# Elliptic Closed Geodesics.

**DEF:**

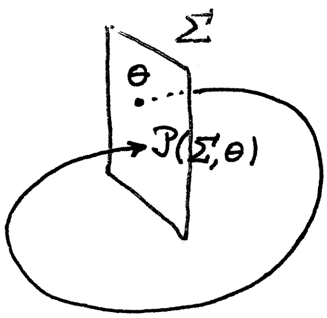
- A periodic orbit is $q$-elliptic iff its Poincaré map has $2q$ eigenvalues of modulus $1$.
- ... is elliptic if it is $q$-elliptic for some $q \geq 1$

Want : $\boxed{\text{Elliptic} + \text{Kupka-Smale} \Rightarrow \exists \text{ horseshoe.}}$

Elliptic = Normally hyperbolic twist map.

Sup. $\theta$ is $q$-elliptic , $1 \leq q \leq n$, periodic pt.

$P = \mathcal{P}(\Sigma, \theta)$ linearized Poincaré map.




$T_\theta \Sigma = E^s \oplus E^c \oplus E^u$

$P$- invariant subspaces s.t.
$$\begin{cases} P|_{E^s} & \text{eigenvalues} & |\mu| < 1 \\ P|_{E^u} & \text{"} & |\lambda| > 1 \\ P|_{E^c} & \text{"} & |\rho| = 1 \end{cases}$$

$\exists$ invariant mflds $W^s, W^u, W^c$ s.t. $T_\theta W^\alpha = E^\alpha$ , $\alpha = s, u, c.$

symplectic form $\omega$ : $\omega|_{W^s} \equiv 0$, $\omega|_{W^u} \equiv 0$, $\omega|_{W^c}$ non-deg.

$\Rightarrow$ $\mathcal{P}|_{W^c}$ is a symplectic map in nbhd of $\theta$.

1 $\boxed{\begin{array}{l} 1\text{-elliptic} \\ +\ \text{Kupka-Smale} \end{array} \Rightarrow \exists \text{ horseshoe}}$

<u>$\mathcal{P}|_{W^c}$ is a K-S exact twist map.</u>

$$\begin{tikzcd}
(x,y) \arrow[r] \arrow[rr, bend left, "G"] & (r,\theta) \arrow[r] & (\frac{1}{2}r^2,\theta) = (R,\theta) \\
\mathbb{D}^* \arrow[r, "P"] \arrow[d, "f_0"'] & \mathbb{R}^+ \times S^1 \arrow[r] \arrow[d] & \mathbb{R}^+ \times S^1 \arrow[d, "T"] \\
\mathbb{D}^* \arrow[r] & \mathbb{R}^+ \times S^1 \arrow[r] & \mathbb{R}^+ \times S^1
\end{tikzcd}$$

$$\mathbb{D}^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$$

$$G(x,y) = (\tfrac{1}{2}r^2, \theta) = (R,\theta)$$

$$G^*(R\,d\theta) = \tfrac{1}{2}(x\,dy - y\,dx) =: \lambda$$

$x = r\cos\theta$ $\qquad dx = \cos\theta\,dr - r\,\mathrm{sen}\,\theta\,d\theta$

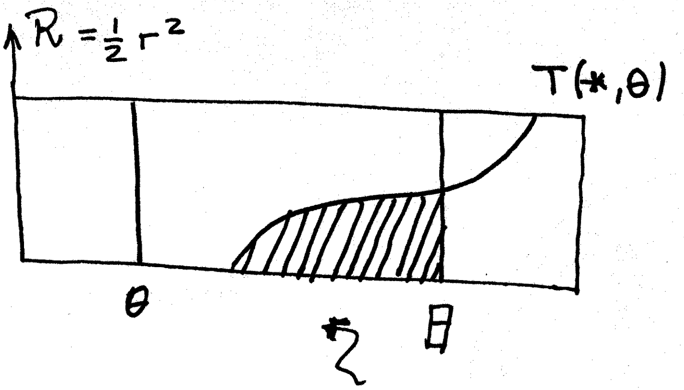
$y = r\,\mathrm{sen}\,\theta$ $\qquad dy = \mathrm{sen}\,\theta\,dr + r\cos\theta\,d\theta$

$$\begin{aligned} x\,dy - y\,dx &= r\cos\theta(\mathrm{sen}\,\theta\,dr + r\cos\theta\,d\theta) \\ &\quad - r\,\mathrm{sen}\,\theta(\cos\theta\,dr - r\,\mathrm{sen}\,\theta\,d\theta) \\ &= \dots = r^2\,d\theta \end{aligned}$$

$d\lambda = dx \wedge dy$ ← *area form in* $\mathbb{D}$

$\mathbb{D}$ *contractible* $\Rightarrow$ $f_0^*(\lambda) - \lambda$ *exact.*

$T^*(R\,d\theta) - R\,d\theta$ <u>exact</u>. $\square$

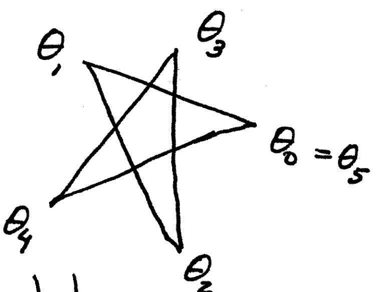



<u>Action</u> $A(\theta, \Theta) =$ area

critical pts of action $\mathbb{A}(\theta_0, \dots, \theta_N) = \sum_{i=0}^{N-1} \mathbb{A}(\theta_i, \theta_{i+1})$
correspond to orbits of $T$ by

$$(\theta_i, \theta_{i+1}) \longrightarrow T(*, \theta_i) \cap (*, \theta_{i+1})$$

For periodic sequences
$\mathbb{A}(\theta_0, \dots, \theta_N)$, $\theta_N = \theta_0$
obtain:

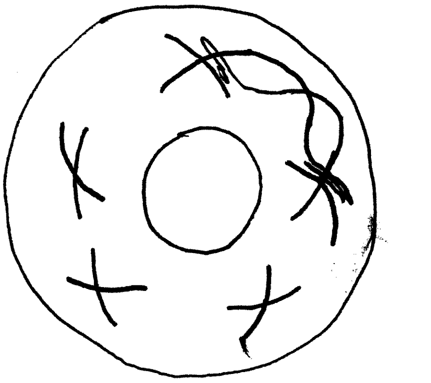



- <u>minimum</u> $\longrightarrow$ hyperbolic periodic orbit (MacKay).
- From $(\theta_0, \dots, \theta_N) \rightsquigarrow (\theta_1, \dots, \theta_N, \theta_1)$
  A $\qquad$ B

obtain mountain pass geometry

B = 1/5 rotation of per. seq. A


<u>minimax</u> $\longrightarrow$ elliptic periodic point

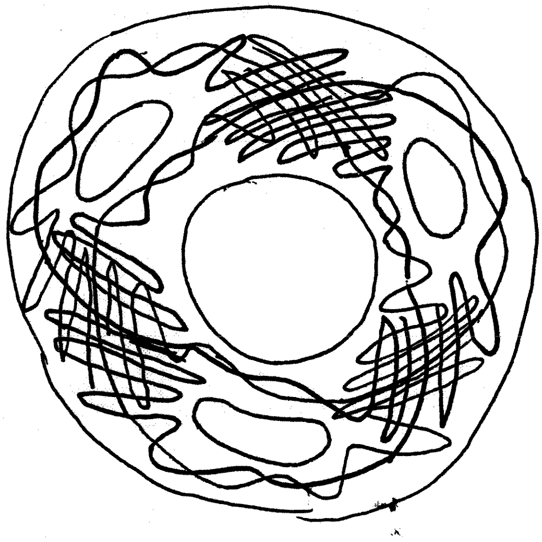


Obtain variationally a homoclinic

min $\mathbb{A}(\varphi_{-N}, \dots, \varphi_N)$

$N \longrightarrow \infty$.

$\left.\begin{matrix}\varphi_{-N} \\ \varphi_N\end{matrix}\right\}$ projections of minim. periodic orbit:
$\varphi_{-N} = \theta_0$, $\varphi_N = \theta_3$



Normally hyperbolic

$\Rightarrow$

hyperbolic orbits for the twist map are also hyperbolic on the ambient manifold $\Sigma$ = Poincaré section

$\pitchfork$ homoclinics for the twist map are $\pitchfork$ homoclinics in ambient mfld.

③

| | | |
|---|---|---|
| $q$-Elliptic fixed point + Kupka-Smale | $\Rightarrow$ | weakly monotonous exact twist map $C^1$ near totally integrable |

$\mathbb{P} : (\mathbb{R}^{2n}, \theta) \circlearrowleft$ $q$-elliptic fixed pt. of symplectic map

$P = d_\theta \mathbb{P}$

$\rho_1, \dots, \rho_q ; \bar{\rho}_1, \dots, \bar{\rho}_q$ eigenvalues of $P$ with modulus 1.

$\theta$ is <u>4-elementary</u> if

$$1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4 \implies \prod_{i=1}^{q} \rho^{\nu_i} \neq 1$$

4-eltary $\Rightarrow$ Birkhoff normal form.

$\exists$ sympl. coords. $(x_1, \dots, x_q; y_1, \dots, y_q)$ in $W^c$ s.t.

$$\omega|_{W^c} = \sum dy_i \wedge dx_i$$

$\mathbb{P}|_{W^c}$ writes as $\mathbb{P}(x,y) = (X,Y)$

$$Z_k = e^{2\pi i \phi_k} z_k + g_k(z)$$

$$\phi_k(z) = a_k + \sum_{\ell=1}^{q} \beta_{k\ell} |z_\ell|^2$$

where $z = x + iy$ ; $Z = X + iY$ ; $\rho_i = e^{2\pi i a_k}$

$g(z) = g(x,y)$ has vanishing 3-Jet at $\theta$

Say $\theta$ is <u>weakly monotonous</u> iff $\det \beta_{k\ell} \neq 0$

$\rightarrow$ (this prop. is indep. of choice of normal form).

In Birkhoff coords. we have

$$
\begin{tikzcd}
(x,y) \arrow[r, Rightarrow] \arrow[rr, bend left, "Q"] & (\theta,\rho) \arrow[r] & (\theta, \rho^2/\epsilon) = (\theta, r) \\
\mathbb{D}^* \arrow[r, "\mathcal{P}"] \arrow[d, "f"'] & \mathbb{T}^q \times \mathbb{R}^q_+ \arrow[r, "R"] & \mathbb{T}^q \times \mathbb{R}^q_+ \arrow[d, "F_\epsilon"] \\
\mathbb{D}^* \arrow[r, "\mathcal{P}"] & \mathbb{T}^q \times \mathbb{R}^q_+ \arrow[r] & \mathbb{T}^q \times \mathbb{R}^q_+
\end{tikzcd}
$$

To preserve usual area form in $\mathbb{T}^q \times \mathbb{R}^q_+$.

$\mathbb{D}^* = \{ (x,y) \in \mathbb{R}^q \times \mathbb{R}^q \mid 0 < |x_i|^2 + |y_i|^2 < 1 \}$

$f = \mathbb{P}|_{W^c}$ in Birkhoff coords.

$\mathbb{T}^q = \mathbb{R}^q / \mathbb{Z}^q$

$\mathcal{P}^{-1}$ is $\quad x_i = \rho_i \cos(2\pi\theta_i)$ , $y_i = \rho_i \sin(2\pi\theta_i)$

$f$ preserves $\omega = dx \wedge dy$

$Q = R \circ \mathcal{P} \qquad Q(x,y) = (\theta, r) \qquad r_i = \rho_i^2/\epsilon$

$Q^*(r\,d\theta) = \frac{1}{2\pi\epsilon}(x\,dy - y\,dx) =: \lambda_\epsilon$

$\mathbb{D}$ simply connected $\Rightarrow f^*(\lambda_\epsilon) - \lambda_\epsilon$ exact

$\Rightarrow F_\epsilon^*(r\,d\theta) - r\,d\theta$ exact.

$G_\epsilon(\theta, r) = (\theta + \alpha + \epsilon\beta r, r)$

↳ 1st term in Birkhoff normal form is

- symplectic diffeo
- "totally integrable"
- weakly monotonous $(\det\beta \neq 0)$
- $C^1$ near $F_\epsilon$.

Will prove

(2)

| | | |
|---|---|---|
| F (weak) Twist map<br>+ exact<br>+ $C^1$ near completely integrable | $\Rightarrow$ | F has a 1-elliptic periodic orbit |

(3)

| | | |
|---|---|---|
| q-Elliptic fixed point<br>+ Kupka-Smale | $\Rightarrow$ | Twist map with conditions (2)<br>$\mathbb{T}^q \times \mathbb{R}^q \circlearrowleft$ x normally hyp. |

## SYMPLECTIC TWIST MAPS ON $\mathbb{T}^n \times \mathbb{R}^n$

Uses techiques from M.C. Arnaud & M. Herman

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$

Liouville 1-form $\lambda = r\,d\theta = \sum r_i\,d\theta_i$ , $(\theta, r) \in \mathbb{T}^n \times \mathbb{R}^n$

$\omega := d\lambda$ symplectic form

$F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ is symplectic iff $F^*\omega = \omega$

In coords. $T(\mathbb{T}^n \times \mathbb{R}^n) \approx \mathbb{R}^n \times \mathbb{R}^n$

$\omega(x,y) = x^* J y \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$F$ symplectic $\iff (dF)^* J (dF) = J$

$F$ <u>exact symplectic</u> iff $F^*\lambda - \lambda$ exact form.

$F$ <u>weakly monotonous</u> iff writing $F(\theta, r) = (\Theta, R)$

$$\det \frac{\partial \Theta}{\partial r} \neq 0$$

<u>Torsion</u>: $b := \dfrac{\partial \Theta}{\partial r}$ not neces. symmetric.

↳ is "*positive definite, neg. def., signature $(p,q)$*"
*if* $b + b^*$ *is pos. def, neg. def., sign. $(p,q)$*

signature $(p,q)$ = $\begin{cases} p \text{ negative e.vs} & - \\ q \text{ positive e.v.} & + \\ n-(p+q) \text{ zero e.v.} & 0. \end{cases}$

$G : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ $C^1$ diffeo is <u>completely integrable</u> iff

$G(\theta, r) = (\theta + \beta(r), r)$ some $\beta \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\beta(0) = 0$.

<u>PROPERTIES</u>

① $\left.\begin{array}{l} G \text{ completely integrable} \\ + \text{ symplectic} \end{array}\right\} \Rightarrow$ *its torsion* $\dfrac{\partial \beta}{\partial r}$ *is symmetric*

② ... $\Rightarrow G^*\lambda - \lambda = r\, d\beta$ *is exact*

*because it is closed form in* $\mathbb{R}^n$.

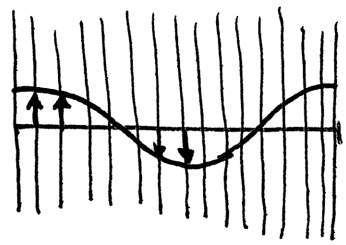
# Fixed points in a nearly integrable twist map.

$F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ weakly monotonous, exact symplectic $C^r$ diffeo, $r \geq 1$, $C^1$ near to a totally integrable map $G$.

- For tot. integ. $G$: zero section $= \mathbb{T}^n \times \{0\} \subset \mathrm{Fix}(G)$.
- Look for fixed pts for $F$

(1) Construct radially transformed torus

$T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$




$\mathcal{Z} = \mathrm{Graph}(\eta)$ solving

$F(\theta, \eta(\theta)) = (\theta, *)$

⌈ Using implicit funct thm in

$\Theta(\theta, \eta(\theta), F) = \theta$

where $F(\theta, r) = (\Theta, R)$, continuing solution $\eta_G \equiv 0$ for $G$

by weak. monot. cond. $\det\left[\frac{\partial \Theta}{\partial r}\right] \neq 0$

$F \in C^r \Rightarrow \eta \in C^r$.

⌈ want $\mathcal{H}(\eta, F) \equiv 0$ where $\mathcal{H}(\eta, F)(\theta) = \Theta(\theta, \eta(\theta), F) - \theta$

need $\frac{\partial \mathcal{H}}{\partial \eta}$ non-singular : $\frac{\partial \mathcal{H}}{\partial \eta} = \frac{\partial \Theta}{\partial r}$ i.e.

$$\frac{\partial \mathcal{H}}{\partial \eta}(\theta) \cdot \zeta(\theta) = \frac{\partial \Theta}{\partial r}(\theta, \eta(\theta), F) \cdot \zeta(\theta) = h(\theta)$$

⌋

(2) F exact symplectic $\Rightarrow$ $\exists$ generating function $S : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$

$$dS = F^*\lambda - \lambda = R\, d\Theta - r\, d\theta$$

on radially transformed torus $\mathfrak{Z}$:

$$dS|_{\mathfrak{Z}} = (R - r)\, d\theta$$

$\therefore$ $\mathrm{Fix}(F|_{\mathfrak{Z}}) \subset \mathrm{Crit}(S|_{\mathfrak{Z}})$

Define radial function $\varphi = \mathcal{L}(F) : \mathbb{T}^n \to \mathbb{R}$

$$\varphi(\theta) := S(\theta, \eta(\theta))$$

- $\varphi \in C^1 \Rightarrow \# \mathrm{Fix}(F) \geq n+1 = \text{cup length}(\mathbb{T}^n)$
- $\varphi$ Morse function $\Rightarrow \# \mathrm{Fix}(F) \geq 2^n$.

KURKA SMALE

$Q \subset J^3_s(n)$ conditions on 3-jet of ell. per. pt.

(i) $\neq$ eigenvalues

(ii) 4-eltary condition $1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4 \Rightarrow \prod_{i=1}^{q} \rho_i^{\nu_i} \neq 1$

(iii) Birkhoff normal form is weakly monotonous. $\rho_i$, $i = 1, \dots, q$ e.v. with $|\rho_i| = 1$.

DEF: $F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ is kupka-Smale iff

(i) $z \in \mathrm{Per}(F)$, $\mathrm{per}(z, F) = m \Rightarrow DF^m(z) \in Q$.

(ii) All heteroclinic intersections are $\pitchfork$.

A. **Lemma** [M. Herman]

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \quad \text{symplectic matrix}$$

$a, b, c, d \in \mathbb{R}^n$, $\det(b) \neq 0$

For $\lambda \in \mathbb{C}$, let

$$M_\lambda := b^{-1}a + db^{-1} - \lambda b^{-1} - \lambda^{-1}(b^{-1})^*$$

$$\Rightarrow \quad \operatorname{rank}(\lambda I - M) = n + \operatorname{rank} M_\lambda$$

In particular

$$\lambda \text{ eigenvalue of } M \iff \det M_\lambda = 0.$$

B. **Lemma**: $\varphi = L(F)$ radial funct on radially transf. Torus $\mathcal{T}$.

$(\theta, \eta(\theta)) \in \mathrm{Fix}(F) \cap \mathcal{T}$, $M = DF(\theta, \eta(\theta))$ &

$$\Rightarrow \quad M_\lambda = D^2\varphi(\theta) + (1-\lambda)b^{-1} + (1-\lambda^{-1})(b^{-1})^*$$

If fixed pts of F non-deg. $\Rightarrow$ $\varphi = L(F)$ is Morse funct.

C. **Lemma** $z \in \mathcal{T} \cap \mathrm{Fix}(F)$

$\Rightarrow \exists$ Poly $P \in \mathbb{R}[x]$ of $\deg P = n$ s.t. $\quad a_n z^n + \dots + a_0 =: P(z).$

$\lambda$ eig.val of $DF(z) \iff P(2 - \lambda - \lambda^{-1}) = 0$

- Leading coef. of $P = a_n = \det b^{-1}$,

  $b = \frac{\partial \Theta}{\partial r}$ the torsion

- Independent term of $P = a_0 = \det D^2\varphi(\theta)$.

**Theorem** $F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ $C^4$ Kupka-Smale, weakly monotonous exact symplectic diffeomorphism

+ $C^1$ - near to a symplectic compl. integr. diffeo $G$

$\Rightarrow$ $F$ has a 1-elliptic periodic point near $\mathbb{T}^n \times 0$

**PROOF:** $n \geq 2$

claim I: $F$ as above $\Rightarrow \exists\, z_0 \in \mathrm{Fix}(F)$ elliptic $\times$ hyperbolic

i.e. $q_0$-elliptic $\quad 1 \leq q_0 < n$

with this

$$F|_{\text{near } z_0} \xrightarrow[\text{Birkhoff normal form}]{} F_{q_0} : \mathbb{T}^{q_0} \times \mathbb{R}^{q_0} \circlearrowleft \text{ twist map. satisf hypot}$$

$F = F_{q_0} \times$ normally hyperbolic

$\Rightarrow$ $F_{q_0}$ has $z_1$ : $q_1$ - elliptic fixed pt.

$1 \leq q_1 < q_0$

$\Rightarrow \cdots \Rightarrow \; n > q_0 > q_1 > \cdots > q_m = 1$

▨ claim I

$G$ completely integrable $\quad G \underset{C^1\text{-near}}{\frown} F$

$G(\theta, r) = (\theta + \beta(r), r)$

$\left.\begin{array}{l} G \text{ symplectic} \\ + \text{compl. int.} \end{array}\right\} \Rightarrow$ its torsion $b_0 = \frac{\partial \beta}{\partial r}$ is symmetric

$(q, n-q) := \underset{\smile}{\text{signature}}$ of $b_0^{-1}$

<u>Claim II</u>

torsion of $F$ $\quad$ "radial function"

$\boxed{A}$ $z \in \mathcal{Z} \cap \mathrm{Fix}(F)$ & $(-1)^n (\det b) \det D^2\varphi(z) < 0$

$\Rightarrow DF(z)$ *has a hyperbolic eigenvalue.*

$\boxed{B}$ For any $0 \leq p \leq n$ $\exists z \in \mathcal{Z} \cap \mathrm{Fix}(F)$ s.t.

- $D^2\varphi(z)$ has signature $(p, n-p)$
- $DF(z)$ has at least $2|p-q|$ eigenvalues of modulus $1$.

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Now take $\sigma := \mathrm{sgn}\left[(-1)^n \det b\right]$

$$\mathrm{sgn}\left[(-1)^n (\det b) \det D^2\varphi(z)\right] = \sigma(-1)^p$$

$\boxed{1}$ If $\sigma < 0$ want $p$ even and $|q-p| \geq 1$

$q \neq 0$ take $p = 0$

$q = 0$ take $p = 2$ $\longleftarrow$ (because $n \geq 2$).

$\boxed{2}$ If $\sigma > 0$ want $p$ odd

$q \neq 1$ take $p = 1$

$q = 1$ & $n \geq 3$ take $p = 3$

$\sigma > 0$, $q = 1$, $n = 2$, take $p = 1$ (special case).

$\omega_i = 2 - \lambda_i - \lambda_i^{-1}$

$\sigma(-1)^p = \omega_1 \omega_2 < 0 \Rightarrow \omega_1, \omega_2 \in \mathbb{R}$ *other wise complex conjugate*

$\omega_1 < 0$, $\omega_2 > 0$

*hyp.* $\quad \hookrightarrow \omega_2 \in [0, 4] \Rightarrow$ *elliptic.*

because close to compl. integ.

$\boxtimes$ Thm.

$\lambda = 1 \iff \omega = 0$
$\lambda \in S^1 \iff \omega \in [0,4]$
$\lambda \in \mathbb{R} \iff \omega \in \mathbb{R} \setminus [0,4]$
$\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup S^1) \iff \omega \in \mathbb{C} \setminus \mathbb{R}$

Proof of claim II

Write $\omega = 2 - \lambda - \lambda^{-1}$

① Compl. integ. $G$ has all e.v. $\lambda = 1$, $\omega = 0$.
$F$ $C^1$ near $G$ $\Longrightarrow$ can assume all $|\omega| < 4$

② $z \in Z \cap \mathrm{Fix}(F)$, $\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}$ = e.v. of $DF(z)$
$\omega_i = 2 - \lambda_i - \lambda_i^{-1}$

$\star$ $(-1)^n (\det b) \det D^2\varphi(z) = \omega_1 \dots \omega_n$
$\omega_i \in \mathbb{C} \setminus \mathbb{R} \Rightarrow \bar{\omega}_i$ also appears and $\omega_i \bar{\omega}_i = |\omega_i|^2 > 0$
$\therefore$ $\star < 0 \Rightarrow 2$ real hyperbolic e.v.
claim A.

③ For $G$, $\varphi_G \equiv 0 \Rightarrow$ can assume $D^2\varphi_F$ near $0$
$D\varphi(\theta) = dS|_Z = R(\theta, \eta(\theta)) - \eta(\theta)$
$D^2\varphi(\theta) = M_{\lambda=1}$ = in terms of $DF(\theta)$
$\therefore$ can assume
$D^2\varphi(\theta) + 2[b^{-1} + (b^{-1})^*]$ has same signature as
$[b^{-1} + (b^{-1})^*]$ $b = \frac{\partial \Theta}{\partial r}$ torsion of $F$.

(4) Since $\varphi$ Morse function on $\mathbb{T}^n$ ($\Leftarrow$ (Lemma C) $DF(z)$ non-deg $\forall z \in \Xi \cap \mathrm{Fix}(F)$).

$\forall\ 0 \le p \le n\ \exists \binom{n}{p}$ crit. pts $\theta$ where $\mathrm{sgn}\, D^2\varphi(\theta) = (p, n-p)$.

$[0,\pi] \ni \alpha \xmapsto{N} M_{e^{i\alpha}}$ $\quad M_\lambda$ from Lemma B.

$N(\alpha) = M_{e^{i\alpha}} = D^2\varphi(\theta) + (1-e^{i\alpha})b^{-1} + (1-e^{-i\alpha})(b^{-1})^*$

hermitian $\Rightarrow$ real e.v.

$N(0) = M_{\lambda=1} = D^2\varphi(\theta) \longrightarrow$ signature $(p, n-p)$

$N(\pi) = M_{\lambda=-1} = D^2\varphi(\theta) + 2[b^{-1} + (b^{-1})^*] \longrightarrow$ sign. $(q, n-q)$.

$\therefore$ $|p-q|$ values of $\lambda = e^{i\alpha}$, $\alpha \in [0,\pi]$

where $\det M_\lambda = 0$ (counting multip.)

$\underset{\text{Lemma A}}{\Longrightarrow}$ $DF(z)$ has $\geq 2|p-q|$ eigenvalues in $S^1$

(considering cx conj. $\bar{\lambda} = e^{-i\alpha}$, $-\alpha \in [\pi, 0]$).