Plan

We will examine the existence of diffusion orbits around double resonances in near-integrable systems on $\mathbb{A}^3$ with convex unperturbed part. The course is divided into the following five (essentially) independent lectures which will all contain a set of open questions complementing our results. Our methods are (as far as possible) geometric, with a view to extensions to non-convex or non-perturbative diffusion using symplectic topology.

1. Dynamics at double resonances: examples and counter-examples. The goal of this first part is to show the crucial role played by double resonances on the near-integrable dynamics at a global level, and to examine that of the usual convexity assumptions. As a particular example we will show the construction of a (simple) perturbation of $\frac{1}{2}\|r\|^2$ with asymptotically dense orbits when the perturbation tends to 0 (joint work with L. Sabbagh). Examples of non-diffusive systems will be given, in order to motivate the introduction of the non-degeneracy assumptions which underly our study.

2. From Morse-Hedlund to generic hyperbolic properties of classical systems on $\mathbb{A}^2$. A classical system on $\mathbb{A}^n$ is the sum of a positive definite quadratic form in the action variables and a potential depending only of the angles. Classical systems on $\mathbb{A}^2$ appear as averaged systems at double resonances and are immediately relevant as approximations in an $O(\sqrt{\varepsilon})$ domain around the double resonance point (where ε is the perturbation parameter). For energies larger that the maximum of the potential, classical systems at fixed energies are essentially geodesic systems. We will use the Morse-Hedlund theory as described by Bangert to make explicit the hyperbolic structure of classical systems under a list of non-degeneracy assumptions. In particular, we will prove the existence of "chains of hyperbolic annuli" lying from the critical energy to the infinite energy.

3. A Birkhoff-smale theorem and more hyperbolic properties near the critical energy. We will prove an adapted version of the Birkhoff-Smale theorem for classical systems on $\mathbb{T}^2$, in the case where the potential admits a unique and non-degenerate maximum. We will then use this result to make precise the generic structure of the previous chains of annuli near the critical energy. We will finally prove the existence of a "singular" annulus, which admits heteroclinic connexions with the lower energy annuli in each previous chain. We will also prove the existence of heteroclinic connections between the former low-energy annuli themselves.

4. Analytic smoothing and flexible normal forms for finitely differentiable systems. We will use classical tools from analytic smoothing theory to obtain normal forms for initely differentiable (or Gevrey) systems from those of Pöschel, with minimal technicalities and a large amount of flexibility in the choice of the various parameters. This in particular makes very simple the reconstruction of the initial system from its averaged models, and the gluing of the hyperbolic objects obtained in different domains.

5. From the averaged system to the initial one. Starting with complete normal forms, we will prove the relevance of the chains of the annuli obtained in the classical systems to obtain (short) diffusion orbits crossing the double resonance along a simple resonance or exchanging from one simple resonance to another one. This necessitates additional non-degeneracy assumptions, which are essentially those introduced by Moeckel in its work on the drift on Cantor sets of annuli. We will relate the anisotropic stucture of the "splitting of separatrices" to the probability of crossing and exchange during a finite time.