On the local minimizers of the Mahler volume

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Abstract

We focus on the analysis of local minimizers of the Mahler volume, that is to say the local solutions to the problem

$$\min\{M(K) := |K||K^\circ| \;/\; K \subset \mathbb{R}^d \text{ open and convex}, K = -K\},$$

where $K^\circ := \{\xi \in \mathbb{R}^d; \forall x \in K, x \cdot \xi < 1\}$ is the polar body of K, and $|\cdot|$ denotes the volume in $\mathbb{R}^d$. According to a famous conjecture of Mahler the cube is expected to be a global minimizer for this problem.

We express the Mahler volume in terms of the support functional of the convex body, which allows us to compute first and second derivatives, and leads to a concavity property of the functional. As a consequence, we prove first that any local minimizer has a Gauss curvature that vanishes at any point where it is defined. Going more deeply into the analysis in the two-dimensional case, we also prove that any local minimizer must be a parallelogram. We thereby retrieve and improve an original result of Mahler, who showed that parallelograms are global minimizers in dimension 2, and also the case of equality of Reisner, who proved that they are the only global minimizers.

Keywords: Shape optimization, convex geometry, Mahler conjecture.

1 Introduction and results

This paper is devoted to the analysis of local minimizers of the Mahler-volume functional. In particular we point out a concavity property of this functional, which supports the usual expectations about the minimizers, according to a well-known conjecture of Mahler.

Notation:

Let $K \subset \mathbb{R}^d$ be a convex body, that is, K is nonempty, open, convex, and bounded. We can define the polar dual body of K:

$$K^\circ := \left\{ \xi \in \mathbb{R}^d \;/\; \forall x \in K,\ x \cdot \xi < 1 \right\}.$$

The polar dual is always another convex body. The *Mahler volume* $M(K)$ of K is defined as the product of the volumes of K and its polar dual:

$$M(K) := |K||K^\circ|,$$

where $|\cdot|$ denotes the volume in $\mathbb{R}^d$.

We say that K is symmetric when K is centrally symmetric, that is $-K = K$. In that case, one can interpret K as the unit ball of a Banach-space norm on $\mathbb{R}^d$, for which K° is simply the unit ball for the dual norm, and is also symmetric.

It is well known that the ball maximizes the Mahler volume among convex symmetric bodies: this is the Blaschke-Santaló inequality:

$$\forall K \text{ convex symmetric body, } M(K) \leq M(B^d),$$

where B^d is the unit Euclidean ball, with equality if and only if K is an ellipsoid.

The corresponding minimization problem is the subject of a notorious and difficult conjecture, which is the main motivation for this paper. Let us recall this conjecture:

The symmetric Mahler conjecture:

The symmetric version of Mahler's conjecture asserts that for all convex symmetric bodies $K \subset \mathbb{R}^d$, we should have

$$M(Q^d) = M(O_d) \leq M(K), \tag{1}$$

where Q^d is the unit cube and $O_d = (Q^d)^\circ = \{x \in \mathbb{R}^d \ / \ \sum_i |x_i| < 1\}$ is the unit octahedron.
In [5], Kuperberg gives the strongest known lower bound:

$$\forall K \text{ symmetric convex body of } \mathbb{R}^d, \ (\pi/4)^{d-1} M(Q^d) \leq M(K).$$

The proof of Mahler's conjecture in dimension 2 appeared already in the original paper of Mahler [9]. It has also been proved by Reisner in [15] that equality in (1) is attained only for parallelograms. In higher dimensions, (1) is still an open question.

It should be remarked at this point that Q^d and $O_d = (Q^d)^\circ$ are not the only expected minimizers. In the first place, the Mahler volume is not only invariant by duality ($M(K) = M(K^\circ)$) but is also an affine invariant, in the sense that if $T : \mathbb{R}^d \to \mathbb{R}^d$ is a linear invertible transformation, then $M(T(K)) = M(K)$. Secondly, there is a sort of invariance with dimension:

$$M(Q^{d_1} \times O_{d_2}) = \frac{M(Q^{d_1}) M(O_{d_2})}{\begin{pmatrix} d_1 + d_2 \\ d_1 \end{pmatrix}} = M(Q^{d_1+d_2}),$$

since $M(Q^d) = \frac{4^d}{d!}$. Therefore products of cubes and octahedra, polar bodies of products of cubes and octahedra, etc. should also be minimizers. However, in dimension 2 and 3, the cube and the octahedra are the only expected minimizers, up to affine transformation. See [17] for further details and remarks.

The nonsymmetric conjecture:

One can also pose a nonsymmetric version of the Mahler conjecture, which is perhaps easier, because we expect the minimizer to be unique, up to invertible affine transformations that preserve a certain choice of the origin. To be precise, we introduce a more suitable version of the Mahler volume in this nonsymmetric setting: since $M(K)$ is not invariant by translation, one can choose the position of K so that it minimizes the Mahler volume:

$$\mathcal{P}(K) = \inf\{|K||(K-z)^\circ|, z \in int(K)\}$$

the minimum being attained at a unique $z = s(K)$, known as the Santaló point of K (see [12]). We refer to this functional as the *nonsymmetric Mahler volume*.

Denoting by Δ_d a d-dimensional simplex, it is conjectured (see [9, 4]) that for every convex body $K \subset \mathbb{R}^d$ containing 0,

$$\mathcal{P}(\Delta_d) \leq \mathcal{P}(K),$$

with equality only if and only if K is a d-dimensional simplex.

In this article we obtain some geometrical information about the minimizers of the Mahler volume in both settings. Our main result is the following;

Theorem 1.1 *Let K^* be a symmetric convex body in $\mathbb{R}^d$, which minimizes the Mahler volume among symmetric convex bodies:*

$$M(K^*) = \min \{ M(K) : \ K \,\text{convex symmetric body} \} . \tag{2}$$

If ∂K^* contains a relatively open set ω of class $\mathcal{C}^2$, then the Gauss curvature of K^* vanishes on ω.

The same result holds if K^* is no longer symmetric, and is a minimizer of the nonsymmetric Mahler volume among convex bodies containing 0:

$$\mathcal{P}(K^*) = \min\{\mathcal{P}(K) : \ \text{convex body} \ni 0\}. \tag{3}$$

Remark 1.1 The results remain true if K^* is only a local minimizer, in the sense given in Remark 3.3.

An easy consequence can be obtained on the regularity of minimizers:

Corollary 1.2 *If K^* is a minimizer for (2) or (3), then K^* cannot be globally $\mathcal{C}^2$.*

In order to prove Theorem 1.1, we apply the calculus of variations to a formulation of the Mahler volume in terms of the support function of the convex body, and we observe that the Mahler functional enjoys a certain concavity property, using the second-order derivative of the functional, see Lemmas 2.2 and 3.1 (these ideas are inspired by some results in [7, 2]). Therefore, it becomes quite natural that the minimizers should saturate the constraint, which is here the convexity of the shape. This result bolsters the conjecture and the intuition that the minimizers should contain flat parts, and that the Mahler volume should capture the roundness of a convex body.

Our conclusion can be considered as a local version of a result in [13], which asserts that if K belongs to the class $\mathcal{C}^2_+$, that is to say K is globally $\mathcal{C}^2$ and has a positive Gauss curvature everywhere on its boundary, then one can find a suitable deformation that decreases the Mahler volume (and preserves the symmetry if K is itself symmetric). Local minimality of the cube and the simplex are also proved in [11] and [4], respectively for problem (2) and (3).

Even though the Mahler conjecture is still a distant hope, we emphasize that our analysis can be strengthened. In particular, a deeper analysis of the concavity properties of the Mahler volume allows us to retrieve a proof of Mahler's conjecture in two dimensions, indeed a slight strengthening of the equality case in (1) of S. Reisner ([15]), dealing not only with global minimizers but also local minimizers:

Theorem 1.3 *In dimension 2, any symmetric* **local** *minimizer of (2) is a parallelogram.*

The word "local" is to be understood in the sense of the H^1-distance between the support functions of the bodies; refer to Remark 3.3 for more details. We note that some results related to Theorems 1.1 and 1.3 have recently appeared in the literature [8, 16]. Our analysis is self-contained and distinguished by the use of the new concavity property of the Mahler functional given in Lemma 2.2. (See also [10] for additional concavity properties.)

In the next section, we prove Theorem 1.1, and in the third section we focus on the 2-dimensional improvement.

2 Proof of the $d-$dimensional result

2.1 A variational formulation of the Mahler volume

If K is a convex body, one can define its support function $h_K : \mathbb{S}^{d-1} \to \mathbb{R}$ by:

$$h_K(\theta) := \sup \{ \ x \cdot \theta, \ x \in K \ \} .$$

It is well-known (see [14] for details, especially sections 1.7 and 2.5) that h_K characterizes the convex body K, and that its positive 1-homogeneous extension $\tilde{h}_K$ to $\mathbb{R}^d$ (that is to say $\tilde{h}_K(\lambda x) = \lambda h_K(x),\ \forall \lambda \geq 0, \forall x \in \mathbb{S}^{d-1}$) is convex. In that case we shall say that h_K is *convex*. Moreover, any functional $h : \mathbb{S}^{d-1} \to \mathbb{R}$, the extension of which is convex, is the support function of a convex body. The volumes of K and K° are conveniently written in terms of h_K:

$$|K| = \frac{1}{d}\int_{\mathbb{S}^{d-1}} h_K \det(h_K'' + h_K Id)d\sigma(\theta) \quad \text{ and } |K^\circ| = \frac{1}{d}\int_{\mathbb{S}^{d-1}} h_K(\theta)^{-d}d\sigma(\theta),$$

where h_K'' denotes the matrix of second covariant derivatives with respect to an orthonormal frame on $\mathbb{S}^{d-1}$. Hence the problem of minimizing the Mahler volume can be formulated as:

$$\min\left\{\frac{1}{d}\int_{\mathbb{S}^{d-1}} h\det(h'' + hId)d\sigma\frac{1}{d}\int_{\mathbb{S}^{d-1}} h^{-d}d\sigma, h : \mathbb{S}^{d-1} \to \mathbb{R} \text{ convex}\right\}.$$

In order to incorporate the symmetry constraint, one simply demands that admissible h be even.

These formulas are only valid if one can make sense of $\det(h'' + hId)$, which is not clear without regularity (one should use the surface area measure of Alexandrov [14]). Furthermore, some care is necessary in using the support function, because it is defined on the Gauss sphere, which is only a one-to-one image of ∂K in the smooth, strictly convex case. Since our argument is local in $\omega \subset \partial K^*$, assumed to be $\mathcal{C}^2$ (where K^* is a minimizer), one can restrict the calculation to $\Omega := \nu_{K^*}(\omega)$, where $\nu_{K^*} : \partial K^* \to \mathbb{S}^{d-1}$ is the Gauss map of the set K^*.

Lemma 2.1 *If K is a convex body, and $\omega \subset \partial K$ such that ω is $\mathcal{C}^2$ and the Gauss curvature is positive on ω, then $\Omega := \nu_K(\omega)$ is a nonempty open set in $\mathbb{S}^{d-1}$, h_K is $\mathcal{C}^2$ in Ω, and $\det(h_K'' + h_K Id) > 0$ on Ω.*

Proof. This is classical, and generally stated for convex bodies which are globally $\mathcal{C}^2$ and with a Gauss curvature everywhere positive, but the proof is actually local and so our lemma follows with usual arguments, see for example [14, Section 2.5, p. 106]. □

Therefore, the problem can be formulated as:

$$J(h_0) = \min\{J(h) := A(h)B(h), h : \mathbb{S}^{d-1} \to \mathbb{R} \text{ convex, } \mathcal{C}^2 \text{ in } \Omega \text{ and even}\}, \tag{4}$$

or, in the nonsymmetric case:

$$J(h_0) = \min\{J(h), h : \mathbb{S}^{d-1} \to \mathbb{R} \text{ convex, } \mathcal{C}^2 \text{ in } \Omega \text{ and positive}\}, \tag{5}$$

where

$$A(h) := \frac{1}{d}\int_\Omega h\det(h'' + hId)d\sigma \quad \text{ and } B(h) = \frac{1}{d}\int_\Omega h^{-d}d\sigma.$$

and $h_0 = h_{K^*}$ is $\mathcal{C}^2$ in Ω and $\det({h_0}'' + h_0 Id) > 0$ on Ω.
Note that an analytical characterization of convexity in terms of second-order derivatives in this context is:

$$\text{If the eigenvalues of } (h'' + hId) \text{ are nonnegative, then } h \text{ is convex.} \tag{6}$$

Remark 2.1 In (5), we drop the translation operation by the Santaló point, since this is an artificial constraint: a local minimizer among convex sets is also a local minimizer among sets whose Santaló point is zero, and reciprocally.

2.2 Concavity of the functional

We prove here a local concavity property of the functional (which implies that the second-order derivative is negative if the deformation has small support); see [7, 2] for similar results.

- **Nonsymmetric case:**

Lemma 2.2 *Let $\Omega \subset \mathbb{S}^{d-1}$ and $h : \mathbb{S}^{d-1} \to \mathbb{R}$ be of class $\mathcal{C}^2$ in Ω, with $det(h'' + hId) > 0$ on Ω. Then for all $U \subset\subset \Omega$, there exist $C = C(h, U)$, and $\alpha = \alpha(h, U) > 0$ such that:*

$$\forall v \in \mathcal{C}_c^\infty(U), \quad J''(h) \cdot (v, v) \leq C\|v\|^2_{L^2(\Omega)} - \alpha |v|^2_{H^1_0(\Omega)},$$

where $|v|^2_{H^1_0(\Omega)} = \int_\Omega |\nabla v|^2 d\sigma$ denotes the standard norm on $H^1_0(\Omega)$.

Proof.

Refering to [3, Proposition 5,6] for more detailed calculations, we get

$$\forall v \in \mathcal{C}_c^\infty(U), \qquad A'(h) \cdot v = \int_{\mathbb{S}^{d-1}} v \det(h'' + hId) d\sigma,$$

$$A''(h) \cdot (v, v) = \int_{\mathbb{S}^{d-1}} v \sum_{i,j \leq d-1} c_{ij}(\partial_{ij} v + \delta_{ij} v) d\sigma = \int_{\mathbb{S}^{d-1}} \left(Tr(c_{ij}) v^2 - \sum_{i,j \leq d-1} c_{ij} \partial_i v \partial_j v \right) d\sigma,$$

where $(c_{ij})_{1 \leq i,j \leq d-1}$ is the cofactor matrix of $(h'' + hId) = (\partial_{ij} h + h\delta_{ij})_{1 \leq i,j \leq d-1}$, that is to say $(c_{ij})_{1 \leq i,j \leq d-1} = \det(h'' + hId)(h'' + hId)^{-1}$. (For the last formula, we integrate by parts and use Lemma 3 in [3].)
Moreover, we easily get

$$B'(h) \cdot v = -\int_{\mathbb{S}^{d-1}} \frac{v}{h^{d+1}} d\sigma, \quad B''(h) \cdot (v, v) = (d+1) \int_{\mathbb{S}^{d-1}} \frac{v^2}{h^{d+2}} d\sigma.$$

Therefore

$$\begin{aligned}
(AB)''(h) \cdot (v, v) &= A''(h) \cdot (v, v) B(h) + 2A'(h) \cdot (v) B'(h) \cdot (v) + B''(h) \cdot (v, v) A(h) \\
&= A(h)(d+1) \int_{\mathbb{S}^{d-1}} \frac{v^2}{h^{d+2}} d\sigma - 2 \int_{\mathbb{S}^{d-1}} v \det(h'' + hId) d\sigma \int_{\mathbb{S}^{d-1}} \frac{v}{h^{d+1}} d\sigma \\
&\quad + B(h) \int_{\mathbb{S}^{d-1}} \left(Tr(c_{ij}) v^2 - \sum_{i,j \leq d-1} c_{ij} \partial_i v \partial_j v \right) d\sigma,
\end{aligned}$$

but the eigenvalues of the matrix (c_{ij}) are κ_i / κ, where κ is the Gauss curvature and κ_i are the principal curvatures [14, Corollary 2.5.2]. Therefore

$$\sum_{i,j \leq d-1} c_{ij} \partial_i v \partial_j v \geq \beta |\nabla v|^2,$$

where $\beta(\theta) = \min_i \kappa_i(\theta) / \kappa(\theta)$. This then leads to the result, with

$$C = (d+1) A(h) \left\| \frac{1}{h^{d+2}} \right\|_{L^\infty(U)} + 2 \left\| \frac{1}{\kappa} \right\|_{L^\infty(U)} \left\| \frac{1}{h^{d+1}} \right\|_{L^\infty(U)} \mathcal{H}^{d-1}(U) + B(h) \left\| \frac{H}{\kappa} \right\|_{L^\infty(U)},$$

where $H = \sum_i \kappa_i$, and $\alpha = B(h) \min_{\theta \in U} \beta(\theta)$, which is positive since $det(h'' + hId) > 0$ on Ω. □

• **Symmetric case:**
Assuming, without restriction, that Ω is included in one hemisphere, for any $v \in \mathcal{C}_c^\infty(\Omega)$, one can consider the following symmetrization of perturbations, which helps by preserving the symmetry constraint:

$$\tilde{v}(\theta) = \begin{cases} v(\theta) \text{ if } \theta \in \Omega, \\ v(-\theta) \text{ if } \theta \in -\Omega, \\ 0 \text{ otherwise.} \end{cases} \tag{7}$$

Lemma 2.3 *Let Ω be contained in one hemisphere of $\mathbb{S}^{d-1}$, and suppose that $h : \mathbb{S}^{d-1} \to \mathbb{R}$ is of class $\mathcal{C}^2$ in Ω, symmetric, and with $det(h'' + hId) > 0$ on Ω. Then for all $U \Subset \Omega$:*

$$\forall v \in \mathcal{C}_c^\infty(U), \quad J''(h) \cdot (\tilde{v}, \tilde{v}) \leq 4C\|v\|^2_{L^2(\Omega)} - 2\alpha |v|^2_{H^1(\Omega)},$$

where $C = C(h, U)$, and $\alpha = \alpha(h, U) > 0$ appears in Lemma 2.2.

Proof. The proof is the same as in Lemma 2.2 after noticing that:

$$(AB)''(h)\cdot(\tilde{v},\tilde{v}) = 2A''(h)\cdot(v,v)B(h) + 8A'(h)\cdot(v)B'(h)\cdot(v) + 2B''(h)\cdot(v,v)A(h),$$

thanks to the symmetry of h and $\tilde{v}$. □

2.3 Conclusion

Proof of Theorem 1.1:

• **Symmetric case:** Let K^* an optimal set for (2), with ω a $\mathcal{C}^2$ subset of ∂K^* where the Gauss curvature is positive. Then with Lemma 2.1, $h_0 = h_{K^*}$ is optimal for problem (4), where $\Omega = \nu_{K^*}(\omega)$. We introduce a nonempty open set $U \Subset \Omega$. Then, for all $v \in \mathcal{C}_c^\infty(U)$, $h_0 + tv$ is still the support function of a convex set for sufficiently small $|t|$: indeed, the eigenvalues of $(h_0+tv)'' + (h_0+tv)Id$ are nonnegative, since they are close to those of $h_0'' + h_0$, which are positive, and we use (6). Therefore we only need to preserve symmetry in order for $h_0 + tv$ to be admissible.

Assuming, without restriction, that Ω is included in one hemisphere, using the symmetrization (7), $h_0 + t\tilde{v}$ is admissible for small $|t|$, and the second-order optimality condition yields

$$\forall v \in \mathcal{C}_c^\infty(U), 0 \leq J''(h_0)\cdot(\tilde{v},\tilde{v}) \leq 4C\|v\|^2_{L^2(\Omega)} - 2\alpha|v|^2_{H_0^1(\Omega)},$$

with Lemma 2.3. This would imply the false imbedding $L^2(U) \subset H_0^1(U)$, which is a contradiction. □

• **Nonsymmetric case:** A similar proof as for the symmetric case applies: indeed, we no longer need to restrict ourselves to symmetric perturbations, and as explained in Remark 2.1, we only ask admissible convex functions to be positive (to ensure that the set K contains 0), a property that is preserved for small, smooth perturbations of $h_0 > 0$. Therefore the same argument as before follows with Lemma 2.2. □

3 Proof of the 2-dimensional result

In this section, we focus on the case $d = 2$, and our analysis enables us to retrieve the results of Mahler and Reisner [9, 15], that is to say inequality (1) with the case of equality, and we even slightly improve them with versions that are local.

3.1 A variational formulation of the Mahler volume

We express the functional in terms of the support function, and since we work in dimension 2, we are now able to write the Mahler volume without any regularity assumption.

Using polar coordinates, we regard θ as in $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ rather than in $\mathbb{S}^1$, and therefore $h_K : \mathbb{T} \to \mathbb{R}$ is viewed as a 2π-periodic function. Therefore,

$$M(K) = \frac{1}{2}\int_{\mathbb{T}} (h_K^2(\theta) - h_K'^2(\theta))d\theta \int_{\mathbb{T}} \frac{1}{2h_K^2(\theta)}d\theta,$$

and the convexity constraint on the set can be written $h_K'' + h_K \geq 0$, in the sense of a periodic distribution on $\mathbb{R}$. This implies for example that $h_K \in W^{1,\infty}(\mathbb{T})$. We are therefore interested in the following optimization problems:

$$J(h_0) = \min\left\{\, J(h) := A(h)B(h),\ h'' + h \geq 0 \text{ and } \forall\, \theta \in \mathbb{T},\ h(\theta) = h(\theta+\pi) \,\right\}, \tag{8}$$

and, in the nonsymmetric case,

$$J(h_0) = \min\left\{\, J(h),\ h'' + h \geq 0 \text{ and } h > 0 \,\right\},$$

with the same notation as in the previous section:

$$A(h) = \frac{1}{2}\int_{\mathbb{T}} (h^2 - h'^2)d\theta, \quad B(h) = \int_{\mathbb{T}} \frac{1}{2h^2}d\theta.$$

3.2 Concavity of the functional

We now prove a 2-dimensional version of Lemmas 2.2 and 2.3, dropping the regularity assumption on h:

- **Nonsymmetric case:**

Lemma 3.1 *If $h \in H^1(\mathbb{T})$ such that $h'' + h \geq 0$, then there exists $C = C(h)$, and $\alpha = \alpha(h) > 0$ such that:*

$$\forall v \in \mathcal{C}^\infty(\mathbb{T}), \quad J''(h) \cdot (v, v) \leq C\|v\|_{L^\infty(\mathbb{T})}\|v\|_{L^1(\mathbb{T})} - \alpha |v|^2_{H^1(\mathbb{T})}.$$

Proof. We easily get:

$$\begin{aligned} A'(h) \cdot v &= \int_{\mathbb{T}} hv - h'v', \quad A''(h) \cdot (v, v) = \int_{\mathbb{T}} v^2 - v'^2, \\ B'(h) \cdot v &= -\int_{\mathbb{T}} \frac{v}{h^3}, \qquad B''(h) \cdot (v, v) = 3\int_{\mathbb{T}} \frac{v^2}{h^4}, \end{aligned}$$

and so

$$\begin{aligned} (AB)'(h) \cdot v &= B(h)\int_{\mathbb{T}} (hv - h'v') - A(h)\int_{\mathbb{T}} \frac{v}{h^3}, \\ (AB)''(h) \cdot (v, v) &= B(h)\int_{\mathbb{T}} (v^2 - v'^2) - 2\int vd(h'' + h)\int_{\mathbb{T}} \frac{v}{h^3} + 3A(h)\int_{\mathbb{T}} \frac{v^2}{h^4}, \end{aligned}$$

where $h'' + h$ is a nonnegative Radon measure on $\mathbb{T}$. The following local concavity estimate follows:

$$(AB)''(h)\cdot(v, v) \leq \left(B(h) + 3A(h)\|1/h^4\|_{L^\infty(\mathbb{T})}\right)\|v\|^2_{L^2(\mathbb{T})} + 2(h''+h)(\mathbb{T})\|1/h^3\|_{L^\infty(\mathbb{T})}\|v\|_{L^\infty(\mathbb{T})}\|v\|_{L^1(\mathbb{T})} - B(h)|v|^2_{H^1(\mathbb{T})},$$

where $(h'' + h)(\mathbb{T})$ is the total mass of the measure $h'' + h$. This leads to the result with $\alpha = B(h) > 0$, since $\|v\|^2_{L^2(\mathbb{T})} \leq \|v\|_{L^\infty(\mathbb{T})}\|v\|_{L^1(\mathbb{T})}$. □

Remark 3.1 One can also conclude that

$$\forall v \in \mathcal{C}^\infty(\mathbb{T}) \text{ such that } (AB)'(h) \cdot v = 0, \quad (AB)''(h) \cdot (v, v) \leq C\|v\|^2_{L^2(\mathbb{T})} - \alpha|v|^2_{H^1(\mathbb{T})},$$

since $(AB)'(h) \cdot v = 0$ implies that the middle term $2(A'(h) \cdot v)(B'(h) \cdot v)$ is nonpositive.

- **Symmetric case:** With the new parametrization of $\mathbb{T}$, the symmetrization procedure of $v \in H^1_0(0, \pi)$ becomes:

$$\tilde{v}(\theta) = \begin{cases} v(\theta) \text{ if } \theta \in (0, \pi) \\ v(-\theta) \text{ if } \theta \in (-\pi, 0). \end{cases} \tag{9}$$

Lemma 3.2 *If $h \in H^1(\mathbb{T})$ is* **symmetric***, and such that $h'' + h \geq 0$, then*

$$\forall v \in \mathcal{C}^\infty(\mathbb{T}) \cap H^1_0(0, \pi), \quad J''(h) \cdot (\tilde{v}, \tilde{v}) \leq 4C\|v\|_{L^\infty(\mathbb{T})}\|v\|_{L^1(\mathbb{T})} - 2\alpha|v|^2_{H^1(\mathbb{T})},$$

where $C = C(h)$, and $\alpha = \alpha(h) > 0$ are as in Lemma 3.1.

Proof. Similar to the proof of Lemma 2.3, with the help of Lemma 3.1. □

3.3 Any local minimal set is a polygon

We cannot directly apply Theorem 2.1 from [7], since our functional is not exactly of the type of the ones considered there, and also because the constraints are slightly different, but one can follow the same argument, as is done in the following lines.

- **Symmetric case:**

Assume for the purpose of contradiction that $\Omega^* = \Omega_{h_0}$, a local minimizer, is not a polygon. Then there must exist an accumulation point θ_0 of $\text{supp}(h_0'' + h_0)$.

Without loss of generality we may assume that $\theta_0 = 0$ and also that there exists a decreasing sequence (ε_n) tending to 0 such that $\text{supp}(h_0'' + h_0) \cap (0, \varepsilon_n) \neq \emptyset$. As in [7] we follow an idea of T. Lachand-Robert and M.A. Peletier (see [6]): for any $n \in \mathbb{N}$, we choose $0 < \varepsilon_n^i < \varepsilon_n$, $i \in [\![1, 4]\!]$, increasing with respect to i, such that $\text{supp}(h_0'' + h_0) \cap (\varepsilon_n^i, \varepsilon_n^{i+1}) \neq \emptyset$, $i = 1, 3$. We consider $v_{n,i}$ solving

$$v_{n,i}'' + v_{n,i} = \chi_{(\varepsilon_n^i, \varepsilon_n^{i+1})}(h_0'' + h_0), \quad v_{n,i} = 0 \text{ in } (0, \varepsilon_n)^c, \ i = 1, 3.$$

Such $v_{n,i}$ exist since ε_n^i have been chosen so as to avoid the spectrum of the Laplace operator with Dirichlet boundary conditions. Next, we look for $\lambda_{n,i}, \ i = 1, 3$ such that $v_n = \sum_{i=1,3} \lambda_{n,i} v_{n,i}$ satisfy

$$v_n'(0^+) = v_n'(\varepsilon_n^-) = 0.$$

The above derivatives exist since $v_{n,i}$ are regular near 0 and ε_n in $(0, \varepsilon_n)$. We can always find such $\lambda_{n,i}$, as they satisfy two linear equations. This implies that v_n'' does not have any Dirac mass at 0 and ε_n, and therefore, $h + tv_n$ is the support function of a convex set, for $|t|$ small enough (n now being fixed). Therefore $h + t\widetilde{v_n}$ is an admissible function for (8) (see (9) for the definition of $\tilde{v}$).

Therefore the second-order optimality condition yields

$$0 \leq J''(h_0) \cdot (\widetilde{v_n}, \widetilde{v_n}) \leq 4C\|v_n\|_{L^\infty(\mathbb{T})}\|v_n\|_{L^1(\mathbb{T})} - 2\alpha|v_n|^2_{H^1(\mathbb{T})} \leq (4C\varepsilon_n^2 - 2\alpha)|v_n|^2_{H^1(\mathbb{T})}$$

using Lemma 3.1 and the Poincaré inequality

$$\forall\, v \in H^1(\mathbb{T}) \text{ such that } \text{supp}(v) \subset [0, \varepsilon], \forall x \in [0, \varepsilon], \ |v(x)| \leq \sqrt{\varepsilon}|v|_{H^1(\mathbb{T})},$$

with $\varepsilon = \varepsilon_n$.
As ε_n tends to 0, this inequality becomes impossible, which proves that $\text{supp}(h_0'' + h_0)$ has no accumulation points. It follows that $h_0'' + h_0$ is a sum of positive Dirac masses, which is to say that Ω^* is a polygon.

Remark 3.2 As in Section 2.3, a similar argument applies in the nonsymmetric case.

3.4 Proof of Theorem 1.3 in the Symmetric case

Let K be a local minimizer of problem (2). By "local," we mean that K is minimal among all convex sets whose support function is close to that of K in the H^1-norm.

Remark 3.3 More precisely, we say that K is a local minimizer for (2) if there exists $\varepsilon > 0$ such that

$$\forall L \text{ convex symmetric body such that } \|h_L - h_K\|_{H^1(\mathbb{T})} \leq \varepsilon, \quad M(K) \leq M(L).$$

Another useful distance is the Hausdorff distance, expressible through the support functions by $\|h_L - h_K\|_{L^\infty(\mathbb{T})}$. It is an easy consequence of the Poincaré inequality that the Hausdorff distance is bounded above by the H^1-distance, up to an universal constant (see [1] for example).

The converse inequality is not clear, but one can prove that the convergence in the sense of Hausdorff implies the convergence in the H^1-distance, and so there is topological equivalence. We give a short sketch of proof of this last property:

if h_n, h_∞ are such that $h_n'' + h_n \geq 0, h_\infty'' + h_\infty \geq 0$, and $h_n \to h_\infty$ in $L^\infty(\mathbb{T})$, then it is easy to see that h_n is bounded in $W^{1,\infty}(\mathbb{T})$ by a constant C (see for example [7, Lemma 4.1]), and therefore that

$$\int_{\mathbb{T}} d|h_n''| \leq \int_{\mathbb{T}} d(h_n'' + h_n) + \int_{\mathbb{T}} d|h_n| \leq 2\int_{\mathbb{T}} d|h_n| \leq 2C.$$

Therefore h_n' is bounded in $BV(\mathbb{T})$, so up to a subsequence, $h_n' \to h'$ a.e. and in $L^1(\mathbb{T})$ (by the compact imbedding of $BV(\mathbb{T})$ in L^1). We conclude with the dominated convergence theorem that $h_n \to h_\infty$ in $H^1(\mathbb{T})$, and by uniqueness of the accumulation point of h_n that the whole sequence converges.

Hence the support function h of K is a local minimizer for problem (8). From Section 3.3, we already know that K is a polygon, that is to say,

$$h'' + h = \sum_{i=0}^{2N-1} a_i \delta_{\theta_i} \text{ for some } N \in \mathbb{N}^*, \theta_i \in \mathbb{T} \text{ and } a_i > 0. \tag{10}$$

We want to prove that K is a parallelogram, that is to say $N = 2$ in (10).

As in Section 3.3, we would like to find a perturbation v such that $J''(h) \cdot (v, v) < 0$, which would be a contradiction. So that $h + tv$ remains admissible for all small t, we need $v'' + v$ to be supported within the support of $h'' + h$.

Remark 3.4 Again we shall symmetrize any perturbation $v \in H_0^1(0, \pi)$ with (9), which gives $J'(h) \cdot \tilde{v} = 2J'(h) \cdot v$ if h is symmetric, and

$$(AB)''(h) \cdot (\tilde{v}, \tilde{v}) = 2A''(h) \cdot (v, v)B(h) + 8A'(h) \cdot (v)B'(h) \cdot (v) + 2B''(h) \cdot (v, v)A(h). \tag{11}$$

• **Another expression for B and its derivatives:**

Since the expression for B is not very tractable from the geometric point of view, we would like to rewrite B and its derivatives when one knows that h is the support function of a polygon, and that v is a deformation such that $v'' + v$ is supported within the discrete set on which $h'' + h$ is nonzero.

Let us denote by $A_i, i = 0 \dots 2N - 1$ the vertices of K. Then the support function h is defined by

$$h(\theta) = \rho_i \cos(\theta - \alpha_i) \quad \text{for } \theta \in (\theta_i, \theta_{i+1}),$$

where $\rho_i = OA_i, \alpha_i = (\vec{e_1}, \overrightarrow{OA_i})$ and (θ_i, θ_{i+1}) are the two angles of the normal vectors of sides adjacent to A_i. Therefore

$$\begin{aligned} B(h) &= \int_0^{2\pi} \frac{d\theta}{2h^2(\theta)} = \sum_{i=0}^{2N-1} \int_{\theta_i}^{\theta_{i+1}} \frac{d\theta}{2\rho_i^2 \cos^2(\theta - \alpha_i)} \\ &= \sum_{i=0}^{2N-1} \frac{1}{2\rho_i^2} \tan(\theta - \alpha_i)\Big|_{\theta=\theta_i}^{\theta=\theta_{i+1}} = \sum_{i=0}^{2N-1} \frac{\sin(\theta_{i+1} - \theta_i)}{2h(\theta_i)h(\theta_{i+1})}. \end{aligned} \tag{12}$$

Now, when we replace h by $h + tv$ where $v'' + v = \sum_i \beta_i \delta_{\theta_i}$, the angles of the new polygon are unchanged, because $(h + tv)'' + (h + tv)$ is a sum of nonnegative Dirac masses at the same points, when t is small enough. Thus we can compute the first and second derivative of $B(h)$ using formula (12), obtaining:

$$B'(h) \cdot v = -\sum_{i=0}^{2N-1} \frac{\sin(\theta_{i+1} - \theta_i)}{2h(\theta_i)h(\theta_{i+1})} \left[\frac{v(\theta_i)}{h(\theta_i)} + \frac{v(\theta_{i+1})}{h(\theta_{i+1})}\right] = -\sum_{i=0}^{2N-1} \left[\frac{\sin(\theta_{i+1} - \theta_i)}{h(\theta_{i+1})} + \frac{\sin(\theta_i - \theta_{i-1})}{h(\theta_{i-1})}\right] \frac{v(\theta_i)}{h^2(\theta_i)}$$

and

$$B''(h) \cdot (v, v) = \sum_{i=0}^{2N-1} \frac{\sin(\theta_{i+1} - \theta_i)}{h(\theta_i)h(\theta_{i+1})} \left[\frac{v^2(\theta_i)}{h^2(\theta_i)} + \frac{v^2(\theta_{i+1})}{h^2(\theta_{i+1})} + \frac{v(\theta_i)v(\theta_{i+1})}{h(\theta_i)h(\theta_{i+1})}\right]. \tag{13}$$

Therefore the first optimality condition becomes: $A(h)B'(h) \cdot v + B(h)A'(h) \cdot v = 0$ for any v symmetric (*i.e.*, for any $v(\theta_i), i \in [\![0, N - 1]\!]$), and we get:

$$B(h)a_i - \frac{A(h)}{2h^2(\theta_i)} \left(\frac{\sin(\theta_{i+1} - \theta_i)}{h(\theta_{i+1})} + \frac{\sin(\theta_i - \theta_{i-1})}{h(\theta_{i-1})}\right) = 0 \quad \text{for } i = 0, \dots N - 1. \tag{14}$$

Remark 3.5 Simple calculations show that this first-order optimality conditions (14) is satisfied by any regular symmetric polygon. This explains why we need to analyze the second-order condition to get the conclusion.

- **Optimality conditions for a simple deformation:**

We choose v such that $v''+v=\alpha\delta_{\theta_1}$ and $v\in H^1_0(\theta_0,\theta_2)$. Therefore equations (14), (9), (11) and (13) give

$$\begin{aligned} J''(h)\cdot(\tilde v,\tilde v) &= 2B(h)\int vd(v''+v)-8\frac{B(h)}{A(h)}\left(\int vd(h''+h)\right)^2+2A(h)\left[\frac{\sin(\theta_2-\theta_1)}{h(\theta_2)}+\frac{\sin(\theta_1-\theta_0)}{h(\theta_0)}\right]\frac{v^2(\theta_1)}{h^3(\theta_1)} \\ &= 2B(h)\alpha v(\theta_1)-8\frac{B(h)}{A(h)}(a_1v(\theta_1))^2+2A(h)\left[\frac{2B(h)h(\theta_1)^2a_1}{A(h)}\right]\frac{v^2(\theta_1)}{h^3(\theta_1)} \\ &= 2B(h)\left[-\frac{\sin(\theta_2-\theta_0)}{\sin(\theta_2-\theta_1)sin(\theta_1-\theta_0)}-4\frac{a_1^2}{A(h)}+2\frac{a_1}{h(\theta_1)}\right]v^2(\theta_1), \qquad (15)\end{aligned}$$

where the last equality is obtained because a straightforward calculation gives $\alpha=-\frac{\sin(\theta_2-\theta_0)}{\sin(\theta_2-\theta_1)\sin(\theta_1-\theta_0)}v(\theta_1)$.

- **Conclusion**

Let us assume, for a contradiction, that K has at least 6 sides. Let $\theta_0,\theta_1,\theta_2$ be the three first angles of the normal, in such a way that the support function of K satisfies

$$h''+h=a_0\delta_{\theta_0}+a_1\delta_{\theta_1}+a_2\delta_{\theta_2}+\dots,$$

and $\theta_2-\theta_0<\pi$.
We recall that the Mahler functional is invariant by affine transformation. Therefore, if K is a local minimizer, the image of K by such a transformation T remains a local minimizer of J, since the neighbors of K are transformed in neighbors of $T(K)$ by T. By a small abuse, we keep the notation h as the support function of $T(K)$. This allows us to study the sign of (15) after a suitable transformation.

Using affine invariance, one can choose $\theta_0=0$, and $\theta_1-\theta_0=\pi/2$, which ensures that the polygon is contained in a rectangle of sides $2h(\theta_0),2h(\theta_1)$. With a further scaling we arrange that $h(\theta_0)=h(\theta_1)=1$ and choose an orientation so that $a_1\le a_0$, see Figure 1. Under these conditions $A<4$ (equality would imply the square, excluded by hypothesis), $\tan(\theta_2)<0$, and a trigonometrical calculation shows that

$$|\tan(\theta_2)|\ge\frac{2-a_1}{2-a_0}.$$

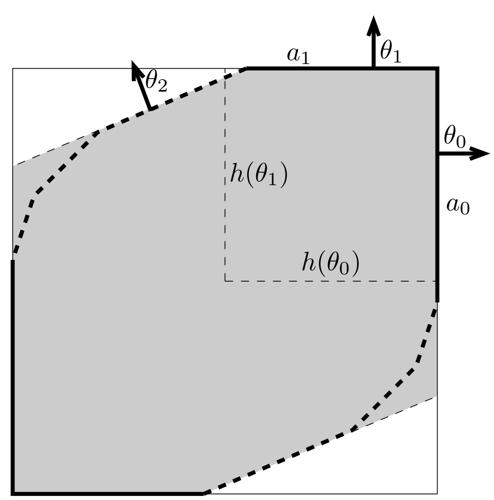

Figure 1: Estimate of (15) with $\theta_0=0,\theta_1=\pi/2,h(\theta_0)=h(\theta_1)=1,0\le a_1\le a_0\le 2$.

Therefore,

$$\begin{aligned}
-\frac{\sin(\theta_2-\theta_0)}{\sin(\theta_2-\theta_1)\sin(\theta_1-\theta_0)} - 4\frac{a_1^2}{A(h)} + 2\frac{a_1}{h(\theta_1)} &< \tan(\theta_2) - a_1^2 + 2a_1 \\
&\le -\frac{2-a_1}{2-a_0} - a_1^2 + 2a_1 \\
&= \frac{2-a_1}{2-a_0}\left(a_1(2-a_0)-1\right)
\end{aligned}$$

The factor $(a_1(2-a_0)-1)$ is a harmonic function, negative on the edges of the triangle $\{0 \le a_1 \le a_0 \le 2\}$ except when $a_1 = a_0 = 1$, where it equals 0. By the maximum principle it is always nonpositive in this triangle. Observing that the inequality in the first line is strict, we conclude that $J''(h)\cdot(\tilde{v},\tilde{v}) < 0$. This contradicts local optimality in the sense of the H^1-distance and concludes the proof of Theorem 1.3. □

Remark 3.6 The invariance of the Mahler functional under affine transformation cannot be simply expressed with the first and second derivatives of J, because the support function of $T(K)$ cannot be simply deduced from the support function of K. Nevertheless, we can prove that the quantity in (15) keeps a constant sign under affine transformation.

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