Wentzell eigenvalues, upper bound and stability

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Isoperimetric Problems Between Analysis and Geometry
16/06/2014, Pisa

Wentzell eigenvalue problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ -\beta\Delta_\tau u + \partial_n u = \lambda u & \text{on } \partial\Omega \end{cases}$$

where $\beta \in \mathbb{R}_+$.

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Remark : it is the spectrum of

$$-\beta \Delta_\tau + \mathcal{D}$$

where $\mathcal{D}$ is the Dirichlet-to-Neumann operator on $\partial\Omega$.

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Question : *Does the ball maximize $\lambda_{1,\beta}$ among smooth open sets of given volume ?*

Variational formulation

$$A_\beta(u,v) = \int_\Omega \nabla u.\nabla v \ dx + \beta \int_{\partial\Omega} \nabla_\tau u.\nabla_\tau v \ d\sigma, \qquad B(u,v) = \int_{\partial\Omega} uv,$$

$$\lambda_{1,\beta}(\Omega) = \min \ \left\{ \frac{A_\beta(v,v)}{B(v,v)}, \ v \in H^{3/2}(\Omega), \ \int_{\partial\Omega} v = 0 \right\}$$

Outline

1. Extremal cases $\beta = 0$ and $\beta = \infty$
2. Generalization of Brock's bound
3. First order analysis
4. Second order analysis

Outline

Steklov eigenvalue problem, $\beta = 0$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_n u = \lambda^{St} u & \text{on } \partial\Omega \end{cases}$$

It has a discrete sprectrum consisting in a sequence

$$\lambda_0^{St}(\Omega) = 0 < \lambda_1^{St}(\Omega) \leq \lambda_2^{St}(\Omega) \ldots \to +\infty$$

Remark :

$$\lambda_1^{St}(\alpha\Omega) = \alpha^{-1}\lambda_1^{St}(\Omega).$$

Bound for λ_1^{St}, $\beta = 0$, Dimension 2

- Weinstock (1954) ; Ω simply connected

$$\lambda_1^{St}(\Omega) \leq \frac{2\pi}{P(\Omega)} \quad \left(\leq \sqrt{\frac{\pi}{|\Omega|}}\right),$$

- Hersch-Payne (1968) ; Ω simply connected

$$\frac{1}{\lambda_1^{St}(\Omega)} + \frac{1}{\lambda_2^{St}(\Omega)} \geq \frac{P(\Omega)}{\pi},$$

- Hersch-Payne-Schiffer (1975) ; Ω simply connected

$$\lambda_1^{St}(\Omega).\lambda_2^{St}(\Omega) \leq \frac{4\pi^2}{P(\Omega)^2},$$

Bound for λ_1^{St}, $\beta = 0$, Dimension d

- Brock (2001) : Ω any smooth set in $\mathbb{R}^d$ such that $\int_{\partial\Omega} x = 0$:

$$\sum_{i=1}^{d} \frac{1}{\lambda_i^{St}(\Omega)} \geq \frac{1}{|\Omega|} \int_{\partial\Omega} |x|^2 \geq d \left(\frac{|\Omega|}{\omega_d} \right)^{\frac{1}{d}} .$$

$$\lambda_1^{St}(\Omega) \leq \frac{d|\Omega|}{\int_{\partial\Omega} |x|^2} \leq \left(\frac{\omega_d}{|\Omega|} \right)^{\frac{1}{d}} .$$

Laplace-Beltrami eigenvalue problem, $\beta = +\infty$

$$-\Delta_\tau u = \lambda u \quad \text{on} \quad \partial\Omega$$

$$\lambda_0^{LB}(\partial\Omega) = 0 < \lambda_1^{LB}(\partial\Omega) \leq \lambda_2^{LB}(\partial\Omega) \ldots \to +\infty$$

Remarks :

- $\lambda_1^{LB}(\alpha\partial\Omega) = \alpha^{-2}\lambda_1^{LB}(\partial\Omega)$.
- $\lambda_1^{LB}(\Omega) = \lim_{\beta\to\infty} \frac{\lambda_{1,\beta}(\Omega)}{\beta}$.

Bound for λ_1^{LB}, $\beta = +\infty$, 2-dimensional surface

- Hersch (1970) : If $\Omega \subset \mathbb{R}^3$ smooth and bounded is such that $\partial\Omega$ is diffeomorphic to the 2-dimensional sphere $\mathbb{S}^2 = \partial B$, then

$$\frac{1}{\lambda_1^{LB}(\partial\Omega)} + \frac{1}{\lambda_2^{LB}(\partial\Omega)} + \frac{1}{\lambda_3^{LB}(\partial\Omega)} \geq \frac{3}{8\pi} P(\Omega).$$

$$\lambda_1^{LB}(\partial\Omega) P(\Omega) \leq \lambda_1^{LB}(\mathbb{S}^2) P(B).$$

Bound for λ_1^{LB}, $\beta = +\infty$, 2-dimensional surface

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$$\frac{1}{\lambda_1^{LB}(\partial\Omega)} + \frac{1}{\lambda_2^{LB}(\partial\Omega)} + \frac{1}{\lambda_3^{LB}(\partial\Omega)} \geq \frac{3}{8\pi} P(\Omega).$$

$$\lambda_1^{LB}(\partial\Omega) P(\Omega) \leq \lambda_1^{LB}(\mathbb{S}^2) P(B).$$

- Colbois-Dryden-El Soufi (2009)

$$\sup_{\Omega \subset \mathbb{R}^3} \lambda_1^{LB}(\partial\Omega) P(\Omega) = \infty$$

where the supremum is taken among smooth compact set Ω.

Bound for λ_1^{LB}, $\beta = +\infty$, 2-dimensional surface, volume constraint

Corollary

If $\Omega \subset \mathbb{R}^3$ smooth and bounded is such that $\partial\Omega$ is diffeomorphic to the 2-dimensional sphere $\mathbb{S}^2 = \partial B$, then

$$\lambda_1^{LB}(\partial\Omega)|\Omega|^{2/3} \leq \lambda_1^{LB}(\mathbb{S}^2)|B|^{2/3}.$$

Bound for λ_1^{LB}, $\beta = +\infty$, m-dimensional manifold

- Colbois-Dodziuk (1994) ; for $m \geq 3$,

$$\sup_M \lambda_1^{LB}(M) Vol(M)^{2/m} = \infty$$

where the supremum is taken among smooth compact manifold of dimension m and fixed smooth structure.

Bound for λ_1^{LB}, $\beta = +\infty$, *m*-dimensional manifold

- Colbois-Dodziuk (1994) ; for $m \geq 3$,

$$\sup_{M} \lambda_1^{LB}(M) Vol(M)^{2/m} = \infty$$

where the supremum is taken among smooth compact manifold of dimension m and fixed smooth structure.

- Colbois-Dryden-El Soufi (2009) ; for $m \geq 3$,

$$\sup_{\Omega \subset \mathbb{R}^{m+1}} \lambda_1^{LB}(\partial\Omega) P(\Omega)^{2/m} = \infty$$

where the supremum is taken among smooth compact set Ω such that $\partial\Omega$ has a fixed smooth structure.

Outline

Generalization of Brock's result

Theorem (Dambrine-Kateb-L. 2014)

Let Ω a smooth set such that $\int_{\partial\Omega} x = 0$. Let $\Lambda[\Omega]$ be the spectral radius of $P[\Omega] = (p_{ij})_{i,j=1,\dots,d}$ defined as

$$p_{ij} = \int_{\partial\Omega} (\delta_{ij} - n_i n_j),$$

where $\mathbf{n}$ is the outward normal vector to $\partial\Omega$. Then if $\beta \geq 0$, one has :

$$\sum_{i=1}^{d} \frac{1}{\lambda_{i,\beta}(\Omega)} \geq \frac{\int_{\partial\Omega} |x|^2}{|\Omega| + \beta\Lambda[\Omega]} \tag{1}$$

Equality holds in (1) if Ω is a ball.

Generalization of Brock's result

Corollary

If $\beta \geq 0$, it holds :

$$\lambda_{1,\beta}(\Omega) \leq d\,\frac{|\Omega| + \beta\Lambda[\Omega]}{\int_{\partial\Omega} |x|^2}$$

Equality holds if Ω is a ball.

- $\beta = 0$ is Brock's result.
- In general, the right-hand side is not maximized by the ball.
- $\beta = \infty$ gives :

$$\lambda_1^{LB}(\partial\Omega) \leq d\,\frac{\Lambda[\Omega]}{\int_{\partial\Omega} |x|^2}$$

Proof of the generalization of Brock's result

Variational formulation

$$A_\beta(u,v) = \int_\Omega \nabla u.\nabla v \ dx + \beta \int_{\partial\Omega} \nabla_\tau u.\nabla_\tau v \ d\sigma, \qquad B(u,v) = \int_{\partial\Omega} uv,$$

$$\sum_{i=1}^{d} \frac{1}{\lambda_{i,\beta}(\Omega)} = \max_{v_1,\cdots,v_d} \sum_{i=1}^{d} \frac{B(v_i,v_i)}{A_\beta(v_i,v_i)},$$

where the functions $(v_i)_{i=1,\ldots,d}$ are non zero functions that are B-orthogonal to the constants and pairwise A_β-orthogonal.

Proof of the generalization of Brock's result

Choice of test functions

We choose coordinates such that

$$\forall i \neq j, \quad \int_{\partial\Omega} x_i = 0 \;\; and \;\; \int_{\partial\Omega} x_i x_j = 0.$$

Let for $i \in [\![1, d]\!]$:

$$w_i = \sum_{j=1}^{d} c_{ij} x_j, \qquad B\text{-orthogonal to the constants.}$$

Proof of the generalization of Brock's result

Choice of test functions

Let us compute $A_\beta(w_i, w_j)$:

$$\int_{\partial\Omega} \nabla_\tau w_i \cdot \nabla_\tau w_j = \sum_{k,m} c_{ik}\, p_{km}\, c_{jm} = (CP[\Omega]C^T)_{ij}.$$

Therefore

$$A_\beta(w_i, w_j) = |\Omega|\, (CC^T)_{ij} + \beta(CP[\Omega]C^T)_{ij}$$

We choose $C \in O(n)$ such that $CP[\Omega]C^T$ is diagonal. Then

- w_i and w_j are A_β-orthogonal if $i \neq j$,
- $A_\beta(w_i, w_i) = |\Omega| + \beta(CP[\Omega]C^T)_{ii} \leq |\Omega| + \beta\Lambda[\Omega]$.

Proof of the generalization of Brock's result

Conclusion

Using

$$B(w_i, w_i) = \sum_{k=1}^{d} c_{ik}^2 \int_{\partial\Omega} x_k^2$$

we obtain :

$$\sum_{i=1}^{d} \frac{1}{\lambda_i(\Omega)} \geq \frac{\sum_{k=1}^{d} \int_{\partial\Omega} x_k^2 \left(\sum_{i=1}^{d} c_{ik}^2 \right)}{|\Omega| + \beta\Lambda[\Omega]} = \frac{\int_{\partial\Omega} |x|^2}{|\Omega| + \beta\Lambda[\Omega]}$$

Outline

Derivative of a multiple eigenvalue

Theorem (Dambrine-Kateb-L. 2014)

Let λ be an eigenvalue of order m and $\Omega_t = (I + t\mathbf{V})(\Omega)$. Then there exist m functions $(t \mapsto \lambda_{k,\beta}(t))_{k\in[\![1,m]\!]}$ such that

- $\lambda_{k,\beta}(0) = \lambda$,
- *for $|t|$ small, $\lambda_{k,\beta}(t)$ is an eigenvalue of Ω_t,*
- *the functions $(t \mapsto \lambda_{k,\beta}(t))_{k\in[\![1,m]\!]}$ admit derivatives and their values at 0 are the eigenvalues of the matrix $M= M_\Omega(V_n)$:*

$$M_{ij} = \int_{\partial\Omega} V_n \Big(\nabla_\tau u_i . \nabla_\tau u_j - \partial_n u_i \partial_n u_j - \lambda H u_i u_j + \beta \left(H I_d - 2 D^2 b_{\partial\Omega} \right) \nabla_\tau u_i . \nabla_\tau u_j \Big) \, d\sigma.$$

where $(u_k)_{k=1,\ldots,m}$ denote the eigenfunctions associated to λ.

First order optimality for the ball

If $\Omega = B$ the unit ball and $\mathbf{V}$ preserves the volume, then :

$$M_{ij} = C(d,\beta)\int_{\mathbb{S}^{d-1}} V_n x_i x_j.$$

Corollary

Any ball B is a critical shape for $\lambda_{1,\beta}$ with volume constraint : for every volume preserving deformations $\mathbf{V}$,

$$\sum_{k=1}^{d} \lambda'_{k,\beta}(0) = \mathrm{Tr}(M_B(V_n)) = 0.$$

Moreover, if $V_n : \mathbb{S}^{d-1} \to \mathbb{R}$ is orthogonal to spherical harmonics of order 2, the directional derivative exists in the usual sense and vanishes.

Outline

Sign of the second order derivative

Theorem (Dambrine-Kateb-L. 2014)

Let B be a ball in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $t \mapsto B_t = I + t\mathbf{V} + O(t^2)$ a volume preserving deformation such that

$$V_n \text{ is orthogonal to spherical harmonics of order 2.}$$

Then the functions $(t \mapsto \lambda_{k,\beta}(t))_{k \in [\![1,d]\!]}$ admit a second derivative and there exists $\alpha > 0$ such that

$$\sum_{k=1}^{d} \lambda''_{k,\beta}(0) \leq -\alpha \|V_n\|^2_{\mathrm{H}^1(\partial B)}.$$

Local optimality

Corollary

If B is a ball in $\mathbb{R}^2$ or $\mathbb{R}^3$, and $t \mapsto B_t = (I + t\mathbf{V} + O(t^2))(B)$ a smooth volume preserving deformation, then

$$\lambda_{1,\beta}(B) \geq \lambda_{1,\beta}(B_t), \quad \text{for } t \text{ small enough.}$$

Perspectives - Open Problems

- Positive answer to the question when $\Omega \subset \mathbb{R}^3$ and $\partial\Omega$ is differomorphic to $\mathbb{S}^2$?
- Study the stability question for Hersch's inequality in a smooth neighborhood :
  - solve the "two-norm discrepancy issue" :

$$S(\Omega) := \sum_{i=1}^{d} \frac{1}{\lambda_i^{LB}(\Omega)} = S(B) + \underbrace{S''(B)(V,V)}_{\leq -\alpha \|V_n\|_{H^1}^2} + o(\|V\|_{C^3}^2).$$

  - prove coercivity for deformation preserving the perimeter.
  - Enlarge the neighborhood/regularization procedure.