Shape optimization under convexity constraint

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Examples I

Isoperimetric inequality :

$$\min_{\Omega \subset \mathbb{R}^N,\ |\Omega| = V_0} P(\Omega).$$

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Faber-Krahn inequality :

$$\min_{\Omega\subset \mathbb{R}^N,\ |\Omega|=V_0} \lambda_1(\Omega).$$

Main goals

We seek to "solve" :

$$\min_{\Omega \in \mathcal{F}_{ad}} J(\Omega),$$

where J is a shape functional, $\mathcal{F}_{ad} \subset \{domains \;\; of \;\; \mathbb{R}^N\}$.

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- "Find" the minimizer (explicitly/numerically).

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What about the case $\mathcal{F}_{ad} \subset \{\textit{convex domains of } \mathbb{R}^N\}$.

Outline

1. Regularity for isoperimetric problems
   - Dimension 2
   - Higher dimension

2. Case of concave functionals
   - Higher Dimension
   - Dimension 2

Outline

Examples II

$$\min_{K \ \ convex,\ K\subset D} \Big[P(K) + F(K) \Big]$$

where $F(K) = f(|K|, \lambda_1(K))$ for some smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

$$\min_{K \ \ convex,\ K\subset D, |K|=V_0} \Big[P(K) + G(K) \Big]$$

where $G(K) = g(\lambda_1(K))$.

Without convexity constraint

Theorem

Let Ω be a quasi-minimizer of the perimeter, i.e.

$$P(\Omega) \leq P(\Omega') + Cr^{N-1+2\alpha}, \quad \forall \Omega' \text{ such that } \Omega \Delta \Omega' \Subset B_r(x)$$

for all x and r small.
Then it exists $\Sigma_{sing} \subset \partial\Omega$ such that

$$\partial\Omega \setminus \Sigma_{sing} \text{ is } C^{1,\alpha}$$

$$\dim_{\mathcal{H}} \Sigma_{sing} \leq N - 8$$

With convexity constraint

Results

$$\min_{K \text{ convex}, \, K\subset D} \Big[P(K) + F(K)\Big] \qquad \min_{K \text{ convex}, K\subset D, |K|=V_0} \Big[P(K) + G(K)\Big]$$

Theorem (L.-Novruzi-Pierre 2011)

$N = 2$. *The solutions of the first order optimality condition are* $C^{1,1}$.

Theorem (L. 2013)

$N \geq 2$. *The minimizers are* $C^{1,1}$.

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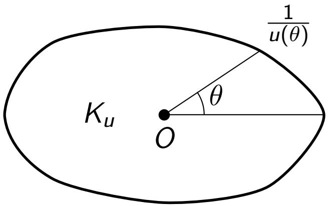
   - Higher Dimension
   - Dimension 2

Linear formulation of the convexity

To a periodic function $u : \mathbb{T} \to \mathbb{R}_+^*$, we associate

$$K_u = \{(r,\theta)\ ;\ 0 \le r < 1/u(\theta)\}.$$



Parametrization of a starshaped set.

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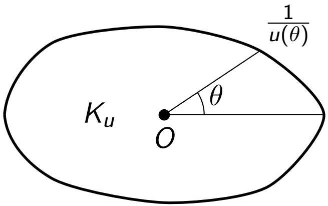


Parametrization of a starshaped set.

Then K_u convex $\Leftrightarrow u'' + u \geq 0$.

Linear formulation of the convexity

There is a one-to-one correspondance

$$\begin{array}{ccc} \{\textit{Convex 2d sets}\} & \xrightarrow{\sim} & \{v > 0 \in H^1(\mathbb{T}) \textit{ such that } v'' + v \geq 0\} \\ K_u & \longmapsto & u \end{array}$$

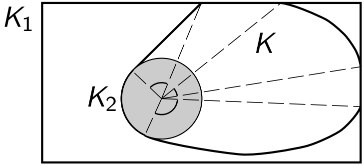
Reformulation of the problem

$$\min_{\substack{K\in\mathcal{F}_{ad}\\ K\ convex}} J(K) \qquad \min_{\substack{u\in S_{ad}\\ u''+u\geq 0}} \Big\{ j(u) := J(K_u) \Big\}$$

where S_{ad} is a functional space taking the other constraints into account.

Examples :

- $S_{ad} = \{u : \mathbb{T} \to \mathbb{R} \ / \ u_2 \leq u \leq u_1\}$



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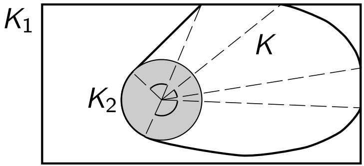

- $S_{ad} = \left\{ u : \mathbb{T} \to \mathbb{R} \ / \ |K_u| = \int_{\mathbb{T}} \frac{1}{2u^2(\theta)} d\theta = V_0 \right\}$

Proof of the regularity

$$\min\left\{\, j(u)=P(K_u)+F(K_u),\ u''+u\geq 0\,\right\}.$$

- Write the first optimality condition :

$$\left\{\begin{array}{l} j'(u)(v)=\langle \zeta''+\zeta, v\rangle_{\mathcal{D}'\times\mathcal{D}} \\ \zeta\geq 0, \qquad \zeta(u''+u)=0 \end{array}\right.$$

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- Observe that

$$j(u) = \int_{\mathbb{T}} G(u, u') + g(u)$$

with $j'(u)(v) = \langle -\partial_{u'u'} G(u, u')u'' + h, v\rangle_{\mathcal{D}'\times\mathcal{D}}$ and $h \in L^\infty$.

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  with $j'(u)(v)=\langle -\partial_{u'u'}G(u,u')u''+h, v\rangle_{\mathcal{D}'\times\mathcal{D}}$ and $h\in L^\infty$.
- Compare the signs of the singular measures in the optimality condition.

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Regularity in calculus of variations

$$\min\left\{\int_\Omega L(x,u(x),\nabla u(x)),\quad u\in H^1_0(\Omega)\right\}$$

$$L\ \ convex\ in\ \ \nabla u.$$

Model case with convexity constraint

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu, \quad u\in H_0^1(\Omega)\cap\mathcal{C}\right\}$$

où $\mathcal{C}=\left\{u:\overline{\Omega}\to\mathbb{R}, \quad u \textit{ convex } \right\}$.

Theorem (Caffarelli-Carlier-Lions 2013)

If f is in L^p $(p>N)$, and u is a minimizer, then u is $C^{1,1-N/p}$.

Proof of the regularity

$$\min\{P(K)+F(K), K \textit{ convex }\}.$$

- Write a minimizer K as a graph that solves locally

$$\min\left\{\int_D \sqrt{1+|\nabla u|^2}+f(u), u \textit{ convex }\right\}.$$

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- Adapt the Caffarelli-Carlier-Lions procedure to a convex Lagrangian : built $K_\varepsilon = graph(u_\varepsilon)$.
- Prove the estimate

$$|F(K)-F(K_\varepsilon)| \le C|K\setminus K_\varepsilon|.$$

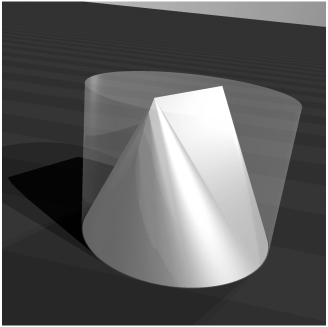
Outline

Example III

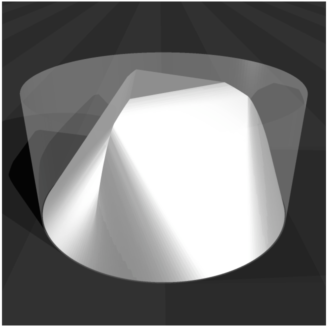
Newton's problem of the body of minimal resistance

$$\min\left\{\int_D \frac{1}{1+|\nabla f|^2} \;\Big/\; f:\{x\in\mathbb{R}^2,\ |x|\leq 1\}\to[0,M], f\ \ concave\right\}$$

Numerical computations : Lachand-Robert, Oudet, 2004 :



$M = 3/2$



$M = 1$

Example IV

Mahler conjecture

Conjecture : is the cube $Q_N := [-1, 1]^N$ solution of

$$\min \left\{ M(K) := |K||K^\circ|, \;\; K \text{ convex of } \mathbb{R}^N, \;\; -K = K \right\} ?$$

$$K^\circ := \left\{ \xi \in \mathbb{R}^N, \;\; \langle \xi, x \rangle \leq 1, \;\; \forall x \in K \right\}.$$

Example V

Conjecture of Pólya-Szegö

The electrostatic capacity of a set $\Omega \subset \mathbb{R}^3$ is defined by

$$Cap(\Omega) := \int_{\mathbb{R}^3 \setminus \Omega} |\nabla u_\Omega|^2 \quad \text{where} \quad \begin{cases} \Delta u_\Omega & = & 0 & \textit{in} & \mathbb{R}^3 \setminus \Omega \\ u_\Omega & = & 1 & \textit{on} & \partial\Omega \\ \lim\limits_{|x|\to+\infty} u_\Omega & = & 0 & & \end{cases}$$

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Is the disk $D \subset \mathbb{R}^3$ solution of :

$$\min_{K \text{ convex of } \mathbb{R}^3,\ P(K) = P_0} Cap(K) \quad ?$$

Example VI : Toy Problem

$$\min_{\substack{D(a)\subset K\subset D(b)\\ K \text{ convex}}} \Big[\mu|K| - P(K)\Big].$$

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$$\min_{\substack{D(a)\subset K\subset D(b)\\ K \text{ convex}}} \Big[\mu|K| - P(K)\Big].$$

- If $\mu = 0$: the solution is the big disk (maximizes the perimeter).
- If "$\mu = +\infty$" : the solution is the small disk (minimizes the area).
- What happens if $\mu \in (0, +\infty)$?

Results

$$\min_{K \text{ convex}} \Big[-P(K) + F(K) \Big] \qquad \min_{K \text{ convex}, |K|=V_0} \Big[-P(K) + G(K) \Big]$$

where $F(K) = f(|K|, \lambda_1(K)), \quad G(K) = g(\lambda_1(K))$.

Theorem (L.-Novruzi-Pierre 2011)

$N = 2$. Minimizers are polygons.

Theorem (Bucur, Fragala, L. 2010)

$N \geq 2$. Minimizers have a zero Gauss curvature where the boundary is C^2.

Outline

$$\min_{K \text{ convex}} \Big[J(K) = -P(K) + F(K) \Big]$$

Theorem (Bucur, Fragalà, L. 2010)

We assume that $\omega \subset \partial K$ is C^2. Then

$$J''(K)(v,v) < 0 \ \ \textit{if} \ \ v : \partial K \to \mathbb{R} \ \ \textit{has a small support in}\ \omega.$$

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Corollary

If K is a minimizer and $\omega \subset \partial K$ is C^2, then the Gauss curvature vanishes on ω.

Application :

- Pólya-Szegö's conjecture : $F(K) = Cap(K)^2$.

Outline

2d result

$$\min_{u''+u\geq 0} j(u) := J(K_u)$$

Theorem (L., Novruzi, Pierre, 2011)

Let u be a minimizer. We assume j smooth and,

$$j''(u)(v,v) < 0, \textit{ if } v \textit{ has a small enough support.}$$

Then $u'' + u$ is a sum of Dirac masses.

Applications :

- $\min \left\{ \mu|K| - P(K), \;\; K \textit{ convex}, \; D(a) \subset K \subset D(b) \right\}$
- Mahler in $\mathbb{R}^2$ (with E. Harrell, A. Henrot) : $[-1,1]^2$.

2d result

Theorem (L., Novruzi, Pierre, 2011)

Let K_u be convex. Then for any $\varepsilon > 0$, there exist C such that

$$|\lambda_1''(K_u)(v,v)| \leq C\|v\|^2_{H^{\frac{1}{2}+\varepsilon}}.$$

Applications :

- $\min\left\{\mu\lambda_1(K) - P(K),\ \ K \text{ convex } \subset D,\ \ |K| = V_0\right\}$

Perspectives - Open problems

- $\max\{\lambda_1(K) \ / \ K \ convex \subset D, \ |K| = V_0\}$ in dimension 2 ?
- Polyhedra in dimension ≥ 3 ?