & (N, G, \hat{\alpha}) & \mapsto & H = N \circ G + R_{\hat{\alpha}}, \end{array}$$

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where $C^\infty_+(X)$ is the quotient of the space of Hamiltonians by the real constants. The Hamiltonian $N \circ G$ is symplectically conjugate to N by G; hence for the flow of $N \circ G$ the torus $G^{-1}(\mathbb{T}^p_0)$ is invariant and α-quasiperiodic. The term $R_{\hat\alpha}$, which tunes the frequency, unfortunately breaks down the dynamical conjugacy; hence I call $(N, G, \hat\alpha)$ a *twisted conjugacy* of H, and in general H does not have an invariant torus. Eventually, define

$$HD_{\gamma,\tau} = \{\alpha \in \mathbb{R}^p, \quad \forall k \in \mathbb{Z}^p \setminus 0 \quad |k \cdot \alpha| \geq \gamma ||k||^{-\tau}\} \quad (\gamma, \tau > 0).$$

Theorem 2 (Twisted conjugacy, M. Herman)**.** *For every $\alpha \in \mathrm{HD}_{\gamma,\tau}$ and for every $N^o \in \mathscr{N}_\alpha$, the map ϕ_α is a local (tame in the sense of Hamilton) C^∞-diffeomorphism in a neighborhood of $(N^o, id, 0) \mapsto N^o$; in particular, the $\mathscr{G}$-orbit of $\mathscr{N}_\alpha$ defines a germ of submanifold of codimension p of $C^\infty_+(X)$. Moreover, the germ of map $(H, \alpha) \mapsto \phi_\alpha^{-1}(H)$ is C^∞-smooth in the sense of Whitney.*

Sketch of proof. We want to solve the equation $\phi_\alpha(N, G, \hat\alpha) = H$ for H close enough to N^o in the C^∞-topology, by using some inverse function theorem. Linearizing the equation reduces the problem to inverting the linear operator $d\phi_\alpha(N, G, \hat\alpha)$. In this setting, small denominators manifest themselves in the loss of differentiability of ϕ_α and its differential, which prevents from choosing Banach norms at the source and target spaces for which these operators are both bounded and coercive. This is easy to see, using Fourier series, for instance on the Lie derivative (which occurs as one of the components of $d\phi_\alpha$)

$$\mathscr{L}_\alpha : C^\infty_0(\mathbb{T}^p) \to C^\infty_*(\mathbb{T}^p), \quad f \mapsto g = df \cdot \alpha,$$

where the index 0 in the source space means $f(0) = 0$ and the index $*$ in the target space means $\int_{\mathbb{T}^p} g = 0$. A way out is to use scaled Fréchet structures and the Nash-Moser inverse function theorem. For the sake of simplicity, the version due to Sergeraert and Hamilton in the C^∞-category can be used. In order to apply this theorem, one needs to invert the linear operator $d\phi_\alpha(N, G, \hat\alpha)$ for $(N, G, \hat\alpha)$ close, but not necessarily equal, to $(N^o, id, 0)$. This inversion is equivalent to one step in the induction of Kolmogorov's original proof of the invariant torus theorem.

In order to get rid of the twist of the conjugacy, a natural idea could be to tune the frequency before conjugating by G i.e., to consider $\psi_\alpha : (N, G, \hat\alpha) \mapsto (N + R_{\hat\alpha}) \circ G$ instead of ϕ_α. But ψ_α is glaringly not a local diffeomorphism – if it were, the property of having an invariant torus would be open in the space of Hamiltonians! We will actually use this idea of relaxing the frequency of the unperturbed Hamiltonian, but in a more sophisticated manner. Let

$$\mathscr{N} = \cup_{\alpha \in \mathbb{R}^p} \mathscr{N}_\alpha = \{\alpha \cdot r + O(r^2)\}_{\alpha \in \mathbb{R}^p}.$$

Corollary 3 (Conditional conjugacy)**.** *For every $N^o \in \mathscr{N}$ there is a (non unique) germ of C^∞-diffeomorphism*

$$\begin{array}{llll} \Theta : & C^\infty_+(X) & \to & \mathscr{N} \times \mathscr{G} \\ & H & \mapsto & (N_H, G_H), \quad N_H = \alpha_H \cdot r + O(r^2), \end{array}$$

at $N^o \mapsto (N^o, id)$ such that for every H the following implication holds :

$$\alpha_H \in HD_{\gamma,\tau} \Longrightarrow H = N_H \circ G_H.$$

I call (N_H, G_H) a *hypothetical conjugacy* of H because the property $H = N_H \circ G_H$ depends on arithmetical conditions involving the unknown frequency α_H.

The idea of the proof is to apply theorem 2 to all possible values of α and to see whether the corresponding $\hat{\alpha}$ vanishes for some adequate choice of α.

Proof. According to theorem 2, the equality $\tilde{\Theta}(H,\alpha) = {\phi_\alpha}^{-1}(H)$ defines a germ of map

$$\tilde{\Theta} : C^\infty_+(X) \times HD_{\gamma,\tau} \to \mathscr{N} \times \mathscr{G} \times \mathbb{R}^p$$

at $N^o = \alpha^o \cdot r + O(r^2)$ which is Whitney-smooth. According to Whitney's extension theorem, this germ extends to a smooth germ

$$\tilde{\Theta} : C^\infty_+(X) \times \mathbb{R}^p \to \mathscr{N} \times \mathscr{G} \times \mathbb{R}^p.$$

Now, the equality $N^o = (N^o + R_{\alpha-\alpha^o}) \circ id + R_{\alpha^o-\alpha}$ shows that

$$\left.\frac{\partial \hat{\alpha}}{\partial \alpha}\right|_{\{G=id\}} = -id_{\mathbb{R}^p}.$$

Hence, the usual implicit function theorem entails that there is a unique germ of function $\alpha = \bar{\alpha}(H)$ such that $\hat{\alpha}(\bar{\alpha}) = 0$. There only remains to set $(\Theta(H), 0) = \tilde{\Theta}(H, \bar{\alpha}(H))$.

Now assume that the perturbed Hamiltonian H depends on some parameter $s \in \mathbb{B}^t$; if H is close to some completely integrable Hamiltonian, s may be the action coordinate and, in the case of Arnold's theorem, s represents the semi-major axes, excentricities and inclinations. By composition with Θ, H determines a frequency map $s \mapsto \alpha_s$, which is C^∞-close to the frequency map $s \mapsto \alpha^o_s$ of the unperturbed Hamiltonian N^o.

Theorem 4 (Arnold, Margulis, Pyartli)**.** *If some real-analytic map $s \in \mathbb{B}^t \mapsto \alpha^o_s \in \mathbb{R}^p$ is non-planar in the sense that its image is nowhere locally contained in some proper vector subspace of $\mathbb{R}^p$, the Lebesgue measure of $\{s \in \mathbb{B}^t,\ \alpha^o_s \in HD_{\gamma,\tau}\}$ is positive provided that γ is small enough and τ large enough.*

There exists a similar statement in the smooth setting, involving finitely many derivatives of the frequency map. By combining the two latter statements and using the fact that being non planar is an open property in the C^∞-topology, we get an invariant tori theorem. Unfortunately, the following holds.

Theorem 5 (M. Herman)**.** *The frequency map α^o of the first order secular system –that is, the Birkhoff normal form of the planetary problem along circular and coplanar Keplerian n-tori–, as a function of the semi major axes, has its image lying entirely in a plane P of codimension 2. Moreover, its image lies in no plane of higher codimension.*

The theorem can be proved by induction on the number of planets and by complexifying the semi-major axes. The first resonance consists in that one of the frequencies is zero. It comes from the Galilean symmetry and disappears when fixing the direction of the angular momentum, e.g. vertically. The second resonance is that the sum of all the secular frequencies is zero. For two planets revolving around the sun, it is consistent with fact, well known of astronomers, that the plane of each ellipse slowly rotates around the vertical axis in the negative direction, whereas the ellipses rotate in their own plane in the positive direction. But for n planets the resonnance is mysterious and seems not to have been noticed before. According to numerical evidence, for small values of n it vanishes in the second order secular system; but one preciely wants to avoid to check this (quoting M. Herman, 'BLC' for 'Bonjour Les Calculs'!).

It turns out that Herman's resonance too disappears, namely in the system fully reduced by the symmetry of rotations. The difficulty is that symplectic coordinates are not known explicitely on the symplectic submanifold of vertical angular momentum. A key remark is that the fully reduced system (i.e. the system with fixed angular momentum, quotiented by rotations around the angular momentum) is non planar if and only if there is a rotating frame in which the partially reduced system (i.e. the system with vertical angular momentum) is non planar. But there is one, and actually infinitely many, such rotating frames, because the trace of the quadratic part of the angular momentum is non zero, as it can easily be seen, again by an argument of analytic continuation.

The $2n-1$ slow frequencies vanish when $\epsilon = 0$. Hence, when ϵ is small, there is a competition between choosing diophantine conditions (1) good enough so that (as a quantitative version of the twisted conjugacy theorem shows) the local image of the operator Θ at the secular system of some high enough order contains the full Hamiltonian of the planetary problem; (2) bad enough so that (as a quantitative version of the Arnold-Margulis-Pyartli theorem shows) the frequency map passes through such diophantine vectors in positive measure in the space of semi major axes. It happens that fixing τ large enough and choosing γ as some power of ϵ fits the bill. The above abstract theory applies to the reduced systems and yields a positive measure of invariant quasiperiodic diophantine $(3n-2)$-tori (or, as a refinement shows, invariant normally elliptic tori of any dimensions between n and $3n-2$), which lift to a positive measure of invariant quasiperiodic $(3n-1)$-tori of the full system (respectively, to invariant normally elliptic tori of dimensions between $n+1$ and $3n-1$).

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References

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