Diameter constraint in Shape Optimization

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Outline

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Main question

We consider the Shape Optimization problem

$$\min \left\{ J(\Omega), \;\; \Omega \in \mathcal{S}_{ad}, \;\; \mathrm{Diam}(\Omega) = \alpha \;\right\}$$

where $\mathcal{S}_{ad}$ is a class of admissible convex domains in $\mathbb{R}^2$, and J is shape functional (rotation and translation invariant).

- Existence of a solution ?
- Geometrical description of the solution.
- Numerical computation of the solution.

Examples

- Isodiametrical inequality :

$$\frac{|\Omega|}{\mathrm{Diam}(\Omega)^N} \leq \frac{|B_1|}{2^N}.$$

- Spectral Gap theorem :

$$\lambda_2(\Omega) - \lambda_1(\Omega) \geq \frac{3\pi^2}{\mathrm{Diam}(\Omega)^2}$$

B. Andrews, J. Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. **24** (2011), 899–916.

- Another problem involving eigenvalues :

$$\min\left\{\sqrt{\lambda_1(\Delta,\Omega)} - \lambda_1(\Delta_\infty,\Omega),\ \ \Omega\ convex,\ \mathrm{Diam}(\Omega) = \alpha\right\}$$

M. Belloni, E. Oudet, *The Minimal Gap Between $\Lambda_2(\Omega)$ and $\Lambda_\infty(\Omega)$ in a Class of Convex Domains*, J. Convex Anal. **15** (2008), no. 3, 507–521.

Trivia

$$\min\left\{J(\Omega),\ \ \Omega\ convex,\ \ \mathrm{Diam}(\Omega)=\alpha\ \right\}$$

- If J is monotone increasing for inclusion, then solutions are segments.

  Examples : $J=|\cdot|$, $J=\mathrm{Per}$, $J=-\lambda_k$.

Trivia

$$\min\left\{ J(\Omega),\ \ \Omega \textit{ convex},\ \ \mathrm{Diam}(\Omega)=\alpha \right\}$$

- If J is monotone increasing for inclusion, then solutions are segments.

  Examples : $J = |\cdot|$, $J = \mathrm{Per}$, $J = -\lambda_k$.

- If J is monotone decreasing for inclusion, then solutions are of constant width α and we have

$$\min\left\{ J(\Omega),\ \ \mathrm{Diam}(\Omega)=\alpha \right\} = \min\left\{ J(\Omega),\ \Omega \textit{ convex of constant width } \alpha \right\}$$

  Examples : $J = -|\cdot|$, $J = -\mathrm{Per}$, $J = \lambda_k$.

Examples

- Generalization of the Gap functional : what shape realizes the minimum in :

$$\inf \left\{ \gamma[-\lambda_1(\Omega)] + \lambda_2(\Omega),\ \Omega \textit{ open and convex}, \mathrm{Diam}(\Omega) = \alpha \right\}$$

where $\gamma > 0$.

- Some reverse isoperimetric inequality : what shape realizes the minimum in :

$$\min \left\{ \gamma|\Omega| - \mathrm{Per}(\Omega),\ \Omega \textit{ convex}, \mathrm{Diam}(\Omega) = \alpha \right\}.$$

Outline

Parametrization of convex domain with its support function

FIGURE : a convex domain C

Definition :

$$h_C : \theta \in \mathbb{T} \mapsto \max_{c \in C} \left\langle c, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\rangle .$$

Geometrical interpretation : distance between the support line orthogonal to $d = (\cos\theta, \sin\theta)$ and the origin.

Examples

- Segment $\Sigma = \{0\} \times [-1, 1]$: $h_\Sigma(\theta) = |\sin\theta|$
- Rectangle R with corners $(\pm a, \pm b)$: $h_R(\theta) = a|\cos\theta| + b|\sin\theta|$
- Ellipse centered at 0 and semi-axes a, b : $h_{\mathcal{E}}(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$.

Second order arguments

Reformulation

The problem is now of the form

$$\min\left\{ j(h),\ \ h''+h\geq 0,\ \ \max_{\theta\in\mathbb{T}}\{h(\theta)+h(\theta+\pi)\}=\alpha\ \ \right\}.$$

where $j(h_\Omega):=J(\Omega)$.

Example

$$\gamma|\Omega|-P(\Omega)=\frac{\gamma}{2}\int_{\mathbb{T}}(h_\Omega^2-h_\Omega'^2)-\int_{\mathbb{T}}h_\Omega.$$

Remark : there is some concavity for j.

Second order arguments

Early work on convexity constraint : Theorem

$$\min\left\{j(h),\ h''+h\geq 0\right\}$$

Theorem (L., Novruzi, Pierre, 2011)

Let h be a minimizer. We assume j smooth and locally concave in the sense that,

$$j''(h)(v,v)<0, \text{ for any } v \text{ having a small enough support.}$$

Then $h''+h$ is a sum of Dirac masses inside the constraint.

Second order arguments

Early work on convexity constraint : Examples

- Reverse isoperimetry in an annulus :

$$\min\Big\{\gamma|\Omega| - P(\Omega),\ \ \Omega\ convex,\ D_a \subset \Omega \subset D_b\Big\}.$$

J. Lamboley, A. Novruzi *Polygon as optimal shapes with convexity constraint*, SIAM Control Optim. **48**, no. 5, (2009), 3003–3025

C. Bianchini, A. Henrot, *Optimal sets for a class of minimization problems with convex constraints*, J. Convex Anal. **19** (2012), no. 2.

- Mahler problem in 2d :

$$\min\{|\Omega||\Omega^\circ|,\ \ \Omega\ convex\}\ \ is\ achieved\ for\ \Omega = [-1,1]^2$$

E. Harrell, A. Henrot, J. Lamboley, *Analysis of the Mahler volume*, Preprint

- Faber-Krahn versus reverse isoperimetry :

$$\min\Big\{\lambda_1(\Omega) - P(\Omega),\ \ \Omega\ convex\ \subset D,\ \ |\Omega| = V_0\Big\},$$

the boundary $\partial\Omega^* \cap D$ is polygonal (difficulty : estimate $\lambda_1''(\Omega)$)

J. Lamboley, A. Novruzi, M. Pierre, *Regularity and singularities of optimal convex shapes in the plane*, Arch. Ration. Mech. Anal. **205**, no. 1 (2012), 311–343.

Second order arguments

Application to diameter constraint

$$\min\left\{J(\Omega),\ \ \Omega\ convex,\ \ \mathrm{Diam}(\Omega)=\alpha\ \right\},\ \ j(h_\Omega)=J(\Omega)$$

Theorem (Henrot, L., Privat, 2013)

Let Ω be a minimizer. We assume j smooth and locally concave. Then any connected component of

$$\partial\Omega\setminus\{x\in\partial\Omega, x \text{ belongs to a diameter of } \Omega\}$$

is made of a finite number of segments.

First order argument

$$\min\Big\{J(\Omega),\ \ \Omega\ convex,\ \ \mathrm{Diam}(\Omega)=\alpha\ \Big\},\ \ j(h_\Omega)=J(\Omega)$$

Theorem (Henrot, L., Privat, 2013)

Let Ω be a minimizer. We assume j is of integral form

$$j(h)=\int_{\mathbb{T}} G(h,h'),$$

and

$$\partial_1 G(\alpha,0)>0 \qquad \textit{or} \qquad \partial_1 G(\alpha,0)>\alpha\partial_{22}G(\alpha,0)$$

Then Ω saturates the diameter constraint at a finite number of points.

Outline

Reverse isoperimetric estimate

For $\gamma > 0$, we seek to minimize

$$J_\gamma(\Omega) = \gamma|\Omega| - \mathrm{Per}(\Omega).$$

among the sets Ω convex such that $\mathrm{Diam}(\Omega) = 1$.

Results

$$j(h) = \gamma \int_{\mathbb{T}} (h^2 - h'^2) - \int_{\mathbb{T}} h.$$

Theorem

Let $\gamma > 0$ and Ω_γ be a solution. Then

- *every point of $\partial\Omega_\gamma$ is either diametrical or belongs to a segment of $\partial\Omega_\gamma$.*
- *If $\gamma > 1/2$, then Ω_γ is a polygon with diametrical corners.*
- *The segment is a solution if and only if $\gamma \geq \frac{4}{\sqrt{3}}$.*
- *If $\gamma \leq \frac{1}{2}$, then the Reuleaux triangle is the unique minimizer.*

Proof : The first two statements : application of the previous section...

Large values of γ

We start with an under-statement : for $\gamma > 4$ then the segment is the unique solution.

- We fix a diameter $[AB]$ chosen as an axis, and the upper part of the shape is a graph of a concave function u

$$\left\{(x,u(x)), x\in\left[-\frac{1}{2},\frac{1}{2}\right]\right\}.$$

- We consider the perturbation $T_t : (x,y) \mapsto (x,(1-t)y)$ for $t \geq 0$ (affinity).
- We write the optimality condition for $t \mapsto J(T_t(\Omega_\gamma))$ minimized in $[0,1]$ by $t=0$, this leads to

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f'^2(x)}{\sqrt{1+f'^2(x)}}dx \geq \gamma \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx.$$

- We prove that for every $f \in H^1_0(-\frac{1}{2},\frac{1}{2})$ concave,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f'^2(x)}{\sqrt{1+f'^2(x)}}dx \leq 4 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx.$$

Large values of γ

$\hookrightarrow$ The previous statement is not optimal.

With more efforts, we can actually prove

Proposition

If $\gamma \geq \frac{4}{\sqrt{3}}$, then :

$$\gamma|\Omega| - \mathrm{Per}(\Omega) \geq -2,$$

for every Ω convex of diameter 1.

This is no longer true for $\gamma < \frac{4}{\sqrt{3}}$.

In other words, the segment is solution if and only if $\gamma \geq \frac{4}{\sqrt{3}}$.

Proof (optimality of the segment Σ)

Step 1 : shape reformulation

Σ is solution if and only if

$$\gamma \geq \gamma^* := \sup\left\{\frac{P(\Omega)-P(\Sigma)}{|\Omega|},\ \Omega \text{ convex }, \mathrm{Diam}(\Omega)=1\right\}$$

Step 2 : 1-sided shape reformulation

We denote $\Sigma=[AB]$ with $A=(-1/2,0)$, $B=(1/2,0)$, and $\mathbb{H}=\mathbb{R}\times\mathbb{R}_+$.
Then,

$$\gamma^* = \sup\left\{\frac{P_{\mathbb{H}}(\Omega)-1}{|\Omega|},\ \Omega \text{ convex }, \Sigma\subset\Omega\subset T\right\},$$

where $T=\{X\in\mathbb{H}, d(X,A)\leq 1,\ d(X,B)\leq 1\}$.

Proof (optimality of the segment Σ)

Step 3 : Angle reformulation in the class of polygons

We optimize the number p and the angles θ_0, θ_1, ..., θ_{p-1} at the left and similarly at the right.
$\hookrightarrow$ We end up with $p \leq 1$.

Proof (optimality of the segment Σ)

Step 4 : optimization in the class of quadrilateral

We optimize the position of L and R.
$\hookrightarrow$ The solutions are $L = R = D$ (equilateral triangle) or $[L = A, R = B]$ (segment)

Small values of γ

- $\gamma = 0$. So $J(\Omega) = -\mathrm{Per}(\Omega) \geq -\pi$ and any set of constant width is optimal.
- If $\gamma \leq \frac{1}{2}$,
  - with a non-local perturbation, we prove that Ω_γ has no segment in its boundary.
  - Using the first point in our Theorem, it implies Ω_γ is of constant width.
  - We conclude with the Blaschke-Lebesgue Theorem.

Transition value $\gamma \in (\frac{1}{2}, \frac{4}{\sqrt{3}})$, incomplete

Let Ω_γ be an optimal shape (which is a polygon).

- Let $[AB]$ be a diameter. Then either A or B is diametrically opposed to at least two points.
  Proof : order two argument.
- Situations which remains to be excluded :

Perspectives

- Complete the description for $\gamma \in (\frac{1}{2}, \frac{4}{\sqrt{3}})$.
- Replace $|\cdot|$ by λ_1.
- Replace Per by λ_1.
- Dimension 3 !