POSITIVE TOPOLOGICAL ENTROPY FOR GENERIC GEODESIC FLOWS

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CIMAT

GEODESIC FLOW

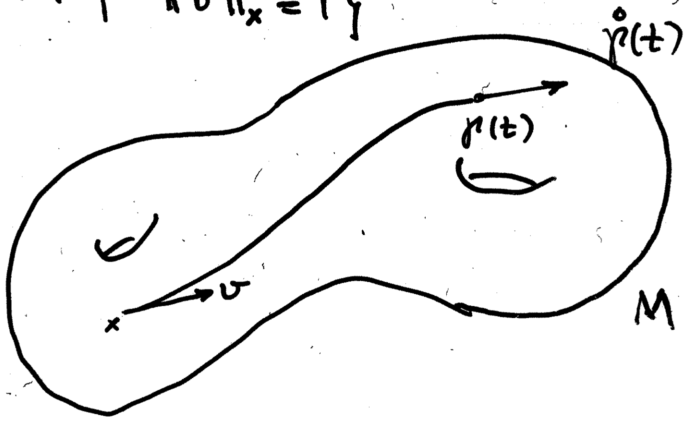
M closed C^∞ manifold [compact, connected, $\partial M = \phi$]

$g = \langle \, , \rangle_x$ C^∞ riemannian metric on M.

Unit tangent bundle = sphere bundle of (M,g)

$$SM = \{ (x,v) \in TM \mid \|v\|_x = 1 \}$$

$$\pi : SM \longrightarrow M$$
$$(x,v) \longmapsto x$$



$(x,v) \in SM$

$\gamma : \mathbb{R} \longrightarrow M$

geodesic s.t. $\gamma(0) = x$, $\dot\gamma(0) = v$

"locally length minimizing curve with $|\dot\gamma| \equiv 1$"

Geodesic Flow

$$\phi_t : SM \longrightarrow SM$$
$$(x,v) \longmapsto (\gamma(t), \dot\gamma(t))$$

TOPOLOGICAL ENTROPY

1. Measures the "complexity" of the orbit structure of the flow.
   Measures the difficulty in predicting the position of an orbit given an approximation of its initial state.

Dynamic Ball: $\theta \in SM$, $\epsilon, T > 0$

$$\mathbb{B}(\theta, \epsilon, T) = \{ w \in SM \mid d(\phi_t \theta, \phi_t w) \leq \epsilon, \ \forall t \in [0,T] \}$$

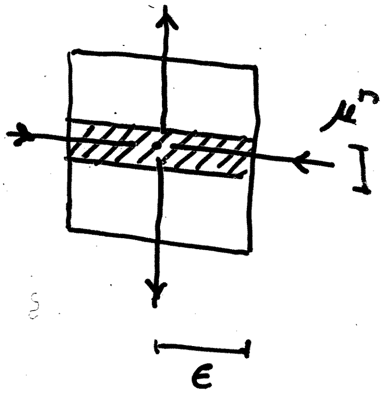
points whose orbit stay near the orbit of θ for times in $[0,T]$

$$N_\epsilon(T) := \min \{ \# \mathcal{C} \mid \mathcal{C} \text{ cover of } SM \text{ by } (\epsilon,T)\text{-dyn. balls} \}$$

$$h_{top}(g) := \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log N_\epsilon(T).$$

$$N_\epsilon(T) \sim e^{h_{top} \cdot T}$$

If $h_{top} > 0$ some dynamic balls must contract exponentially at least in one direction



② For C^∞ riemannian metrics

Mañé

$$h_{top}(g) = \lim_{T\to\infty} \frac{1}{T} \operatorname{Log} \int_{M\times M} n_T(x,y)\, dx\, dy$$

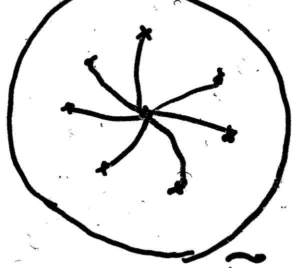
$n_T(x,y) := \#\{$geod. arcs $x \to y$ of length $\leq T\}$

$h_{top} > 0 \implies$ positive measure of (x,y) s.t. $n_T(x,y)$ is exponentially large.

TOPOLOGY $\Longrightarrow$ Some manifolds have always $h_{top}(g) > 0$

- Dinaburg: $\pi_1(M)$ exponential growth $\implies h_{top} > 0$

  [# dyn balls growths expo]



$\widetilde{M}$

Also if $\lim_{R\to\infty} \frac{1}{R} \log \operatorname{Vol}(\widetilde{B}(x,R)) > 0$ "volume entropy"

- Paternain - Petean: If $H_*(\text{Loop Space}, x)$ growths exponentially $\implies h_{top} > 0$.

GEOMETRY

sectional curvatures $K < 0 \Rightarrow \phi_t$ Anosov $\Rightarrow h_{top} > 0$

$K > 0$ not clear.

If the geod. flow ϕ_t contains a "horseshoe"
= a non-trivial hyperbolic basic set
$\Longrightarrow$ $h_{top}(g) > 0$.

$\exists$ hyperbolic periodic orbit with transversal homoclinic point. $\Longleftrightarrow$ $\exists$ horseshoe.

$\mathcal{R}^2(M) :=$ C^∞ riemannian metrics on M with the C^2 topology

THEOREM

$\dim M \geq 2$

$\exists\, \mathcal{U} \subset \mathcal{R}^2(M)$ open and dense s.t.

$g \in \mathcal{U} \Longrightarrow \phi_t^g$ has a horseshoe.

Previous Work:

- Proved for $\dim M = 2$ Paternain & C. JDG 2002
- $\dim M = 2$ & C^∞ topology Knieper & Weiss JDG 2002

Application:

A. Delshams, R. de la Llave, T. Seara:

Initial system that allows Arnold's diffusion by perturbation with generic non-autonomous potentials.

mp_arc

Comparison with other systems:

1. General Hamiltonian Systems

S. Newhouse: (M^{2n}, ω) closed symplectic manifold

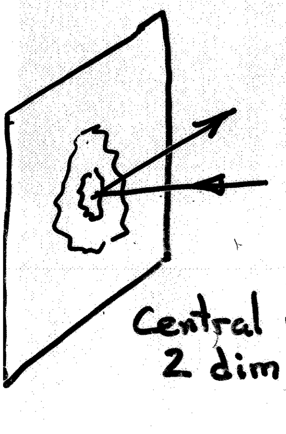
$\exists\, \mathcal{R} \subset C^2(M, \mathbb{R})$ residual s.t.

$H \in \mathcal{R} \Rightarrow$ Hamiltonian flow of H → Anosov
→ Has a generic 1-elliptic periodic orbit

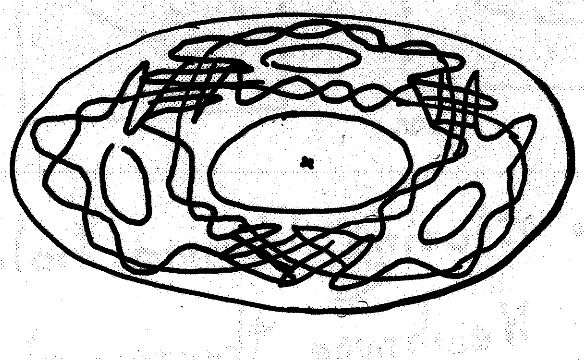
1-elliptic = 2 (elliptic) eigenvalues of modulus 1
1 eigenvalue $\lambda = 1$ (direct. of Ham. vect. field)
1 eigenvalue $\lambda = 1$ ($\pitchfork$ direct. to energy level)
$2n-4$ hyperbolic eigenvalues.

In this case:

Poincaré map restricted to energy level is Twist map × normally hyperbolic.



Central mfld
2 dim

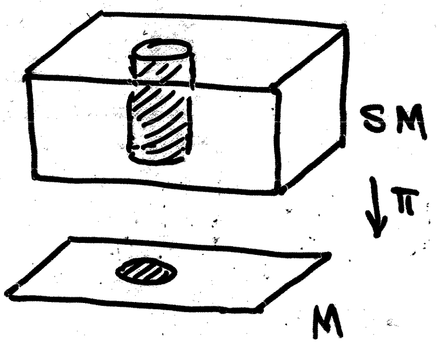


⟹ *homoclinic orbits*

Newhouse thm uses the closing lemma.

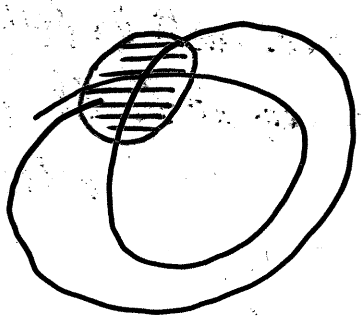
<u>Closing</u> <u>Lemma</u> is not known for geodesic flows.

<u>reason</u> : Proof uses <u>local</u> perturbations.



Perturbations of riemannian metrics $g_{ij}(x)$ are never local in phase space = SM.

"the orbit to close could have passed through the cylinder before comming back"



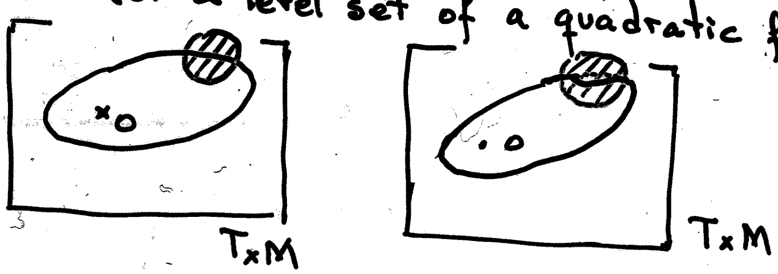
Newhouse theorem for geodesic flows is only known for $M = S^2$ or $\mathbb{RP}^2$

ETDS 2004.

<u>General Finsler Metrics</u>

= norm $\| \ \|_x$ on tangent spaces $T_x M$

unit sphere does not need to be symmetric (or a level set of a quadratic form)



- Closing Lemma holds
- Newhouse theorem should hold.

INGREDIENTS OF THE PROOF

① Kupka - Smale Theorem (for Geod. Flows)

M^{n+1}

$J_s^k(n) = \left\{ \begin{array}{l} \text{k-Jets of symplectic} \\ \text{diffeos } f: (\mathbb{R}^{2n}, 0) \circlearrowleft \end{array} \right\}$

$Q \subset J_s^k(n)$ is invariant iff

$$\sigma Q \sigma^{-1} = Q \quad \forall \sigma \in J_s^k(n)$$

$\mathcal{R}^r(M) = C^\infty$ riem. metrics on M with C^r topology.

Theorem

If $Q \subset J_s^k(n)$ is open, dense and invariant

$\Rightarrow \forall r \geq k+1 \ \exists \mathcal{G} \subset \mathcal{R}^r(M)$ residual s.t.

(a) [Anosov, Klingenberg - Takens]
Poincaré maps of all periodic orbits of ϕg are in Q.

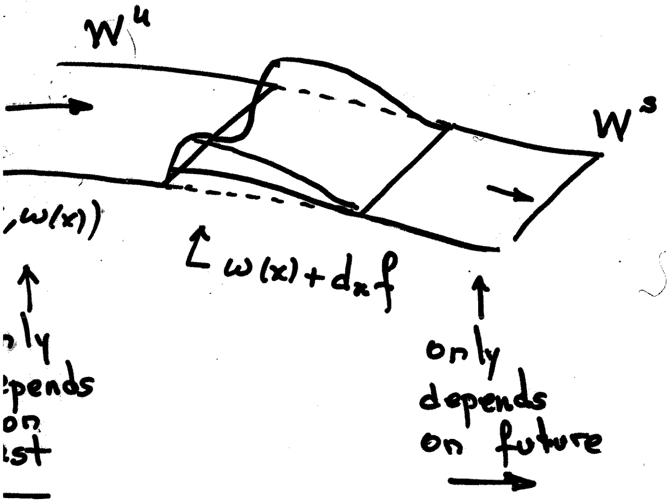
(b) All heteroclinic intersections are transversal.

OBS:

(a) Also holds for Q residual and invariant.

(b) Donnay for $n=2$, Petroll $n>2$ show how to perturb a single non-transverse intersection.
But perhaps this is not enough.

Simple Proof of (b) [$\pitchfork$ inters.]:



- W^s is Lagrangian in (T^*M, ω_0)
- Choose place where is locally a lagrangian graph
- Deform to another lagrangian graph (by adding a d_xf)

  $\omega_0 = dp\wedge dx$ fixed cannonical sympl. form.
- Change metric s.t. $H(\text{new } W^s) = 1$

$\Rightarrow$ New W^s is invariant.

(2) Elliptic Fixed Points

Symplectic Diffeomorphism $F: (\mathbb{R}^{2n}, 0) \circlearrowleft$ will be Poincaré map of closed orbit.

elliptic periodic point $:=$ non-hyperbolic.

If q-elliptic $\Rightarrow \exists$ $2q$-dim. central manifold which is normally hyperbolic.

We choose $Q \subset J^3_s(n)$: 3-Jets of sympl. C^∞ diffeos
$F : (\mathbb{R}^{2n}, 0) \circlearrowleft$
s.t. map restricted to central manifold is "weakly monotonous" twist map.

i.e. (a) Elliptic eigenvalues $\rho_1, \dots, \rho_q, \bar\rho_1, \dots, \bar\rho_q$ are 4-elementary:

$$1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4 \implies \prod_{i=1}^{q} \rho_i^{\nu_i} \neq 1.$$

(b) Birkhoff normal form

$$Z_k = e^{2\pi i \phi_k} + f_k(z)$$

$$\phi_k = a_k + \sum_{l=1}^{q} \beta_{kl} |z_l|^2$$

satisfies $\det[\beta_{kl}] \neq 0$.

Using techniques of Moser, Herman, M.C. Arnaud

Theorem: If $F : (\mathbb{R}^{2n}, 0) \circlearrowleft$ germ of sympl. diffeo s.t. (a) F is Q-Kupka-Smale.
(b) 0 is elliptic fixed point.

$\Rightarrow$ F has a 1-elliptic periodic point.

In particular, F has a $\pitchfork$ homoclinic orbit.

③ <u>Rademacher Theorem</u>

$\exists\ \mathcal{D} \subset \mathcal{R}^\infty(M)$ Residual set s.t.

$g \in \mathcal{D} \Rightarrow (M, g)$ has infinitely many prime closed geodesics.

Moreover, one can take

$\mathcal{D}$ = bumpy metrics = eigenvalues of Poincaré maps are not roots of 1.

④ <u>Theory of Dominated Splittings</u> [Mañé]

"If one can not perturb in C^2 topology to create an elliptic periodic orbit
$\Rightarrow$ closure of hyperbolic per. orbits is uniformly hyperbolic."

[$\Rightarrow$ (Spectral Decomposition Thm) contains a horseshoe]

Theory of Dominated Splittings

$Sp(n) :=$ symplectic linear isom. of $\mathbb{R}^{2n}$

sequence $\xi : \mathbb{Z} \to Sp(n)$ is periodic if $\exists m \quad \xi_{i+m} = \xi_i \quad \forall i \in \mathbb{Z}$] will be time 1 Poincaré map

A Periodic sequence ξ is hyperbolic if $\prod_{i=1}^{m} \xi_i$ is hyperbolic.

Family of periodic sequences $\xi = \{\xi^\alpha\}_{\alpha \in a}$
is bounded if $\exists B > 0 \quad \| \xi_i^\alpha \| < B \quad \forall i \in \mathbb{Z}, \forall \alpha \in a$
is hyperbolic if ξ^α is hyp. $\forall \alpha \in a$.

Families $\xi = \{\xi^\alpha\}_{\alpha \in a}$, $\eta = \{\eta^\alpha\}_{\alpha \in a}$ are periodically equivalent iff $\forall \alpha$ ξ^α, η^α have same periods.

Families ξ, η period. equiv. define

$$\| \xi - \eta \| := \sup \{ \| \xi_n^\alpha - \eta_n^\alpha \| : \alpha \in a, n \in \mathbb{Z} \}$$

This determines how to perturb:
up to a fixed amount in each time 1 - Poincaré map.
$\Rightarrow$ Following theorem would be usefull only in C^1- topology of flow
$= C^2$- topology of metric (or Hamiltonian)

Family ξ is <u>stably hyperbolic</u> iff
$\exists \epsilon > 0$ s.t. If η family period. equiv. to ξ
$\|\eta - \xi\| < \epsilon \Rightarrow \eta$ is hyperbolic.

Family ξ is <u>uniformly hyperbolic</u> iff
$\exists M > 0$ s.t.

$$\left\| \prod_{i=0}^{M} \xi^{\alpha}_{i+j} \Big|_{E^s_j(\xi^\alpha)} \right\| < 1/2 \ , \quad \left\| \left(\prod_{i=0}^{M} \xi^{\alpha}_{i+j} \Big|_{E^u_j(\xi^\alpha)} \right)^{-1} \right\| < 1/2$$

$\forall \alpha \in \mathcal{A}, \ \forall j \in \mathbb{Z}$.

Theorem

ξ Bounded periodic family
is stably hyperbolic
$\Longrightarrow \xi$ uniformly hyperbolic.

<u>Remark</u>:

- Families in $Sp(n)$: stably hyp $\Rightarrow$ unif. hyp.
- Families in $GL(\mathbb{R}^n)$: stably hyp $\Rightarrow$ dominated splitting

i.e.
$$\left\| \prod^{M} \xi^{\alpha}_{i+j} \Big|_{E^s} \right\| \cdot \left\| \left(\prod^{M} \xi^{\alpha}_{i+j} \Big|_{E^u} \right)^{-1} \right\| < 1/2$$

⑥ Perturbation Lemma: "Franks Lemma".

Example: statement for Diffeos $f: M \to M$.

$\exists \epsilon_0 > 0 \quad \forall \epsilon \in [0, \epsilon_0] \quad \exists \delta > 0$ s.t. if

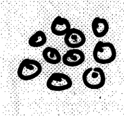
$\mathcal{F} = \{x_1, \dots, x_N\} \subset M$ any finite set

U any neighbourhood of $\mathcal{F}$

$A_i \in L(T_{x_i} M, T_{f(x_i)} M)$ "candidate for Df"

$\| Df(x_i) - A_i \| < \epsilon$

$\Rightarrow \exists g \in \mathrm{Diff}(M)$ s.t.



$$g|_{M \setminus U} = f|_{M \setminus U}$$

$$g(x_i) = f(x_i) \quad \forall x_i \in \mathcal{F}$$

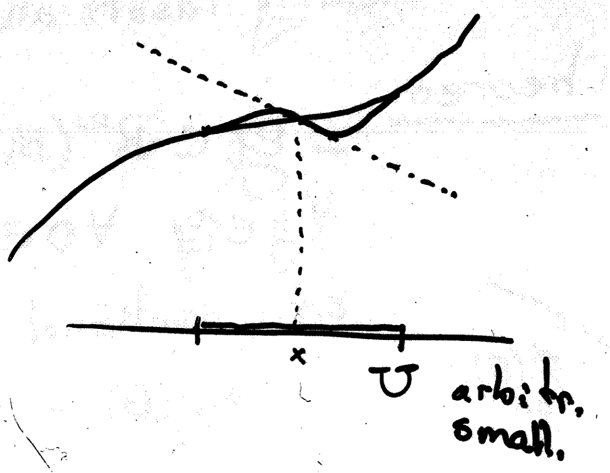
$$Dg(x_i) = A_i$$

$$\| f - g \|_{C^1} < \delta$$

arbitrarily small support

no size problem as in closing lemma.

Example: dimension 1



Analogous for geodesic flows:

realize any perturbation in $Sp(n)$ of a fixed distance of the derivative of the Poincaré map of any geodesic segment of length 1

- fixing the geodesic
- with support in an arbitrarily narrow strip U
- Outside small neighb. of given finitely many transversal segments

By a metric which is C^2 close.

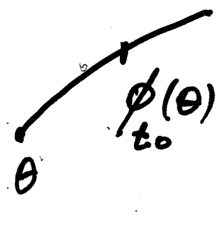
The perturbation is done on nbhd of one point.

Following result allowed to pass from dim 2 to dim $n>2$

Theorem $\exists\, \mathcal{G} \subset \mathcal{R}^\infty(M)$ residual s.t. $\forall g \in \mathcal{G}$ $\forall \theta \in SM$ $\exists\, t_0 \in [0, 1/2]$ s.t. sectional curvature matrix

$$K_{ij}(\theta) = \langle R(\theta, e_i)\theta, e_j \rangle$$

has no repeated eigenvalues.



The Perturbation Lemma

Derivative of the geodesic flow

$$d\phi_t(J(0), \dot{J}(0)) = (J(t), \dot{J}(t))$$

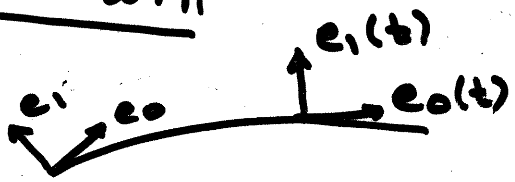
$J(t)$ = Jacobi field orthogonal to the geodesic $\gamma = \pi \circ \phi_t(\theta)$.

Jacobi Equation: $\ddot{J} + K(t)\, J = 0$

$$K(u,v) = \langle R(u,\dot\gamma)v, \dot\gamma \rangle$$

① Can change the Jacobi eq. at will

Use Fermi coordinates:



$e_0 = \dot\gamma$, $e_1, \dots, e_n$ = parallel transp. of orthonormal basis along γ.

$$F(t = x_0, x_1, \dots, x_n) = \exp\left(\sum_{i=1}^{n} x_i\, e_i(t)\right)$$

exp for a fixed initial metric g_0

Our general perturbation of the metric g^0 is:

$$g_{00}(t,x) = \left[g^0(t,x)\right]_{00} + \sum_{i,j=1}^{n} \alpha_{ij}(t,x)\, x_i\, x_j$$

$$g_{ij}(t,x) = \left[g^0(t,x)\right]_{ij} \qquad \text{if } (i,j) \neq (0,0)$$

This perturbation:

(1) Preserves the geodesic γ.

(2) Preserves the metric along γ.

(orthogonal vect. fields along γ are still $\perp$)

(3) Changes the curvature along γ by

$$K(t) = K_0(t) - \alpha(t,x)$$

(4) If the perturbation term is

$$x^* \alpha\, x = \varphi(x)\; x^* P(t)\, x$$

φ ← bump funct. in $x_1, \dots, x_n$

and $\operatorname{supp}(\varphi)$ is sufficiently small

$$\Rightarrow \; \| x^* \alpha\, x \|_{C^2} \sim \| P(t) \|_{C^0}$$

② **Estimate the perturbation in the solutions of the Jacobi equation.**

$$\ddot{J} + K(t)\,J = 0$$

$$\begin{bmatrix} J \\ \dot{J} \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} J \\ \dot{J} \end{bmatrix}$$

$$\mathring{X} = \mathbb{A} X, \quad X \in \mathbb{R}^{n\times n}$$

$$X(0) = I \Rightarrow X(t) = d\phi_t \leftarrow \text{Fundamental solution}$$

OBS:

- Can only perturb on K not on whole matrix $\mathbb{A}$
- Only perturbations $K \mapsto K + \alpha$ ← symmetric matrices

[because it was $x^* \alpha x$]

The solutions X are symplectic linear maps

$$Sp(n) = \{X \in \mathbb{R}^{n\times n} \mid X^* J X = J\}, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$T_X Sp(n) = \{XY \mid Y^* J + JY = 0\}$$

$$= \left\{ XY \mid Y = \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix} \quad \begin{matrix} \alpha, \gamma \text{ symmetric} \\ \beta \text{ arbitrary} \end{matrix} \right\}$$

We will use

$$x^{*} \angle\, x = h(t)\, \varphi(x)\, x^{*} \mathcal{I}(t)\, x$$

$\varphi(x)$ = bump function in $x_1, \dots, x_n$

$h(t)$ = *approximation of characteristic function of* $[0,1] \setminus F^{-1}(\mathcal{F}_1)$

i.e. $0 \le h(t) \le 1$, $\operatorname{supp}(h) \cap \left(\begin{smallmatrix}\text{intersecting}\\ \text{points}\end{smallmatrix}\ \mathcal{L}\right) = \phi$

$$\int_0^1 h \geq 1 - \epsilon$$

[only $\|h\|_{C^0} \leq 1$ counts if $\operatorname{supp}(\varphi)$ small]

$\mathcal{I}(t) = a\,\delta(t) + b\,\delta'(t) + c\,\delta''(t) + d\,\delta'''(t)$

$a, b, c, d \in \mathrm{Sym}(n \times n) =: S(n)$ $d_{ii} = 0$ $S^{*}(n)$

$\delta(t)$ = *approximation of Dirac δ at some point τ near $1/2$ where $K(t)$ has no repeated eigenvalues:*

$$\min_{i \neq j} |\lambda_i - \lambda_j| > \eta = \eta(\mathcal{U}) > 0$$

$\uparrow$ nbhd of g_0.

Strategy:

Think on 1-parameter family of metrics $s \mapsto g_s$

$$s \longmapsto K_s(t) = K(t) + s\,\alpha(t)$$

$$s \longmapsto X_s(t) = d\phi_t^{g_s}$$

$\alpha(t) = \alpha(t, E)$ $\qquad E = (a, b, c, d) \in S(n)^3 \times S^*(n)$

$S(n)$: sym$(n \times n)$; $S^*(n)$: sym$(n\times n)$, diag $\equiv 0$

same dim as $T_X Sp(n)$

Take the derivative

$$Z_s = \frac{dX_s}{ds} = \frac{d}{ds}\left(d\phi_t^s\right)$$

Prove that

$$\| Z_s(1) \| \geq k \| E \| \sim k \, \| x^* \alpha x \|_{C^2}$$

with $k = k(\mathcal{U})$

uniform for every geod. segment of length 1 and $\forall g \in \mathcal{U}$

$\Rightarrow \{ d\phi_1^g \mid g \in \mathcal{U} \}$ covers a neighbourhood of the original linearized Poincaré map $d\phi_1^{g_0}$ of size depending only on the C^2 norm of the perturbation

Derivative of the Jacobi equation

$$\mathring{X}_s = \mathbb{A}_s X_s$$

$$Z = \frac{dX_s}{ds}, \quad \mathbb{A}_s = \mathbb{A} + s\mathbb{B}, \quad \mathbb{B} = \begin{bmatrix} 0 & 0 \\ I(t) & 0 \end{bmatrix}$$

$$\mathring{Z} = \mathbb{A} Z + \mathbb{B} \qquad I(t) = a\delta(t) + b\delta'(t) + \dots$$

"variation of parameters": $Z = XY$

$$X\mathring{Y} = \mathbb{B} X$$

$$Y(t) = \int_0^t X^{-1} \mathbb{B} X$$

$$Z(1) = X(1) \underbrace{\int_0^1 X^{-1} \mathbb{B}(t) X \, dt}$$

Integrating by parts: want this to cover $\begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix}$ ← arbitrary

$$\int_0^1 X^{-1} \mathbb{B}(t) X \, dt \approx$$

$$\approx X_\tau^{-1} \left\{ \begin{bmatrix} & \\ a & \end{bmatrix} + \begin{bmatrix} b & \\ & -b \end{bmatrix} + \begin{bmatrix} & -2c \\ -(Kc + cK) & \end{bmatrix} + \begin{bmatrix} -Kd - 3dK & \\ & 3Kd + dK \end{bmatrix} \right\} X_\tau$$

↑ symmetric, not arbitrary

To solve

$$b - (Kd + 3dK) = \beta$$

↑ symm ↑ sym ↑ arbitrary

is equiv. to solve $Ke - eK = f$

↑ sym ↑ antisym

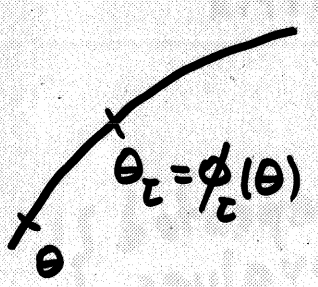
may not have solution <u>unless</u> K has no repeated eigenval

A generic condition on the curvature

Theorem $\exists \mathcal{G} \subset \mathcal{R}^\infty(M)$ residual s.t. $\forall g \in \mathcal{G}$ $\forall \theta \in SM$ $\exists \tau \in [0, 1/2]$ s.t. the Jacobi matrix

$$K_{ij}(\theta_\tau) = \langle R(\theta_\tau, e_i)\theta_\tau, e_j \rangle$$

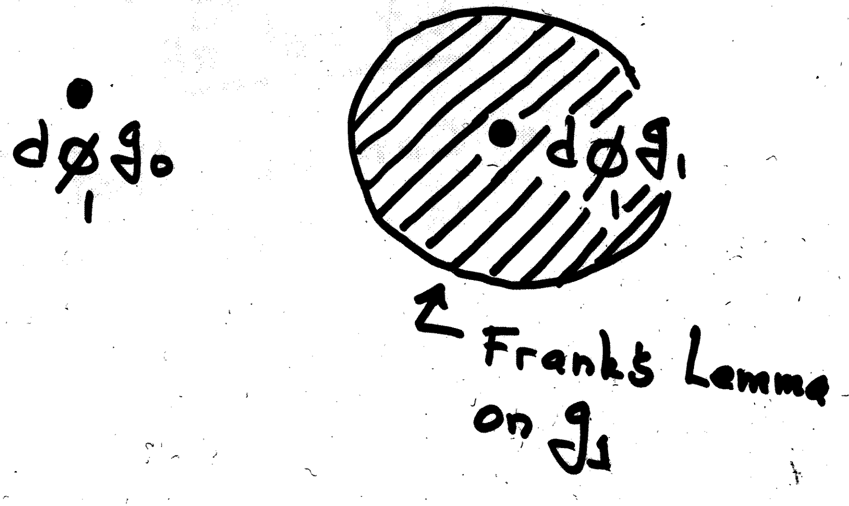
has no repeated eigenvalue.



- Why need this and not just a preliminary perturbation?

preliminary perturb. to separate the eigenvalues $\longrightarrow$ Franks lemma depends on amount of separation of e.v.'s

$$g_0 \longrightarrow g_1$$



<u>Strategy</u>: Use a transversality argument.

Know: can perturb Jacobi matrix (curvature) at will.

$$\Sigma = \{ A \in \underset{S(n)}{\underset{\shortparallel}{\mathrm{Sym}(n \times n)}} \mid A \text{ has repeated eigenvalues} \}$$

it is an algebraic set <u>with</u> singularities

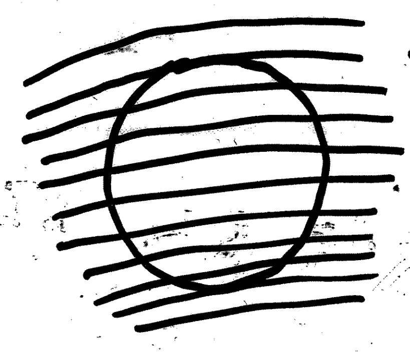
$$A \in \Sigma \iff \det P_A(A) = \prod (\lambda_i - \lambda_j)^2 = 0.$$

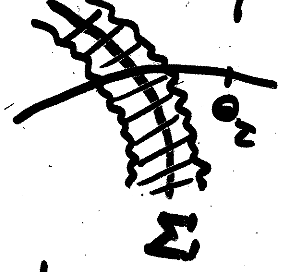
Enough to show that $\quad P_A(x) = \det[xI - A]$.

geodesic vector field "crosses Σ transversally".

<u>Example</u>:

- Flow in $\mathbb{R}^2$ without sing.
- $\Sigma = S^1$





- Can ask that a chosen orbit segment is $\pitchfork S^1$ <u>but not all</u>.
- Can ask that tangency is not of order 2.

If Σ were a smooth manifold:

$J^k\Sigma$ = k-jets of curves inside Σ.

$\dim J^k\Sigma = (k+1)\dim\Sigma$

coefs Taylor series $t \mapsto a_0 + a_1 t + \dots + a_k t^k$
in local chart $\quad a_i \in \mathbb{R}^\sigma, \ \sigma = \dim\Sigma$

$\dim J^k S(n) = (k+1)\dim S(n)$

$\operatorname{codim}_{S(n)} \Sigma = r \geq 1$

$\operatorname{codim}_{J^kS(n)} J^k\Sigma = (k+1)r \longrightarrow \infty$
when $k \to \infty$

$$F : \mathcal{R}^\infty(M) \times SM \times]0,1[\longrightarrow J^kS(n)$$
$$(g, \theta, \tau) \longmapsto J^k_\tau K(g, \phi_t\theta)$$

ϕ_t ← Jacobi matrix

J^k_τ ← k-jet at $t = \tau$

If $F \pitchfork J^k\Sigma$

$\Rightarrow \exists$ residual $\mathcal{G} \subset \mathcal{R}^\infty(M)$ s.t.

$g \in \mathcal{G} \Rightarrow F(g,\cdot,\cdot) \pitchfork J^k\Sigma$.

$\underline{k \text{ large}} \Rightarrow \text{codim } J^k\Sigma > \dim(SM \times]0,1[)$

$\pitchfork \Rightarrow$ no intersection

+ compacity argument $\Rightarrow$ required bounds on eigenvalues

use

$$\min_{\theta \in SM} \max_{t \in [0,1]} \prod |\lambda_i - \lambda_j|^2 > 0$$

when $\pitchfork$

But Σ has singularities

Algebraic Jet space

$\mathcal{L}_K(\Sigma) =$ polynoms $a_0 + a_1 t + \dots + a_k t^k = p(t)$
s.t. $f \circ p(t) \equiv 0 \pmod{t^{k+1}}$

Arc space

$\mathcal{L}_\infty(\Sigma) =$ formal power series $p(t)$
s.t. $f \circ p \equiv 0$

$\pi_k : \mathcal{L}_\infty(\Sigma) \longrightarrow \mathcal{L}_k(\Sigma)$ truncation

$\mathcal{L}_k(\Sigma)$ is an algebraic variety.

$\pi_k(\mathcal{L}_\infty(\Sigma)) \subset \mathcal{L}_k(\Sigma)$ is a finite union of algebraic subsets.

(it is "constructible")

$J^k\Sigma$ = k-jets of C^∞ curves in Σ

$\Longrightarrow \quad J^k\Sigma \subset \pi_k(\mathcal{L}_\infty(\Sigma)) \subset \mathcal{L}_k(\Sigma).$

<u>Denef & Loeser</u>:

$$\dim \pi_k(\mathcal{L}_\infty(\Sigma)) \leq (k+1) \dim \Sigma.$$

(same bound as in smooth case).