CHAPTER 1
VARIATIONAL SOLUTION FOR PARABOLIC EQUATION

1. Introduction

In this chapter we will focus on the question of existence (and uniqueness) of a solution $f = f(t,x)$ to the (linear) evolution PDE of "parabolic type"

$$\partial_t f = \Lambda f \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

where Λ is the following integro-differential operator

$$(\Lambda f)(x) = \Delta f(x) + a(x) \cdot \nabla f(x) + c(x)\, f(x) + \int_{\mathbb{R}^d} b(y,x)\, f(y)\, dy, \tag{1.2}$$

that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d. \tag{1.3}$$

Here $t \geq 0$ stands for the *"time"* variable, $x \in \mathbb{R}^d$ stands for the *"position"* variable, $d \in \mathbb{N}^*$.

In order to develop the variational approach for the equation (1.1)-(1.2), we make the strong assumption that

$$f_0 \in L^2(\mathbb{R}^d) =: H, \quad \text{which is an Hilbert space,}$$

and that the coefficients satisfy

$$a \in W^{1,\infty}(\mathbb{R}^d), \quad c \in L^\infty(\mathbb{R}^d), \quad b \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

The **main result** we will present in this chapter is the existence of a weak (variational) solution (which sense will be specified below)

$$f \in X_T := C([0,T); L^2) \cap L^2(0,T; H^1) \cap H^1(0,T; H^{-1}),$$

to the evolution equation (1.1), (1.3). We mean variational solution because the space of "test functions" is the same as the space in which the solution lives. It also refers to the associated stationary problem which is of "variational type" (see [1, chapter VIII & IX]).

The existence of solutions issue is tackled by following a scheme of proof that we will repeat for all the other evolution equations that we will consider in the next chapters.

(1) We look for **a priori estimates** by performing (formal) differential and integral calculus.

(2) We deduce a possible natural **functional space** in which lives a solution and we propose a **definition of a solution**, that is a (weak) sense in which we may understand the evolution equation.

(3) We state and prove the associated **existence theorem**. For the existence proof we typically argue as follows: we introduce a *"regularized problem"* for which we are able to construct a solution and we are allowed to rigorously perform the calculus leading to the *"a priori estimates"*, and then we pass to the limit in the sequence of regularized solutions.

2. A priori estimates

Define $V = H^1(\mathbb{R}^d)$. We first observe that for any $f \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}\langle \Lambda f, f\rangle &= -\int |\nabla f|^2 + \int a\cdot\nabla_x \frac{f^2}{2} + \int c\, f^2 + \iint b(y,x)\, f(x) f(y)\, dx dy.\\ &\le -\|f\|_V^2 + \Big(1 + \|(c - \frac{1}{2}\operatorname{div} a)_+\|_{L^\infty} + \|b_+\|_{L^2}\Big)\, \|f\|_H^2.\end{aligned}$$

We also observe that for any $f, g \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}|\langle \Lambda f, g\rangle| &\le \|\nabla f\|_{L^2}\, \|\nabla g\|_{L^2} + \|a\|_{L^\infty}\, \|\nabla f\|_{L^2}\, \|g\|_{L^2} + (\|c\|_\infty + \|b\|_{L^2})\, \|f\|_{L^2}\, \|g\|_{L^2}\\ &\le (1 + \|a\|_\infty + \|c\|_{L^\infty} + \|b\|_{L^2})\, \|f\|_V \|g\|_V.\end{aligned}$$

We easily deduce from the two preceding estimates that our parabolic operator falls into the following abstract variational framework.
Abstract variational framework. We consider a Hilbert space H endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$. We identify H with its dual $H' = H$. We consider another Hilbert space V endowed with a norm $\|\cdot\|_V$ and we denote $\langle .,. \rangle$ the duality product on V. We assume $V \subset H$ with dense and bounded embedding so that $V \subset H \subset V'$.

We consider a linear operator $\Lambda : V \to V'$ which is bounded (or continuous), which means

(i) $\exists\, M > 0$ such that

$$|\langle \Lambda g, h\rangle| \le M\, \|g\|\, \|h\| \qquad \forall\, g, h \in V;$$

and such that Λ is *"coercive+dissipative"*[1] in the sense

(ii) $\exists\, \alpha > 0$, $b \in \mathbb{R}$ such that

$$\langle \Lambda g, g\rangle \le -\alpha\, \|g\|^2 + b\, |g|^2 \qquad \forall\, g \in V;$$

and we consider the abstract evolution equation

$$\frac{dg}{dt} = \Lambda g \quad \text{in} \quad (0,T), \tag{2.1}$$

for a solution $g : [0,T) \to H$, with prescribed initial value

$$g(0) = g_0 \in H. \tag{2.2}$$

[1]We commonly say that (the bilinear form associated to) $-\Lambda$ is coercive if (ii) holds with $\alpha > 0$ and $b = 0$, and that $\Lambda - b$ is dissipative if (ii) holds with $\alpha = 0$ and $b \in \mathbb{R}$. Our assumption (ii) is then more general than a coercivity condition (on $-\Lambda$) but less general than a dissipativity condition (on Λ).

A priori bound in the abstract variational framework. With the above assumptions and notations, any solution g to the abstract evolution equation (2.1) (formally) satisfies the following estimate

$$|g(T)|_H^2 + 2\alpha \int_0^T \|g(s)\|_V^2 \, ds \le e^{2bT} \, |g_0|_H^2 \quad \forall T. \tag{2.3}$$

We (formally) prove (2.3). Using just the *coercivity+dissipativity* assumption (ii), we have

$$\frac{d}{dt} \frac{|g(t)|_H^2}{2} = \langle \Lambda g, g \rangle \le -\alpha \|g(t)\|_V^2 + b \, |g(t)|_H^2,$$

and we conclude thanks to the Gronwall lemma, that we recall now.

Lemma 2.1 (Gronwall). *Consider* $0 \le u \in C^1([0,T])$, $0 \le v \in C([0,T])$ *and* $\alpha, b \ge 0$ *such that*

$$u' + 2\alpha v \le 2bu \quad \text{in a pointwise sense on } (0,T), \tag{2.4}$$

or more generally $0 \le u \in C([0,T])$ *and* $0 \le v \in L^1(0,T)$ *which satisfies* (2.4) *in the distributional sense, namely*

$$u(t) + 2\alpha \int_0^t v(s) \, ds \le 2b \int_0^t u(s) \, ds + u(0) \quad \forall t \in (0,T). \tag{2.5}$$

Then, the following estimate holds true

$$u(t) + 2\alpha \int_0^t v(s) \, ds \le e^{2bt} \, u(0) \quad \forall t \in (0,T). \tag{2.6}$$

Proof of Lemma 2.1. Since (2.4) clearly implies (2.5), we just have to prove that (2.5) implies (2.6). We introduce the C^1 function

$$w(t) := 2b \int_0^t u(s) \, ds + u(0).$$

Differentiate w, we get thanks to (2.5)

$$w'(t) = 2bu(t) \le 2b \, w(t)$$

so that

$$w(t) \le e^{2bt} \, w(0) = e^{2bt} \, u(0).$$

We conclude by coming back to (2.5). □

From the formal/natural/physics estimate (2.3) together with equation (2.1) and the continuity estimate (i) on Λ, we deduce

$$\Big\| \frac{dg}{dt} \Big\|_{V'} = \|\Lambda g\|_{V'} \le M \, \|g\|_V \in L^2(0,T),$$

and we conclude with

$$g \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'). \tag{2.7}$$

3. Variational solutions

Definition 3.1. *For any given $g_0 \in H$, $T > 0$, we say that*

$$g = g(t) \in X_T := C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V')$$

is a **variational solution** *to the Cauchy problem* (2.1), (2.2) *on the time interval* $[0,T]$ *if it is a solution in the following weak sense*

$$(g(t),\varphi(t))_H = (g_0,\varphi(0))_H + \int_0^t \big\{ \langle \Lambda g(s), \varphi(s)\rangle_{V',V} + \langle \varphi'(s), g(s)\rangle_{V',V} \big\}\, ds \tag{3.1}$$

for any $\varphi \in X_T$ and any $0 \le t \le T$.
We say that g is a global solution if it is a solution on $[0,T]$ for any $T > 0$.

Theorem 3.2 (J.L. Lions)**.** *With the above notations and assumptions for any $g_0 \in H$, there exists a unique global variational solution to the Cauchy problem* (2.1), (2.2)*. As a consequence, any solution satisfies* (2.3) *and the application $g_0 \mapsto g(t)$ defines a C_0-semigroup on H.*

We start with some remarks and we postpone the proof of the existence part of Theorem 3.2 to the next section.

3.1. **Parabolic equation.** As a consequence of Theorem 3.2, for any $f_0 \in L^2(\mathbb{R}^d)$ there exists a unique function

$$f = f(t) \in C([0,T];L^2) \cap L^2(0,T;H^1) \cap H^1(0,T;H^{-1}), \quad \forall T > 0,$$

which is a solution to the parabolic equation (1.1)-(1.2) in the variational sense.

3.2. **About the functional space.** The space obtained thanks to the a priori estimates established on g is nothing but X_T as consequence of the following result.

Lemma 3.3. *The following inclusion*

$$L^2(0,T;V) \cap H^1(0,T;V') \subset C([0,T];H)$$

holds true. Moreover, for any $g \in L^2(0,T;V) \cap H^1(0,T;V')$ there holds

$$t \mapsto |g(t)|_H^2 \in W^{1,1}(0,T)$$

and

$$\frac{d}{dt}|g(t)|_H^2 = 2\,\langle g'(t), g(t)\rangle_{V',V} \quad \text{a.e. on} \quad (0,T).$$

Proof of Lemma 3.3. Step 1. We define $\bar g = g$ on $[0,T]$, $\bar g = 0$ on $\mathbb{R}\backslash[0,T]$, and for a mollifier ρ with compact support included in $(-1,-1/2)$, we define the approximation to the identity sequence (ρ_ε) by setting $\rho_\varepsilon(t) := \varepsilon^{-1}\rho(\varepsilon^{-1}t)$ and then the sequence $g_\varepsilon(t) := \bar g *_t \rho_\varepsilon$ where $*$ stands for the usual convolution operator on $\mathbb{R}$. We observe that $g_\varepsilon \in C^1(\mathbb{R};H)$, $g_\varepsilon \to g$ a.e. on $[0,T]$ and in $L^2(0,T;V)$. For a fixed $\tau \in (0,T)$ and for any $t \in (0,\tau)$ and any $0 < \varepsilon < T - \tau$, we have $s \mapsto \rho_\varepsilon(t-s) \in \mathcal{D}(0,T)$, since $\operatorname{supp} \rho_\varepsilon(t-\cdot) \subset [t+\varepsilon/2, t+\varepsilon] \subset [\varepsilon/2, \tau+\varepsilon]$, and we compute

$$\begin{aligned} g'_\varepsilon &= \int_{\mathbb{R}} \partial_t \rho_\varepsilon(t-s)\, \bar g(s)\, ds \\ &= -\int_0^T (\partial_s \rho_\varepsilon(t-s))\, g(s)\, ds \\ &= \int_0^T \rho_\varepsilon(t-s)\, g'(s)\, ds = \rho_\varepsilon * (\overline{g'}). \end{aligned}$$

As a consequence $g'_\varepsilon \to g'$ a.e. and in $L^1(0,\tau;V')$.

Step 2. We fix $\tau \in (0,T)$ and $\varepsilon, \varepsilon' \in (0, T-\tau)$, and we compute

$$\frac{d}{dt} |g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 = 2\,\langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle_{V',V},$$

so that for any $t_1, t_2 \in [0,\tau]$

$$|g_\varepsilon(t_2) - g_{\varepsilon'}(t_2)|_H^2 = |g_\varepsilon(t_1) - g_{\varepsilon'}(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle ds. \tag{3.2}$$

Since $g_\varepsilon \to g$ a.e. on $[0,\tau]$ in H, we may fix $t_1 \in [0,\tau]$ such that

$$g_\varepsilon(t_1) \to g(t_1) \quad \text{in } H. \tag{3.3}$$

As a consequence of (3.2), (3.3) as well as $g_\varepsilon \to g$ in $L^2(0,\tau;V)$ and $g'_\varepsilon \to g'$ in $L^2(0,\tau;V')$, we have

$$\limsup_{\varepsilon,\varepsilon'\to 0} \sup_{[0,\tau]} |g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 \le \lim_{\varepsilon,\varepsilon'\to 0} \int_0^\tau \|g'_\varepsilon - g'_{\varepsilon'}\|_{V'}\, \|g_\varepsilon - g_{\varepsilon'}\|_V\, ds = 0,$$

so that (g_ε) is a Cauchy sequence in $C([0,\tau];H)$, and then g_ε converges in $C([0,\tau];H)$ to a limit $\tilde g \in C([0,\tau];H)$. That proves $g = \tilde g$ a.e. and $g \in C([0,\tau];H)$. We prove similarly that $g \in C([\tau,T];H)$ for any $\tau \in (0,T)$ and thus $g \in C([0,T];H)$.

Step 3. Similarly as for (3.2), we have

$$|g_\varepsilon(t_2)|_H^2 = |g_\varepsilon(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon, g_\varepsilon\rangle ds,$$

and passing to the limit $\varepsilon \to 0$ we get

$$|g(t_2)|_H^2 = |g(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g', g\rangle ds.$$

Using again that $\langle g', g\rangle \in L^1(0,T)$, we easily deduce from the above identity the two remaining claims of the Lemma. □

3.3. **A posteriori estimate, uniqueness and C_0-semigroup.** Taking $\varphi = g \in X_T$ as a test function in (3.1), we deduce from Lemma 3.3,

$$\begin{aligned} \frac12 |g(t)|_H^2 - \frac12 |g_0|_H^2 &= |g(t)|_H^2 - |g_0|_H^2 - \int_0^t \langle g'(s), g(s)\rangle\, ds \\ &= \int_0^t \langle \Lambda g, g\rangle\, ds \\ &\le \int_0^t (-\alpha\, \|g\|_V^2 + b\, |g|_H^2)\, ds, \end{aligned}$$

and we then obtain (2.3) as an **a posteriori estimate** thanks to the Gronwall lemma 2.1.

Let us prove now the **uniqueness of the variational solution** g associated to a given initial datum $g_0 \in H$. In order to do so, we consider two variational solutions g and f associated to the same initial datum. Since the equation (2.1), (2.2) is linear, or more precisely, the variational formulation (3.1) is linear in the solution, the function $g - f$ satisfies the same variational formulation (3.1) but associated to the initial datum $g_0 - f_0 = 0$. The a posteriori estimate (2.3) then holds for $g - f$ and implies that $g - f = 0$.

We explain how we may associate a C_0-semigroup to the evolution equation (2.1), (2.2) as a mere consequence of the linearity of the equation and of the existence and uniqueness result.

Definition 3.4. *Consider X a Banach space, and denote by $\mathscr{B}(X)$ the set of linear and bounded operators on X. We say that $S = (S_t)_{t\ge 0}$ is a strongly continuous semigroup of linear operators on X, or just a C_0-semigroup on X, we also write $S(t) = S_t$, if*

(i) $\forall t \ge 0, \quad S_t \in \mathscr{B}(X)$ (one parameter family of operators);

(ii) $\forall f \in X, \quad t \mapsto S_t\, f \in C([0,\infty), X)$ (continuous trajectories);

(iii) $S_0 = I; \quad \forall\, s,\, t \ge 0 \quad S_{t+s} = S_t\, S_s$ (semigroup property).

Proposition 3.5. *The operator Λ generates a semigroup on H defined in the following way. For any $g_0 \in H$, we set $S_t g := g(t)$ where $g(t)$ is the unique variational solution associated to g_0 and given by Theorem 3.2. We also denote $S_\Lambda(t) = e^{\Lambda t} = S_t$ for any $t \ge 0$.*

• *S satisfies (i).* By linearity of the equation and uniqueness of the solution, we clearly have

$$S_t(g_0 + \lambda f_0) = g(t) + \lambda f(t) = S_t g_0 + \lambda S_t f_0$$

for any $g_0, f_0 \in H$, $\lambda \in \mathbb{R}$ and $t \ge 0$. Thanks to estimate (2.3) we also have $|S_t g_0| \le e^{bt}\,|g_0|$ for any $g_0 \in H$ and $t \ge 0$. As a consequence, $S_t \in \mathscr{B}(H)$ for any $t \ge 0$.

• *S satisfies (ii).* Thanks to lemma 3.3 we have $t \mapsto S_t g_0 \in C(\mathbb{R}_+; H)$ for any $g_0 \in H$.

• *S satisfies (iii).* For $g_0 \in H$ and $t_1, t_2 \ge 0$ denote $g(t) = S_t g_0$ and $\tilde g(t) := g(t+t_1)$. Making the difference of the two equations (3.1) written for $t = t_1$ and $t = t_1 + t_2$, we see that $\tilde g$ satisfies

$$\begin{aligned}
(\tilde g(t_2), \tilde\varphi(t_2)) &= (g(t_1+t_2), \varphi(t_1+t_2)) \\
&= (g(t_1), \varphi(t_1)) + \int_{t_1}^{t_1+t_2} \big\{ \langle \Lambda g(s), \varphi(s)\rangle + \langle \varphi'(s), g(s)\rangle \big\}\, ds \\
&= (\tilde g(0), \tilde\varphi(0)) + \int_0^{t_2} \big\{ \langle \Lambda \tilde g(s), \tilde\varphi(s)\rangle + \langle \tilde\varphi'(s), \tilde g(s)\rangle \big\}\, ds,
\end{aligned}$$

for any $\varphi \in X_{t_1+t_2}$ with the notation $\tilde\varphi(t) := \varphi(t+t_1) \in X_{t_2}$. Since the equation on the functions $\tilde g$ and $\tilde\varphi$ is nothing but the variational formulation associated to the equation (2.1), (2.2) with initial datum $\tilde g(0)$, we obtain

$$S_{t_1+t_2} g_0 = g(t_1+t_2) = \tilde g(t_2) = S_{t_2}\tilde g(0) = S_{t_2} g(t_1) = S_{t_2} S_{t_1} g_0.$$

4. Proof of the existence part of Theorem 3.2.

We first prove thanks to a compactness argument in step 1 to step 3 that there exists a function $g \in L^2(0,T;V)$ such that

$$\langle g_0, \varphi(0)\rangle + \int_0^t \left\{ \langle \Lambda g(s), \varphi(s)\rangle_{V',V} + \langle \varphi'(s), g(s)\rangle_{V',V} \right\} ds = 0 \tag{4.1}$$

for any $\varphi \in C^1_c([0,T);V)$. We then deduce by some "regularization tricks" in step 4 and step 5 that the above weak solution is a variational solution.

Step 1. For a given $g_0 \in H$ and $\varepsilon > 0$, we seek $g_1 \in V$ such that

$$g_1 - \varepsilon \Lambda g_1 = g_0. \tag{4.2}$$

We introduce the bilinear form $a : V \times V \to \mathbb{R}$ defined by

$$a(u,v) := (u,v) - \varepsilon \, \langle \Lambda u, v\rangle.$$

Thanks to the assumptions made on Λ, we have

$$|a(u,v)| \le |u| \, |v| + \varepsilon \, M \, \|u\| \, \|v\|,$$

and

$$a(u,u) \ge |u|^2 + \varepsilon \, \alpha \, \|u\|^2 - \varepsilon \, b \, |u|^2 \ge \varepsilon \, \alpha \, \|u\|^2,$$

whenever $\varepsilon \, b < 1$, what we assume from now. On the other hand, the mapping $v \in V \mapsto (g_0, v)$ is a linear and continuous form. We may thus apply the Lax-Milgram theorem, and we get

$$\exists! \, g_1 \in V \qquad (g_1, v) - \varepsilon \langle \Lambda g_1, v\rangle = (g_0, v) \quad \forall \, v \in V.$$

Step 2. Fix $\varepsilon > 0$ as in the preceding step and build by induction the sequence (g_k) in $V \subset H$ defined by the family of equations

$$\forall \, k \qquad \frac{g_{k+1} - g_k}{\varepsilon} = \Lambda \, g_{k+1}. \tag{4.3}$$

Observe that from the identity

$$(g_{k+1}, g_{k+1}) - \varepsilon \, \langle \Lambda g_{k+1}, g_{k+1}\rangle = (g_k, g_{k+1}),$$

we deduce

$$|g_{k+1}|^2 + \varepsilon \, \alpha \, \|g_{k+1}\|^2 - \varepsilon \, b \, |g_{k+1}|^2 \le |g_k| \, |g_{k+1}|.$$

On the one hand, it implies that

$$|g_k| \le \frac{1}{1-\varepsilon b} \, |g_{k-1}| \le \frac{1}{(1-\varepsilon b)^k} \, |g_0| \le A^{k\,\varepsilon} \, |g_0| \quad \forall \, k \ge 0,$$

with $A := e^b$ (where we have used that $\ln(1+\varepsilon b) \le \varepsilon b$ in the last inequality).
On the other hand, using Young inequality and the last two inequalities, we obtain for any $n \ge 1$

$$\begin{aligned}
\alpha \sum_{k=1}^n \varepsilon \, \|g_k\|^2 &\le \sum_{k=1}^n \frac{1}{2}(|g_{k-1}|^2 - |g_k|^2) + b \sum_{k=1}^n \varepsilon \, |g_k|^2 \\
&\le \frac{1}{2} \, |g_0|^2 + b \sum_{k=1}^n \varepsilon \, |g_k|^2 \\
&\le \frac{1}{2} \, |g_0|^2 + b \, n\varepsilon \, A^{2n\varepsilon} \, |g_0|^2.
\end{aligned}$$

We fix $T > 0$, $n \in \mathbb{N}^*$ and we define

$$\varepsilon := T/n, \quad t_k = k\,\varepsilon, \quad g^\varepsilon(t) := g_k \text{ on } [t_k, t_{k+1}).$$

The two precedent estimates write then

$$\sup_{[0,T]} |g^\varepsilon|_H^2 + \alpha \int_0^T \|g^\varepsilon\|_V^2\, dt \le \Big(\frac{3}{2} + b\, T\, A^{2T}\Big)|g_0|^2. \tag{4.4}$$

Step 3. Consider a test function $\varphi \in C_c^1([0,T);V)$ and define $\varphi_k := \varphi(t_k)$, so that $\varphi_n = \varphi(T) = 0$. Multiplying the equation (4.3) by φ_k and summing up from $k = 0$ to $k = n$, we get

$$-(\varphi_0, g_0) - \sum_{k=1}^n \langle \varphi_k - \varphi_{k-1}, g_k\rangle = \sum_{k=0}^n \varepsilon\,\langle \Lambda g_{k+1}, \varphi_k\rangle,$$

where in the LHS we use the duality production $\langle,\rangle$ in $V' \times V$ instead of the scalar product $(,)$ in H thanks to the inclusions $V \subset H = H' \subset V'$. Introducing the two functions $\varphi^\varepsilon, \varphi_\varepsilon : [0,T) \to V$ defined by

$$\varphi^\varepsilon(t) := \varphi_{k-1} \quad \text{and} \quad \varphi_\varepsilon(t) := \frac{t_{k+1} - t}{\varepsilon}\,\varphi_k + \frac{t - t_k}{\varepsilon}\,\varphi_{k+1} \qquad \text{for} \quad t \in [t_k, t_{k+1}),$$

in such a way that

$$\varphi_\varepsilon'(t) = \frac{\varphi_{k+1} - \varphi_k}{\varepsilon} \qquad \text{for} \quad t \in (t_k, t_{k+1}),$$

the above equation also writes

$$-\langle \varphi(0), g_0\rangle - \int_\varepsilon^T \langle \varphi_\varepsilon', g^\varepsilon\rangle\, dt = \int_0^T \langle \Lambda g^\varepsilon, \varphi^\varepsilon\rangle\, dt. \tag{4.5}$$

On the one hand, from (4.4) we know that up to the extraction of a subsequence, there exists $g \in X_T$ such that $g^\varepsilon \rightharpoonup g$ weakly in $L^2(0,T;V)$. On the other hand, from the above construction, we have $\varphi_\varepsilon' \to \varphi'$ and $\varphi_\varepsilon \to \varphi$ both strongly in $L^2(0,T;V)$. We may then pass to the limit as $\varepsilon \to 0$ in (4.5) and we get (4.1).

Step 4. We prove that $g \in X_T$. Taking $\varphi := \chi(t)\,\psi$ with $\chi \in C_c^1((0,T))$ and $\psi \in V$ in equation (4.1), we get

$$\Big\langle \int_0^T g\chi' dt, \psi\Big\rangle = \int_0^T \langle g, \psi\rangle \chi'\, dt = -\int_0^T \langle \Lambda g, \psi\rangle \chi\, dt = \Big\langle -\int_0^T \Lambda g \chi\, dt, \psi\Big\rangle.$$

This equation holding true for any $\psi \in V$, it is equivalent to

$$\int_0^T g\chi' dt = -\int_0^T \Lambda g \chi\, dt \quad \text{in} \quad V' \quad \text{for any} \quad \chi \in \mathcal{D}(0,T),$$

or in other words

$$g' = \Lambda g \ \text{ in the sense of distributions in } V'.$$

Since $g \in L^2(0,T;V)$, we get that $\Lambda g \in L^2(0,T;V')$ and the above relation precisely means that $g \in H^1(0,T;V')$. We conclude thanks to Lemma 3.3 that $g \in X_T$.

Step 5. Assume first $\varphi \in C_c([0,T);H) \cap L^2(0,T;V) \cap H^1(0,T;V')$. We define $\varphi_\varepsilon(t) := \varphi *_t \rho_\varepsilon$ for a mollifier (ρ_ε) with compact support included in $(0,\infty)$ so that $\varphi_\varepsilon \in C_c^1([0,T);V)$ for any $\varepsilon > 0$ small enough and

$$\varphi_\varepsilon \to \varphi \quad \text{in} \quad C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V').$$

Writing the equation (4.1) for φ_ε and passing to the limit $\varepsilon \to 0$ we get that (4.1) also holds true for φ.

Assume next that $\varphi \in X_T$. We fix $\chi \in C^1(\mathbb{R})$ such that $\operatorname{supp} \chi \subset (-\infty, 0)$, $\chi' \le 0$, $\chi' \in C_c(]-1,0[)$ and $\int_{-1}^0 \chi' = -1$, and we define $\chi_\varepsilon(t) := \chi((t-T)/\varepsilon)$ so that $\varphi_\varepsilon := \varphi \chi_\varepsilon \in C_c([0,T);H)$ and $\chi_\varepsilon \to \mathbf{1}_{[0,T]}$, $\chi'_\varepsilon \to -\delta_T$ as $\varepsilon \to 0$. Equation (4.1) for the test function φ_ε writes

$$-\langle g_0, \varphi(0)\rangle - \int_0^t \chi'_\varepsilon \langle \varphi, g\rangle \, ds = \int_0^T \chi_\varepsilon \big\{ \langle \Lambda g, \varphi\rangle + \langle \varphi', g\rangle \big\} \, ds,$$

and we obtain the variational formulation (3.1) for $t_1 = 0$ and $t_2 = T$ by passing to the limit $\varepsilon \to 0$ in the above equation. □

5. Exercises

Exercice 5.1. *Prove that $f \ge 0$ if $f_0 \ge 0$ for the solution of the parabolic equation* (1.1). *(Hint. Show that the sequence (g_k) defined in step 2 of the proof of the existence part is such that $g_k \ge 0$ for any $k \in \mathbb{N}$).*

Exercice 5.2. *Prove the existence of a solution $g \in X_T$ to the equation*

$$\frac{dg}{dt} = \Lambda g + G \quad \text{in} \quad (0,T), \quad g(0) = g_0, \tag{5.1}$$

for any initial datum $g_0 \in H$ and any source term $G \in L^2(0,T;V')$.
(Ind. Repeat the same proof as for the Theorem 3.2 where for the a priori bound one can use

$$\int_0^T \langle g, G\rangle \, dt \le \frac{\alpha}{2} \int_0^T \|g(t)\|_V^2 \, dt + \frac{1}{2\alpha} \int_0^T \|G(t)\|_{V'}^2 \, dt,$$

and for the approximation scheme one can define

$$\varepsilon^{-1} (g_{k+1} - g_k) = \Lambda g_{k+1} + G_k, \quad G_k := \int_{t_k}^{t_{k+1}} G(s) \, ds).$$

Exercice 5.3. *Generalize the existence and uniqueness result to the PDE equation*

$$\partial_t f = \partial_i (a_{ij} \, \partial_j f) + b_i \, \partial_i f + c f + \int k(t,x,y) \, f(t,y) \, dy + G \tag{5.2}$$

where a_{ij}, b_i, c and k are times dependent coefficients and where a_{ij} is uniformly elliptic in the sense that

$$\forall \, t \in (0,T), \; \forall \, x \in \mathbb{R}^d, \; \forall \, \xi \in \mathbb{R}^d \quad a_{ij}(t,x) \, \xi_i \xi_j \ge \alpha \, |\xi|^2, \quad \alpha > 0. \tag{5.3}$$

More precisely, establish the following result:

Theorem 5.4 (J.L. Lions - the time dependent case)**.** *Assume that*

$$b \in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d)), \quad a, \, c \in L^\infty((0,T) \times \mathbb{R}^d), \quad k \in L^\infty(0,T;L^2(\mathbb{R}^d \times \mathbb{R}^d)),$$

and that a satisfies the uniformly elliptic condition (5.3)*. For any $g_0 \in L^2(\mathbb{R}^d)$ and $G \in L^2(0,T;\, H^{-1}(\mathbb{R}^d))$, there exists a unique variational solution to the Cauchy problem associated to* (5.2) *in the sense that*

$$f \in X_T := C([0,T);L^2) \cap L^2(0,T;H^1) \cap H^1(0,T;H^{-1}),$$

such that for any $\varphi \in X_T$ and any $t \in (0,T)$ there holds

$$\int_{\mathbb{R}^d} g(t)\varphi(t) \, dx = \int_{\mathbb{R}^d} g_0 \varphi(0) \, dx + \int_0^t \int_{\mathbb{R}^d} (G\varphi + g \partial_t \varphi) \, dx ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \{(b_i \, \partial_i f + cf) \, \varphi - a_{ij} \, \partial_j f \partial_i \varphi\} \, dx ds + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} k(t,x,y) \, f(t,y) \, \varphi(s,x) \, dx dy ds \tag{5.4}$$

Exercice 5.5. *Generalize the existence and uniqueness result to the PDE equation* (5.2) *set in an open set* $\Omega \subset \mathbb{R}^d$ *with Dirichlet, Neuman or Robin boundary condition.*

Exercice 5.6. *Let* $\Omega \subset \mathbb{R}^d$ *an open connected set or the torus. We define*

$$H := \{u \in L^2(\mathbb{R}^d)^d;\ div\, u = 0\}, \quad V := \{u \in H^1(\mathbb{R}^d)^d;\ div\, u = 0\}.$$

1) - Prove that for any $u_0 \in H$ *there exists a unique function* $u \in X_T$ *solution of the variational equation*

$$\int_\Omega u(T)\cdot\varphi(T) - \int_\Omega u_0\cdot\varphi(0) = \int_0^T\int_\Omega Du : D\varphi\, dx \quad \forall\, \varphi \in X_T. \tag{5.5}$$

2) (a) - Prove that $T \in \mathcal{D}'(\Omega)$*,* $\nabla T = 0$ *implies* $T = C$.
(b) - Prove the Poincaré-Wirtinger inequality

$$\forall\, v \in H^1(\Omega) \quad \|u - \bar u\|_{L^2} \le C\,\|\nabla u\|_{L^2}, \quad \bar u := \int_\Omega u\, dx.$$

(c) - Assume Ω *bounded and deduce the following inequality*

$$\forall T \in H^{-1}(\Omega),\ T \perp H, \quad \exists p \in L^2(\Omega),\ T = \nabla p,\ \|p\|_{L^2} \le C\,\|T\|_{H^{-1}}.$$

3) (a) - Assume that Ω *is the torus and prove that the solution* u *of* (5.5) *satisfies*

$$\int_\Omega u(T)\cdot\varphi(T) - \int_\Omega u_0\cdot\varphi(0) = \int_0^T\int_\Omega Du : D\varphi\, dx \quad \forall\, \varphi \in L^2(0,T;H^1)\cap C([0,T];L^2).$$

(Ind. Define $\Pi := I + \nabla\,(-\Delta)^{-1}\, div$ *the projector on divergence-free vectors and observe that for any* $\varphi \in H^1(\Omega)$ *and any* $u \in H$ *there holds* $\langle u, \varphi\rangle = \langle u, \Pi\varphi\rangle$*).*
(b) Deduce that there exists a function $p \in L^2((0,T)\times\Omega)$ *such that* u *satisfies*

$$\partial_t u = \Delta u + \nabla p \quad in \quad (0,T)\times\Omega.$$

Exercice 5.7. *(1) Prove the existence of a (weak in the sense of distributions) solution* $f \in L^\infty(0,T;L^2(\mathbb{R}^d))$ *to the first order equation*

$$\partial_t f = a(x)\cdot\nabla f(x) + c(x)\, f(x) + \int_{\mathbb{R}^d} b(y,x)\, f(y)\, dy,$$

with the usual assumptions on a*,* c*,* b *by the vanishing viscosity method: that is by passing to the limit in the familly of equation*

$$\partial_t f_\varepsilon = \varepsilon\Delta f_\varepsilon + a\cdot\nabla f_\varepsilon + c\, f_\varepsilon + \int_{\mathbb{R}^d} b(y,x)\, f_\varepsilon(y)\, dy,$$

as $\varepsilon \to 0$.
(2) Using a similar vanishing viscosity method, prove the existence of a weak solution for positive time to the wave equation

$$\partial^2_{tt} f = \partial^2_{xx} f, \quad f(0,.) = f_0,\ \partial_x f(0,.) = g_0,$$

for any $f_0, g_0 \in L^2(\mathbb{R})$*, and to the Schrödinger equation*

$$i\partial_t f + \Delta_x f = 0, \quad f(0,.) = f_0,$$

for any $f_0 \in L^2(\mathbb{R}^d)$.
(Hint. For the wave equation prove the existence of a solution $(f_\varepsilon, g_\varepsilon)$ *to the system of equations*

$$\partial_t f = g + \varepsilon\partial^2_{xx} f, \quad \partial_t g = \partial^2_{xx} f + \varepsilon\partial^2_{xx} g,$$

and pass to the limit $\varepsilon \to 0$*).*

Exercice 5.8. *We consider the nonlinear McKean-Vlasov equation*

$$\partial_t f = \Lambda[f] := \Delta f + div(F[f] f), \quad f(0) = f_0, \tag{5.6}$$

with

$$F[f] := a * f, \quad a \in W^{1,\infty}(\mathbb{R}^d)^d.$$

1) Prove the a priori estimates

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} \quad \forall\, t \ge 0, \quad \|f(t)\|_{L^2_k} \le e^{Ct}\, \|f_0\|_{L^2_k} \quad \forall\, t \ge 0,$$

for any $k > 0$ and a constant $C := C(k, \|a\|_{W^{1,\infty}}, \|f_0\|_{L^1})$, where we define the weighted Lebesgue space L^2_k by its norm $\|f\|_{L^2_k} := \|f \langle x \rangle^k\|_{L^2}$, $\langle x \rangle := (1 + |x|^2)^{1/2}$.
2) We set $H := L^2_k$, $k > d/2$, and $V := H^1_k$, where we define the weighted Sobolev space H^1_k by its norm $\|f\|^2_{H^1_k} := \|f\|^2_{L^2_k} + \|\nabla f\|^2_{L^2_k}$. Observe that for any $f \in V$ the distribution $\Lambda[f]$ is well defined in V' thanks to the identity

$$\langle \Lambda[f], g \rangle := - \int_{\mathbb{R}^d} (\nabla f + (a * f) f) \cdot \nabla (g \langle x \rangle^{2k})\, dx \quad \forall\, g \in V.$$

(Hint. Prove that $L^2_k \subset L^1$). Write the variational formulation associated to the nonlinear McKean-Vlasov equation. Establish that if moreover the variational solution to the nonlinear McKean-Vlasov equation is nonnegative then it is mass preserving, that is $\|f(t)\|_{L^1} = \|f_0\|_{L^1}$ for any $t \ge 0$. (Hint. Take $\chi_M \langle x \rangle^{-2k}$ as a test function in the variational formulation, with $\chi_M(x) := \chi(x/M)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$).
3) Prove that for any $0 \le f_0 \in H$ and $g \in C([0,T]; H)$ there exists a unique mass preserving variational solution $0 \le f \in X_T$ to the linear McKean-Vlasov equation

$$\partial_t f = \Delta f + div(F[g] f), \quad f(0) = f_0.$$

Prove that the mapping $g \mapsto f$ is a contraction in $C([0,T]; H)$ for $T > 0$ small enough. Conclude to the existence and uniqueness of a global (in time) variational solution to the nonlinear McKean-Vlasov equation.

Exercice 5.9. *Define $H := L^2$, $V := H^1$. For $a \in W^{1,\infty}$, $c \in L^\infty$, $f_0 \in L^2$, we consider a variational solution $f \in X_T$ to the linear parabolic equation*

$$\partial_t f = \Lambda f := \Delta f + a \cdot \nabla f + c f, \quad f(0) = f_0. \tag{5.7}$$

1) Prove that for $\gamma \in C^1(\mathbb{R})$, $\gamma(0) = 0$, $\gamma' \in L^\infty$, there holds $\gamma(f) \in H$ for any $f \in H$ and $\gamma(f) \in V$ for any $f \in V$.
2) Consider an increasing function $\gamma \in C^1(\mathbb{R})$, $\gamma(0) = 0$, $\gamma' \in L^\infty$ and define the functions $\beta, \Gamma \in C^1(\mathbb{R})$ by

$$\beta(s) := s\, \gamma(s), \quad \Gamma(s) := \int_0^s \gamma(\sigma)\, d\sigma.$$

Prove that the variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f(t))\, dx \le \int_{\mathbb{R}^d} \beta(f_0)\, dx + \int_0^t \int_{\mathbb{R}^d} \{c\, \beta(f) - (div\, a)\, \Gamma(f)\}\, dx ds \quad \forall\, t \ge 0.$$

Deduce that for some constant $C := C(a, c, \gamma)$ there holds

$$\int_{\mathbb{R}^d} \beta(f(t))\, dx \le e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\, dx \quad \forall\, t \ge 0.$$

(Hint. Observe that $\Gamma'(s) \le \beta'(s)$ for any $s \ge 0$, $\Gamma'(s) \ge \beta'(s)$ for any $s \le 0$ and use the Gronwall lemma). Deduce that for any $p \in [1, \infty]$ and for some constant $C := C(a, c, p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p} \quad \forall\, t \ge 0.$$

(Hint. Take γ as the even function which vanishes in 0 and is the primitve of

$$\gamma'(s) := s\, \mathbf{1}_{s \in (0,\varepsilon)} + (s^{p-1} - \varepsilon^{p-1} + \varepsilon)\, \mathbf{1}_{s \in (\varepsilon, R)} + (R^{p-1} - \varepsilon^{p-1} + \varepsilon)\, \mathbf{1}_{s \ge R},$$

for some $R > \varepsilon > 0$ and $p \in (1,2]$ in the preceding estimate, and pass to the limit $\varepsilon \to 0$, $R \to \infty$. Generalize to $2 < p < \infty$ by choosing γ adequately. Conclude the proof by passing to the limit $p \to 1$, $p \to \infty$).
3) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (5.7). (Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0,T];L^p)$. Conclude the proof by passing to the limit $p \to \infty$).
4) Extend the above result to an equation with an integral term and/or a source term.
5) Prove the existence of a weak solution to the McKean-Vlasov equation (5.6) for any initial datum $f_0 \in L^1(\mathbb{R}^d)$.
6) Recover the positivity result of exercice 5.1. (Hint. Choose $\beta(s) := s_-$).

Exercice 5.10. *We denote by S_t the semigroup in H generated by a coercive+dissipative operator $\Lambda : V \subset H \to V'$.*
1) Prove that for any $g_0 \in H$ and $\varphi \in V$ the function $t \mapsto (S_t g_0, \varphi)$ belongs to $H^1(0,T)$ and

$$\frac{d}{dt}(S_t g_0, \varphi) = (\Lambda S_t g_0, \varphi) \quad \text{in} \quad H^{-1}(0,T).$$

2) Prove that for any $G \in C([0,T];H)$ and $\varphi \in V$ there holds

$$\frac{d}{dt}\int_0^t (S_{t-s}G(s), \varphi)\, ds = (G(t), \varphi) + \int_0^t (\Lambda S_{t-s}G(s), \varphi)\, ds \quad \text{in} \quad H^{-1}(0,T).$$

3) Establish the Duhamel formula, namely that for $g_0 \in H$ and $G \in C([0,T];H)$ the function

$$g(t) := S_t g_0 + \int_0^t S_{t-s}G(s)\, ds$$

is a weak (make precise the sense) solution to the evolution equation with source term

$$\frac{dg}{dt} = \Lambda g + G \ \text{ on } \ [0,\infty), \quad g(0) = g_0.$$

6. Further results: evolution equation with source term and Duhamel formula

In that last section, we come back on Exercices 5.2 and 5.10.

• For $g_0 \in H$ and $G \in L^2(0,T;V')$ a function $g \in X_T$ is a variational solution to the evolution equation with source term

$$\frac{dg}{dt} = \Lambda g + G \ \text{ on } \ [0,T], \quad g(0) = g_0, \tag{6.8}$$

if that equation holds in V', namely if for any $\varphi \in V$ there holds

$$\frac{d}{dt}(g(t), \varphi) = \langle \Lambda g(t), \varphi\rangle + \langle G(t), \varphi\rangle \quad \text{in the sense of} \quad \mathcal{D}'(0,T;\mathbb{R}).$$

That is equivalent to

$$\langle \frac{d}{dt} g(t), \varphi\rangle = \langle \Lambda g(t), \varphi\rangle + \langle G(t), \varphi\rangle \quad \textit{a.e. } t \in (0,T),$$

or more explicitely

$$\int_0^T (g(t), \varphi)\, \chi'\, dt - (g_0, \varphi)\, \chi(0) = \int_0^T \{\langle \Lambda g(t), \varphi\rangle + \langle G(t), \varphi\rangle\} \chi(t)\, dt$$

for any $\chi \in C^1_c([0,T))$ and $\varphi \in V$. One can then deduce from the last formulation and by density of the separate variables functions $\mathcal{D}(0,T) \otimes V$ into X_T, or just by taking the next formulation as a definition of a variational solution, that for any $\varphi \in X_T$

$$[\langle g, \varphi \rangle]_0^T - \int_0^T \langle \frac{d}{dt}\varphi, g \rangle dt = \int_0^T \langle \Lambda g + G, \varphi \rangle dt. \tag{6.9}$$

• When $g_0 \in H$ and $G \in C([0,T];H)$ one can define thanks to Proposition 3.5 the following function

$$g(t) := e^{\Lambda\, t}\, g_0 + \int_0^t e^{\Lambda\,(t-s)}\, G(s)\, ds. \tag{6.10}$$

We see that $g \in C([0,T];H)$, and using the estimate (2.3)

$$\int_0^T \|e^{\Lambda\, t} f\|_V^2 \, dt \le \frac{e^{2bT}}{2\alpha}\, |f|_H^2,$$

we easily find

$$\int_0^T \|g(t)\|_V^2 \, dt \le C_1(T)\, |g_0|_H^2 + C_2(T) \int_0^T |G(s)|_H^2 \, ds.$$

Finaly, since

$$t \mapsto e^{\Lambda\, t} f \in H^1(0,T;V'), \quad \|\partial_t (e^{\Lambda\, t} f)\|_{L^2(V')} \le C_3(T)\, |f|_H^2,$$

we deduce that $g \in H^1(0,T;V')$ with explicit estimates. We may then compute in $L^2(0,T;V')$

$$\partial_t g = \Lambda e^{\Lambda t} g_0 + G(t) + \int_0^t \Lambda e^{\Lambda\,(t-s)}\, G(s)\, ds = \Lambda g + G(t),$$

(see also Exercice 5.10) and we obtain that $g(t)$ is a variational solution to the evolution equation with source term (6.8).

• When $g_0 \in H$ and $G \in L^2([0,T];V')$ the sense of the Duhamel formula is less clear. One can however prove the existence of a variational solution by just repeating the proof used to tackle the sourceless evolution equation (1.1). More precisely, we consider the following discrete scheme: we build (g_k) iteratively by setting

$$\frac{g_{k+1} - g_k}{\varepsilon} = \Lambda g_{k+1} + G_k, \quad G_k := \int_{t_k}^{t_{k+1}} G(s)\, ds.$$

We compute

$$\begin{aligned} |g_{k+1}|^2 (1 - \varepsilon b) + \varepsilon\, \alpha\, \|g_{k+1}\|_V^2 \quad &\le \quad |g_k|\, |g_{k+1}| + \varepsilon\, \|g_{k+1}\|_V\, \|G_k\|_{V'} \\ &\le \quad \frac{1}{2}\, |g_k|^2 + \frac{1}{2}\, |g_{k+1}|^2 + \varepsilon\, \frac{\alpha}{2}\, \|g_{k+1}\|_V^2 + \frac{\varepsilon}{2\,\alpha}\, \|G_k\|_{V'}^2, \end{aligned}$$

and then

$$|g_{k+1}|^2 (1 - 2\varepsilon b) + \varepsilon\, \alpha\, \|g_{k+1}\|_V^2 \quad \le \quad |g_k|^2 + \frac{\varepsilon}{\alpha}\, \|G_k\|_{V'}^2.$$

We get an estimate on $|g_{k+1}|^2$ which is uniform on k when $k\,\varepsilon \le T$, for $T > 0$ fixed, by using a discrete version of the Gronwall lemma. We conclude as in the proof of Theorem 3.2.

• We may argue in a different way. When $g_0 \in H$ and $G \in C([0,T];H)$ the Duhamel formula (6.10) gives a variational solution to the evolution equation with source term in the sense (6.9). Making the choice $\varphi = g$, we get

$$\begin{aligned} \frac{1}{2} [|g|^2]_0^T \quad &= \quad \int_0^T (\langle \Lambda g, g \rangle + \langle G, g \rangle)\, dt \\ &\le \quad \int_0^T \{-\alpha\, \|g\|_V^2 + b\, |g|^2 + \|G\|_{V'}\, \|g\|_V\}\, dt \\ &\le \quad -\frac{\alpha}{2} \int_0^T \|g\|_V^2 \, dt + \frac{b^2}{2\alpha} \int_0^T \|G\|_{V'}^2 \, dt, \end{aligned}$$

and thanks to the Gronwall lemma, we obtain

$$|g(T)|^2 + \alpha \int_0^T \|g\|_V^2 \, dt \le e^{bT} |g_0|_H^2 + C_T \int_0^T \|G\|_{V'}^2 \, dt.$$

We conlcude to the existence by smoothing the source term G (what it is always possible in the explicit examples $H = L^2$, $V = H^1$) and by passing to the limit in the variational formulation.

7. References

Theorems 3.2 and 5.4 are stated in [1, Théorème X.9] where the quoted reference for a proof is [2].

- [1] Brezis, H. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].
- [2] Lions, J.-L., and Magenes, E. *Problèmes aux limites non homogènes et applications. (3 volumes).* Travaux et Recherches Mathématiques. Dunod, Paris, 1968 & 1969.