Semigroups in Banach spaces and applications to (nonlinear) evolution PDEs

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Abstract - The aim of the course is to present some recent progresses in the theory of semigroups which have been motivated by the application to several classes of PDE coming from the kinetic theory of gases and the biological modeling.
We present efficient versions of classical results such as the spectral mapping theorem, Weyls theorems, Krein-Rutman theorem, stability under perturbation theorem. The proofs are based on a factorization approach both for the spectral analysis of semigroup generators and semigroup growth estimates. More precisely, we systematically use iterated Duhamel formula and its resolvent counterpart in the spirit of Dyson series approach.
The abstract theory is motivated and illustrated by the applications to several PDEs. We will present an uniform treatment of the convergence to the equilibrium for the discrete and classical Fokker-Planck equation. We will also investigate the long-time asymptotic of solutions to the Keller-Segel equation for chemotaxis as well as of solutions to a time elapsed neuron network model.

Program

Lecture 1 - Introduction

I - Introduction

II - Basic tools & classical caracterization of growth estimates

Lecture 2 - Growth estimates

III - Factorization & growth estimates

IV - Examples of evolution PDEs

Lecture 3 - Spectral Analysis

V - Basic facts about spectral anlysis of opeartors and semigroups

VI - Factorization & spectral analysis

Lecture 4 - Application to nonlinear PDEs

VII - The Keller-Segel equation

VIII - A neural network model

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Excercices

Excercice 1. Consider the rescaled Keller-Segel equation on the plane

$$\partial_t g = \Delta g + \nabla(xg + (K * g)g),$$

$g = g(t,x)$, $t \geq 0$, $x \in \mathbb{R}^2$ and $K(z) := \nabla\kappa(z) = (2\pi)^{-1} z/|z|^2$, $\kappa(z) := (2\pi)^{-1} \log |z|$.
(1) We admit that

$$\mathcal{E}(g) := \int g \log g + \int g\langle x\rangle^2 + \frac{1}{2}\int\int g(x)g(y)\kappa(x-y) \geq 0.$$

Prove that

$$\frac{d}{dt}\mathcal{E}(g) = -\mathcal{D}(g)$$

with

$$\mathcal{D}(g) := \int g\Big|\nabla\Big(\log g + \frac{|x|^2}{2} + \kappa * g\Big)\Big|^2.$$

(2) Admit the log-Sobolev inequality

$$\lambda\, H(f|\gamma) \leq \frac{1}{2} I(f|\gamma) \quad \forall\, f \in D,$$

with $E(f|\gamma) = E(f) - E(\gamma)$,

$$H(f) = \int f \log f, \quad I(f) = \int \frac{|\nabla f|^2}{f}, \quad \gamma(x) = (2\pi)^{-d/2}\exp(-|x|^2/2),$$

$$D := \Big\{ f \in L^1(\mathbb{R}^d); \quad f \geq 0, \quad \int f = 1, \quad \int f\, x = 0, \quad \int f\, |x|^2 = d \Big\}$$

and deduce the Poincaré inequality

$$(\lambda + d)\, \|h\|^2_{L^2(\gamma^{-1/2})} \leq \int |\nabla h|^2\, \gamma^{-1} \quad \forall\, h \in \mathcal{D}(\mathbb{R}^d),\ \langle h[1, x, |x|^2]\,\rangle = 0,$$

(Hint : choose $f = \gamma + \varepsilon h$ and make $\varepsilon \to 0$)
(3) Admit that there exists a unique minimizer G to $\mathcal{E}$ on the class of functions with mass $M \in (0, 8\pi)$. Mimicking the argument of question (2), prove that

$$Q_1[f] := \int f^2 G^{-1} + \int\int f(x)f(y)\kappa(x-y)\, dxdy$$

is a Lyapunov functional for the linearized equation

$$\partial_t f = \Delta f + \nabla(xf + (K * f)G + (K * G)f).$$

Excercice 2. For a familly $(S_t)_{t\geq 0}$ of operators acting on a Banach space X, we introduce the following properties :

(i) $\forall t \geq 0,\ f \mapsto S_t f$ is linear and continuous on X ;

(ii) $\forall f \in X,\ t \mapsto S_t\, f \in C([0,\infty), X)$;

(ii′) $S(t) f \to f$ when $t \searrow 0$, for any $f \in X$.

(iii) $S_0 = I$; $\quad \forall\, s,\, t \geq 0,\ S_{t+s} = S_t\, S_s$;

(iv) $\exists\, b \in \mathbb{R},\ \exists\, M \geq 1,$

$$\|S_t\|_{\mathscr{B}(X)} \leq M\, e^{bt} \quad \forall\, t \geq 0.$$

We define a new familly $(T_t)_{t\geq 0}$ and the new norm $|||\cdot|||$ on X by

$$T(t) := e^{-\omega\, t}\, S(t) \qquad \text{and} \qquad |||f||| := \sup_{t\geq 0} \|T(t)\, f\|, \tag{0.1}$$

where

$$\omega := \limsup_{t\to\infty} \frac{1}{t} \log \|S(t)\| = \inf\{b \in \mathbb{R};\ (iv) \text{ holds}\}.$$

Here are questions :

(1) Prove that (i), (ii′) and (iii) imply (iv). (Hint. Use a contradiction argument and the Banach-Steinhaus Theorem).

(2) Prove that (i)-(ii′)-(iii) implies (i)-(ii)-(iii). (Hint. Use (1)).

(3) Prove the two norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent and that $T(t)$ is a semigroup of contractions for the new norm.

(4) Prove that if S_t satisfies (iii) as well as the continuity properties (i) & (ii) in the sense of the weak topology $\sigma(X, X')$, then S_t is a strongly continuous semigroup.

Excercice 3. Consider two Banach spaces X, Y and a function $u : \mathbb{R}_+ \to \mathscr{B}(X) + \mathscr{B}(Y)$. For $\Theta \geq 0$, $a^*, b \in \mathbb{R}$, $a^* < b$, we assume

(a) $ue^{-at} \in L^\infty(0,\infty; \mathscr{B}(X) \cap \mathscr{B}(Y))$ for any $a > a^*$;

(b) $ue^{-bt} t^{-\Theta} \in L^\infty(0,\infty; \mathscr{B}(X, Y))$.

Prove that

(c) for any $a > a^*$, there exists n such that $u^{(*n)} e^{-at} \in L^\infty(0,\infty; \mathscr{B}(X, Y))$.

$\Big($Hint. Consider as a first step the case $n = 2$, and split

$$(u * u)(t) = \int_0^{t/2} u(t-s) u(s)\, ds + \int_{t/2}^{t} u(t-s) u(s)\, ds\Big).$$

Excercice 4. We consider two Banach spaces E and $\mathcal{E}$ such that $E \subset \mathcal{E}$ with dense and continuous embedding, and we consider L the generator of a semigroup $\mathcal{S}_L(t) := e^{tL}$ on E, $\mathcal{L}$ the generator of a semigroup $\mathcal{S}_\mathcal{L}(t) := e^{t\mathcal{L}}$ on $\mathcal{E}$ with $\mathcal{L}_{|E} = L$.
We assume that there exist some operators $A, B \in \mathscr{C}(E)$, $A \preceq B$, $\mathcal{A}, \mathcal{B} \in \mathscr{C}(\mathcal{E})$, $\mathcal{A} \preceq \mathcal{B}$, such that

$$\mathcal{L} = \mathcal{A} + \mathcal{B},\ \ L = A + B,\ \ \mathcal{A}_{|E} = A,\ \ \mathcal{B}_{|E} = B. \tag{0.2}$$

We also assume that following properties hold for some real number $a \in \mathbb{R}$:

$$\begin{aligned}
&\forall\, \ell \in \mathbb{N}, && \|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}(t)\|_{\mathcal{E}\to\mathcal{E}} \le C_\ell\, e^{at}, \\
&\forall\, \ell \in \mathbb{N}, && \|S_B * (A S_B)^{(*\ell)}(t)\|_{E\to E} \le C_\ell\, e^{at}, \\
&\exists\, n \ge 1, && \int_0^\infty \|(\mathcal{A} S_\mathcal{B})^{(*n)}(t)\|_{\mathcal{E}\to E}\, e^{-at}\, dt \le C_n, \\
&\exists\, n \ge 1, && \int_0^\infty \|(S_\mathcal{B} \mathcal{A})^{(*n)}(t)\|_{\mathcal{E}\to E}\, e^{-at}\, dt \le C_n,
\end{aligned}$$

and there exists an operator Π such that

$$\Pi \in \mathscr{B}(\mathcal{E}, E) \text{ is a projector which commutes with } \mathcal{L}.$$

Prove that the following equivalence holds :

(1) The semigroup S_L satisfies the splitting structure and the growth estimate

$$S_L = S_L \Pi_{|E} + S_L(I - \Pi_{|E}) \quad \text{and} \quad \Big\| S_L(t)(I - \Pi_{|E}) \Big\|_{\mathscr{B}(E)} \le C_{L,a}\, e^{a\,t},$$

for any $t \ge 0$ and some constant $C_{L,a} > 0$;

(2) The semigroup $S_\mathcal{L}$ satisfies the splitting structure and the growth estimate

$$S_L = S_L \Pi + S_L(I - \Pi) \quad \text{and} \quad \Big\| S_\mathcal{L}(t)(I - \Pi) \Big\|_{\mathscr{B}(\mathcal{E})} \le C_{\mathcal{L},a}\, e^{a\,t},$$

for any $t \ge 0$ and some constant $C_{\mathcal{L},a} > 0$.

$\Big($Hint. For **(1)** $\Rightarrow$ **(2)**, use the iterated Duhamel formula

$$(I - \Pi) S_\mathcal{L} = (I - \Pi) \sum_{\ell=0}^{n-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} + [(I - \Pi_{|E}) S_L] * (\mathcal{A} S_\mathcal{B})^{(*n)} \Big).$$